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Research Article

Local Fractional Series Expansion Method for Solving Wave and Diffusion Equations on Cantor Sets

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We proposed a local fractional series expansion method to solve the wave and diffusion equations on Cantor sets. Some examples are given to illustrate the efficiency and accuracy of the proposed method to obtain analytical solutions to differential equations within the local fractional derivatives.

1. Introduction

Fractional calculus theory [1–3] has been applied to a wide class of complex problems encompassing physics, biology, mechanics, and interdisciplinary areas [4–9]. Various methods, for example, the Adomian decomposition method [10], the Rach-Adomian-Meyers modified decomposition method [11], the variational iteration method [12, 13], the homotopy perturbation method [13, 14], the fractal Laplace and Fourier transforms [15], the homotopy analysis method [16], the heat-balance integral method [17–19], the fractional variational iteration method [20–22], the fractional subequation method [23, 24], and the generalized Exp-function method [25], have been utilized to solve fractional differential equations [3, 15].

The characteristics of fractal materials have local and fractal behaviors well described by nondifferential functions. However, the classic fractional calculus is not valid for differential equation on Cantor sets due to its no-local nature. In contrast, the local fractional calculus is one of the best candidates for dealing with such problems [26–44]. The local fractional calculus theory has played crucial applications in several fields, such as theoretical physics, transport problems in fractal media described by nondifferential functions. There are some versions of the local fractional calculus where different approaches in definition of the local fractional derivative exist, among them the local fractional derivative of Kolwankar et al. [32–38], the fractal derivative of Chen et al. [39, 40], the fractal derivative of Parvate et al. [41, 42], the modified Riemann-Liouville of Jumarie [43, 44], and versions described in [45–52].

In order to deal with local fractional ordinary and partial differential equations, there are some developed technologies, for example, the local fractional variational iteration method [45, 46], the local fractional Fourier series method [47, 48], the Cantor-type cylindrical-coordinate method [49], the Yang-Fourier transform [50, 51], and the Yang-Laplace transform [52].

The local fractional derivative is defined as follows [26–31, 45–52]:

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}, \quad (1)$$

where $\Delta^\alpha(f(x)-f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x)-f(x_0))$, and $f(x)$ is satisfied with the condition [26, 47]

$$|f(x)-f(x_0)| \le \tau^\alpha |x-x_0|^\alpha \quad (2)$$

so that [26–31]

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{3}$$

with $U:|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$.

The main idea of this paper is to present the local fractional series expansion method for effective solutions of wave and diffusion equations on Cantor sets involving local fractional derivatives. The paper has been organized as follows. Section 2 gives a local fractional series expansion method. Some illustrative examples are shown in Section 3. The conclusions are presented in Section 4.

2. Analysis of the Method

Let us consider the local fractional differential equation

$$u_t^{n\alpha}=L_\alpha u, \tag{4}$$

where L is a linear local operator with respect to x, $n\in\{1,2\}$.

In accordance with the results in [28, 47], there are multiterm separated functions of independent variables t and x, namely,

$$u(x,t)=\sum_{i=0}^{\infty}T_i(t)X_i(x), \tag{5}$$

where $T_i(t)$ and $X_i(x)$ are local fractional continuous functions.

Moreover, there is a nondifferential series term

$$T_i(t)=p_i\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}, \tag{6}$$

where p_i is a coefficient.

In view of (6), we may present the solution in the form

$$u(x,t)=\sum_{i=0}^{\infty}p_i\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}X_i(x). \tag{7}$$

Then, following (7), we have

$$u(x,t)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}X_i(x). \tag{8}$$

Hence,

$$\begin{gathered}u_t^{n\alpha}=\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}X_{i+1}(x)=\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}X_{i+n}(x),\\ L_\alpha u=L_\alpha\left[\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}X_i(x)\right]=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}\left(L_\alpha X_i\right)(x).\end{gathered} \tag{9}$$

In view of (9), we have

$$\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}X_{i+n}(x)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}\left(L_\alpha X_i\right)(x). \tag{10}$$

Hence, from (10) we can obtain a recursion; namely,

$$X_{i+n}(x)=\left(L_\alpha X_i\right)(x), \tag{11}$$

with $n=1$; we arrive at the following relation:

$$X_{i+1}(x)=\left(L_\alpha X_i\right)(x), \tag{12}$$

with $n=2$; we may rewrite (11) as

$$X_{i+2}(x)=\left(L_\alpha X_i\right)(x). \tag{13}$$

By the recursion formulas, we can obtain the solution of (4) as

$$u(x,t)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}X_i(x). \tag{14}$$

The convergent condition is

$$\lim_{n\to\infty}\left[\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}X_i(x)\right]=0. \tag{15}$$

This approach is termed *the local fractional series expansion method* (LFSEM)

3. Applications to Wave and Diffusion Equations on Cantor Sets

In this section, four examples for wave and diffusion equations on Cantor sets will demonstrate the efficiency of LFSEM.

