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Local flux mimetic finite difference methods

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Abstract We develop a local flux mimetic finite difference method for second order elliptic equations with full tensor coefficients on polyhedral meshes. To approximate the velocity (vector variable), the method uses two degrees of freedom per element edge in two dimensions and n degrees of freedom per n-gonal mesh face in three dimensions. To approximate the pressure (scalar variable), the method uses one degree of freedom per element. A specially chosen quadrature rule for the L^2-product of vector-functions allows for a local flux elimination and reduction of the method to a cell-centered finite difference scheme for the pressure unknowns. Under certain assumptions, first-order convergence is proved for both variables and second-order convergence is proved for the pressure. The assumptions are verified on simplicial meshes for a particular quadrature rule that leads to a symmetric method. For general polyhedral meshes, non-symmetric methods are constructed based on quadrature rules that are shown to satisfy some of the assumptions. Numerical results confirm the theory.

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1 Introduction

The mimetic finite difference (MFD) method has been successfully employed for solving problems of continuum mechanics [37], electromagnetics [27], gas dynamics [18], and linear diffusion on polygonal and polyhedral meshes in both the Cartesian and polar coordinates [28,36,41]. The MFD method mimics essential properties of the continuum equations, such as conservation laws, solution symmetries, and the fundamental identities and theorems of vector and tensor calculus. For second-order elliptic problems, which are considered in this paper, the MFD method mimics the Gauss divergence theorem, preserves the null space of the gradient operator, and keeps the adjoint relationship between the gradient and the divergence operators. This leads to a symmetric and locally conservative finite difference scheme. However, the resulting algebraic system is of saddle-point type and couples the velocity (vector variable) and the pressure (scalar variable) unknowns. The elimination of the velocity unknowns results in a cell-centered discretization scheme with a non-local stencil. In this paper we develop a MFD method that can be reduced to a cell-centered scheme with a *local* stencil.

A close relationship between the MFD method and the mixed finite element (MFE) method with the lowest order Raviart–Thomas elements RT_0 [42] has been established in [9]. There, it is shown that the spaces of discrete mimetic degrees of freedom on triangles and quadrilaterals are isomorphic to the RT_0 spaces; moreover, the MFD method can be viewed as a MFE method with a quadrature rule for calculating the velocity mass matrix. This relationship is explored in [9–11] to establish convergence and superconvergence for the MFD approximations on simplicial and quadrilateral elements. An alternative approach for analyzing the MFD method is developed in [16,17], where the error in appropriate discrete norms is estimated. The main advantage of this approach is that the analysis applies to more general polyhedral meshes.

The MFE method, like the MFD method, leads to a saddle-point problem. Several approaches have been proposed to handle this issue, including hybridization [7] and reduction to cell-centered finite differences (CCFD) [4,5,8,39,43,46]. These methods, however, either lead to a more expensive face-centered stencil [7], or limited to diagonal tensor coefficients [8,39,43,46], or exhibit deterioration of convergence for discontinuous coefficients [4,5]. More recent works [29,30,47] establish relationships between the MFE method and the multipoint flux approximation (MPFA) method introduced by the petroleum reservoir simulation community [1,2,21], see also [12,20,34] for closely related methods. The MPFA method, which is formulated as a finite volume method, utilizes sub-edge fluxes and reduces to a cell-centered pressure scheme through local flux elimination. Papers [30] and [47] study the convergence properties of the MPFA method and related MFE methods with broken RT_0 and BDM_1 [14] spaces, respectively. More recently [31] analyzes the convergence of a non-symmetric MPFA method on general quadrilateral grids.

In this paper, we employ a MPFA-type construction and analysis inspired by [16] to develop new cell-centered discretization methods on polyhedral meshes for diffusion problems with full tensor coefficients. To approximate the velocity, we use two degrees of freedom per mesh edge in two dimensions and n degrees of freedom per mesh face (which is n-gon) in three dimensions. To approximate the pressure, we use one degree of freedom per element. This choice of unknowns is similar to that in the MPFA method. A specially chosen quadrature rule for the L^2-product of vector-functions couples the velocity unknowns into small groups around mesh vertices and allows for their *local* elimination, thus reducing the method to a cell-centered finite difference scheme for the pressure unknowns.

Under a few constructive assumptions, we prove first-order convergence for both the velocity and the pressure variables, as well as second-order superconvergence for the pressure variable in discrete L^2 norms. For simplicial meshes, we employ a *symmetric* quadrature rule introduced in [40] and similar to the vector inner product used in [47], and prove that the constructive assumptions hold. These results can be extended to smooth quadrilateral and hexahedral meshes. For general polyhedral meshes, we extend techniques from [17] to construct *non-symmetric* quadrature rules that satisfy a consistency assumption and discuss sufficient conditions on the mesh and tensor coefficient under which the optimal convergence rate can be proved.

The proposed new method compares favorably with existing MFD methods, since it reduces to a cell-centered scheme and is therefore more efficient. On the other hand, our approach is more general than the one in [30,31,47] for MPFA and related methods, since the analysis there relies on finite element techniques and is limited to simplicial and quadrilateral meshes. We estimate the errors directly in the norms of the discrete mimetic spaces without the use of finite element polynomial extensions, except in the pressure superconvergence proof. In terms of computational cost, our method is comparable to finite volume methods [22]. However, the latter are either limited to diagonal tensor coefficients, or require certain orthogonality properties of the grid [23], or need to be augmented with face-centered pressures [24], which increases their cost.

The paper outline is as follows. The new MFD method is developed in Sect. 2. In Sect. 3, we prove convergence estimates for the pressure and the velocity variables under certain assumptions. In Sect. 4, we develop symmetric and non-symmetric methods on simplicial and general grids, respectively. Results of numerical experiments confirming the theoretical estimates are presented in Sect. 5.

2 Mimetic finite difference method

Let X_1 and X_2 be Hilbert spaces and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear operators, $\mathcal{L}_i : X_i \to Y_i$, $i = 1, 2$, which satisfy some fundamental identity:

$$\mathcal{I}(\mathcal{L}_1, \mathcal{L}_2; f_1, f_2) = 0 \quad \forall f_1 \in X_1,\ f_2 \in X_2.$$

Suppose that discrete approximation spaces X_{ih}, Y_{ih}, $i = 1, 2$, and the discrete operator $\mathcal{L}_{1h}$ are given. The idea of the mimetic discretization is to find a discrete operator $\mathcal{L}_{2h}$ such that a discrete analog of the fundamental identity holds, i.e

$$\mathcal{I}_h(\mathcal{L}_{1,h}, \mathcal{L}_{2,h}; f_{1h}, f_{2h}) = 0 \quad \forall f_{1h} \in X_{1h}, f_{2h} \in X_{2h}. \tag{2.1}$$

This implies that operators $\mathcal{L}_1$ and $\mathcal{L}_2$ cannot be discretized independently from each other. For a given $\mathcal{L}_{1,h}$, formula (2.1) is the implicit definition of the operator $\mathcal{L}_{2,h}$.

Let $\Omega \subset \Re^d$ be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain with a Lipschitz continuous boundary and let $f \in L^2(\Omega)$. We consider the second-order elliptic problem written as a system of two first order equations

$$\begin{aligned} \vec{u} &= -\mathcal{K}\nabla p \quad \text{in} \quad \Omega, \\ \operatorname{div} \vec{u} &= f \qquad\quad\ \text{in} \quad \Omega, \end{aligned} \tag{2.2}$$

subject to appropriate boundary conditions. For simplicity, we consider the homogeneous Dirichlet boundary condition (see [26] for more general boundary conditions)

$$p = 0 \qquad \text{on} \quad \partial\Omega. \tag{2.3}$$

The coefficient $\mathcal{K}$ is a symmetric and uniformly positive definite tensor satisfying the following assumption.

[A1] There exist positive constants k_0 and k_1 such that for any $x \in \Omega$

$$k_0 \xi^T \xi \le \xi^T \mathcal{K}(x) \xi \le k_1 \xi^T \xi \qquad \forall \xi \in \Re^d. \tag{2.4}$$

Following the terminology established in porous media applications, we refer to p as the pressure, to $\vec{u}$ as the velocity, and to $\mathcal{K}$ as the permeability tensor.

In the problem of interest (2.2), the operators are $\mathcal{L}_1 = \operatorname{div}$ and $\mathcal{L}_2 = \mathcal{K}\nabla$, the spaces are $X_1 = H(\operatorname{div}; \Omega)$, $Y_1 = L^2(\Omega)$, $X_2 = H_0^1(\Omega)$ and $Y_2 = (L^2(\Omega))^d$, and $\mathcal{I}$ is the Green's formula,

$$\mathcal{I}(\mathcal{L}_1, \mathcal{L}_2; \vec{u}, p) = \int_\Omega p \operatorname{div} \vec{u} \, \mathrm{d}x + \int_\Omega \vec{u} \cdot \mathcal{K}^{-1}(\mathcal{K}\nabla p) \, \mathrm{d}x = 0. \tag{2.5}$$

Note that, due to the homogeneous Dirichlet boundary condition (2.3), there is no boundary integral in the above equation. For other types of boundary conditions, appropriate boundary integrals need to be added to (2.5).

2.1 The local flux MFD method

The MFD method has four steps. First, we define degrees of freedom for the pressure and the velocity. Second, we discretize the easiest of the two operators; depending on the chosen degrees of freedom, it could be either of them. Third, we discretize the Green's formula using quadrature rules for each of the integrals in (2.5). Some minimal approximation properties for these quadratures are required to prove the optimal convergence rates. Fourth, we derive a discrete formula for the other operator.

Let Ω_h be a conforming shape-regular partition [19] of the computational domain into polygonal or polyhedral elements. Let

$$h = \max_{E \in \Omega_h} h_E,$$

where h_E is the diameter of element E. In two dimensions, we split each edge into two *sub-edges* using the mid-point. In three dimensions, we split each face into several quadrilateral *facets*, for instance, by connecting the face center of mass with the edge midpoints. To simplify the presentation, we shall refer to the sub-edges as facets. The boundaries of facets are marked by thin lines in Fig. 1.

We denote the area (volume in 3D) of an element E by $|E|$. Similarly, for each facet e, we denote by $|e|$ its length (area in 3D). Let $\vec{n}_e$ be a unit normal vector assigned to a facet e. To distinguish between faces (edges in 2D) and facets, we shall write $\tilde{e}(e)$, or simply $\tilde{e}$ for the mesh face (edge in 2D) containing facet e. Let $\vec{n}_{\tilde{e}}$ be a unit normal vector assigned to $\tilde{e}$. Finally, let $|\ell|$ denote the length of edge ℓ.

For each element E, we denote by m_E the number of its vertices and by k_E the number of its facets. In the following, ∂E denotes either the union of all edges (faces in 3D) or the union of all facets of E, depending on the context. Let $\vec{n}_E$ be a unit *external* normal vector to ∂E and $\chi_E^e = \vec{n}_e \cdot \vec{n}_E$. Note that χ_E^e is either 1 or -1.

With each vertex of an element E we associate a corner that is formed by all facets sharing the vertex. Let c denote a mesh corner. The angle between facets e and e' forming the corner c is denoted by $\gamma^c_{e,e'}$. The angle between edges ℓ and ℓ' with one common point at the corner c is denoted by $\gamma^c_{\ell,\ell'}$

Let ρ_E be the radius of the largest sphere that can be inscribed in E. Similarly, let $\rho_{\tilde{e}}$ be the radius of the largest disk contained in face $\tilde{e}$. We make the following mesh regularity assumption.

[A2] Partition Ω_h consists of non-degenerate elements and it is shape-regular in the sense that there exist positive constants ρ_* and $\gamma_* < \pi$ independent of h and such that for every $E \in \Omega_h$, every face $\tilde{e}$, corner c and edge ℓ of E,

$$\rho_E,\ \rho_{\tilde{e}},\ |\ell| \geq \rho_* h_E \quad \text{and} \quad \pi - \gamma_* \geq \gamma^c_{e,e'}, \quad \gamma^c_{\ell,\ell'} \geq \gamma_*. \tag{2.6}$$

Remark 2.1 For simplicial elements in can be shown that the angle conditions follow from the conditions on ρ_E and $\rho_{\tilde{e}}$.

The discrete pressure space Q_h consists of one degree of freedom per element approximating the pressure value at the center of mass. The dimension of Q_h equals

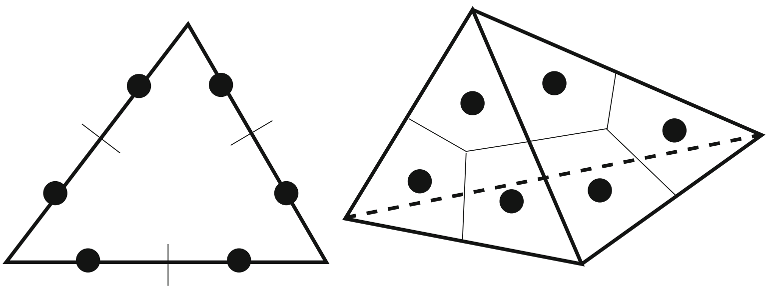

Fig. 1 Velocity degrees of freedom marked by *solid circles* for a triangle ($m_E = 3$, $k_E = 6$) and a tetrahedron ($m_E = 4$, $k_E = 12$). The boundaries of the facets are marked by *thin lines*

the number of elements, N_Q. For $\mathbf{q} \in Q_h$, we shall denote by q_E (or $(\mathbf{q})_E$) its constant value on element E.

The discrete velocity space X_h is similar to the one used in the MPFA methods [1,2,20,21] and consists of one degree of freedom per facet approximating the average normal flux $\frac{1}{|e|}\int_e \vec{u} \cdot \vec{n}_e$. Location of velocity degrees of freedom is shown in Fig. 1. The dimension of X_h equals the total number of facets, N_X. For $\mathbf{v} \in X_h$, we shall denote by $\mathbf{v}_E$ the restriction of $\mathbf{v}$ to element E, and by v_E^e (or $(\mathbf{v})_E^e$) its (constant) value on facet e. We shall write $\mathbf{v}_E \in X_{E,h}$ where $X_{E,h}$ is the restriction of X_h to E. Similarly, $\mathbf{v}_c$ will be the restriction of $\mathbf{v}$ to corner c, and v_c^e (or $(\mathbf{v})_c^e$) will be its value on facet e.

