Research Article

Local Fractional Adomian Decomposition and Function Decomposition Methods for Laplace Equation within Local Fractional Operators

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We perform a comparison between the local fractional Adomian decomposition and local fractional function decomposition methods applied to the Laplace equation. The operators are taken in the local sense. The results illustrate the significant features of the two methods which are both very effective and straightforward for solving the differential equations with local fractional derivative.

1. Introduction

Many problems of physics and engineering are expressed by ordinary and partial differential equations, which are termed boundary value problems. We can mention, for example, the wave, the Laplace, the Klein-Gordon, the Schrodinger, the Advection, the Burgers, the Boussinesq, and the Fisher equations, and others [1].

Several analytical and numerical techniques were successfully applied to deal with differential equations, fractional differential equations, and local fractional differential equations [1–10]. The techniques include the heat-balance integral [11], the fractional Fourier [12], the fractional Laplace transform [12], the harmonic wavelet [13, 14], the local fractional Fourier and Laplace transform [15], local fractional variational iteration [16–18], the local fractional decomposition [19], and the generalized local fractional Fourier transform [20] methods.

In this paper, we investigate the application of local fractional Adomian decomposition method and local fractional function decomposition method for solving the local fractional Laplace equation [21, 22] with the different fractal conditions.

This paper is organized as follows. In Section 2, the basic mathematical tools are reviewed. Section 3 presents briefly the local fractional Adomian decomposition method and the local fractional function decomposition method. Section 4 presents solutions to the local fractional Laplace equation with differential fractal conditions.

2. Mathematical Fundamentals

We recall in this section the notations and some properties of the local fractional operators [15–20, 23, 24].

Definition 1 (see [15–20, 23, 24]). The function $f(x)$ is local fractional continuous at $x = x_0$, if it is valid for

$$\left| f(x) - f(x_0) \right| < \varepsilon^\alpha, \quad 0 < \alpha \leq 1 \tag{1}$$

with $|x - x_0| < \delta$, for $\varepsilon > 0$ and $\varepsilon \in R$. For $x \in (a, b)$, it is so called local fractional continuous on the interval (a, b), denoted by $f(x) \in C_\alpha(a, b)$.

We notice that there are existence conditions of local fractional continuities that operating functions are right-hand

and left-hand local fractional continuities. Meanwhile, the right-hand local fractional continuity is equal to its left-hand local fractional continuity. For more details, see [20].

Definition 2 (see [15–20, 23, 24]). The local fractional derivative of $f(x)$ at $x = x_0$ is defined as

$$
D_x^\alpha f(x_0) = \left. \frac{d^\alpha}{dx^\alpha} f(x) \right|_{x=x_0}
$$

$$
= f^{(\alpha)}(x) = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{2}
$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)\Delta(f(x) - f(x_0))$.

Local fractional derivative of high order is written in the form

$$
f^{(k\alpha)}(x) = \overbrace{D_x^\alpha D_x^\alpha \cdots D_x^\alpha}^{k \text{ times}} f(x). \tag{3}
$$

And local fractional partial derivative of high order is written in the form

$$
\frac{\partial^{k\alpha} f(x,y)}{x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x,y). \tag{4}
$$

Definition 3 (see [15–20, 23, 24]). A partition of the interval $[a,b]$ is denoted by (t_j, t_{j+1}), $j = 0, \ldots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots\}$. Local fractional integral of $f(x)$ in the interval $[a,b]$ is given by

$$
{}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha
$$

$$
= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha. \tag{5}
$$

If the functions are local fractional continuous, then the local fractional derivatives and integrals exist. Some properties of local fractional derivative and integrals are given in [20].

Definition 4. Let $f(x)$ be $2l$-periodic. For $k \in Z$ and $f(x) \in C_\alpha(a,b)$, the local fraction Fourier series of $f(x)$ is defined as (see [15, 25])

$$
f(x) = \frac{a_0}{2} + \sum_{k=1}^\infty \left(a_k \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} + b_k \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} \right), \tag{6}
$$

where $a_k = (1/l^\alpha) \int_{-1}^1 f(x)\cos_\alpha(\pi^\alpha(kx)^\alpha/l^\alpha)(dx)^\alpha$

$$
b_k = \frac{1}{l^\alpha} \int_{-1}^1 f(x) \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha}(dx)^\alpha \tag{7}
$$

are local fractional Fourier coefficients.

