Research Article

Local Fractional Fourier Series with Application to Wave Equation in Fractal Vibrating String

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We introduce the wave equation in fractal vibrating string in the framework of the local fractional calculus. Our particular attention is devoted to the technique of the local fractional Fourier series for processing these local fractional differential operators in a way accessible to applied scientists. By applying this technique we derive the local fractional Fourier series solution of the local fractional wave equation in fractal vibrating string and show the fundamental role of the Mittag-Leffler function.

1. Introduction

Fractional calculus arises in many problems of physics, continuum mechanics, visco-elasticity, and quantum mechanics, and other branches of applied mathematics and nonlinear dynamics have been studied [1–7]. In general, the fractional analogues are obtained by changing the classical time derivative by a fractional one, which can be Riemann-Liouville, Caputo, or another one. Many classical partial differential equations possess a fractional analogue, like the fractional diffusion-wave equation [8–12], the fractional diffusion equation [13–16], the fractional wave equation [17, 18], the fractional Schrödinger equation [19, 20], the fractional heat equation [21], the fractional KdV equation [22], the fractional Fokker-Planck equations [23], the fractional Fick's law [24], the fractional evolution equation [25], the Fractional Heisenberg equation [26], the fractional Ginzburg-Landau equation [27], Fractional hydrodynamic equation [28], the fractional seepage flow equation [29], and the fractional KdV-Burgers equation [30].

There also are other methods for solving fractional differential equations, for example, the fractional variational iteration method [31, 32] and the fractional complex transform [33–37]. In all of the methods mentioned above, the solutions of the fractional differential equations should be analytical if the fractional derivative is in the Caputo or Riemann-Liouville sense. However, some solutions to ordinary and partial differential equations are fractal curves. As a result, we cannot employ the classical Fourier series, which requires that the defined functions should be differentiable, to describe some solutions to ordinary and partial differential equations in fractal space. However, based on the modified Riemann-Liouville derivative, Jumarie structured a Jumurie's calculus of fractional order [38] (which is one of useful tools to deal with everywhere continuous but nowhere differentiable functions) and its applications were taken into account in Probability calculus of fractional order [39], Laplace transform of fractional order via the Mittag-Leffler function (in convenient Hilbert space) [40], and adomian decomposition method for nonsmooth initial value problems [41]. Local fractional calculus is revealed as one of useful tools to deal with everywhere continuous but nowhere differentiable functions in areas ranging from fundamental science to engineering [42–57]. For these merits, local fractional calculus was successfully applied in the local fractional Laplace problems [53, 54], local fractional Fourier analysis [53, 54], local fractional short time transform [53, 54], local fractional wavelet transform [53–55], fractal signal [55, 56], and local fractional variational calculus [57].

In this paper we introduce a local fractional wave equation in fractal vibrating string which is described as

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + a^{2\alpha}\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \tag{1.1}$$

with fractal boundary conditions

$$u(0,t) = u(l,t) = 0,$$

$$u(0,t) = \frac{\partial^{\alpha} u(l,t)}{\partial x^{\alpha}} = 0,$$

$$u(x,0) = f(x),$$

$$\frac{\partial^{\alpha} u(l,0)}{\partial x^{\alpha}} = g(x), \tag{1.2}$$

where $\partial^{2\alpha} u(x,t)/\partial t^{2\alpha}$, $\partial^{2\alpha} u(x,t)/\partial x^{2\alpha}$, $\partial^{\alpha} u(l,0)/\partial x^{\alpha}$, and $\partial^{\alpha} u(l,t)/\partial x^{\alpha}$ are local fractional partial differential operator, and where $u(x,t)$ is local fractional continuous (for more details, see [53, 54]). We study the technique of the local fractional Fourier series for treating the local fractional wave equation in fractal vibrating string. This paper is organized as follows. In Section 2, we specify and investigate the concepts of local fractional calculus and local fractional Fourier series. In Section 3, we present the solving process for local fractional wave equation with local fractional derivative. In Section 4, we study the expression solution with Mittag-Leffler functions in fractal space. Finally, Section 5 is conclusions.