The choice of velocity degrees of freedom as normal fluxes allows for a simple discretization of the divergence operator $\mathcal{DIV} : X_h \to Q_h$. Integrating $\operatorname{div}\vec{u}$ over element E, applying the divergence theorem, and using that u_E^e approximates $\frac{1}{|e|}\int_e \vec{u} \cdot \vec{n}_e$, we let

$$(\mathcal{DIV}\,\mathbf{u})_E = \frac{1}{|E|}\sum_{e\in\partial E} \chi_E^e\,|e|\,u_E^e. \tag{2.7}$$

A similar formula appears in other locally conservative methods, like the finite volume, MPFA, and MFE methods. The essential difference in the proposed method will be in the discretization of the first equation in (2.2).

The following interpolants will be used in the analysis. For any $q \in L^1(\Omega)$, we define $q^I \in Q_h$ such that

$$(q^I)_E = \frac{1}{|E|}\int\limits_E q(x)\,\mathrm{d}x \quad \forall E \in \Omega_h. \tag{2.8}$$

For any bounded domain D, we define the following space:

$$\mathcal{V}(D) = \left\{\vec{v}:\ \vec{v} \in (L^s(D))^d,\ s > 2,\quad \operatorname{div}\vec{v} \in L^2(D)\right\} \tag{2.9}$$

and set $\mathcal{V} = \mathcal{V}(\Omega)$. To define the interpolant in $\mathcal{V}$ we need the following trace result, where $W^{k,p}$ denotes the usual Sobolev space.

Lemma 2.1 *Let $D \subset \Re^d$ be a bounded domain with a Lipschitz continuous boundary and $\mathcal{V}(D)$ be the space defined in (2.9). Furthermore, let*

$$\tilde{s} = \begin{cases} s, & d = 2,\\ \min(s,6), & d = 3,\end{cases}$$

and $1/\tilde{s} + 1/\tilde{s}' = 1$. Then, there exists a unique continuous map

$$\gamma_n : \mathcal{V}(D) \to \left(W^{1/\tilde{s},\tilde{s}'}(\partial D)\right)^*$$

such that for smooth vector functions $\gamma_n \vec{v} = \vec{v} \cdot \vec{n}$. Moreover, the Green's formula

$$\int_{\partial D} \gamma_n \vec{v}\, w \,\mathrm{d}s = \int_D \mathrm{div}\vec{v}\, w \,\mathrm{d}x + \int_D \vec{v} \cdot \nabla w \,\mathrm{d}x$$

holds for all $\vec{v} \in \mathcal{V}(D)$ and $w \in W^{1,\tilde{s}'}(D)$.

Proof The proof is a generalization of the classical normal trace theorem [45]. For $\vec{v} \in \mathcal{V}(D)$, define $\gamma_n \vec{v}$ by

$$\langle \gamma_n \vec{v}, \phi \rangle_{\partial D} := \int_D \mathrm{div}\vec{v}\, w \,\mathrm{d}x + \int_D \vec{v} \cdot \nabla w \,\mathrm{d}x \quad \forall \phi \in W^{1/\tilde{s},\tilde{s}'}(\partial D),$$

where $w \in W^{1,\tilde{s}'}(D)$ is a continuous extension of ϕ in D. By the Sobolev imbedding theorem [3], $w \in L^2(D)$, therefore

$$\begin{aligned} |\langle \gamma_n \vec{v}, \phi \rangle_{\partial D}| &\le C \left(\|\mathrm{div}\vec{v}\|_{L^2(D)} + \|\vec{v}\|_{(L^{\tilde{s}}(D))^d} \right) \|w\|_{W^{1,\tilde{s}'}(D)} \\ &\le C \left(\|\mathrm{div}\vec{v}\|_{L^2(D)} + \|\vec{v}\|_{(L^{\tilde{s}}(D))^d} \right) \|\phi\|_{W^{1/\tilde{s},\tilde{s}'}(\partial D)}, \end{aligned}$$

hence the map $\gamma_n : \mathcal{V}(D) \to (W^{1/\tilde{s},\tilde{s}'}(\partial D))^*$ is continuous. The use of the Green's formula for smooth functions implies that for such functions $\gamma_n \vec{v} = \vec{v} \cdot \vec{n}$. Since the space of smooth vector functions is dense in $\mathcal{V}(D)$ [45], the map $\gamma_n \vec{v}$ is uniquely defined. □

In the following, for a vector $\vec{v} \in \mathcal{V}$ we will use the notation $\vec{v} \cdot \vec{n}$, understanding this in the sense of distributions.

We are now ready to define the interpolant in $\mathcal{V}$. For any $\vec{v} \in \mathcal{V}$, we define $\vec{v}^I \in X_h$ such that

$$(\vec{v}^I)_E^e = \frac{1}{|e|} \int_e \vec{v} \cdot \vec{n}_e \,\mathrm{d}s \quad \forall E \in \Omega_h, \quad \forall e \subset \partial E. \tag{2.10}$$

Note that the edge integral in (2.10) is well defined for any $\vec{v} \in \mathcal{V}$, due to Lemma 2.1, since the functions in $W^{1/\tilde{s},\tilde{s}'}(\partial E)$ can be discontinuous and one can take $\phi = 1$ on e and $\phi = 0$ on $\partial E \backslash e$.

Let us now discretize each integral in the Green's identity (2.5). Introducing $\mathbf{p} = p^I$ and $\mathbf{q} = q^I$ from Q_h, the first integral is approximated with the central-point quadrature rule:

$$\int_\Omega p(x)\, q(x) \,\mathrm{d}x \approx \sum_{E \in \Omega_h} [\mathbf{p},\, \mathbf{q}]_{Q,E} \equiv [\mathbf{p},\, \mathbf{q}]_Q, \quad [\mathbf{p},\, \mathbf{q}]_{Q,E} = |E|\, p_E\, q_E. \tag{2.11}$$

To discretize the second term in (2.5), we introduce $\mathbf{u} = \vec{u}^I$ and $\mathbf{v} = \vec{v}^I$ in X_h and write formally a quadrature rule:

$$\int\limits_{\Omega} \mathcal{K}^{-1}\vec{u}(x) \cdot \vec{v}(x)\,\mathrm{d}x \approx \sum_{E \in \Omega_h} [\mathbf{u},\ \mathbf{v}]_{X,E} \equiv [\mathbf{u},\ \mathbf{v}]_X, \tag{2.12}$$

where

$$[\mathbf{u},\ \mathbf{v}]_{X,E} = \sum_{c \in E} [\mathbf{u},\ \mathbf{v}]_{X,E,c}, \quad [\mathbf{u},\ \mathbf{v}]_{X,E,c} = \sum_{e,e' \in c} (\mathbf{M}_c)_{e,e'}\, u_E^e v_E^{e'}. \tag{2.13}$$

Let $\mathbf{M}_c$ be the matrix with entries $(\mathbf{M}_c)_{e,e'}$. The size of $\mathbf{M}_c$ equals the number of facets that form the corner c. Letting $\langle \cdot\,,\ \cdot \rangle$ be the usual dot product, we have

$$[\mathbf{u},\ \mathbf{v}]_{X,E,c} = \langle \mathbf{M}_c \mathbf{u}_c, \mathbf{v}_c \rangle.$$

Similarly,

$$[\mathbf{u},\ \mathbf{v}]_{X,E} = \langle \mathbf{M}_E \mathbf{u}_E, \mathbf{v}_E \rangle,$$

where $\mathbf{M}_E$ is a matrix of size k_E. It is clear from (2.13) that $\mathbf{M}_E$ is block-diagonal with as many blocks as there are corners in E, having a block $\mathbf{M}_c$ for each corner c. We assume the following.

[A3] For each element E, $\mathbf{M}_E$ is positive definite and there exist two positive constants α_0 and α_1 independent of h such that

$$\alpha_0 |E|\, \xi^T \xi \le \xi^T \mathbf{M}_E\, \xi \le \alpha_1 |E|\, \xi^T \xi \qquad \forall \xi \in \Re^{k_E} \tag{2.14}$$

and

$$\xi^T \mathbf{M}_E^T\, \mathbf{M}_E\, \xi \le \alpha_1^2 |E|^2\, \xi^T \xi \qquad \forall \xi \in \Re^{k_E}. \tag{2.15}$$

Note that (2.14) is equivalent to stating that the symmetric part of $\mathbf{M}_E$, $\mathbf{M}_{E,s} = \frac{1}{2}(\mathbf{M}_E + \mathbf{M}_E^T)$, is positive definite and satisfies the same inequalities. Consequently, $\|\mathbf{M}_{E,s}^{1/2} \xi\| \le \sqrt{\alpha_1 |E|}\, \|\xi\|$, which implies $\|\mathbf{M}_{E,s} \xi\| \le \alpha_1 |E| \|\xi\|$, where $\|\cdot\|$ denotes the Euclidean norm in $\Re^{k_E}$. Condition (2.15) gives a similar bound on $\mathbf{M}_E$, and therefore also bounds the non-symmetric part, $\mathbf{M}_{E,n}$, of matrix $\mathbf{M}_E$:

$$\|\mathbf{M}_{E,n}\, \xi\| = \|\mathbf{M}_E\, \xi - \mathbf{M}_{E,s}\, \xi\| \le \|\mathbf{M}_E\, \xi\| + \|\mathbf{M}_{E,s}\, \xi\| \le 2\alpha_1 |E| \|\xi\|.$$

We approximate $\mathcal{K}$ by a positive definite piecewise constant tensor $\overline{\mathcal{K}}$ that is equal to the mean value $\mathcal{K}_E$ of $\mathcal{K}$ on E. Now, we restrict the admissible set of quadrature rules (2.14)–(2.15) by the following assumption.

[A4] For every E in Ω_h, every linear function q^1, and every $\mathbf{v} \in X_h$ the following discrete Green's formula holds:

$$\left[(\mathcal{K}_E \nabla q^1)^I,\, \mathbf{v}\right]_{X,E} = -\left[(\mathcal{DIV}\,\mathbf{v})_E,\, (q^1)^I\right]_{Q,E} + \sum_{e\in\partial E} \chi_E^e\, |e|\, v_E^e\, q^1(x_e), \tag{2.16}$$

where x_e is the center of mass of e^o, a subset of edge (face in 3D) $\tilde{e}(e)$ satisfying

$$|e^o| \geq \sigma_* |e|, \tag{2.17}$$

where σ_* is a positive constant independent of h.

If the matrix $\mathbf{M}_E$ is symmetric, assumption (2.15) follows from (2.14). In general, we do not assume symmetry of matrix $\mathbf{M}_E$. This allows us to formulate and analyze new MPFA-type MFD methods. It also allows to consider problems with non-symmetric tensor $\mathcal{K}$. A symmetric matrix $\mathbf{M}_E$ satisfying assumptions **A3** and **A4** can be built for simplicial meshes (Sect. 4.1). The analysis there can be extended to uniformly refined quadrilateral and hexahedral meshes. The construction of non-symmetric matrices $\mathbf{M}_E$ satisfying assumptions **A3** and **A4** on general polyhedral grids is discussed in Sect. 4.2.

Assumption **A4** resembles the one used in [17]; however, the point x_e is no longer the center of mass of e and only (2.17) is required to hold. This provides more flexibility in the construction of the matrix $\mathbf{M}_E$. In Sect. 3, we show that assuming (2.17) is enough to prove optimal convergence estimates.

With the discrete divergence and quadrature rules for approximating L^2 inner products defined, the discrete gradient operator is derived from the discrete Green's formula (cf. (2.5))

$$[\mathbf{q},\, \mathcal{DIV}\,\mathbf{v}]_Q + [\mathcal{GRAD}\,\mathbf{q},\, \mathbf{v}]_X = 0 \quad \forall\, \mathbf{q} \in Q_h,\ \forall\, \mathbf{v} \in X_h. \tag{2.18}$$

Note that the homogeneous Dirichlet boundary condition (2.3) is incorporated into the definition of operator $\mathcal{GRAD}$. Other types of boundary conditions could lead to an additional boundary integral in (2.18) [26].

Lemma 2.2 *If* (2.14) *in assumption* **A3** *holds, then formula* (2.18) *gives a unique definition for operator* $\mathcal{GRAD}$.

Proof Let $\mathbf{D}$ and $\mathbf{M}$ be the matrices associated with quadrature rules (2.11) and (2.12) through the usual dot product $\langle\cdot\,,\,\cdot\rangle$:

$$[\mathbf{p},\, \mathbf{q}]_Q = \langle \mathbf{D}\,\mathbf{p},\, \mathbf{q}\rangle \quad \text{and} \quad [\mathbf{u},\, \mathbf{v}]_X = \langle \mathbf{M}\,\mathbf{u},\, \mathbf{v}\rangle. \tag{2.19}$$

Here $\mathbf{D}$ is a diagonal matrix, $\mathbf{D} = \mathrm{diag}\{|E_1|, \ldots, |E_{N_Q}|\}$, and $\mathbf{M}$ is a $N_X \times N_X$ matrix assembled from the element matrices $\mathbf{M}_E$. Formula (2.18) is equivalent to

$$\mathcal{DIV}^T\,\mathbf{D} + \mathbf{M}\,\mathcal{GRAD} = 0,$$

where, by abuse of notation, $\mathcal{DIV}$ and $\mathcal{GRAD}$ denote the matrices associated with the discrete operators. Since

$$\langle \mathbf{M}\,\mathbf{u},\ \mathbf{v}\rangle = \sum_{E\in\Omega_h} \langle \mathbf{M}_E\,\mathbf{u}_E,\ \mathbf{v}_E\rangle,$$

the left inequality in (2.14) implies that $\mathbf{M}$ is nonsingular. Therefore $\mathcal{GRAD}$ is defined uniquely as

$$\mathcal{GRAD} = -\mathbf{M}^{-1}(\mathcal{DIV})^T\mathbf{D}. \tag{2.20}$$

□

In Sect. 2.3 we show that the operator $\mathcal{GRAD}$ has a local stencil. The local flux MFD method reads: find $\mathbf{u}_h \in X_h$ and $\mathbf{p}_h \in Q_h$ such that

$$\begin{aligned} \mathbf{u}_h &= -\mathcal{GRAD}\,\mathbf{p}_h, \\ \mathcal{DIV}\,\mathbf{u}_h &= \mathbf{f}, \end{aligned} \tag{2.21}$$

where $\mathbf{f} = f^I$.

2.2 Well-posedness of the method

The following lemma is an immediate result of the definition of matrix $\mathbf{M}_E$.

Lemma 2.3 *If* (2.14) *in assumption* **A3** *holds, then*

$$\alpha_0|E| \sum_{e\in\partial E} |v_E^e|^2 \le [\mathbf{v}_E,\ \mathbf{v}_E]_{X,E} \le \alpha_1|E| \sum_{e\in\partial E} |v_E^e|^2 \tag{2.22}$$

for any $E \in \Omega_h$ *and any* $\mathbf{v}_E \in X_{E,h}$.