Definition 5. Let $(1/\Gamma(1+\alpha)) \int_0^\infty |f(x)|(dx)^\alpha < k < \infty$. The Yang-Laplace transforms of $f(x)$ are given by [15, 22]

$$
L_\alpha \{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha,
$$

$$
0 < \alpha \le 1, \tag{8}
$$

where the latter integral converges and $s^\alpha \in R^\alpha$.

Definition 6. The inverse formula of the Yang-Laplace transforms of $f(x)$ is given by [15, 22]

$$
L_\alpha^{-1} \{f_s^{L,\alpha}(s)\}
$$

$$
= f(t) = \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha, \tag{9}
$$

$$
0 < \alpha \le 1,
$$

where $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$; fractal imaginary unit i^α and $\mathrm{Re}(s) = \beta > 0$.

3. Analytical Methods

In order to illustrate two analytical methods, we investigate the nonlinear local fractional equation of order 2α as follows:

$$
\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + k_1 \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + k_2 \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + k_3 \frac{\partial^\alpha u(x,t)}{\partial x^\alpha}
$$

$$
= f(x,t) \tag{10}
$$

with constants $k_1, k_2, k_3, 0 < \alpha \le 1$ and with boundary and initial conditions

$$
u(0,t) = u(l,t) = 0
$$

$$
u(x,0) = \varphi(x) \tag{11}
$$

$$
\frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = \psi(x).
$$

3.1. Local Fractional Adomian Decomposition Method. We rewrite (10) in the following form:

$$
L_{tt}^{(2\alpha)} u(x,t) + k_1 L_t^{(\alpha)} u(x,t) + k_2 L_{xx}^{(2\alpha)} u(x,t) + k_3 L_x^{(\alpha)} u(x,t)
$$

$$
= f(x,t). \tag{12}
$$

Applying the inverse operator $L_{tt}^{(-2\alpha)}$ to both sides of (12) yields

$$L_{tt}^{(-2\alpha)} L_{tt}^{(2\alpha)} u(x,t)$$

$$= L_{tt}^{(-2\alpha)} \left(-k_1 L_s^{(\alpha)} u(x,s) - k_2 L_{xx}^{(2\alpha)} u(x,s) \right.$$

$$\left. - k_3 L_x^{(\alpha)} u(x,s) + f(x,s) \right)$$

$$\tag{13}$$

$$u(x,t) = r(x,t) + L_{tt}^{(-2\alpha)} \left(f(x,s) \right)$$

$$+ L_{tt}^{(-2\alpha)} \left(-k_1 L_s^{(\alpha)} u(x,s) - k_2 L_{xx}^{(2\alpha)} u(x,s) \right.$$

$$\left. - k_3 L_x^{(\alpha)} u(x,s) \right),$$

where the term $r(x,t)$ is to be determined from the fractal initial conditions.

Now, we decompose the unknown function $u(x,t)$ as a sum of components defined by the series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).\tag{14}$$

The components $u_n(x,t)$ are obtained by the recursive formula:

$$u_0(x,t) = r(x,t) + L_{tt}^{(-2\alpha)} \left(f(x,s) \right)$$

$$u_{n+1}(x,t) = L_{tt}^{(-2\alpha)} \left(-k_1 L_s^{(\alpha)} u_n(x,s) - k_2 L_{xx}^{(2\alpha)} u_n(x,s) \right.$$

$$\left. - k_3 L_x^{(\alpha)} u_n(x,s) \right), \quad n \geq 0.$$

$$\tag{15}$$

3.2. Local Fractional Function Decomposition Method. According to the decomposition of the local fractional function, with respect to the system $\{\sin_\alpha n^\alpha(\pi x/l)^\alpha\}$, the following functions coefficients can be given by

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha$$

$$\varphi(x) = \sum_{n=1}^{\infty} C_n \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha$$

$$\psi(x) = \sum_{n=1}^{\infty} D_n \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha,$$

$$\tag{16}$$

where

$$f_n(t) = \frac{2}{l^\alpha} \int_0^1 f(x,t) \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha (dx)^\alpha$$

$$C_n = \frac{2}{l^\alpha} \int_0^1 \varphi(x) \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha (dx)^\alpha \tag{17}$$

$$D_n = \frac{2}{l^\alpha} \int_0^1 \varphi(x) \sin_\alpha n^\alpha \left(\frac{\pi x}{l} \right)^\alpha (dx)^\alpha.$$