2. Preliminaries

In this section we start with local fractional continuity of functions, and we introduce the notions of local fractional calculus and local fractional Fourier series.

2.1. Local Fractional Continuity of Functions

In order to discuss the local fractional continuity of nondifferential functions on fractal sets, we first consider the following results.

Lemma 2.1 (see [57]). *Let F be a subset of the real line and be a fractal. If $f : (F, d) \rightarrow (\Omega', d')$ is a bi-Lipschitz mapping, then there are for constants $\rho, \tau > 0$, and $F \subset R$,*

$$\rho^s H^s(F) \leq H^s(f(F)) \leq \tau^s H^s(F), \tag{2.1}$$

such that for all $x_1, x_2 \in F$,

$$\rho^\alpha |x_1 - x_2|^\alpha \leq |f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha. \tag{2.2}$$

As a direct result of Lemma 2.1, we have, [57],

$$|f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha, \tag{2.3}$$

such that

$$|f(x_1) - f(x_2)| < \varepsilon^\alpha, \tag{2.4}$$

where α is fractal dimension of F. The result that is directly deduced from fractal geometry is related to fractal coarse-grained mass function $\gamma^\alpha[F, a, b]$, which reads, [57],

$$\gamma^\alpha[F, a, b] = \frac{H^\alpha(F \cap (a, b))}{\Gamma(1 + \alpha)}, \tag{2.5}$$

with

$$H^\alpha(F \cap (a, b)) = (b - a)^\alpha, \tag{2.6}$$

where H^α is α dimensional Hausdorff measure.

Notice that we consider the dimensions of any fractal spaces (e.g., Cantor spaces or like-Cantor spaces) as a positive number. It looks like Euclidean space because its dimension is also a positive number. The detailed results had been considered in [53, 54, 57].

Definition 2.2. If there exists, [53, 57],

$$|f(x) - f(x_0)| < \varepsilon^\alpha, \tag{2.7}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. $f(x)$ is called local fractional continuous on the interval (a, b), denoted by

$$f(x) \in C_\alpha(a, b), \tag{2.8}$$

if (2.7) is valid for $x \in (a, b)$.

Definition 2.3. If a function $f(x)$ is called a nondifferentiable function of exponent α, $0 < \alpha \leq 1$, which satisfies Hölder function of exponent α, then for $x, y \in X$ such that, [54, 57],

$$|f(x) - f(y)| \leq C|x - y|^\alpha. \tag{2.9}$$

Definition 2.4. A function $f(x)$ is called to be continuous of order α, $0 < \alpha \leq 1$, or shortly α continuous, when we have that, [54, 57],

$$f(x) - f(x_0) = o((x - x_0)^\alpha). \tag{2.10}$$

Remark 2.5. Compared with (2.10), (2.7) is standard definition of local fractional continuity. Here (2.9) is unified local fractional continuity [57].

2.2. Local Fractional Derivatives and Integrals

Definition 2.6 (let $f(x) \in C_\alpha(a, b)$). Local fractional derivative of $f(x)$ of order α at $x = x_0$ is given, [53–57],

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{2.11}$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1 + \alpha)\Delta(f(x) - f(x_0))$.

For any $x \in (a, b)$, there exists, [53–57],

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x), \tag{2.12}$$

denoted by

$$f(x) \in D_x^{(\alpha)}(a, b). \tag{2.13}$$

Local fractional derivative of high order is derived as, [57],

$$f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)} \cdots D_x^{(\alpha)}}^{k \text{ times}} f(x), \tag{2.14}$$

and local fractional partial derivative of high order, [57],

$$\frac{\partial^{k\alpha} f(x)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k \text{ times}} f(x). \tag{2.15}$$

Definition 2.7 (let $f(x) \in C_{\alpha}(a,b)$). Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is given by, [53–57],

$$_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha}, \tag{2.16}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$, and $[t_j, t_{j+1}]$, $j = 0, \ldots, N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a,b]$.

