Commun. Math. Phys. 84, 329–331 (1982)

Communications in
Mathematical
Physics

Local Extensions in Singular Space-Times II

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Abstract. Previous results of the author are corrected by reformulating them in space-times whose Riemann tensor satisfies a Hölder condition.

1. Introduction

In an earlier paper with this title [2] I showed the existence of local extensions through quasiregular singularities (in the terminology of [5]) by (implicitly) assuming that a spacetime with a C^{k-2} Riemann tensor had a C^k metric. This assumption may not be correct (the alleged proof which I gave in [3] being invalid). The basic results do hold, however, if one uses $C^{k,\alpha}$ conditions (a Hölder condition with exponent α, $0<\alpha<1$, on the k^{th} derivative). The technical tools needed to modify the proof are given in detail in [4]; the aim of the present paper is to outline their application to local extensions.

We first clarify the term "local extension" of a spacetime (M, g), of which there are two definitions in the literature. Here, and in [5], it means an isometry $\phi: U \to M'$, where $U \subset M$ and (M', g') is a spacetime, such that

(i) U contains a curve γ which is incomplete with respect to a generalised affine parameter and inextendible in M.

(ii) $\phi \circ \gamma$ is extendible in M'.

Hawking and Ellis [6], on the other hand, replace (i) by the condition that $\bar{U}$ is not compact in M, and (ii) by the condition that $\phi(U)$ is compact in M'. This has the undesirable consequence that Minkowski space is locally extendible [1]. With the author's definition, certain compact space-times having trapped geodesics may be locally extendible.

2. Results

The theorem will be formulated for the case where γ in the definition above is a broken geodesic. Since any rectifiable curve can be approximated by a broken geodesic this is no loss of generality, and it enables us to give a concrete description of the set U that can be extended. In addition we impose a restriction ((iv) below) that corresponds to the non-spiral condition imposed in the earlier paper [2]. The theorem will only be proved for $C^{0,\alpha}$ Riemann tensors; but it is clear that the procedure extends to $C^{k,\alpha}$.

0010-3616/82/0084/0329/$01.00

Theorem. *Let* $\gamma:(0,1]\to(M,g)$ *be an incomplete curve such that* $\gamma|(t_i,t_{i-1})$ *is a geodesic for some sequence* $0<\ldots<t_n<\ldots<t_1<t_0=1$. *Let there be given a sequence* $(a_n)_{n\in\mathbb{N}}$ *and a frame* $(\underset{i}{E})_{i=0}^3$ *(not necessarily orthonormal) parallely propagated on* γ *such that the following are satisfied:*

(i) $a_n\to 0\,(n\to\infty)$.

(ii) *The map* $T_n:\mathbb{R}^4\ni\xi\mapsto\exp(\xi^i\underset{i}{E}(t_n))$ *is defined and* $1-1$ *in the ball* $B_n^r:=\{\xi|\,\|\xi\|<ra_n\}$ *for all* $r\leqq 1$, *and its image contains both* $\gamma(t_{n-1})$ *and* $\gamma(t_{n+1})$.

(iii) *There is a bounded (with respect to the b-metric [6]) section of the frame bundle over the set* $U:=\bigcup_n T_n(B_n^{1/2})$, *the section containing* $(\underset{i}{E})$, *on which the components* R^m_{npq} *of the Riemann tensor satisfy a uniform Hölder condition with exponent* α.

(iv) *There is a constant* $K>0$ *such that* $\dot\gamma^0(s)>K$ *(where* $\dot\gamma=\dot\gamma^i\underset{i}{E}$*) for all* $s\in(0,1]$.

Then there is a local extension $\phi:U\to M'$, *where* (M',g') *is a spacetime whose Riemann tensor satisfies a Hölder condition with exponent* α.

Proof. (In the following we merely outline certain operations that have been fully described in [4]). Because of (i) and (iii) (the latter implying continuity of R^m_{npq}), for large enough N the dimensionless quantity $a_n\|R\|^{1/2}(n>N)$ is so small that no geodesics constructed in $T_n(B'_m)$ will focus or intersect (cf. §§3.2, 3.3 of [4]). We define $U_n=\bigcup_{i=N}^{n}T_i(B_i^{1/2})$ and proceed by induction on n: we assume U_n to be extended; then append $T_{n+1}(B_{n+1}^{1/2})$; and finally let $n\to\infty$.