Substituting (16) into (10) implies that

$$\frac{\partial^{2\alpha} v_n(t)}{\partial t^{2\alpha}} + k_1 \frac{\partial^\alpha v_n(t)}{\partial t^\alpha} + k_2 \left(\frac{n\pi}{l} \right)^{2\alpha} v_n(t) + k_3 \left(\frac{n\pi}{l} \right)^\alpha v_n(t)$$

$$= f_n(t)$$

$$v_n(0) = C_n, \qquad v_n'(0) = D_n.$$

$$\tag{18}$$

Suppose that the Yang-Laplace transforms of functions $v_n(t)$ and $f_n(t)$ are $V_n(s)$ and $F_n(s)$, respectively. Then, we obtain

$$s^{2\alpha} V_n(s) - C_n s^\alpha - D_n + k_1 \left(s^\alpha V_n(s) - C_n \right) + k_2 \left(\frac{n\pi}{l} \right)^{2\alpha} V_n(s)$$

$$+ k_3 \left(\frac{n\pi}{l} \right)^\alpha V_n(s) = F_n(s).$$

$$\tag{19}$$

That is,

$$V_n(s) = \frac{D_n + k_1 C_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha}$$

$$+ \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha}.$$

$$\tag{20}$$

Hence, we have

$$v_n(t)$$

$$= L_\alpha^{-1} \{ V_n(s) \}$$

$$= \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha \left(s^\alpha x^\alpha \right) V_n(s) (ds)^\alpha$$

$$= \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha \left(s^\alpha x^\alpha \right)$$

$$\times \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} (ds)^\alpha$$

$$+ \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha \left(s^\alpha x^\alpha \right)$$

$$\times \frac{D_n + k_1 C_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} (ds)^\alpha.$$

$$\tag{21}$$

Let $v_n(t) = v_{1,n}(t) + v_{2,n}(t)$

$$v_{1,n}(t)$$
$$= \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha\left(s^\alpha x^\alpha\right)$$
$$\times \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha}(ds)^\alpha$$
$$(22)$$

$$v_{2,n}(t)$$
$$= \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha\left(s^\alpha x^\alpha\right)$$
$$\times \frac{D_n + k_1 C_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha}(ds)^\alpha.$$
$$(23)$$

Hence, we get

$$V_{1,n}(s) = \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha}$$
$$(24)$$
$$V_{2,n}(s) = \frac{D_n + k_1 C_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha}.$$

Then, making use of (8) and (9) and rearranging integration sequence, we have the following several formulas about $v_{1,n}(t)$ and $v_{2,n}(t)$.

If $-(1/4)k_1^2 + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha > 0$, then

$$s^{2\alpha} + k_1 s + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^\alpha = \left(s^\alpha + \frac{k_1}{2}\right)^2 + D_n',$$
$$(25)$$

where $D_n' = \sqrt{-(1/4)k_1^2 + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha}$.

Then, we get

$$v_{1,n}(t) = \frac{1}{\Gamma(1+\alpha)D_n'}$$
$$\times \int_0^t E_\alpha\left(\frac{-k_1 \tau^\alpha}{2^\alpha}\right) \sin_\alpha\left(D_n'\tau^\alpha\right) f_n(t-\tau)(d\tau)^\alpha$$

$$v_{2,n}(t) = C_n E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right)\cos_\alpha\left(D_n' t^\alpha\right)$$
$$+ \left(D_n + k_1 C_n - \frac{k_1}{2}\right) E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right)\sin_\alpha\left(D_n' t^\alpha\right).$$
$$(26)$$

In case $-(1/4)k_1^2 + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha < 0$ and $-(1/4)k_1^2 + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^\alpha = 0$, see [26].

4. Solutions of Local Fractional Laplace Equation in Fractal Time-Space

In this section, two examples for Laplace equation are presented in order to demonstrate the simplicity and the efficiency of the above methods.

The local fractional Laplace equation (see [21]) is one of the important differential equations with local fractional derivatives. In the following, we consider solutions to local fractional Laplace equations in fractal time-space.