The definitions (2.8) and (2.10) of the interpolants and the divergence theorem imply the following simple result.

Lemma 2.4 *Let* $\vec{v} \in \mathcal{V}$. *Then for every element* $E \in \Omega_h$, *we have*

$$(\mathcal{DIV}\,\vec{v}^I)_E = (\operatorname{div}\vec{v})_E^I. \tag{2.23}$$

We are now ready to prove the solvability of (2.21).

Lemma 2.5 *Let* (2.14) *in assumption* **A3** *hold. Then, the discrete problem* (2.21) *has a unique solution.*

Proof It is convenient to rewrite (2.21) in the equivalent variational form

$$\begin{aligned} [\mathbf{u}_h, \mathbf{v}]_X - [\mathbf{p}_h, \mathcal{DIV}\,\mathbf{v}]_Q &= 0, \qquad \forall\,\mathbf{v} \in X_h, \\ [\mathcal{DIV}\,\mathbf{u}_h, \mathbf{q}]_Q &= [\mathbf{f}, \mathbf{q}]_Q, \quad \forall\,\mathbf{q} \in Q_h, \end{aligned} \tag{2.24}$$

where we have used the discrete Green's formula (2.18). Since (2.24) is a square system, it suffices to show uniqueness for the homogeneous problem. Letting $\mathbf{f} = 0$, $\mathbf{v} = \mathbf{u}_h$, and $\mathbf{q} = \mathbf{p}_h$, we conclude that $[\mathbf{u}_h, \mathbf{u}_h]_X = 0$. Hence, due to (2.22), $\mathbf{u}_h = 0$.

Next, we construct $\mathbf{v} \in X_h$ such that $\mathcal{DIV}\,\mathbf{v} = \mathbf{p}_h$. Let p_h be a piecewise constant function such that $p_h|_E = (\mathbf{p}_h)_E$. Let B be an open ball containing Ω and let $\tilde{p}_h$ be the extension of p_h by zero on B. Consider the auxiliary problem

$$\begin{aligned} \Delta\phi &= \tilde{p}_h \quad \text{in} \quad B, \\ \phi &= 0 \quad \text{on} \quad \partial B. \end{aligned} \tag{2.25}$$

Since $\tilde{p}_h \in L^2(B)$ and ∂B is smooth, by elliptic regularity [35], $\phi \in H^2(B)$. Therefore $\nabla\phi \in (H^1(\Omega))^d \subset \mathcal{V}$, then $(\nabla\phi)^I$ is well defined. Using (2.23), we have that

$$\mathcal{DIV}\,\mathbf{v} = \mathcal{DIV}\,(\nabla\phi)^I = (\operatorname{div}\nabla\phi)^I = (p_h)^I = \mathbf{p}_h.$$

Therefore taking $\mathbf{v} = (\nabla\phi)^I$ in (2.24) implies $[\mathbf{p}_h, \mathbf{p}_h]_Q = 0$ and $\mathbf{p}_h = 0$. □

2.3 Reduction to a cell-centered scheme

The matrix $\mathbf{M}$ introduced in Sect. 2.1 satisfies

$$\langle \mathbf{M}\mathbf{u}, \mathbf{v}\rangle = \sum_{E\in\Omega_h}\sum_{c\in E}\langle \mathbf{M}_c\mathbf{u}_c, \mathbf{v}_c\rangle;$$

therefore $\mathbf{M}$ is a block-diagonal matrix with as many blocks as there are mesh nodes. Each block of $\mathbf{M}$ has nonzero entries that describe the interaction of neighboring velocity unknowns on all facets sharing a mesh node. In two dimensions, each block is a tridiagonal cyclic matrix. For instance, the block corresponding to the interior node shown on the left picture in Fig. 2 is a 5×5 matrix.

Recall the formula for $\mathcal{GRAD}$ (2.20). Due to the special structure of matrix $\mathbf{M}$, its inverse is also a block-diagonal matrix and can be easily computed. As the product of sparse matrices, the discrete gradient operator is also sparse (contrary to other MFD methods). Substituting the first equation in (2.21) into the second one, we get a cell-centered discretization with a local stencil:

$$-\,\mathcal{DIV}\;\mathcal{GRAD}\,\mathbf{p}_h = \mathbf{f}. \tag{2.26}$$

Examples of the stencils for the operators $\mathcal{GRAD}$ and $\mathcal{DIV}\;\mathcal{GRAD}$ are shown in Fig. 2a, b, respectively.

The matrix for problem (2.26) appears on the right in the identity

$$\left[-\mathcal{DIV}\;\mathcal{GRAD}\,\mathbf{p}, \mathbf{q}\right]_Q = \left\langle \mathbf{D}\,\mathcal{DIV}\,\mathbf{M}^{-1}(\mathcal{DIV})^T\mathbf{D}\,\mathbf{p}, \mathbf{q}\right\rangle.$$

As shown in the proof of Lemma 2.5, $\mathcal{DIV}^T\mathbf{q} = 0$ implies $\mathbf{q} = 0$. Therefore, the resulting algebraic system has a positive definite matrix when all $\mathbf{M}_E$ satisfy (2.14)

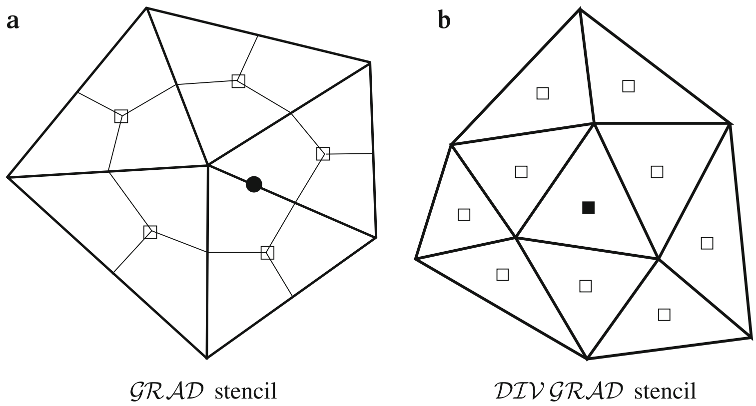

Fig. 2 Stencils for operators $\mathcal{GRAD}$ and $\mathcal{DIV}\,\mathcal{GRAD}$ on a triangular mesh. On the *left*, the equation for the velocity unknown at the position marked by a *solid circle* involves pressure unknowns at the positions marked by *squares*. On the *right*, the pressure marked by a *solid square* is coupled with the pressures marked by squares

in assumption **A3**. When the matrices $\mathbf{M}_E$ are symmetric, the coefficient matrix of problem (2.26) is symmetric and positive definite.

3 Convergence analysis

Throughout the paper, C and C_i denote generic positive constants which are independent of h but may depend on various constants appearing in assumptions **A1**–**A7** and (3.2). To prove optimal convergence estimates we need additional assumptions on the tensor $\mathcal{K}$.

[A5] We assume that $\mathcal{K} \in (W^{1,\infty}(E))^{d\times d}$ for all $E \in \Omega_h$ and that $\max_{E\in\Omega_h} \|\mathcal{K}\|_{1,\infty,E}$ is uniformly bounded independently of h.

In the above $\|\mathcal{K}\|_{1,\infty,E} = \max\limits_{1\le i,j\le d} \|\mathcal{K}_{ij}\|_{W^{1,\infty}(E)}$. The Taylor's theorem and assumption **A5** imply that

$$\max_{x\in E} \left|\mathcal{K}_{ij}(x) - \mathcal{K}_{E,ij}\right| \le C\, h_E\, \|\mathcal{K}_{ij}\|_{W^{1,\infty}(E)}, \quad 1 \le i, j \le d. \tag{3.1}$$

Using assumption **A1** and (3.1), it can also be shown that there exists a constant $C_{\mathcal{K}}$ depending on k_0 and the constant in (3.1) such that

$$\max_{x\in E} \left|\mathcal{K}^{-1}_{ij}(x) - \mathcal{K}^{-1}_{E,ij}\right| \le C_{\mathcal{K}}\, h_E \|\mathcal{K}\|_{1,\infty,E}, \quad 1 \le i, j \le d. \tag{3.2}$$

We shall use repeatedly the following approximation result [13, Lemma 4.3.8]. For every element E, if $\phi \in W^{m+1,p}(E)$, $p \ge 1$, there exists ϕ^m, a polynomial of degree at most m, such that

$$|\phi - \phi^m|_{W^{k,p}(E)} \le C h_E^{m+1-k} |\phi|_{W^{m+1,p}(E)}, \quad k = 0, \ldots, m+1. \tag{3.3}$$

In particular, if $\phi \in H^{1+q}(E)$, $0 \le q \le 1$, then there exists a linear function ϕ_E^1 such that

$$\|\phi - \phi_E^1\|_{L^2(E)} \le C\, h_E^{1+q}\, |\phi|_{H^{1+q}(E)}, \quad \|\phi - \phi_E^1\|_{H^1(E)} \le C\, h_E^{q}\, |\phi|_{H^{1+q}(E)}. \tag{3.4}$$

We will also make use of the trace inequality [6]:

$$\|\chi\|^2_{L^2(\tilde{e})} \le C \left(h_E^{-1} \|\chi\|^2_{L^2(E)} + h_E\, |\chi|^2_{H^1(E)} \right) \quad \forall \chi \in H^1(E), \tag{3.5}$$

where $\tilde{e}$ is any edge (face in 3D) of E. The constant C depends only on the constants appearing in assumption **A2**. Applying (3.5) to the difference $\phi - \phi_E^1$ and using (3.4), we have

$$\left\| \phi - \phi_E^1 \right\|^2_{L^2(\tilde{e})} \le C\, h_E^{1+2q}\, |\phi|^2_{H^{1+q}(E)}, \quad 0 \le q \le 1. \tag{3.6}$$

The estimate also holds for any facet e of E.

Let

$$\tilde{\mathcal{V}}(E) = \left\{ \vec{v} :\ \vec{v} \in (H^{\tilde{q}}(E))^d, \quad 0 < \tilde{q} \le 1, \quad \operatorname{div} \vec{v} \in L^2(E) \right\}.$$

It was shown in [38] that the map

$$\vec{v} \cdot \vec{n} : \tilde{\mathcal{V}}(E) \to H^{\tilde{q}-1/2}(\tilde{e}), \quad 0 < \tilde{q} \le 1,$$

is continuous, where $H^{\tilde{q}-1/2}(\tilde{e}) = (H^{1/2-\tilde{q}}(\tilde{e}))^*$. The following result is proved in Appendix A using a scaling argument.

Lemma 3.1 *Let $0 < \tilde{q} \le 1$ and $\vec{v} \in \tilde{\mathcal{V}}(E)$. Then, for any face $\tilde{e}$ of E, we have*

$$\|\vec{v} \cdot \vec{n}\|^2_{H^{\tilde{q}-1/2}(\tilde{e})} \le C \left(h_E^{-1} \|\vec{v}\|^2_{(L^2(E))^d} + h_E^{2\tilde{q}-1}\, |\vec{v}|^2_{(H^{\tilde{q}}(E))^d} + h_E \|\operatorname{div}\vec{v}\|^2_{L^2(E)} \right). \tag{3.7}$$

The error estimates are derived in the mesh dependent norms:

$$|||\mathbf{q}|||_Q = [\mathbf{q},\ \mathbf{q}]_Q^{1/2} \quad \text{and} \quad |||\mathbf{v}|||_X = [\mathbf{v},\ \mathbf{v}]_X^{1/2} \equiv \left\langle \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)\mathbf{v},\ \mathbf{v} \right\rangle^{1/2}.$$

It is easy to see that $|||\mathbf{v}|||_X$ is indeed a norm, since (2.14) in assumption **A3** implies that $\mathbf{M}_s = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)$ is symmetric and positive definite. Moreover, if both (2.14) and (2.15) hold, the following Cauchy-Schwarz type inequality is true:

$$[\mathbf{u},\ \mathbf{v}]_X \le \frac{\alpha_1}{\alpha_0} |||\mathbf{u}|||_X |||\mathbf{v}|||_X \quad \forall\, \mathbf{u},\ \mathbf{v} \in X_h. \tag{3.8}$$

3.1 Optimal velocity estimate

In this section, we prove the optimal estimate for the velocity.

Theorem 3.1 *Let pairs* $(p, \vec{u})$ *and* $(\mathbf{p}_h, \mathbf{u}_h)$ *be the solutions to problems* (2.2)–(2.3), and (2.21), *respectively. Assume that* $p \in H^{1+q}(\Omega)$, $0 \le q \le 1$, *and* $\vec{u} \in (H^{\tilde{q}}(\Omega))^d$, $0 < \tilde{q} \le 1$. *Under assumptions* **A1**–**A5**, *there exists a constant* C *independent of* h *such that*

$$\|\!|\vec{u}^I - \mathbf{u}_h\|\!|_X \le C\left(h^q\,|p|_{H^{1+q}(\Omega)} + h^{\tilde{q}}\,|\vec{u}|_{(H^{\tilde{q}}(\Omega))^d} + h\|f\|_{L^2(\Omega)}\right).$$

Proof Let $\mathbf{v} = \vec{u}^I - \mathbf{u}_h$. Lemma 2.4 implies that

$$\mathcal{DIV}\,\mathbf{v} = \mathcal{DIV}\,(\vec{u}^I - \mathbf{u}_h) = f^I - f^I = 0.$$

Then, using (2.24), we get

$$\|\!|\vec{u}^I - \mathbf{u}_h\|\!|_X^2 = \left[\vec{u}^I - \mathbf{u}_h,\ \mathbf{v}\right]_X = \left[\vec{u}^I,\ \mathbf{v}\right]_X - [\mathbf{p}_h, \mathcal{DIV}\,\mathbf{v}]_Q = \left[\vec{u}^I,\ \mathbf{v}\right]_X.$$

Let p^1 be a discontinuous piecewise linear function satisfying (3.4) on every element E. Adding and subtracting $(\overline{\mathcal{K}}\nabla p^1)^I$, we have

$$\left[\vec{u}^I - \mathbf{u}_h,\ \mathbf{v}\right]_X = \left[\vec{u}^I + (\overline{\mathcal{K}}\nabla p^1)^I,\ \mathbf{v}\right]_X - \left[(\overline{\mathcal{K}}\nabla p^1)^I,\ \mathbf{v}\right]_X \equiv I_1 + I_2.$$