Suppose, then, that we have a map $\phi_n:U_n\to M'=\mathbb{R}^4$ which is an isometry for a metric $g_{(n)}$ on M'. The extension of this to ϕ_{n+1} is performed by using a special coordinate system on $T_{n+1}(B_{n+1}^{1/2})$, as follows. Choose four independent vectors $\underset{0}{Z},\ldots,\underset{3}{Z}\in\mathbb{R}^4$ which are such that each of $\underset{i}{Z}^a\underset{a}{E}$ (for $i=0,\ldots,3$) is future-pointing and timelike.

Define

$$\begin{aligned}
P_i &:= \exp(a_{n+1}\underset{i}{Z}^a\underset{a}{E}(t_{n+1})),\\
P'_i &= \exp(-a_{n+1}\underset{i}{Z}^a\underset{a}{E}(t_{n+1})),\\
S_i &:= \{x\in T_{n+1}(B'_n)|d(P_i,x)=d(P'_i,x)\},\\
z^i(x) &:= d(x,S_i)+\int_1^{t_{n+1}} g_{ab}\ddot\gamma^a\underset{i}{z}^b\,\mathrm{d}s.
\end{aligned}$$

Here d denotes the supremum of the geodesic distance for timelike geodesics, the quantities only being defined where this is finite, and g_{ab} are the components of the metric in the frame $(\underset{a}{E})$.

The functions z^i are the required coordinates. They are constructed to achieve the following two properties:

(a) The Hölder constants of components of the metric, and their first derivatives, are bounded in terms of the Hölder constant of the Riemann tensor.

(b) Coordinates defined from adjacent n-values (i.e. in $T_n(B_n^{1/2})$ and $T_{n+1}(B_{n+1}^{1/2})$) approximately agree in the overlap of their domains.

The verification of (a) is very similar to the proof of 6.1 in [4]. We let $\overset{i}{V}$ denote the tangent vector at x to the geodesic from x to S_i that minimises the distance, with proper-time parametrisation. Then it can be shown that $g^{ij} = g(\overset{i}{V}, \overset{j}{V})$ and that

$$g^{ij}{}_{,k} = g_{km}(g(\nabla_{\overset{m}{V}} \overset{i}{V}, \overset{j}{V}) + g(\overset{i}{V}, \nabla_{\overset{m}{V}} \overset{j}{V})).$$

The quantity $\nabla_{\overset{m}{V}} \overset{i}{V}$ can then be computed from the Riemann tensor, using Jacobi's equation for the variation in the geodesic defining $\overset{i}{V}$.

Calculations for (b) are similar, though a bit more involved. We replace the "t_{n+1}" in the definitions of z^i, P_i and P_i' by a variable parameter s, and vary s from t_{n+1} to t_n. As s varies, P_i and P_i' describe curves, whose tangent vectors are related to the tangent vector on γ by Jacobi's equation. These variations in turn give rise to changes in the vector field $\overset{i}{V}$, from which the variation of Z^i as s is varied can be calculated: it turns out to involve only integrals of the Riemann tensor.

In this way we can construct coordinates z^i in $T_{n+1}(B^{1/2}_{n+1})$, and related coordinates – $z^{i'}$, say – in M', using $\phi_{n*}\underset{a}{E}(t_{n+1})$ instead of $\underset{a}{E}(t_{n+1})$. A map $\bar{\phi}_{n+1} : T_{n+1}(B^{1/2}_{n+1}) \to M'$ is then defined by relating points with equal z^i and $z^{i'}$ coordinates.

The extended map ϕ_{n+1} is then defined by "patching" smoothly together ϕ_n and $\bar{\phi}_{n+1}$. Condition (iv) ensures that the image of $\bar{\phi}_{n+1}$ overlaps at most a fixed finite number of the previously constructed coordinate domains, and in these property (b) of the coordinates ensures that the Hölder constants of $g_{ij,k}$ and the Riemann tensor for the metric $(\phi^*_{n+1})^{-1}g$ is of the order of magnitude of that of the Riemann tensor, measured in the frame bundle.

We now have a map ϕ_{n+1} on U_{n+1}, inducing a metric on its image whose Hölder constants are known. The metric $g_{(n+1)}$ is then defined by extending the difference between the induced connections ($\Gamma_{(n)}$ and $\Gamma_{(n+1)}$) from $\phi_{n+1}(U_{n+1})$ to the whole of M', followed by a redefinition of the metric exactly as in [4], §8.3. It is here that the use of Hölder conditions, rather than mere continuity, is essential.

Convergence of $g_{(n)ij,k}$ and $R_{(n)}{}^i{}_{jkl}$ is then easily verified, with the Hölder conditions being respected.

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Communicated by S.-T. Yau
Received October 27, 1981