Example 7. Consider the following local fractional Laplace equation:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^2 u(x,t)}{\partial x^{2\alpha}} = 0 \qquad (27)$$

subject to the fractal value conditions

$$u(x,0) = -E_\alpha\left(x^\alpha\right), \qquad \frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = 0. \qquad (28)$$

According to formula (15), we have

$$u_0(x,t) = r(x,t) + L_{tt}^{(-2\alpha)}\left(f(x,s)\right),$$
$$u_{n+1}(x,t) = L_{tt}^{(-2\alpha)}\left(-k_1 L_s^{(\alpha)} u_n(x,s) - k_2 L_{xx}^{(2\alpha)} u_n(x,s)\right.$$
$$\left. - k_3 L_x^{(\alpha)} u_n(x,s)\right),$$
$$(29)$$

where

$$u_0(x,t) = -E_\alpha\left(x^\alpha\right). \qquad (30)$$

Hence, from (29) we obtain

$$u_{n+1}(x,t) = L_{tt}^{(-2\alpha)}\left\{-L_{xx}^{(2\alpha)} u_n(x,s)\right\},$$
$$= {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)}\left\{-\frac{\partial^{2\alpha} u_n(x,s)}{\partial x^{2\alpha}}\right\}, \qquad (31)$$
$$n \geq 1,$$

where

$$u_0(x,t) = -E_\alpha\left(x^\alpha\right). \qquad (32)$$

Making use of (31), we present

$$u_1(x,t) = {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)}\left\{-\frac{\partial^{2\alpha} u_0(x,s)}{\partial x^{2\alpha}}\right\}$$
$$= {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)}\{E_\alpha\left(x^\alpha\right)\}$$
$$= \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha\left(x^\alpha\right)$$

$$u_2(x,t) = {}_0I_t^{(\alpha)} \, {}_0I_t^{(\alpha)} \left\{ -\frac{\partial^{2\alpha} u_1(x,s)}{\partial x^{2\alpha}} \right\}$$

$$= {}_0I_t^{(\alpha)} \, {}_0I_t^{(\alpha)} \left\{ -\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha(x^\alpha) \right\}$$

$$= -\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} E_\alpha(x^\alpha)$$

$$u_3(x,t) = {}_0I_t^{(\alpha)} \, {}_0I_t^{(\alpha)} \left\{ -\frac{\partial^{2\alpha} u_2(x,s)}{\partial x^{2\alpha}} \right\}$$

$$= {}_0I_t^{(\alpha)} \, {}_0I_t^{(\alpha)} \left\{ \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} E_\alpha(x^\alpha) \right\}$$

$$= \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} E_\alpha(x^\alpha).$$

$$(33)$$

Proceeding in this manner, we get

$$u_n(x,t) = E_\alpha(x^\alpha)(-1)^{n+1} \frac{t^{2n\alpha}}{\Gamma(1+2n\alpha)}. \qquad (34)$$

Thus, the final solution reads as follows:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= E_\alpha(x^\alpha) \left[-1 + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \right.$$

$$\left. + \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} \cdots \right]$$

$$= -E_\alpha(x^\alpha) \left[1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \right.$$

$$\left. - \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} \cdots \right]$$

$$= -E_\alpha(x^\alpha) \cos_\alpha(t^\alpha).$$

$$(35)$$

Now, we solve Example 7 by using the local fractional function decomposition method.

We suppose that

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t) E_\alpha(n^\alpha x^\alpha)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) E_\alpha(n^\alpha x^\alpha)$$

$$(36)$$

$$\varphi(x) = \sum_{n=1}^{\infty} C_n E_\alpha(n^\alpha x^\alpha)$$
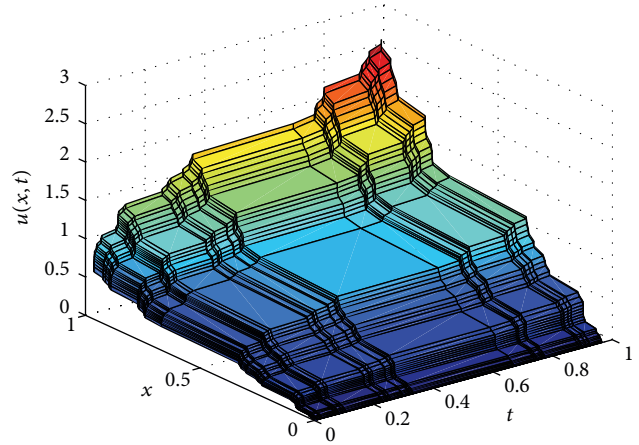
$$\psi(x) = \sum_{n=1}^{\infty} D_n E_\alpha(n^\alpha x^\alpha)$$