Using the Cauchy–Schwarz inequality (3.8), (2.22), (3.7), and assumptions **A1** and **A2**, we bound I_1 as follows:

$$\begin{aligned}
|I_1| &\le \frac{\alpha_1}{\alpha_0}\|\!|(\vec{u} + \overline{\mathcal{K}}\nabla p^1)^I\|\!|_X \|\!|\mathbf{v}\|\!|_X \\
&\le \frac{\alpha_1}{\alpha_0}\left(\alpha_1 \sum_{E\in\Omega_h}\sum_{e\in\partial E}\left[((\vec{u} + \overline{\mathcal{K}}\nabla p^1)^I)_E^e\right]^2 |E|\right)^{1/2} \|\!|\mathbf{v}\|\!|_X \\
&= \frac{\alpha_1}{\alpha_0}\left(\alpha_1 \sum_{E\in\Omega_h}\sum_{e\in\partial E}\left[\frac{1}{|e|}\int_e \left(\vec{u} + \overline{\mathcal{K}}\nabla p^1\right)\cdot \vec{n}_e\,\mathrm{d}s\right]^2 |E|\right)^{1/2} \|\!|\mathbf{v}\|\!|_X \\
&\le C\left(\sum_{E\in\Omega_h}\left[h_E^{-1}\|\vec{u} + \overline{\mathcal{K}}\nabla p^1\|^2_{(L^2(E))^d} + h_E^{2\tilde{q}-1}\,|\vec{u}|^2_{(H^{\tilde{q}}(E))^d}\right.\right. \\
&\qquad \left.\left. + h_E\|\mathrm{div}\vec{u}\|^2_{L^2(E)}\right] h_E\right)^{1/2} \|\!|\mathbf{v}\|\!|_X, \quad 0 < \tilde{q} \le 1. \qquad (3.9)
\end{aligned}$$

For the first term on the right above we have, using assumption **A1**, (3.4) and (3.1),

$$\begin{aligned}\left\|\vec{u}+\overline{\mathcal{K}}\nabla p_E^1\right\|_{(L^2(E))^d} &\le \left\|\mathcal{K}\nabla(p-p_E^1)\right\|_{(L^2(E))^d}+\left\|(\mathcal{K}-\overline{\mathcal{K}})\nabla p_E^1\right\|_{(L^2(E))^d}\\ &\le C\left(h_E^q|p|_{H^{1+q}(E)}+h_E\|\nabla p_E^1\|_{(L^2(E))^d}\right)\\ &\le C\left(h_E^q|p|_{H^{1+q}(E)}+h_E|p|_{H^1(E)}\right),\quad 0\le q\le 1,\quad (3.10)\end{aligned}$$

where in the last inequality we used that (3.4) implies

$$\|\nabla p_E^1\|_{(L^2(E))^d}\le\|\nabla p\|_{(L^2(E))^d}+\|\nabla(p-p_E^1)\|_{(L^2(E))^d}\le C|p|_{H^1(E)}.$$

Inequality (3.10), combined with (3.9), implies

$$|I_1|\le C\left(h^q|p|_{H^{1+q}(\Omega)}+h^{\tilde{q}}|\vec{u}|_{(H^{\tilde{q}}(\Omega))^d}+h\|f\|_{L^2(\Omega)}\right)|||\mathbf{v}|||_X,\quad 0\le q\le 1,\ \ 0<\tilde{q}\le 1. \tag{3.11}$$

To estimate I_2, we use assumption **A4** and $\mathcal{DIV}\,\mathbf{v}=0$ to obtain

$$I_2=-\sum_{E\in\Omega_h}\sum_{e\in\partial E}\chi_E^e\,|e|\,p_E^1(x_e)\,v_E^e.$$

Recall that the point x_e is the mid-point of e^o, a subset of edge (face in 3D) $\tilde{e}(e)$, such that (2.17) holds. For the linear function p_E^1, we get

$$p_E^1(x_e)=\frac{1}{|e^o|}\int_{e^o}p_E^1(s)\,\mathrm{d}s.$$

Using the continuity of p, (2.17), the approximation result (3.6), and (2.22), we have

$$\begin{aligned}|I_2|&=\left|\sum_{E\in\Omega_h}\sum_{e\in\partial E}\chi_E^e\,v_E^e\,\frac{|e|}{|e^o|}\int_{e^o}(p_E^1-p)\,\mathrm{d}s\right|\\ &\le\sigma_*^{-1}\sum_{E\in\Omega_h}\sum_{e\in\partial E}|e|^{1/2}|v_E^e|\,\|p_E^1-p\|_{L^2(e^o)}\\ &\le C\sum_{E\in\Omega_h}h_E^q\left(|E|\sum_{e\in\partial E}|v_E^e|^2\right)^{1/2}|p|_{H^{1+q}(E)}\\ &\le C\,h^q\,|p|_{H^{1+q}(\Omega)}\,|||\mathbf{v}|||_X,\quad 0\le q\le 1.\end{aligned}\tag{3.12}$$

Combining the estimates for I_1 and I_2, we prove the assertion of the theorem. □

3.2 Optimal pressure estimate

To prove optimal convergence for the pressure variable, we first show that an *inf–sup* condition holds. Let us define the mesh dependent H_{div} norm:

$$|||\mathbf{v}|||_{div}^2 = |||\mathbf{v}|||_X^2 + |||\mathcal{DIV}\,\mathbf{v}|||_Q^2.$$

Lemma 3.2 *If assumption* **A2** *and* (2.14) *in assumption* **A3** *hold, then there exists a positive constant* β *independent of* h *such that for any* $\mathbf{q} \in Q_h$

$$\sup_{\mathbf{v}\in X_h,\ \mathbf{v}\neq 0} \frac{[\mathcal{DIV}\,\mathbf{v},\ \mathbf{q}]_Q}{|||\mathbf{v}|||_{div}} \geq \beta |||\mathbf{q}|||_Q. \tag{3.13}$$

Proof Let $\mathbf{q} \in Q_h$ and let q_h be the piecewise-constant function which is equal to $(\mathbf{q})_E$ on E. We shall construct $\vec{v} \in (H^1(\Omega))^d$ such that $\operatorname{div}\vec{v} = q_h$ and

$$\|\vec{v}\|_{(H^1(\Omega))^d} \ \leq\ C_1 \|q_h\|_{L^2(\Omega)}, \tag{3.14}$$

where C_1 is a positive constant independent of h. Let $\phi \in H^2(B)$ be the solution to the auxiliary problem (2.25) from Lemma 2.5, but with a right hand side $\tilde{q}_h$, the extension of q_h by zero on B. Let $\vec{v} = \nabla\phi$. By construction $\operatorname{div}\vec{v} = q_h$ in Ω and by elliptic regularity [35]

$$\|\vec{v}\|_{(H^1(\Omega))^d} \leq \|\vec{v}\|_{(H^1(B))^d} \leq C_1\|\tilde{q}_h\|_{L^2(B)} = C_1\|q_h\|_{L^2(\Omega)},$$

implying (3.14).

Let $\mathbf{v} = \vec{v}^I$. Using (2.22), (3.5), and assumption **A2**, we get

$$\begin{aligned}
[\mathbf{v}, \mathbf{v}]_{X,E} &\leq \alpha_1\,|E| \sum_{e\in\partial E} \left|v_E^e\right|^2 \\
&\leq C \sum_{e\in\partial E} \frac{|E|}{|e|} \left(\left(h_E^{-1}\|\vec{v}\|^2_{(L^2(E))^d} + h_E|\vec{v}|^2_{(H^1(E))^d}\right)\right) \\
&\leq C \sum_{e\in\partial E} \left(\|\vec{v}\|^2_{(L^2(E))^d} + h_E^2\,|\vec{v}|^2_{(H^1(E))^d}\right) \\
&\leq C_2\|\vec{v}\|^2_{(H^1(E))^d}.
\end{aligned} \tag{3.15}$$

Therefore, using (3.14),

$$|||\mathbf{v}|||_X^2 \leq C_2\|\vec{v}\|^2_{(H^1(\Omega))^d} \leq C_1^2 C_2 |||\mathbf{q}|||_Q^2.$$

Further, Lemma 2.4 implies

$$\mathcal{DIV}\,\mathbf{v} = (\operatorname{div}\vec{v})^I = q_h^I = \mathbf{q}.$$

The last two estimates imply that

$$|||\mathbf{v}|||_{div} \leq \sqrt{1 + C_1^2 C_2}\, |||\mathbf{q}|||_Q,$$

thus the assertion of the lemma follows with $\beta = 1/\sqrt{1 + C_1^2 C_2}$. □

We will need the following result.

Lemma 3.3 *Let assumption* **A2** *hold. For any element E and any $\vec{v} \in (H^1(E))^d$, let $\vec{v}_0$ be its L^2 projection on the space of constant vector functions on E. Then there exists a constant C independent of h such that*

$$|||\vec{v}^I - \vec{v}_0^I|||_{X,E} \leq C\, h_E\, |\vec{v}|_{(H^1(E))^d}. \tag{3.16}$$

Proof The proof follows from the argument used in the derivation of (3.15) and the L^2 projection bound

$$\|\vec{v} - \vec{v}_0\|_{(L^2(E))^d} \leq C h_E |\vec{v}|_{(H^1(E))^d}, \tag{3.17}$$

which follows from (3.3). □

Theorem 3.2 *Let $(p, \vec{u})$ and $(\mathbf{p}_h, \mathbf{u}_h)$ be the solutions to problems (2.2)–(2.3) and (2.21), respectively. Assume that $p \in H^{1+q}(\Omega)$, $0 \leq q \leq 1$, and $\vec{u} \in (H^{\tilde{q}}(\Omega))^d$, $0 < \tilde{q} \leq 1$. Under assumptions* **A1**–**A5**, *there exists a constant C independent of h such that*

$$|||p^I - \mathbf{p}_h|||_Q \leq C \left(h^q\, |p|_{H^{1+q}(\Omega)} + h^{\tilde{q}}\, |\vec{u}|_{(H^{\tilde{q}}(\Omega))^d} + h \|f\|_{L^2(\Omega)} \right).$$

Proof Using Lemma 3.2, we have

$$|||p^I - \mathbf{p}_h|||_Q \leq \frac{1}{\beta} \sup_{\mathbf{v} \in X_h, \mathbf{v} \neq 0} \frac{[\mathcal{DIV}\, \mathbf{v},\ p^I - \mathbf{p}_h]_Q}{|||\mathbf{v}|||_{\text{div}}}. \tag{3.18}$$

To estimate the nominator, we first add and subtract $(p^1)^I$ where p^1 is the discontinuous piecewise linear approximation to p satisfying (3.4), and then apply assumption **A4**:

$$\begin{aligned}
\left[\mathcal{DIV}\, \mathbf{v},\ p^I - \mathbf{p}_h\right]_Q &= \left[\mathcal{DIV}\, \mathbf{v},\ (p - p^1)^I\right]_Q + \left[\mathcal{DIV}\, \mathbf{v},\ (p^1)^I\right]_Q - [\mathbf{u}_h,\ \mathbf{v}]_X \\
&= \left[\mathcal{DIV}\, \mathbf{v},\ (p - p^1)^I\right]_Q + \sum_{E \in \Omega_h} \sum_{e \in \partial E} \chi_E^e\, |e|\, p_E^1(x_e) v_E^e \\
&\quad - \sum_{E \in \Omega_h} \left[(\mathcal{K}_E \nabla p_E^1)^I,\ \mathbf{v}\right]_{X,E} - [\mathbf{u}_h,\ \mathbf{v}]_X \\
&\equiv I_3 + I_4 - I_5 - I_6.
\end{aligned}$$

The term I_3 is estimated using (3.4):

$$|I_3| \leq Ch^{1+q} \, |\!|\!|\mathbf{v}|\!|\!|_{div} \, |p|_{H^{1+q}(\Omega)}, \quad 0 \leq q \leq 1. \tag{3.19}$$

The second term is estimated as the similar term in the proof of Theorem 3.1:

$$|I_4| \leq C\, h^q \, |\!|\!|\mathbf{v}|\!|\!|_X \, |p|_{H^{1+q}(\Omega)}, \quad 0 \leq q \leq 1. \tag{3.20}$$

The last two terms are treated by adding and subtracting $\vec{u}^I$:

$$I_5 + I_6 = \left[(\overline{\mathcal{K}}\nabla p^1)^I + \vec{u}^I, \, \mathbf{v} \right]_X - \left[\vec{u}^I - \mathbf{u}_h, \, \mathbf{v} \right]_X \equiv I_{56}^a + I_{56}^b.$$

The first term is the same as term I_1 in the proof of Theorem 3.1; therefore

$$|I_{56}^a| \leq C \left(h^q |p|_{H^{1+q}(\Omega)} + h^{\tilde{q}} |\vec{u}|_{(H^{\tilde{q}}(\Omega))^d} + h\|f\|_{L^2(\Omega)} \right) |\!|\!|\mathbf{v}|\!|\!|_X. \tag{3.21}$$

The term I_{56}^b is estimated using (3.8) and Theorem 3.1:

$$\begin{aligned} |I_{56}^b| &\leq \frac{\alpha_1}{\alpha_0} |\!|\!|\vec{u}^I - \mathbf{u}_h|\!|\!|_X \, |\!|\!|\mathbf{v}|\!|\!|_X \\ &\leq C \left(h^q |p|_{H^{1+q}(\Omega)} + h^{\tilde{q}} |\vec{u}|_{(H^{\tilde{q}}(\Omega))^d} + h\|f\|_{L^2(\Omega)} \right) |\!|\!|\mathbf{v}|\!|\!|_X. \end{aligned} \tag{3.22}$$

The proof is completed by combining (3.18)–(3.22). □

3.3 Superconvergence of the pressure

In this section we restrict our attention to symmetric quadrature rules and prove a second-order convergence estimate for the pressure variable. We make two additional assumptions.

[A6] We assume that for every E in Ω_h, there exist a lifting operator $\mathcal{R}_E$ from $X_{h,E}$ to $H(\mathrm{div}; E)$ such that

$$\mathrm{div}\,(\mathcal{R}_E(\mathbf{v}_E)) = \mathcal{DIV}\,\mathbf{v}_E \qquad \forall \mathbf{v}_E \in X_{h,E}, \tag{3.23}$$

$$\|\mathcal{R}_E(\mathbf{v}_E)\|_{(L^2(E))^d} \leq C\, |\!|\!|\mathbf{v}_E|\!|\!|_{X,E} \qquad \forall \mathbf{v}_E \in X_{h,E}, \tag{3.24}$$

and

$$\mathcal{R}_E((\vec{v}_0^I)_E) = \vec{v}_0 \tag{3.25}$$

for every constant vector $\vec{v}_0$. Moreover, for any edge (face in 3D) $\tilde{e}$ shared by elements E_1 and E_2, we assume that

$$\mathcal{R}_{E_1}(\mathbf{v}_{E_1}) \cdot \vec{n}_{\tilde{e}} = \mathcal{R}_{E_2}(\mathbf{v}_{E_2}) \cdot \vec{n}_{\tilde{e}} \quad \forall \mathbf{v} \in X_h. \tag{3.26}$$

Note that the lifting operator never appears in the implementation of the method. It is a useful tool to prove convergence estimates; therefore, we only need to prove its existence.