Example 1. Let us consider the diffusion equation on Cantor set

$$u_t^{\alpha}(x,t)-u_x^{2\alpha}(x,t)=0,\quad 0<\alpha\leq 1 \tag{16}$$

with the initial condition

$$u(x,0)=\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{17}$$

Following (12), we have recursive formula

$$\begin{gathered}X_{i+1}(\mathrm{x})=\frac{\partial^{2\alpha}X_i(x)}{\partial x^{2\alpha}},\\ X_0(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}.\end{gathered} \tag{18}$$

Hence, we get

$$\begin{gathered}X_0(x)=\frac{x^\alpha}{\Gamma(1+\alpha)},\\ X_1(x)=0,\\ X_2(x)=0,\\ \vdots\end{gathered} \tag{19}$$

and so on.

Therefore, through (19) we get the solution

$$u(x,t)=\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{20}$$

Example 2. Let us consider the diffusion equation on Cantor set

$$u_t^{\alpha}(x,t) - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \cdot u_x^{2\alpha}(x,t) = 0, \quad 0 < \alpha \le 1 \tag{21}$$

with the initial condition

$$u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{22}$$

Following (12), we get

$$X_{i+1}(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}}, \qquad X_0(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{23}$$

By using the recursive formula (23), we get consequently

$$X_0(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \quad X_1(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \quad X_2(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \quad \vdots \tag{24}$$

As a direct result of these recursive calculations, we arrive at

$$u(x,t) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} E_{\alpha}(t^{\alpha}). \tag{25}$$

Example 3. Let us consider the following wave equation on Cantor sets:

$$u_t^{2\alpha}(x,t) - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \cdot u_x^{2\alpha}(x,t) = 0, \quad 0 < \alpha \le 1 \tag{26}$$

with the initial condition

$$u(x,0) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}. \tag{27}$$

In view of (14), we obtain

$$X_{i+2}(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}}, \quad X_0(x) = u(x,0) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \quad X_{i+2}(x) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}}, \quad X_1(x) = u_x^{(\alpha)}(x,0) = 1. \tag{28}$$

Hence, using the relations (29), the recursive calculations yield

$$X_0(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \quad X_1(x) = 1, \tag{29}$$

$$X_2(x) = 0, \quad X_3(x) = 0, \quad X_4(x) = 0, \quad \vdots \tag{30}$$

and so on.

Finally, we obtain

$$u(x,t) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{31}$$

Example 4. Let us consider the wave equation on Cantor sets [26, 30]

$$u_t^{2\alpha}(x,t) - c u_x^{2\alpha}(x,t) = 0, \quad 0 < \alpha \le 1, \tag{32}$$

where c is a constant.

The initial condition is

$$u(x,0) = E_{\alpha}(x^{\alpha}). \tag{33}$$

By using (14) we have

$$X_{i+2}(x) = c\frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}}, \quad X_0(x) = u(x,0) = E_{\alpha}(x^{\alpha}), \quad X_{i+2}(x) = c\frac{\partial^{2\alpha} X_i(x)}{\partial x^{2\alpha}}, \quad X_0(x) = u_x^{(\alpha)}(x,0) = E_{\alpha}(x^{\alpha}). \tag{34}$$

Then, through the iterative relations (35), we have

$$X_0(x) = E_{\alpha}(x^{\alpha}), \quad X_1(x) = E_{\alpha}(x^{\alpha}), \tag{35}$$

$$X_2(x) = cE_{\alpha}(x^{\alpha}), \quad X_3(x) = cE_{\alpha}(x^{\alpha}), \quad X_4(x) = c^2 E_{\alpha}(x^{\alpha}), \quad \vdots \tag{36}$$

Therefore, we obtain

$$\begin{aligned} u(x,t) &= E_\alpha(x^\alpha)\sum_{i=0}^{\infty} c^i \frac{t^{2i\alpha}}{\Gamma(1+2i\alpha)} \\ &\quad + E_\alpha(x^\alpha)\sum_{i=0}^{\infty} c^i \frac{t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} \\ &= E_\alpha(x^\alpha)\left[\cosh_\alpha(ct^\alpha) + \sin_\alpha(ct^\alpha)\right], \end{aligned} \tag{37}$$

where

$$\begin{aligned} \cosh_\alpha(t^\alpha) &= \sum_{i=0}^{\infty} \frac{t^{2i\alpha}}{\Gamma(1+2i\alpha)}, \\ \sinh_\alpha(t^\alpha) &= \sum_{i=0}^{\infty} \frac{t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)}. \end{aligned} \tag{38}$$

For more details concerning (38), we refer to [26–28].

4. Conclusions

In this work, the local fractional series expansion method is demonstrated as an effective method for solutions of a wide class of problems. Analytical solutions of the wave and diffusion equations on Cantor sets involving local fractional derivatives are successfully developed by recurrence relations resulting in convergent series solutions. In this context, the suggested method is a potential tool for development of approximate solutions of local fractional differential equations with fractal initial value conditions, which, of course, draws new problems beyond the scope of the present work.

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