FIGURE 1: Exact solution for local fractional Laplace equation with fractal dimension $\alpha = \ln 2 / \ln 3$.

which leads to

$$f_n(t) = 0, \quad \forall n; \qquad C_n = 0, \quad n \neq 1; \qquad C_1 = -1;$$

$$D_n = 0, \quad \forall n.$$

$$(37)$$

Contrasting (28) with (36), we directly get $k_1 = 0$, $k_2 = 1$, and $k_3 = 0$ and

$$D'_n = 0, \quad n \neq 1; \qquad D'_1 = 1$$

$$v_n(t) = 0, \quad n \neq 1$$

$$(38)$$

$$v_{1,1}(t) = \frac{1}{\Gamma(1+\alpha) D'_1}$$

$$\times \int_0^t E_\alpha\left(\frac{-k_1 \tau^\alpha}{2^\alpha}\right) \sin_\alpha\left(D'_1 \tau^\alpha\right) f_1(t-\tau)(d\tau)^\alpha = 0$$

$$(39)$$

$$v_{2,1}(t) = C_1 E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right) \cos_\alpha\left(D'_1 t^\alpha\right)$$

$$+ \left(D_1 + k_1 C_1 - \frac{k_1}{2}\right) E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right) \sin_\alpha\left(D'_1 t^\alpha\right)$$

$$= -\cos_\alpha(t^\alpha).$$

$$(40)$$

Conclusively, we get

$$v_1(t) = v_{1,1}(t) + v_{2,1}(t) = -\cos_\alpha(t^\alpha). \qquad (41)$$

Thus, we obtain

$$u(x,t) = -E_\alpha(x^\alpha) \cos_\alpha(t^\alpha) \qquad (42)$$

and its graph is shown in Figure 1.

Example 8. We consider the following local fractional Laplace equation:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0 \qquad (43)$$

subject to the fractal value conditions

$$u(x, 0) = 0, \quad \frac{\partial^\alpha u(x, 0)}{\partial t^\alpha} = -E_\alpha(x^\alpha). \tag{44}$$

Now we can structure the same local fractional iteration procedure (15). Hence, we have

$$u_0(x, t) = -\frac{t^\alpha}{\Gamma(1 + \alpha)} E_\alpha(x^\alpha)$$

$$u_1(x, t) = {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)} \left\{ -\frac{\partial^{2\alpha} u_0(x, s)}{\partial x^{2\alpha}} \right\}$$

$$= {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)} \left\{ \frac{t^\alpha}{\Gamma(1 + \alpha)} E_\alpha(x^\alpha) \right\}$$

$$= \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} E_\alpha(x^\alpha)$$

$$u_2(x, t) = {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)} \left\{ -\frac{\partial^{2\alpha} u_1(x, s)}{\partial x^{2\alpha}} \right\}$$

$$= {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)} \left\{ -\frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} E_\alpha(x^\alpha) \right\} \tag{45}$$

$$= -\frac{t^{5\alpha}}{\Gamma(1 + 5\alpha)} E_\alpha(x^\alpha)$$

$$u_3(x, t) = {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)} \left\{ -\frac{\partial^{2\alpha} u_2(x, s)}{\partial x^{2\alpha}} \right\}$$

$$= {}_0I_t^{(\alpha)} {}_0I_t^{(\alpha)} \left\{ \frac{t^{5\alpha}}{\Gamma(1 + 5\alpha)} E_\alpha(x^\alpha) \right\}$$

$$= \frac{t^{7\alpha}}{\Gamma(1 + 7\alpha)} E_\alpha(x^\alpha).$$

Finally, we can obtain the local fractional series solution as follows:

$$u_n(x, t) = (-1)^{n+1} \frac{t^{(2n+1)\alpha}}{\Gamma(1 + (2n+1)\alpha)} E_\alpha(x^\alpha). \tag{46}$$

Thus, the final solution reads as follows:

$$u(x, t)$$

$$= \sum_{n=0}^\infty u_n(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots$$

$$= E_\alpha(x^\alpha) \left[-\frac{t^\alpha}{\Gamma(1 + \alpha)} + \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} - \frac{t^{5\alpha}}{\Gamma(1 + 5\alpha)} \cdots \right]$$

$$= -E_\alpha(x^\alpha) \left[\frac{t^\alpha}{\Gamma(1 + \alpha)} - \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} + \frac{t^{5\alpha}}{\Gamma(1 + 5\alpha)} \cdots \right]$$

$$= -E_\alpha(x^\alpha) \sin_\alpha(t^\alpha). \tag{47}$$

Now, we solve Example 8 by using the local fractional function decomposition method.