For convenience, we assume that

$$_aI_a^{(\alpha)} f(x) = 0 \quad \text{if } a = b, \qquad _aI_b^{(\alpha)} f(x) = -_bI_a^{(\alpha)} f(x) \quad \text{if } a < b. \tag{2.17}$$

For any $x \in (a,b)$, we get, [53, 54, 57],

$$_aI_x^{(\alpha)} f(x), \tag{2.18}$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a,b). \tag{2.19}$$

Remark 2.8. If $f(x) \in D_x^{(\alpha)}(a,b)$, or $I_x^{(\alpha)}(a,b)$, we have that, [46, 47, 50],

$$f(x) \in C_{\alpha}(a,b). \tag{2.20}$$

2.3. Special Functions in Fractal Space

Definition 2.9. The Mittag-Leffler function in fractal space is defined by, [53, 57],

$$E_{\alpha}(x^{\alpha}) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}, \quad x \in R, \ 0 < \alpha \le 1. \tag{2.21}$$

Definition 2.10. The sine function in fractal space is given by the expression, [54, 57],

$$\sin_{\alpha} x^{\alpha} := \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]}, \quad x \in R, \ 0 < \alpha \le 1. \tag{2.22}$$

Definition 2.11. The cosine function in fractal space is given, [54, 57],

$$\cos_\alpha x^a := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}, \quad x \in R, \ 0 < \alpha \le 1. \tag{2.23}$$

The following rules hold [54, 57]:

$$E_\alpha(x^\alpha)E_\alpha(y^\alpha) = E_\alpha((x+y)^\alpha), \qquad E_\alpha(x^\alpha)E_\alpha(-y^\alpha) = E_\alpha((x-y)^\alpha),$$

$$E_\alpha(i^\alpha x^\alpha)E_\alpha(i^\alpha y^\alpha) = E_\alpha(i^\alpha(x+y)^\alpha), \qquad E_\alpha(i^\alpha x^\alpha) = \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha,$$

$$\sin_\alpha x^\alpha = \frac{E_\alpha(i^\alpha x^\alpha) - E_\alpha(-i^\alpha x^\alpha)}{2i^\alpha}, \qquad \cos_\alpha x^\alpha = \frac{E_\alpha(i^\alpha x^\alpha) + E_\alpha(-i^\alpha x^\alpha)}{2},$$

$$\cos_\alpha(-x)^\alpha = \cos_\alpha x^\alpha, \qquad \sin_\alpha(-x)^\alpha = -\sin_\alpha x^\alpha,$$

$$\cos_\alpha^2 x^\alpha + \sin^2_\alpha x^\alpha = 1, \qquad \sin_\alpha^2 x^\alpha = \frac{1 - \cos_\alpha(2x)^\alpha}{2},$$

$$\cos_\alpha^2 x^\alpha = \frac{1 + \cos_\alpha(2x)^\alpha}{2}, \qquad \tan_\alpha x^\alpha = \frac{\sin_\alpha(2x)^\alpha}{1 + \cos_\alpha(2x)^\alpha} = \frac{1 - \cos_\alpha(2x)^\alpha}{\sin_\alpha(2x)^\alpha},$$

$$\sin_\alpha(2x)^\alpha = 2\sin_\alpha x^\alpha \cos_\alpha x^\alpha, \qquad \cos_\alpha(2x)^\alpha = \cos_\alpha^2 x^\alpha - \sin_\alpha^2 x^\alpha,$$

$$\tan_\alpha(2y)^\alpha = \frac{2\tan_\alpha y^\alpha}{1 + \tan_\alpha^2 x^\alpha}, \qquad \sin_\alpha(2x)^\alpha = \frac{2\tan_\alpha x^\alpha}{1 + \tan_\alpha^2 x^\alpha},$$

$$\cos_\alpha(2x)^\alpha = \frac{1 - \tan_\alpha^2 x^\alpha}{1 + \tan_\alpha^2 x^\alpha}, \qquad \tan_\alpha(x+y)^\alpha = \frac{\tan_\alpha x^\alpha + \tan_\alpha y^\alpha}{1 + \tan_\alpha x^\alpha \tan_\alpha y^\alpha},$$