[A7] Let $\mathcal{R}_E$ be a lifting operator satisfying assumption **A6**. Define $\sigma_E(\mathcal{K}^{-1};\, \mathbf{u}_E,\, \mathbf{v}_E)$ as follows:

$$\sigma_E\left(\mathcal{K}^{-1};\, \mathbf{u}_E,\, \mathbf{v}_E\right) = [\mathbf{u}_E,\, \mathbf{v}_E]_{X,E} - \int\limits_E \mathcal{K}^{-1}\mathcal{R}_E(\mathbf{u}_E)\cdot\mathcal{R}_E(\mathbf{v}_E)\,\mathrm{d}x.$$

We assume that

$$\left|\sigma_E(\mathcal{K}^{-1};\, (\vec{u})^I_E,\, (\vec{v})^I_E)\right| \le C h_E^2\, \|\vec{u}\|_{(H^1(E))^d}\, \|\vec{v}\|_{(H^1(E))^d} \qquad (3.27)$$

for all $\vec{u},\ \vec{v} \in (H^1(E))^d$.

For a given $\mathcal{K}$, $\sigma_E(\mathcal{K}^{-1};\ \mathbf{u}_E,\ \mathbf{v}_E)$ is a bilinear form with respect to $\mathbf{u}_E$ and $\mathbf{v}_E$. The following lemma illustrates some of the properties of the lifting operator $\mathcal{R}_E$. For each edge (face in 3D) $\tilde{e}$, we define the space $\mathcal{P}_l(\tilde{e})$ of polynomials of degree $\le l$.

Lemma 3.4 *Let assumption* **A4** *hold and let the lifting operator $\mathcal{R}_E$ satisfy assumptions* **A6–A7**. *For any element E, let $\mathbf{v}_E \in X_{h,E}$ and assume that for each edge (face in 3D) $\tilde{e}$ there exist an integer l such that*

$$\begin{aligned} r_{\tilde{e}} &\equiv \mathcal{R}_E(\mathbf{v}_E)\cdot\vec{n}_{\tilde{e}} \in \mathcal{P}_l(\tilde{e}),\\ r_{\tilde{e}}(x_e) &= \chi_E^e\, v_E^e \quad \forall e \in \tilde{e}. \end{aligned}$$

Furthermore, let x_e, $e \subset \tilde{e}$, be the quadrature points for exact integration of polynomials in $\mathcal{P}_{l+1}(\tilde{e})$ with corresponding weights $|e|$, i.e.,

$$\int\limits_{\tilde{e}} p_{l+1}(s)\mathrm{d}s = \sum_{e\in\tilde{e}} |e|\, p_{l+1}(x_e) \quad \forall p_{l+1} \in \mathcal{P}_{l+1}(\tilde{e}).$$

Let $\vec{u}_0$ be a constant vector and $\mathbf{u}_0 = \vec{u}_0^I$. Then,

$$\int\limits_E \mathcal{K}_E^{-1}\mathcal{R}_E(\mathbf{u}_{0,E})\cdot\mathcal{R}_E(\mathbf{v}_E)\,\mathrm{d}x = [\mathbf{u}_{0,E},\, \mathbf{v}_E]_{X,E} \quad \forall \mathbf{v}_E \in X_{h,E}. \qquad (3.28)$$

Proof Note that $\vec{u}_0 = \mathcal{K}_E \nabla \varphi^1$ for some linear function φ^1. Then, assumption **A6**, integration by parts, and assumption **A4** give

$$\begin{aligned}
\int_E \mathcal{K}_E^{-1} \mathcal{R}_E(\mathbf{u}_{0,E}) \cdot \mathcal{R}_E(\mathbf{v}_E)\,\mathrm{d}x &= -\int_E \varphi^1 \operatorname{div}(\mathcal{R}_E(\mathbf{v}_E))\,\mathrm{d}x + \sum_{\tilde{e}\in\partial E} \int_{\tilde{e}} \varphi^1 r_{\tilde{e}}\,\mathrm{d}s \\
&= -\mathcal{DIV}\,\mathbf{v}_E \int_E \varphi^1\,\mathrm{d}x + \sum_{e\in\partial E} \chi_E^e\,|e|\,\varphi^1(x_e) v_E^e \\
&= \Big[(\mathcal{K}_E \nabla \varphi^1)^I,\ \mathbf{v}_E\Big]_{X,E}.
\end{aligned}$$

This proves the assertion of the lemma. □

An example of the above lemma is when $l = 0$ and x_e is the center of mass of facet e. Another technique for proving (3.28) for simplicial meshes and a particular inner product on $X_{h,E}$ is shown in the next section.

In the theorem below we employ a duality argument to derive a superconvergence estimate for $|||p^I - \mathbf{p}_h|||_Q$.

Theorem 3.3 *Assume that problem* (2.2)–(2.3) *is H^2-regular and $f \in H^1(\Omega)$. Let the pairs $(p, \vec{u})$ and $(\mathbf{p}_h, \mathbf{u}_h)$ be the solutions of problems* (2.2)–(2.3) *and* (2.21), *respectively. Assume also that the quadrature rule $[\cdot,\cdot]_X$ defined in* (2.12) *is symmetric. Under assumptions* **A1**–**A7**, *there exists a constant C independent of h such that*

$$|||p^I - \mathbf{p}_h|||_Q \le C\,h^2\,(\|\vec{u}\|_{(H^1(\Omega))^d} + |p|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}).$$

Proof Sufficient conditions for H^2-regularity can be found in [25]. For example, it holds if $\mathcal{K} \in (W^{1,\infty}(\Omega))^{d\times d}$ and Ω is a convex domain.

Let $\mathcal{R}(\mathbf{v})$ be such that $\mathcal{R}(\mathbf{v})|_E = \mathcal{R}_E(\mathbf{v}_E)$. Let q_h be the piecewise constant function such that $q_h|_E = p_E^I - (\mathbf{p}_h)_E$. We consider the following auxiliary problem:

$$\begin{aligned}
-\operatorname{div} \mathcal{K}\nabla\varphi &= q_h \qquad \text{in} \quad \Omega, \\
\varphi &= 0 \qquad \text{on} \quad \partial\Omega.
\end{aligned}$$

The H^2-regularity assumption implies that

$$\|\varphi\|_{H^2(\Omega)} \le C\|q_h\|_{L^2(\Omega)} = C|||p^I - \mathbf{p}_h|||_Q. \tag{3.29}$$

Let $\vec{v} = -\mathcal{K}\nabla\varphi$ and $\mathbf{v} = \vec{v}^I$. Using Lemma 2.4, the first equation in (2.24), assumption **A6**, and integration by parts, we get

$$\begin{aligned}
|||p^I - \mathbf{p}_h|||_Q^2 &= \left[\mathbf{p}_h - p^I,\, \mathcal{DIV}\,\mathbf{v}\right]_Q \\
&= [\mathbf{u}_h,\, \mathbf{v}]_X - \int_\Omega p\,\mathrm{div}\,(\mathcal{R}(\mathbf{v}))\;\mathrm{d}x \\
&= [\mathbf{u}_h,\, \mathbf{v}]_X + \int_\Omega \mathcal{K}^{-1}\mathcal{K}\nabla\, p \cdot \mathcal{R}(\mathbf{v})\,\mathrm{d}x \\
&= [\mathbf{u}_h - \vec{u}^I,\, \mathbf{v}]_X + \sum_{E\in\Omega_h} \sigma_E\left(\mathcal{K}^{-1};\; (\vec{u})_E^I,\; (\vec{v})_E^I\right) \\
&\quad + \int_\Omega \mathcal{K}^{-1}\left(\mathcal{R}(\vec{u}^I) - \vec{u}\right)\mathcal{R}(\mathbf{v})\,\mathrm{d}x = J_1 + J_2 + J_3. \qquad (3.30)
\end{aligned}$$

To estimate J_1, we first define $\mathbf{w} = \mathbf{u}_h - \vec{u}^I$. Then, using the definition of $\mathbf{v}$ and adding and subtracting the term $(\overline{\mathcal{K}}\nabla\varphi^1)^I$, we have

$$J_1 = \left[\mathbf{w},\, \vec{v}^I + (\overline{\mathcal{K}}\nabla\varphi^1)^I\right]_X - \left[\mathbf{w},\; (\overline{\mathcal{K}}\nabla\varphi^1)^I\right]_X \equiv J_{11} + J_{12},$$

where φ_1 is the piecewise linear approximation to φ satisfying (3.4) on every element E, and $\overline{\mathcal{K}}$ is the piecewise constant approximation to $\mathcal{K}$ defined in Sect. 2. The terms J_{11} is estimated similarly to term I_1 that appeared in the proof of Theorem 3.1. We have

$$|J_{11}| \le C\,h\,\left(\|\varphi\|_{H^2(\Omega)} + \|q_h\|_{L^2(\Omega)}\right)|||\mathbf{w}|||_X.$$

Since the quadrature rule is symmetric, the term J_{12} can be bounded similarly to the term I_2 from the proof of Theorem 3.1:

$$|J_{12}| \le C\,h\,|\varphi|_{H^2(\Omega)}|||\mathbf{w}|||_X.$$

With the above two bounds, applying Theorem 3.1 and regularity result (3.29), we get

$$|J_1| \le C\,h^2\left(|p|_{H^2(\Omega)} + |\vec{u}|_{(H^1(\Omega))^d} + \|f\|_{L^2(\Omega)}\right)|||p^I - \mathbf{p}_h|||_Q. \qquad (3.31)$$

To estimate J_2, we use assumption **A7** and (3.29):

$$|J_2| \le Ch^2\|\vec{u}\|_{(H^1(\Omega))^d}\|\vec{v}\|_{(H^1(\Omega))^d} \le Ch^2\|\vec{u}\|_{(H^1(\Omega))^d}\;|||p^I - \mathbf{p}_h|||_Q. \qquad (3.32)$$

To estimate J_3, we add and subtract $\vec{v}$, then integrate by parts and use assumption **A6**:

$$\begin{aligned} J_3 &= \int_\Omega \mathcal{K}^{-1} \left(\mathcal{R}(\vec{u}^I) - \vec{u}\right) \left(\mathcal{R}(\vec{v}^I) - \vec{v}\right) \mathrm{d}x + \int_\Omega \mathcal{K}^{-1} \left(\mathcal{R}(\vec{u}^I) - \vec{u}\right) \vec{v}\, \mathrm{d}x \\ &= J_{31} - \int_\Omega \left(\mathcal{R}(\vec{u}^I) - \vec{u}\right) \nabla\varphi\, \mathrm{d}x = J_{31} + \int_\Omega \varphi\, \mathrm{div} \left(\mathcal{R}(\vec{u}^I) - \vec{u}\right) \mathrm{d}x \\ &= J_{31} + \int_\Omega \left(f^I - f\right) \varphi\, \mathrm{d}x \\ &= J_{31} + \int_\Omega \left(f^I - f\right) \left(\varphi - \varphi^I\right) \mathrm{d}x = J_{31} + J_{32}. \end{aligned} \tag{3.33}$$

Let $\vec{u}_0$ be the L^2 projection of $\vec{u}$ on the space of piecewise constant vector functions. The triangle inequality, (3.25), (3.24), (3.16), and (3.17) imply that

$$\begin{aligned} \|\mathcal{R}(\vec{u}^I) - \vec{u}\|_{(L^2(\Omega))^d} &\le \|\mathcal{R}(\vec{u}^I) - \vec{u}_0\|_{(L^2(\Omega))^d} + \|\vec{u} - \vec{u}_0\|_{(L^2(\Omega))^d} \\ &\le C\, |\!|\!|\vec{u}^I - \vec{u}_0^I|\!|\!|_X + \|\vec{u} - \vec{u}_0\|_{(L^2(\Omega))^d} \\ &\le C\, h\, |\vec{u}|_{(H^1(\Omega))^d}. \end{aligned}$$

The bound on $\|\mathcal{R}(\vec{v}^I) - \vec{v}\|_{(L^2(\Omega))^d}$ is similar. Therefore

$$|J_{31}| \le Ch^2 |\vec{u}|_{(H^1(\Omega))^d} |\vec{v}|_{(H^1(\Omega))^d} \le Ch^2 |\vec{u}|_{(H^1(\Omega))^d} |\!|\!| p^I - \mathbf{p}_h |\!|\!|_Q, \tag{3.34}$$

where we have used assumption **A5** and (3.29) for the last inequality.

The scalar version of the approximation property (3.17) gives the estimates

$$\|f^I - f\|_{L^2(\Omega)} \le C\, h |f|_{H^1(\Omega)} \tag{3.35}$$

and

$$\|\varphi - \varphi^I\|_{L^2(\Omega)} \le C\, h |\varphi|_{H^1(\Omega)} \le C\, h\, |\!|\!| p^I - \mathbf{p}_h |\!|\!|_Q. \tag{3.36}$$

Inserting estimates (3.34)–(3.36) into (3.33) and combining the resulting estimate with (3.30)–(3.32), we complete the proof of the theorem. □

Remark 3.1 Using techniques from the proofs of Theorem 3.1 and Theorem 3.2, assuming that $p \in H^{1+q}(\Omega)$, $0 \le q \le 1$, and $\vec{u} \in (H^{\tilde{q}}(\Omega))^d$, $0 < \tilde{q} \le 1$, we can obtain the following reduced regularity superconvergence bound:

$$|\!|\!| p^I - \mathbf{p}_h |\!|\!|_Q \le C\, (h^q + h^{\tilde{q}})^2 \left(\|\vec{u}\|_{(H^{\tilde{q}}(\Omega))^d} + |p|_{H^{1+q}(\Omega)} + \|f\|_{H^q(\Omega)} \right).$$

4 Analysis of particular quadrature rules

In this section we consider symmetric and non-symmetric quadrature rules (2.13). We show that on simplicial meshes a symmetric quadrature exists that satisfies the assumptions made above. For general polyhedral meshes, a convergent non-symmetric method can be build whenever assumption **A3** holds.

4.1 Symmetric methods

Throughout this section, we assume that the meshes satisfy the following condition:

$$\text{each corner } c \text{ of } \Omega_h \text{ is formed by exactly } d \text{ facets.}$$

Note that in 2D all meshes satisfy this condition. We give an explicit symmetric formula for matrices $\mathbf{M}_c$ in (2.13) which defines elemental matrices $\mathbf{M}_E$, and verify assumptions **A3**i for polyhedral meshes and assumptions **A4**, **A6**, and **A7** for simplicial meshes.