We suppose that

$$u(x, t) = \sum_{n=1}^\infty v_n(t) E_\alpha(n^\alpha x^\alpha)$$

$$f(x, t) = 0 = \sum_{n=1}^\infty f_n(t) E_\alpha(n^\alpha x^\alpha)$$

$$\varphi(x) = 0 = \sum_{n=1}^\infty C_n E_\alpha(n^\alpha x^\alpha) \tag{48}$$

$$\psi(x) = -E_\alpha(x^\alpha) = \sum_{n=1}^\infty D_n E_\alpha(n^\alpha x^\alpha)$$

which leads to

$$f_n(t) = 0, \quad \forall n; \qquad D_n = 0, \quad n \neq 1; \qquad D_1 = -1;$$

$$C_n = 0, \quad \forall n. \tag{49}$$

Contrasting (28) with (36), we directly get $k_1 = 0$, $k_2 = 1$, and $k_3 = 0$ and

$$D_n' = 0, \quad n \neq 1; \qquad D_1' = 1$$

$$v_n(t) = 0, \quad n \neq 1$$

$$v_{1,1}(t) = \frac{1}{\Gamma(1 + \alpha) D_1'}$$

$$\times \int_0^t E_\alpha \left(\frac{-k_1 \tau^\alpha}{2^\alpha} \right) \sin_\alpha \left(D_1' \tau^\alpha \right) f_1(t - \tau) (d\tau)^\alpha$$

$$= 0$$

$$v_{2,1}(t) = C_1 E_\alpha \left(\frac{-k_1 t^\alpha}{2^\alpha} \right) \cos_\alpha \left(D_1' t^\alpha \right)$$

$$+ \left(D_1 + k_1 C_1 - \frac{k_1}{2} \right) E_\alpha \left(\frac{-k_1 t^\alpha}{2^\alpha} \right) \sin_\alpha(t^\alpha)$$

$$= -\sin_\alpha(t^\alpha). \tag{50}$$

Conclusively, we get

$$v_1(t) = v_{1,1}(t) + v_{2,1}(t) = -\sin_\alpha(t^\alpha). \tag{51}$$

Thus, we obtain

$$u(x, t) = -E_\alpha(x^\alpha) \sin_\alpha(t^\alpha) \tag{52}$$

and its graph is given in Figure 2.

5. Conclusions

In this work solving the Laplace equations using the local fractional function decomposition method with local fractional operators is discussed in detail. Two examples of
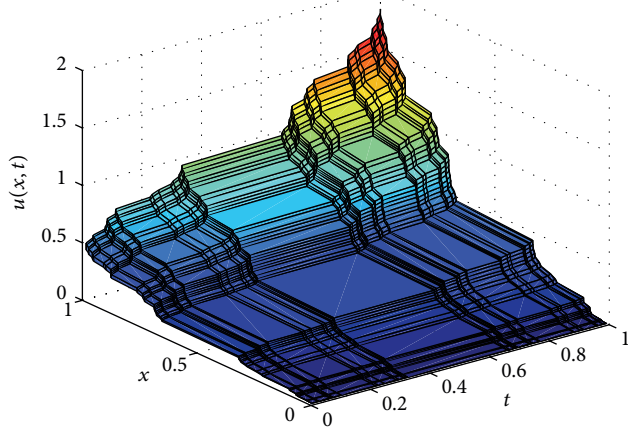
FIGURE 2: The plot of solution to local fractional Laplace equation with fractal dimension $\alpha = \ln 2/\ln 3$.

applications of the local fractional Adomian decomposition method and local fractional function decomposition method to the local fractional Laplace equations are investigated in detail. The reliable obtained results are complementary with the ones presented in the literature.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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