$$\cos_\alpha x^\alpha + \cos_\alpha y^\alpha = 2\cos_\alpha\left(\frac{x+y}{2}\right)^\alpha \cos_\alpha\left(\frac{x-y}{2}\right)^\alpha,$$

$$\cos_\alpha x^\alpha - \cos_\alpha y^\alpha = -2\sin_\alpha\left(\frac{x+y}{2}\right)^\alpha \sin_\alpha\left(\frac{x-y}{2}\right)^\alpha,$$

$$\sin_\alpha x^\alpha + \sin_\alpha y^\alpha = 2\sin_\alpha\left(\frac{x+y}{2}\right)^\alpha \cos_\alpha\left(\frac{x-y}{2}\right)^\alpha,$$

$$\sin_\alpha x^\alpha - \sin_\alpha y^\alpha = 2\cos_\alpha\left(\frac{x+y}{2}\right)^\alpha \sin_\alpha\left(\frac{x-y}{2}\right)^\alpha,$$

$$\cos_\alpha(x+y)^\alpha = \cos_\alpha x^\alpha \cos_\alpha y^\alpha - \sin_\alpha x^\alpha \sin_\alpha y^\alpha,$$

$$\cos_\alpha(x-y)^\alpha = \cos_\alpha x^\alpha \cos_\alpha y^\alpha + \sin_\alpha x^\alpha \sin_\alpha y^\alpha,$$

$$\sin_\alpha(x+y)^\alpha = \sin_\alpha x^\alpha \cos_\alpha y^\alpha + \cos_\alpha x^\alpha \sin_\alpha y^\alpha,$$

$$\cos_\alpha x^\alpha \cos_\alpha y^\alpha = \frac{\cos_\alpha(x+y)^\alpha + \cos_\alpha(x-y)^\alpha}{2},$$

$$\sin_\alpha x^\alpha \sin_\alpha y^\alpha = -\frac{\cos_\alpha (x+y)^\alpha - \cos_\alpha (x-y)^\alpha}{2},$$

$$\sin_\alpha x^\alpha \cos_\alpha y^\alpha = \frac{\sin_\alpha (x+y)^\alpha + \sin_\alpha (x-y)^\alpha}{2},$$

$$\sin_\alpha (mx)^\alpha \sin_\alpha (nx)^\alpha = \frac{\cos_\alpha ((m-n)x)^\alpha - \cos_\alpha ((m+n)x)^\alpha}{2},$$

$$\cos_\alpha (nx)^\alpha \sin_\alpha (mx)^\alpha = \frac{\sin_\alpha ((m+n)x)^\alpha - \sin_\alpha ((m-n)x)^\alpha}{2},$$

$$E_\alpha (i^\alpha (nx)^\alpha) = (\cos_\alpha (nx)^\alpha + i^\alpha \sin_\alpha (nx)^\alpha)^n,$$

$$\sum_{k=1}^{n} \sin_\alpha (nx)^\alpha = \frac{\sin_\alpha (nx/2)^\alpha}{\sin_\alpha (x/2)^\alpha} \sin_\alpha \left(\frac{(n+1)x}{2}\right)^\alpha, \quad \sin_\alpha \left(\frac{x}{2}\right)^\alpha \neq 0,$$

$$\sum_{k=1}^{n} \cos_\alpha (nx)^\alpha = \frac{\sin_\alpha (nx/2)^\alpha}{\sin_\alpha (x/2)^\alpha} \cos_\alpha \left(\frac{(n+1)x}{2}\right)^\alpha, \quad \sin_\alpha \left(\frac{x}{2}\right)^\alpha \neq 0,$$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos_\alpha (nx)^\alpha = \frac{\sin_\alpha ((2n+1)x/2)^\alpha}{2\sin_\alpha (x/2)^\alpha}, \quad \sin_\alpha \left(\frac{x}{2}\right)^\alpha \neq 0.$$

$$(2.24)$$

Remark 2.12. i^α is fractal imaginary unit, for more details, see [53–57].