Given $\mathbf{v}_E \in X_{E,h}$, let $\vec{v}_E(c) \in \Re^d$ be a vector associated with corner c of E such that its normal component on any facet e that forms the corner is equal to v_E^e. Since each corner is formed by exactly d non-planar facets, the vector $\vec{v}_E(c)$ is uniquely determined. If the corner c is formed by facets $e_1, \ldots, e_d$ with normals $\vec{n}_e$, then

$$\vec{v}_E(c) = \mathbf{N}_c^{-T} \left(v_E^{e_1}, \ldots, v_E^{e_d}\right)^T, \quad \mathbf{N}_c = \left[\vec{n}_{e_1}; \ldots ; \vec{n}_{e_d}\right]. \tag{4.1}$$

We refer to $\vec{v}_E(c)$ as the *recovered* vector.

For every corner c of E, using the recovered vectors, we define

$$[\mathbf{u},\, \mathbf{v}]_{X,E,c} = \gamma_E w_c \mathcal{K}_E^{-1} \vec{u}_E(c) \cdot \vec{v}_E(c), \quad \gamma_E^{-1} = \frac{1}{|E|} \sum_{c \in E} w_c, \tag{4.2}$$

where w_c are positive weights. In this section, we choose equal weights, $w_c = |E|/m_E$, m_E is the number of vertices of E, implying $\gamma_E = 1$. With the above definition, the corner quadrature rule matrix $\mathbf{M}_c$ in (2.13) can be written as

$$\mathbf{M}_c = \frac{|E|}{m_E} \mathbf{N}_c^{-1} \mathcal{K}_E^{-1} \mathbf{N}_c^{-T}. \tag{4.3}$$

The next lemma shows that $[\cdot, \cdot]_X$ build from (4.2) satisfies assumption **A3**.

Lemma 4.1 *Let assumptions* **A1** *and* **A2** *hold. Then, assumption* **A3** *is satisfied for the matrix* $\mathbf{M}_E$ *defined through* (2.13) *and* (4.2).

Proof According to (2.13), it is sufficient to show (2.14) for every corner of E. Note that

$$\|\mathbf{N}_c^T \vec{v}\|^2 = \sum_{i=1}^{d} (\vec{n}_{e_i} \cdot \vec{v})^2 \le \sum_{i=1}^{d} \|\vec{v}\|^2 = d \|\vec{v}\|^2 \quad \forall \vec{v} \in \Re^d,$$

which implies that

$$\|\mathbf{N}_c^{-T}\vec{v}\|^2 \geq \frac{1}{d}\|\vec{v}\|^2 \quad \forall \vec{v} \in \Re^d.$$

Using (4.3), assumption **A1**, and the above inequality, it is easy to see that the left inequality in (2.14) holds with

$$\alpha_0 = \frac{1}{m_E\, k_1} \min_{c\in E} \min_{\vec{v}\in\Re^d} \frac{\|\mathbf{N}_c^{-T}\vec{v}\|^2}{\|\vec{v}\|^2} \geq \frac{1}{d\, m_E\, k_1}.$$

Similarly, to estimate α_1, we need an upper bound for $\|\mathbf{N}_c^{-T}\vec{v}\|$. Note that all entries in matrix $\mathbf{N}_c^{-T}$ are bounded by $|det(\mathbf{N}_c)|^{-1}$. Thus,

$$\|\mathbf{N}_c^{-T}\vec{v}\|^2 \leq \sum_{i=1}^{d} \frac{d\|\vec{v}\|^2}{|det(\mathbf{N}_c)|^2} \leq \frac{d^2\,\|\vec{v}\|^2}{|det(\mathbf{N}_c)|^2} \quad \forall \vec{v} \in \Re^d.$$

In 2D, $|det(\mathbf{N}_c)| = |\vec{n}_{e_1} \times \vec{n}_{e_2}| \geq \sin\gamma_*$, using **A2**. In 3D, let ℓ_2 be the edge between faces $\tilde{e}_1$ and $\tilde{e}_3$. Similarly, let ℓ_3 be the edge between faces $\tilde{e}_1$ and $\tilde{e}_2$. The vector product $\vec{n}_{e_1} \times \vec{n}_{e_2}$ is aligned with the direction of ℓ_3 and the vector product $\vec{n}_{e_1} \times \vec{n}_{e_3}$ is aligned with the direction of ℓ_2. Using a formula for the volume of the tetrahedron formed by the three normals and assumption **A2**, we get

$$|det(\mathbf{N}_c)| = 6\frac{2}{3}\frac{|\vec{n}_{e_1} \times \vec{n}_{e_2}|\,|\vec{n}_{e_1} \times \vec{n}_{e_3}|}{4\,|\vec{n}_{e_1}|}|\sin\gamma^c_{\ell_3,\ell_2}| \geq \sin^3\gamma_*.$$

Using (4.3), assumption **A1**, and the above inequalities, we have that the upper bound in (2.14) holds with

$$\alpha_1 = \frac{1}{m_E\, k_0} \max_{c\in E} \max_{\vec{v}\in\Re^d} \frac{\|\mathbf{N}_c^{-T}\vec{v}\|^2}{\|\vec{v}\|^2} \leq \frac{d^2}{m_E\, k_0\, \sin^{4d-6}(\gamma_*)}.$$

Bound (2.15) is trivially satisfied, since $\mathbf{M}_E$ is symmetric. This proves the assertion of the lemma. □

Remark 4.1 Since $\mathbf{M}_E$ is symmetric, it can be shown easily, using Lemma 4.1, that $[\cdot,\cdot]_X$ is an inner product in X_h.

We proceed with verifying assumption **A4** for (4.2) and simplicial meshes. In two dimensions, for each edge with end points a_1 and a_2, we define two new points

$$a_{12} = \frac{1}{3}(2a_1 + a_2) \quad \text{and} \quad a_{21} = \frac{1}{3}(a_1 + 2a_2) \tag{4.4}$$

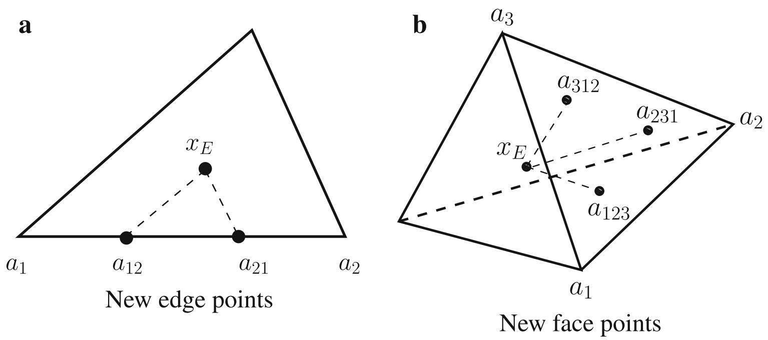

Fig. 3 Auxiliary edge and face points

which are interior points of the two facets, see Fig. 3a. In three dimensions, for each face (which is a triangle) with vertices a_1, a_2 and a_3, we define three new points

$$a_{123} = \frac{1}{4}(2a_1 + a_2 + a_3), \quad a_{231} = \frac{1}{4}(a_1 + 2a_2 + a_3), \quad a_{312} = \frac{1}{4}(a_1 + a_2 + 2a_3), \tag{4.5}$$

which are interior points of three facets, see Fig. 3b. Note that the d new points are the projections of the center of mass, x_E, onto the edge (face in 3D) along directions parallel to the other d edges. We use notation x_e for the new point inside facet e.

Lemma 4.2 *Let Ω_h be a simplicial partition. Then assumption* **A4** *holds with points x_e defined by* (4.4) *in 2D and* (4.5) *in 3D.*

Proof According to (2.13), the matrix $\mathbf{M}_E$ corresponding to $[\cdot,\ \cdot]_{X,E}$ is block diagonal with $d+1$ blocks. Thus, to prove (2.16), it is sufficient to show it for every corner c of E. Recall that corner c is formed by facets $e_1, \dots, e_d$. Assume for simplicity that the normal vectors $\vec{n}_{e_i}$ are outward to E, which gives $\chi_E^{e_i} = 1$. Let $\vec{v}_E(c)$ be the vector recovered at corner c. Note that on simplicial meshes $w_c = \frac{|E|}{d+1}$. Since the constant vector ∇q^1 is recovered exactly, (2.16) reduces to

$$\frac{|E|}{d+1} \left(\mathcal{K}_E^{-1} \vec{v}_E(c)\right) \cdot \left(\mathcal{K}_E \nabla q^1\right) = \sum_{i=1}^{d} |e_i| \left(q^1(x_{e_i}) - q^1(x_E)\right) v_E^{e_i}. \tag{4.6}$$

Using formula (4.1) for the recovered vector $\vec{v}_E(c)$, (4.6) is equivalent to

$$\frac{|E|}{d+1} \nabla q^1 = \sum_{i=1}^{d} |e_i| \, \vec{n}_{e_i} \, q^1(x_{e_i} - x_E). \tag{4.7}$$

To prove (4.7), recall that points x_e are defined by (4.4) in 2D and (4.5) in 3D. For illustration, let us consider the triangle E shown in Fig. 4, although the proof is valid in 3D as well. The shaded triangle $\hat{E}$ is similar to E with ratio $\frac{d}{d+1}$ and $|\hat{E}| = \left(\frac{d}{d+1}\right)^d |E|$.

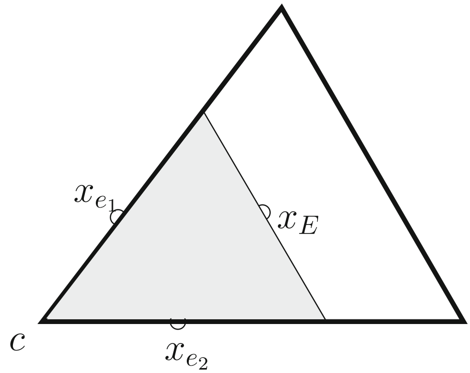

Fig. 4 The similar triangles E and $\hat{E}$ (*shaded*)

The points x_{e_1}, x_{e_2} and x_E are the mid-points of the edges of $\hat{E}$. Using that the midpoint quadrature rule is exact for linear functions and applying the Green's formula to the right hand side of (4.7), we get

$$\sum_{i=1}^{d} |e_i|\, \vec{n}_{e_i}\, q^1(x_{e_i} - x_E) = \beta(d) \int\limits_{\partial \hat{E}} \vec{n}_{\hat{E}}\, q^1(s - x_E)\, \mathrm{d}s = \beta(d) \int\limits_{\hat{E}} \nabla q^1\, \mathrm{d}x$$
$$= \frac{|E|}{d+1}\, \nabla q^1,$$

where $\beta(d) = \frac{1}{d}\left(\frac{d}{d+1}\right)^{1-d}$. We conclude the proof by noting that (2.17) holds with $\sigma_* = \frac{1}{\beta(d)}$. □

Now we verify assumptions **A6** and **A7** on simplicial grids. Consider the lowest order Brezzi–Douglas–Marini mixed finite element space BDM_1 consisting of piecewise linear vector functions with continuous normal components [14]. A BDM_1 vector is uniquely defined by the values of its normal component at d points on each edge (face in 3D). Let $\mathcal{R}_E(\mathbf{v}_E)$ be the BDM_1 interpolant satisfying for each facet e

$$\mathcal{R}_E(\mathbf{v}_E)(c) \cdot \vec{n}_e = v_E^e,$$

where c is the corner associated with e. This lifting operator preserves constant vector functions and has a continuous normal component across mesh interfaces [14]. Note that

$$\mathcal{DIV}\,\mathbf{v}_E = \frac{1}{|E|} \sum_{e \in \partial E} |e|\, v_E^e = \frac{1}{|E|} \sum_{\tilde{e} \in \partial E} \frac{|\tilde{e}|}{d} \sum_{c \text{ of } \tilde{e}} \vec{v}_E(c) \cdot \vec{n}_{\tilde{e}},$$

where $\vec{v}_E(c)$ is the vector recovered at corner c and the last sum includes only corners associated with $\tilde{e}$. By construction, $\vec{v}_E(c) = \mathcal{R}_E(\mathbf{v}_E)(c)$. Since, the last sum is the quadrature rule for exact integration of linear functions, we get

$$\mathcal{DIV}\,\mathbf{v}_E = \frac{1}{|E|}\sum_{\tilde{e}\in\partial E}\int_{\tilde{e}} \mathcal{R}_E(\mathbf{v}_E)\cdot\vec{n}_E\,\mathrm{d}s = \mathrm{div}(\mathcal{R}_E(\mathbf{v}_E)).$$

Thus $\mathcal{R}_E(\mathbf{v}_E)$ satisfies (3.23). The definition of $\mathcal{R}_E(\mathbf{v}_E)$ easily implies (3.24). Therefore assumption **A6** holds on simplicial grids.

The following lemma verifies assumption **A7**.