2.4. Local Fractional Fourier Series

Definition 2.13. Suppose that $f(x) \in C_\alpha(-\infty, \infty)$ and $f(x)$ be $2l$-periodic. For $k \in Z$, local fractional Fourier series of $f(x)$ is defined as, [53–55],

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_n \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} + b_n \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} \right), \tag{2.25}$$

where

$$a_k = \frac{1}{l^\alpha} \int_{-l}^{l} f(x) \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} (dx)^\alpha,$$

$$b_k = \frac{1}{l^\alpha} \int_{-l}^{l} f(x) \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} (dx)^\alpha \tag{2.26}$$

are the local fractional Fourier coefficients.

For local fractional Fourier series (2.25), the weights of the fractional trigonometric functions are calculated as

$$
\begin{aligned}
a_k &= \frac{\int_{-l+t_0}^{l+t_0} f(x)\cos_\alpha\left(\pi^\alpha(kx)^\alpha/l^\alpha\right)(dx)^\alpha}{\int_{-l+t_0}^{l+t_0} \cos_\alpha^2\left(\pi^\alpha(kx)^\alpha/l^\alpha\right)(dx)^\alpha}, \\
a_k &= \frac{\int_{-l+t_0}^{l+t_0} f(x)\sin_\alpha\left(\pi^\alpha(kx)^\alpha/l^\alpha\right)(dx)^\alpha}{\int_{-l+t_0}^{l+t_0} \sin_\alpha^2\left(\pi^\alpha(kx)^\alpha/l^\alpha\right)(dx)^\alpha}.
\end{aligned}
\tag{2.27}
$$

Definition 2.14. Suppose that $f(x) \in C_\alpha(-\infty, \infty)$ and $f(x)$ be $2l$-periodic. For $k \in Z$, complex generalized Mittag-Leffler form of local fractional Fourier series of $f(x)$ is defined as, [53, 54],

$$
f(x) = \sum_{k=-\infty}^{\infty} C_k E_\alpha\left(\frac{\pi^\alpha i^\alpha(kx)^\alpha}{l^\alpha}\right),
\tag{2.28}
$$

where the local fractional Fourier coefficients is

$$
C_k = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_\alpha\left(\frac{-\pi^\alpha i^\alpha(kx)^\alpha}{l^\alpha}\right)(dx)^\alpha \quad \text{with } k \in Z.
\tag{2.29}
$$

The above generalized forms of local fractional series are valid and are also derived from the generalized Hilbert space [53, 54].

For local fractional Fourier series (2.28), the weights of the Mittag-Leffler functions are written in the form

$$
C_k = \frac{(1/(2l)^\alpha) \int_{-l+t_0}^{l+t_0} f(x) E_\alpha\left(-\pi^\alpha i^\alpha(kx)^\alpha/l^\alpha\right)(dx)^\alpha}{(1/(2l)^\alpha) \int_{-l+t_0}^{l+t_0} E_\alpha\left(-\pi^\alpha i^\alpha(kx)^\alpha/l^\alpha\right)\overline{E_\alpha\left(-\pi^\alpha i^\alpha(kx)^\alpha/l^\alpha\right)}(dx)^\alpha}.
\tag{2.30}
$$

Above is generalized to calculate local fractional Fourier series.

3. Solutions to Wave Equation with Fractal Vibrating String

Now we look for particular solutions of the form

$$
u(x, t) = \phi(x)T(t),
\tag{3.1}
$$

and arrive at the equations

$$
\phi^{(2\alpha)} + \lambda^{2\alpha}\phi = 0,
\tag{3.2}
$$

$$
T^{(2\alpha)} + a^{2\alpha}\lambda^{2\alpha}T = 0,
\tag{3.3}
$$

with the boundary conditions

$$\phi(0) = \phi^{(\alpha)}(l) = 0. \tag{3.4}$$

Equation has the solution

$$\phi(x) = C_1 \cos_\alpha \lambda^\alpha x^\alpha + C_2 \sin_\alpha \lambda^\alpha x^\alpha \quad (C_1 = \cos t,\ C_2 = \cos t). \tag{3.5}$$

According to (3.4), for $x = 0$ and $x = l$ we derive as

$$\phi(0) = C_1 = 0,$$
$$\phi(l) = \phi(x)\big|_{x=l} = C_2 \sin_\alpha \lambda^\alpha l^\alpha = 0. \tag{3.6}$$

Assuming that $C_2 \neq 0$, since otherwise $\phi(x)$ is identically zero, we find that

$$\lambda_n^\alpha l^\alpha = n^\alpha \pi^\alpha, \tag{3.7}$$

where n is an integer; we write

$$\lambda_n^\alpha = \left(\frac{n\pi}{l}\right)^\alpha \quad (n = 0, 1, 2, \ldots),$$
$$\phi_n(x) = \sin_\alpha \lambda_n^\alpha x^\alpha = \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha = 0 \quad (n = 0, 1, 2, \ldots). \tag{3.8}$$

For $\lambda^\alpha = \lambda_n^\alpha$ equation (3.3) leads to

$$T_n(t) = A_n \cos_\alpha a^\alpha \lambda_n^\alpha t^\alpha + B_n \sin_\alpha a^\alpha \lambda_n^\alpha t^\alpha \quad (n = 0, 1, 2, \ldots), \tag{3.9}$$

and therefore

$$u_n(x,t) = (A_n \cos_\alpha a^\alpha \lambda_n^\alpha t^\alpha + B_n \sin_\alpha a^\alpha \lambda_n^\alpha t^\alpha) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha \quad (n = 0, 1, 2, \ldots). \tag{3.10}$$

To solve our problem, we form the local fractional Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$
$$= \sum_{n=1}^{\infty} (A_n \cos_\alpha a^\alpha \lambda_n^\alpha t^\alpha + B_n \sin_\alpha a^\alpha \lambda_n^\alpha t^\alpha) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha, \tag{3.11}$$

and require that

$$u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} A_n \sin_\alpha n \left(\frac{\pi x}{l}\right)^\alpha = f(x),$$

$$\frac{\partial^\alpha u(l,0)}{\partial x^\alpha} = \sum_{n=1}^{\infty} (-A_n a^\alpha \lambda_n^\alpha \sin_\alpha a^\alpha \lambda_n^\alpha t^\alpha + B_n a^\alpha \lambda_n^\alpha \cos_\alpha a^\alpha \lambda_n^\alpha t^\alpha) \sin_\alpha n \left(\frac{\pi x}{l}\right)^\alpha \Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n a^\alpha \lambda_n^\alpha \sin_\alpha n \left(\frac{\pi x}{l}\right)^\alpha$$

$$= g(x).$$

(3.12)

A calculation of local fractional Fourier coefficients of $f(x)$ and $g(x)$ with respect to the system $\{\sin_\alpha n^\alpha (\pi x/l)^\alpha\}$ is given by

$$A_n = \frac{\int_0^l f(x) \sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{\int_0^l \sin_\alpha^2 n^\alpha (\pi x/l)^\alpha (dx)^\alpha} \quad (n = 0, 1, 2, \ldots),$$

(3.13)

$$B_n a^\alpha \lambda_n^\alpha = \frac{\int_0^l g(x) \sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{\int_0^l \sin_\alpha^2 n^\alpha (\pi x/l)^\alpha (dx)^\alpha} \quad (n = 0, 1, 2, \ldots).$$

(3.14)

But $\int_0^l \sin_\alpha^2 n^\alpha (\pi x/l)^\alpha (dx)^\alpha = l^\alpha/2$ and therefore

$$A_n = \frac{2}{l^\alpha} \int_0^l f(x) \sin_\alpha \lambda_n^\alpha x^\alpha (dx)^\alpha \quad (n = 0, 1, 2, \ldots),$$