Lemma 4.3 *Let Ω_h be a simplicial partition. Let the tensor $\mathcal{K}$ satisfy assumption* **A5** *and let the lifting operator $\mathcal{R}_E$ be the BDM_1 interpolation operator defined above. Then, assumption* **A7** *holds.*

Proof Let $\vec{v},\ \vec{u}\in(H^1(E))^d$. Let $\vec{v}_{0,E}$ be the L^2 projection of $\vec{v}$ on the space of constant vector functions on E, $\mathbf{v}_{0,E}=(\vec{v}_0^I)_E$, and $\mathbf{u}_E=(\vec{u}^I)_E$. Similarly, we define $\vec{u}_{0,E}$ and $\mathbf{u}_{0,E}$. Then, definition on the inner product (4.2) on $X_{h,E}$ and the quadrature rule for exact integration of linear functions give

$$\begin{aligned}[\mathbf{v}_{0,E},\ \mathbf{u}_E]_{X,E} &= \frac{|E|}{d+1}\sum_{c\in E}\mathcal{K}_E^{-1}\mathcal{R}_E(\mathbf{v}_{0,E})(c)\cdot\mathcal{R}_E(\mathbf{u}_E)(c)\\ &= \int_E \mathcal{K}_E^{-1}\mathcal{R}_E(\mathbf{v}_{0,E})\cdot\mathcal{R}_E(\mathbf{u}_E)\,\mathrm{d}x.\end{aligned}$$

The above identity implies that

$$\sigma_E\left(\mathcal{K}_E^{-1};\ \mathbf{v}_{0,E},\ \mathbf{u}_E\right)=0. \tag{4.8}$$

Using the definition of $\sigma_E(\mathcal{K}^{-1};\ \mathbf{v}_E,\ \mathbf{u}_E)$, we write

$$\begin{aligned}\sigma_E\left(\mathcal{K}^{-1};\ \mathbf{v}_E,\mathbf{u}_E\right) &= \sigma_E\left(\mathcal{K}_E^{-1};\ \mathbf{v}_E,\mathbf{u}_E\right)+\int_E\left(\mathcal{K}_E^{-1}-\mathcal{K}^{-1}\right)\mathcal{R}_E(\mathbf{v}_E)\cdot\mathcal{R}_E(\mathbf{u}_E)\,\mathrm{d}x\\ &= I_1+I_2.\end{aligned}$$

Using (4.8), then (3.16) and (3.17), we bound I_1 as follows:

$$|I_1| = \left|\sigma_E(\mathcal{K}_E^{-1};\ \mathbf{v}_E-\mathbf{v}_{0,E},\ \mathbf{u}_E-\mathbf{u}_{0,E})\right| \le h_E^2\,\|\vec{u}\|_{(H^1(E))^d}\|\vec{v}\|_{(H^1(E))^d}. \tag{4.9}$$

The integral I_2 can be broken into three integrals

$$
\begin{aligned}
I_2 = &\int_E \left(\mathcal{K}_E^{-1} - \mathcal{K}^{-1}\right) \mathcal{R}_E(\mathbf{v}_E - \mathbf{v}_{0,E}) \cdot \mathcal{R}_E(\mathbf{u}_E)\, \mathrm{d}x \\
&+ \int_E \left(\mathcal{K}_E^{-1} - \mathcal{K}^{-1}\right) \mathcal{R}_E(\mathbf{v}_{0,E}) \cdot \mathcal{R}_E(\mathbf{u}_E - \mathbf{u}_{0,E})\, \mathrm{d}x \\
&+ \int_E \left(\mathcal{K}_E^{-1} - \mathcal{K}^{-1}\right) \mathcal{R}_E(\mathbf{v}_{0,E}) \cdot \mathcal{R}_E(\mathbf{u}_{0,E})\, \mathrm{d}x = I_{21} + I_{22} + I_{23}.
\end{aligned}
$$

Using (3.2), (3.24), and (3.16), we bound the first two integrals:

$$
|I_{21} + I_{22}| \le C h_E^2 \|\vec{v}\|_{(H^1(E))^d} \|\vec{u}\|_{(H^1(E))^d}. \tag{4.10}
$$

To bound the third integral, we use property (3.25), the fact that the constant tensor $\mathcal{K}_E$ is the mean value of $\mathcal{K}$ on E, then estimates (3.1) and (3.2):

$$
\begin{aligned}
|I_{23}| &= \left| \int_E (\mathcal{K} - \mathcal{K}_E)\mathcal{K}_E^{-1} \vec{v}_{0,E} \cdot \mathcal{K}^{-1} \vec{u}_{0,E}\, \mathrm{d}x \right| \\
&= \left| \int_E (\mathcal{K} - \mathcal{K}_E)\mathcal{K}_E^{-1} \vec{v}_{0,E} \cdot \left(\mathcal{K}^{-1} - \mathcal{K}_E^{-1}\right) \vec{u}_{0,E}\, \mathrm{d}x \right| \\
&\le C h_E^2 \|\vec{v}\|_{(L^2(E))^d} \|\vec{u}\|_{(L^2(E))^d}.
\end{aligned} \tag{4.11}
$$

A combination of (4.8)–(4.11) completes the proof of the lemma. □

Remark 4.2 The analysis developed in this section can be extended to uniformly refined quadrilateral and hexahedral meshes via a mapping to a reference element, using techniques developed in [30,47].

4.2 Non-symmetric methods

In this section, we consider unstructured polygonal and polyhedral meshes. We give explicit formula for matrices $\mathbf{M}_c$ in (2.13) such that assumption **A4** is automatically satisfied. Analysis of sufficient conditions for assumptions **A3**, **A6** and **A7** will be the topic of future research.

The derivation of matrix $\mathbf{M}_c$ follows essentially the path developed in [17]. It is sufficient to verify assumption **A4** for $d + 1$ linearly independent basis functions in $\mathcal{P}_1(E)$, for example, 1 and x_i, $i = 1, \ldots, d$, where $(\mathrm{x}_1, \ldots, \mathrm{x}_d)$ denote the Cartesian coordinate system in $\Re^d$. Note that both sides of (2.16) are zero when $q^1 = 1$. For $q^1 = \mathrm{x}_i$, the right-hand side of (2.16) is a linear functional of $\mathbf{v}$ and therefore it can be represented as $\mathbf{r}_i^T\, \mathbf{v}$, where $\mathbf{r}_i \in X_{h,E}$. The entries of $\mathbf{r}_i$ are the i-th coordinates of the k_E vectors $x_{e_1} - x_E, \ldots, x_{e_{k_E}} - x_E$, where $e_1, \ldots, e_{k_E}$ are the facets of E. Thus, we get d linear equations for the unknown matrix $\mathbf{M}_E$:

$$\mathbf{M}_E \mathbf{n}_i = \mathbf{r}_i, \quad i = 1, \ldots, d, \tag{4.12}$$

where $\mathbf{n}_i = (\mathcal{K}_E \nabla \mathrm{x}_i)_E^I$. If we define $k_E \times d$ matrices $\tilde{\mathbf{N}}_E$ and $\mathbf{R}_E$ as

$$\tilde{\mathbf{N}}_E = [\mathbf{n}_1; \ \ldots ; \mathbf{n}_d] \quad \text{and} \quad \mathbf{R}_E = [\mathbf{r}_1; \ \ldots ; \mathbf{r}_d],$$

then (4.12) can be written in the compact form

$$\mathbf{M}_E \tilde{\mathbf{N}}_E = \mathbf{R}_E. \tag{4.13}$$

We refer to [17] for more details.

The matrix $\mathbf{M}_E$ is block diagonal with as many blocks as there are corners in E. Let us consider a particular corner c of E. Without loss of generality, we assume that $e_1, \ldots, e_{k_c}$ are the facets that form this corner. It follows from (4.13) that

$$\mathbf{M}_c \tilde{\mathbf{N}}_c = \mathbf{R}_c, \tag{4.14}$$

where $\tilde{\mathbf{N}}_c$ and $\mathbf{R}_c$ are $k_c \times d$ matrices formed by k_c rows of matrices $\tilde{\mathbf{N}}_E$ and $\mathbf{R}_E$, respectively. When the corner c is formed by exactly d facets, $k_c = d$,

$$\tilde{\mathbf{N}}_c = \mathbf{N}_c^T \mathcal{K}_E,$$

where $\mathbf{N}_c = \left[\vec{n}_{e_1}; \ldots ; \vec{n}_{e_{k_c}}\right]$. In this case, the solution to (4.14) is

$$\mathbf{M}_c = \mathbf{R}_c \tilde{\mathbf{N}}_c^{-1}. \tag{4.15}$$

If $k_c > d$, matrix $\tilde{\mathbf{N}}_c^T$ has a non-empty null space. Let $\mathbf{D}_c$ be a matrix with columns that span this null space, i.e. $\tilde{\mathbf{N}}_c^T \mathbf{D}_c = 0$. Then,

$$\mathbf{M}_c = \mathbf{R}_c \left(\tilde{\mathbf{N}}_c^T \tilde{\mathbf{N}}_c\right)^{-1} \tilde{\mathbf{N}}_c^T + \mathbf{D}_c \mathbf{U}_c \mathbf{D}_c^T, \tag{4.16}$$

where $\mathbf{U}_c$ is an arbitrary symmetric positive definite matrix of size $k_c - d$. This implies that there exists a family of solutions to (4.14) which is described by $(k_c - d)(k_c - d + 1)/2$ parameters.

Finding sufficient conditions for assumption **A3** is a non-trivial task (see, e.g., [31] where MPFA methods on quadrilateral meshes are analyzed) since the geometry of E is coupled with the tensor properties of the permeability coefficient $\mathcal{K}_E$. The proposed methodology is reduced to analysis of only $k_c \times k_c$ matrices.

We consider in more detail the two-dimensional case, where $k_c = d = 2$. We introduce some additional notation as shown in Fig. 5. Let $\vec{a}_i$, $i = 1, 2$, be the vector pointing from point x_E to point x_{e_i}. Let $\vec{t}_i$, $i = 1, 2$, be the unit vectors tangential to

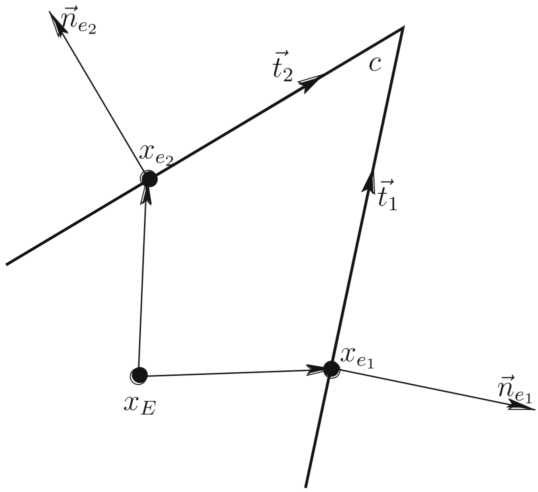

Fig. 5 Geometric interpretation of rows of the matrices $\mathbf{R}_c$, $\mathbf{N}_c$ and columns of the matrix $\mathbf{N}_c^{-1}$ in two dimensions

facets e_i and pointing to the corner c. Then, the 2×2 matrices $\mathbf{R}_c$ and $\mathbf{N}_c$ have the following structure:

$$\mathbf{R}_c^T = [\vec{a}_1;\ \vec{a}_2], \quad \mathbf{N}_c = [\vec{n}_{e_1};\ \vec{n}_{e_2}], \quad \text{and} \quad \mathbf{N}_c^{-1} = \frac{1}{\sin \gamma_{e_1,e_2}^c}[\vec{t}_2;\ \vec{t}_1].$$

Now, formula (4.15) implies that

$$\mathbf{M}_c = \frac{1}{\sin \gamma_{e_1,e_2}^c} \begin{bmatrix} \vec{a}_1^T \mathcal{K}_E^{-1} \vec{t}_2 & \vec{a}_1^T \mathcal{K}_E^{-1} \vec{t}_1 \\ \vec{a}_2^T \mathcal{K}_E^{-1} \vec{t}_2 & \vec{a}_2^T \mathcal{K}_E^{-1} \vec{t}_1 \end{bmatrix}. \tag{4.17}$$

For a mesh consisting of parallelograms, formula (4.17) resembles the $\mathcal{K}$-orthogonality result from [1] derived for a transmissibility matrix. When $\vec{a}_i$ is collinear with $\vec{t}_{3-i}$, $i = 1, 2$, $\vec{a}_i^T \mathcal{K}_E^{-1} \vec{t}_i = 0$ describes a mesh orthogonal in a metric.

Lemma 4.4 *Let $d = 2$, $\mathcal{K}$ be a scalar tensor, and Ω_h be a centroidal Voronoi polygonal mesh. If the points x_e are defined as the intersection of a dual Delaunay mesh with the edges of the Voronoi mesh, then the matrices $\mathbf{M}_c$ defined by (4.15) are diagonal and assumption* **A3** *holds.*

Proof The diagonality of $\mathbf{M}_c$ follows from the definition of the centroidal Voronoi mesh—the vectors $\vec{a}_i$, $i = 1, 2$, are orthogonal to facets e_i. Assumption **A3** then follows from the non-degeneracy of the Voronoi mesh. □

We also note that, for general meshes, the flexibility in the locations of points x_e can be exploited in the construction of a matrix $\mathbf{M}_E$ satisfying assumption **A3**. We conclude this discussion with the following result, which is a corollary of Theorems 3.1 and 3.2.

Theorem 4.1 *Let the matrix* $\mathbf{M}_c$ *in* (2.13) *be given by* (4.15) *or* (4.16)*. Let assumption* **A3** *hold for that matrix. Let pairs* $(p, \vec{u})$ *and* $(\mathbf{p}_h, \mathbf{u}_h)$ *be solutions of problems* (2.2) *and* (2.21)*, respectively. Assume that* $p \in H^{1+q}(\Omega)$, $0 \leq q \leq 1$, *and* $\vec{u} \in (H^{\tilde{q}}(\Omega))^d$, $0 < \tilde{q} \leq 1$. *Under assumptions* **A1**, **A2** *and* **A5**, *there exists a constant* C *independent of* h *such that*

$$|||\vec{u}^I - \mathbf{u}_h|||_X \leq C \left(h^q \, |p|_{H^{1+q}(\Omega)} + h^{\tilde{q}} \, |\vec{u}|_{(H^{\tilde{q}}(\Omega))^d} + h \|f\|_{L^2(\Omega)} \right)$$

and

$$|||p^I - \mathbf{p}_h|||_Q \leq C \left(h^q \, |p|_{H^{1+q}(\Omega)} + h^{\tilde{q}} \, |\vec{u}|_{(H^{\tilde{q}}(\Omega))^d} + h \|f\|_{L^2(\Omega)} \right).$$

Remark 4.3 Extension of the pressure superconvergence result from Theorem 3.3 to polygonal and polyhedral elements requires verifying assumptions **A6** and **A7**. One could construct appropriate interpolation operators on such elements by extending the results from [32,33] on piecewise Raviart–Thomas spaces to piecewise BDM_1 spaces.

5 Numerical experiments

In this section, we present results of numerical experiments using quadrature rules defined in (4.2). As we mentioned in Sect. 2, the velocity unknown can be eliminated from the discrete system resulting in a cell-centered discretization with a symmetric positive definite matrix. This problem is solved with the preconditioned conjugate gradient (PCG) method. In the numerical experiments, we used one V-cycle of the algebraic multigrid method [44] as a preconditioner. The stopping criterion for the PCG method is the relative decrease in the residual norm by a factor of 10^{-12}.

Let us consider the 2D problem (2.2) in the unit square with the known analytical solution

$$p(x, y) = x^3 y^2 + x \sin(2\pi x y) \sin(2\pi y)$$

and the tensor coefficient

$$\mathcal{K} = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}.$$

In the *first* set of experiments, we consider the sequence of smooth triangular meshes generated from uniform square meshes by splitting each square cell into four equal triangles; see Fig. 6. The convergence rates are shown in Table 1 for the discrete L^2 norms defined earlier, as well as for the discrete L^∞ norms defined as the maximum component absolute values of the algebraic vectors. We use a linear regression algorithm to estimate the convergence rates. We observe second-order convergence rate (superconvergence) of the pressure variable and first-order convergence rate of the flux variable in the discrete L^2 norms.