(3.15)

$$B_n = \frac{2}{a^\alpha \lambda_n^\alpha l^\alpha} \int_0^l g(x) \sin_\alpha \lambda_n^\alpha x^\alpha (dx)^\alpha \quad (n = 0, 1, 2, \ldots).$$

(3.16)

Thus, the solution of our problem is given by formula (3.11), where local fractional Fourier coefficients are determined. From (3.14) and (3.16), we get the harmonic vibrations

$$u_n(x,t) = (A_n \cos_\alpha a^\alpha \lambda_n^\alpha t^\alpha + B_n \sin_\alpha a^\alpha \lambda_n^\alpha t^\alpha) \sin_\alpha \lambda_n^\alpha x^\alpha,$$

(3.17)

where

$$A_n = \frac{2}{l^\alpha} \int_0^l f(x) \sin_\alpha \lambda_n^\alpha x^\alpha (dx)^\alpha \quad (n = 0, 1, 2, \ldots),$$

$$B_n = \frac{1}{2/a^\alpha \lambda_n^\alpha l^\alpha} \int_0^l g(x) \sin_\alpha \lambda_n^\alpha x^\alpha (dx)^\alpha \quad (n = 0, 1, 2, \ldots).$$

(3.18)

4. Expression Solutions with Mittag-Leffler Functions in Fractal Space

Taking into account the relations, [57],

$$\sin_\alpha x^\alpha = \frac{E_\alpha(i^\alpha x^\alpha) - E_\alpha(-i^\alpha x^\alpha)}{2i^\alpha},$$

$$\cos_\alpha x^\alpha = \frac{E_\alpha(i^\alpha x^\alpha) + E_\alpha(-i^\alpha x^\alpha)}{2}, \tag{4.1}$$

we obtain the harmonic vibration with the Mittag-Leffler functions in fractal space

$$
\begin{aligned}
u_n(x,t) &= \left(A_n \frac{E_\alpha(i^\alpha a^\alpha \lambda_n^\alpha t^\alpha) + E_\alpha(-i^\alpha a^\alpha \lambda_n^\alpha t^\alpha)}{2} + B_n \frac{E_\alpha(i^\alpha a^\alpha \lambda_n^\alpha t^\alpha) - E_\alpha(-i^\alpha a^\alpha \lambda_n^\alpha t^\alpha)}{2i^\alpha} \right) \\
&\quad \times \frac{E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha)}{2i^\alpha} \\
&= \left[\frac{A_n + B_n}{2} E_\alpha(i^\alpha a^\alpha \lambda_n^\alpha t^\alpha) + \frac{A_n - B_n}{2} E_\alpha(-i^\alpha a^\alpha \lambda_n^\alpha t^\alpha) \right] \\
&\quad \times \frac{E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha)}{2i^\alpha} \\
&= \frac{A_n + B_n}{4i^\alpha} \left\{ E_\alpha \left[i^\alpha \lambda_n^\alpha (at+x)^\alpha \right] - E_\alpha \left[i^\alpha \lambda_n^\alpha (at-x)^\alpha \right] \right\} \\
&\quad + \frac{A_n - B_n}{4i^\alpha} \left\{ E_\alpha \left[i^\alpha \lambda_n^\alpha (x-at)^\alpha \right] - E_\alpha \left[-i^\alpha \lambda_n^\alpha (at+x)^\alpha \right] \right\},
\end{aligned}
\tag{4.2}
$$

where its coefficients are

$$
\begin{aligned}
A_n &= \frac{2}{l^\alpha} \int_0^l f(x) \sin_\alpha \lambda_n^\alpha x^\alpha (dx)^\alpha \\
&= \frac{2}{l^\alpha} \int_0^l f(x) \frac{E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha)}{2i^\alpha} (dx)^\alpha \\
&= \frac{1}{i^\alpha l^\alpha} \int_0^l f(x) (E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha))(dx)^\alpha \quad (n = 0,1,2,\ldots), \\
B_n &= \frac{2}{a^\alpha \lambda_n^\alpha l^\alpha} \int_0^l g(x) \sin_\alpha \lambda_n^\alpha x^\alpha (dx)^\alpha \\
&= \frac{2}{a^\alpha \lambda_n^\alpha l^\alpha} \int_0^l g(x) \frac{E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha)}{2i^\alpha} (dx)^\alpha \\
&= \frac{1}{a^\alpha \lambda_n^\alpha l^\alpha i^\alpha} \int_0^l g(x) (E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha))(dx)^\alpha \quad (n = 0,1,2,\ldots).
\end{aligned}
\tag{4.3}
$$

Hereby, we always find that

$$\left|f(x) - f(x_0)\right| < \varepsilon^\alpha, \qquad \left|g(x) - g(x_0)\right| < \varepsilon^\alpha \tag{4.4}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$.

Hence the boundary conditions are fractal and solution with Mittag-Leffler functions in fractal space is given by

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\
&= \sum_{n=1}^{\infty} \frac{A_n + B_n}{4i^\alpha} \left\{ E_\alpha\left[i^\alpha \lambda_n^\alpha (at + x)^\alpha\right] - E_\alpha\left[i^\alpha \lambda_n^\alpha (at - x)^\alpha\right] \right\} \\
&\quad + \sum_{n=1}^{\infty} \frac{A_n - B_n}{4i^\alpha} \left\{ E_\alpha\left[i^\alpha \lambda_n^\alpha (x - at)^\alpha\right] - E_\alpha\left[-i^\alpha \lambda_n^\alpha (at + x)^\alpha\right] \right\},
\end{aligned} \tag{4.5}$$

where its coefficients are derived as

$$\begin{aligned}
A_n &= \frac{1}{i^\alpha l^\alpha} \int_0^l f(x)(E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha))(dx)^\alpha \quad (n = 0, 1, 2, \ldots), \\
B_n &= \frac{1}{a^\alpha \lambda_n^\alpha l^\alpha i^\alpha} \int_0^l g(x)(E_\alpha(i^\alpha \lambda_n^\alpha x^\alpha) - E_\alpha(-i^\alpha \lambda_n^\alpha x^\alpha))(dx)^\alpha \quad (n = 0, 1, 2, \ldots).
\end{aligned} \tag{4.6}$$

5. Conclusions

We applied the technique of the local fractional Fourier series to treat with the local fractional wave equation in fractal vibrating string. When contrasted with other analytical methods, such as the heat-balance integral method, the homotopy perturbation method [11], the variational iteration method [29], the exp-function method [58], the fractional variational iteration method [31, 32], the fractional complex method [33–37], and others [59–61], the present method combines the following two advantages. The boundary conditions to the governing equations are local fractional continuous (the functions are nondifferential functions in fractal space) because we employ the local fractional Fourier series, derived from local fractional calculus, to deal with them. The governing equations with fractal behaviors in media are structured based on the local fractional calculus. The way plays a crucial role in local fractional calculus. This technique is efficient for the applied scientists to process these differential equations with the local fractional differential operators in fractal space. This paper that is an outstanding example of application of local fractional Fourier series to the local fractional differential operators is given to elucidate the solution processes and reliable results.
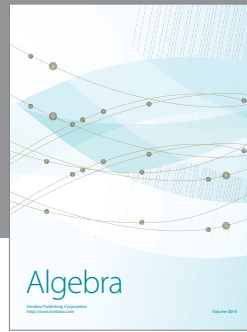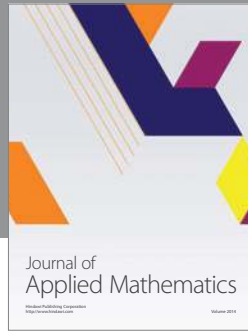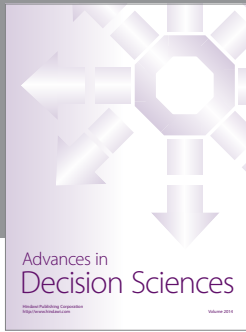
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