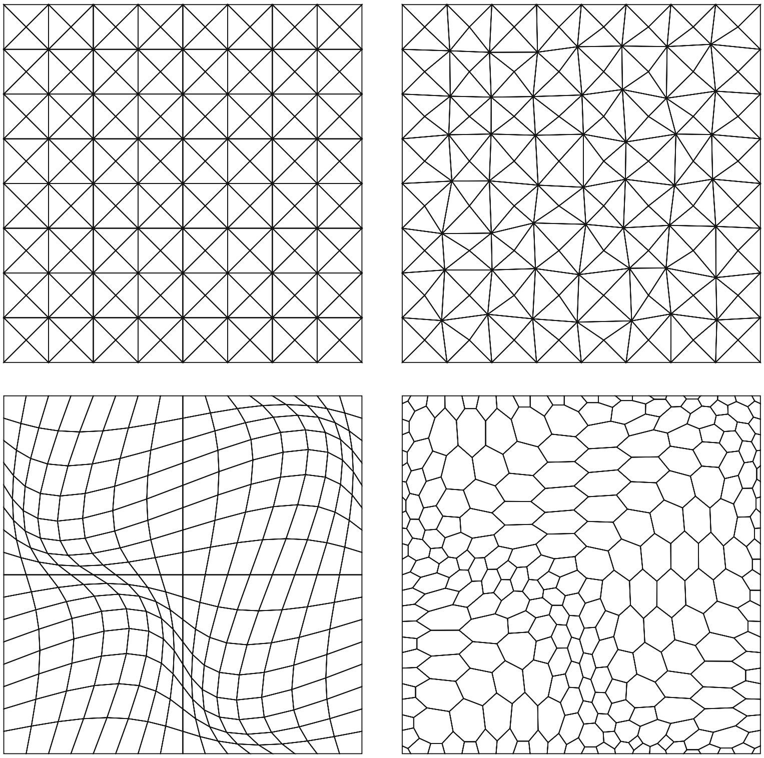

Fig. 6 Examples of meshes used in first (*top left*), second (*top right*), third (*bottom left*), and fourth (*bottom-right*) experiments. The meshes in the *top row* correspond to $h = 1/8$. The meshes in the *bottom row* correspond to $h = 1/16$

Table 1 Convergence rates in the first set of experiments

$1/h$	$\lVert p^I - \mathbf{p}_h \rVert_Q$	$\lVert p^I - \mathbf{p}_h \rVert_\infty$	$\lVert \vec{u}^I - \mathbf{u}_h \rVert_X$	$\lVert \vec{u}^I - \mathbf{u}_h \rVert_\infty$
8	2.22e–3	3.82e–3	2.08e–2	2.17e–1
16	5.50e–4	1.04e–3	9.96e–3	1.11e–1
32	1.37e–4	2.73e–4	4.91e–3	5.62e–2
64	3.43e–5	7.12e–5	2.45e–3	2.82e–2
128	8.59e–6	1.83e–5	1.22e–3	1.42e–2
Rate	2.00	1.93	1.02	0.98

In the *second* set of experiments, we take the meshes generated above and perturb randomly the positions of the mesh nodes. More precisely, we move each of the mesh nodes into a random position inside a square of size $h/2$ centered at the node; see Fig. 6. The convergence rates are shown in Table 2. As in the first example, we observe second-order convergence of the pressure and first-order convergence of the flux. Both experiments confirm the theoretical results proved in the previous sections.

Table 2 Convergence rates in the second set of experiments

$1/h$	$\|\|\|p^I - \mathbf{p}_h\|\|\|_Q$	$\|\|\|p^I - \mathbf{p}_h\|\|\|_\infty$	$\|\|\|\vec{u}^I - \mathbf{u}_h\|\|\|_X$	$\|\|\|\vec{u}^I - \mathbf{u}_h\|\|\|_\infty$
8	2.25e–3	4.21e–3	2.89e–2	2.17e–1
16	5.65e–4	1.05e–3	1.42e–2	1.11e–1
32	1.42e–4	3.26e–4	7.70e–3	5.65e–2
64	3.54e–5	9.25e–5	3.83e–3	3.44e–2
128	8.85e–6	2.49e–5	1.94e–3	1.70e–2
Rate	2.00	1.83	0.97	0.90

Table 3 Convergence rates in the third set of experiments

$1/h$	$\|\|\|p^I - \mathbf{p}_h\|\|\|_Q$	$\|\|\|p^I - \mathbf{p}_h\|\|\|_\infty$	$\|\|\|\vec{u}^I - \mathbf{u}_h\|\|\|_X$	$\|\|\|\vec{u}^I - \mathbf{u}_h\|\|\|_\infty$
8	5.24e–3	1.81e–2	4.54e–1	3.81e–0
16	1.25e–3	6.80e–3	2.48e–1	2.61e–0
32	3.95e–4	1.87e–3	1.27e–1	1.44e–0
64	9.99e–5	4.84e–4	6.37e–2	7.47e–1
128	2.50e–5	1.23e–4	3.19e–2	3.80e–1
Rate	1.91	1.82	0.96	0.85

In the *third* set of experiments we consider a sequence of smooth quadrilateral meshes. On each refinement level the mesh is obtained from a square mesh via the mapping

$$\mathbf{x} := \mathbf{x} + 0.1 \sin(2\pi x) \sin(2\pi y), \tag{5.1}$$

see the bottom picture in Fig. 6. The discrete L^∞ and L^2 norms of the errors are shown in Table 3. The convergence rates are close to those for triangular meshes. The slight reduction in convergence rates is due to slower convergence on coarse meshes.

In the *fourth* set of experiments, we consider a sequence of polygonal median meshes. A polygonal median mesh (see the bottom-right picture in Fig. 6) is built in two steps. First, we generate the Voronoi tessellation for the set of points given by (5.1) applied to nodes of a square mesh. Second, we move each interior mesh node to the center of mass of a triangle formed by the centers of three Voronoi cells sharing the node. The results are shown in Table 4. We observe the second-order convergence of the pressure and the first-order convergence of the flux.

Tables 3 and 4 provide a qualitative comparison of symmetric and non-symmetric methods, since meshes in both sequences have roughly the same number of elements and these elements are distributed with the same mapping (5.1). The non-symmetric method provides more accurate fluxes which is due to the fact that assumption **A4** does not hold exactly for quadrilateral meshes.

Table 4 Convergence rates in the fourth set of experiments

$1/h$	$\vert\vert\vert p^I - \mathbf{p}_h \vert\vert\vert_Q$	$\vert\vert\vert p^I - \mathbf{p}_h \vert\vert\vert_\infty$	$\vert\vert\vert \vec{u}^I - \mathbf{u}_h \vert\vert\vert_X$	$\vert\vert\vert \vec{u}^I - \mathbf{u}_h \vert\vert\vert_\infty$
8	1.40e–2	2.71e–2	1.73e–1	8.18e–1
16	2.67e–3	6.17e–3	5.88e–2	3.77e–1
32	5.74e–4	1.33e–3	2.92e–2	2.09e–1
64	1.33e–4	3.12e–4	1.53e–2	1.39e–1
128	3.19e–5	7.88e–5	7.90e-3	8.44e–2
Rate	2.19	2.12	1.08	0.80

6 Conclusions

We have developed a local flux mimetic finite difference method, which reduces to cell-centered finite differences for the pressure. The method uses *facet fluxes*, which are eliminated from the algebraic system by solving small local systems for each mesh vertex. The method is defined on general polyhedral meshes. We present analysis showing optimal convergence for both variables and superconvergence for the pressure variable under certain constructive assumptions on the L^2 quadrature rule. Our analysis is based on discrete space arguments and does not rely on finite element polynomial extensions, with the exception of the pressure superconvergence proof. A symmetric method that satisfies these assumptions is developed for simplicial meshes. The analysis is extendable to uniformly refined quadrilateral and hexahedral meshes. A non-symmetric method is developed for general polyhedral grids. Both methods satisfy the consistency assumption **A4** by construction. The symmetric method satisfies the coercivity assumption **A3**. The validity of this assumption for the non-symmetric method depends on the shape regularity of the grid and the anisotropy of the tensor permeability coefficient.

Appendix

Proof of Lemma 3.1

Proof Without loss of generality we assume that e is either a segment in 2D or a triangle in 3D. Note that in 3D any facet can be represented as a finite union of shape-regular triangles. If (3.7) is shown for each of these triangles, it is easy to see that it holds for the whole facet. We can also assume that E is a simplex that has e as one of its edges (faces in 3D). In the general polyhedral case, such simplex that is contained in E and satisfies the shape-regularity conditions from assumption **A2** can be constructed easily. Let $\hat{E}$ be the reference simplex. There exists a bijection mapping $F_E : \hat{E} \to E$. Denote the Jacobian matrix by DF_E and let $J_E = |\det(DF_E)|$. It is easy to see that the shape-regularity of E implies that

$$\|DF_E\| \sim h_E \quad \text{and} \quad J_E \sim h_E^d, \tag{A.1}$$

where $\|\cdot\|$ denotes the matrix norm associated with the Euclidean norm in $\Re^d$ and the notation $a \sim b$ means that there exist positive constants C_0 and C_1 independent of h_E such that $C_0 b \le a \le C_1 b$.

We will make use of the transformations

$$\vec{v} \leftrightarrow \hat{\vec{v}} : \vec{v} = \frac{1}{J_E} DF_E \hat{\vec{v}} \circ F_E^{-1}, \quad w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1}.$$

The vector transformation is known as the Piola transformation. It is designed to preserve the normal components of the velocity vectors on the edges (faces) and satisfies the important properties [15]

$$\int_E \mathrm{div}\vec{v}\, w\, \mathrm{d}x = \int_{\hat{E}} \mathrm{div}\hat{\vec{v}}\, \hat{w}\, \mathrm{d}\hat{x} \quad \text{and} \quad \int_e \vec{v}\cdot\vec{n}_e\, w\, \mathrm{d}s = \int_{\hat{e}} \hat{\vec{v}}\cdot\hat{\vec{n}}_{\hat{e}}\, \hat{w}\, \mathrm{d}\hat{s}. \tag{A.2}$$

For the standard change of variables, we have

$$\int_e w\, \mathrm{d}s = \int_{\hat{e}} \hat{w}\, J_e\, \mathrm{d}\hat{s}, \quad J_e = J_E \|DF_E^{-T}\hat{\vec{n}}_{\hat{e}}\|. \tag{A.3}$$

The above relationships imply that for $r \ge 0$ and all sufficiently regular functions

$$|\hat{\vec{v}}|_{(H^r(\hat{E}))^d} \le C\|DF_E^{-1}\|\,\|DF_E\|^r J_E^{1/2} |\vec{v}|_{(H^r(E))^d} \le C h_E^{r-1+d/2} |\vec{v}|_{(H^r(E))^d}, \tag{A.4}$$

$$|\mathrm{div}\hat{\vec{v}}|_{H^r(\hat{E})} \le C\|DF_E\|^r J_E^{1/2} |\mathrm{div}\vec{v}|_{H^r(E)} \le C h_E^{r+d/2} |\mathrm{div}\vec{v}|_{H^r(E)}, \tag{A.5}$$

$$|\hat{w}|_{H^r(\hat{e})} \le C\|DF_E\|^r J_e^{-1/2} |w|_{H^r(e)} \le C h_E^{r+1/2-d/2} |w|_{H^r(e)}. \tag{A.6}$$

It was shown in [38] that for all $\hat{\vec{v}} \in \tilde{\mathcal{V}}(\hat{E})$, $\hat{\vec{v}}\cdot\hat{\vec{n}}_{\hat{e}} \in H^{\tilde{q}-1/2}(\hat{e})$, $0 < \tilde{q} \le 1$, and

$$\|\hat{\vec{v}}\cdot\hat{\vec{n}}_{\hat{e}}\|_{H^{\tilde{q}-1/2}(\hat{e})} \le C\left(\|\hat{\vec{v}}\|_{(H^{\tilde{q}}(\hat{E}))^d} + \|\mathrm{div}\hat{\vec{v}}\|_{L^2(\hat{E})}\right). \tag{A.7}$$

We will give the argument for $0 < \tilde{q} < 1/2$, in which case $\hat{\vec{v}}\cdot\hat{\vec{n}}_{\hat{e}}$ is defined in the sense of distributions. The argument in the case $1/2 \le \tilde{q} \le 1$ is similar. Let $w \in H^{1/2-\tilde{q}}(e)$.

Using (A.2) and (A.4)–(A.7), we have

$$
\begin{aligned}
\int\limits_{e} \vec{v}\cdot\vec{n}_e\, w\,\mathrm{d}s &= \int\limits_{\hat e} \hat{\vec{v}}\cdot\hat{\vec{n}}_{\hat e}\,\hat w\,\mathrm{d}\hat s \\
&\le \|\hat{\vec{v}}\cdot\hat{\vec{n}}_{\hat e}\|_{H^{\tilde q-1/2}(\hat e)}\|\hat w\|_{H^{1/2-\tilde q}(\hat e)} \\
&\le C\left(\|\hat{\vec{v}}\|_{(L^2(\hat E))^d} + |\hat{\vec{v}}|_{(H^{\tilde q}(\hat E))^d} + \|\mathrm{div}\hat{\vec{v}}\|_{L^2(\hat E)}\right) \\
&\qquad \times\left(\|\hat w\|_{L^2(\hat e)} + |\hat w|_{H^{1/2-\tilde q}(\hat e)}\right) \\
&\le C\left(h_E^{-1+d/2}\|\vec{v}\|_{(L^2(E))^d} + h_E^{\tilde q-1+d/2}|\vec{v}|_{(H^{\tilde q}(E))^d} + h_E^{d/2}\|\mathrm{div}\vec{v}\|_{L^2(E)}\right) \\
&\qquad \times\left(h_E^{1/2-d/2}\|w\|_{L^2(e)} + h_E^{1-\tilde q-d/2}|w|_{H^{1/2-\tilde q}(e)}\right) \\
&\le C\left(h_E^{-1/2}\|\vec{v}\|_{(L^2(E))^d} + h_E^{\tilde q-1/2}\,|\vec{v}|_{(H^{\tilde q}(E))^d} + h_E^{1/2}\|\mathrm{div}\vec{v}\|_{L^2(E)}\right) \\
&\qquad \times\left(\|w\|_{L^2(e)} + h_E^{1/2-\tilde q}|w|_{H^{1/2-\tilde q}(e)}\right),
\end{aligned}
$$

which implies the assertion of the lemma. □

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