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Local higher integrability for parabolic quasiminimizers in metric spaces

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Abstract Using purely variational methods, we prove in metric measure spaces local higher integrability for minimal p-weak upper gradients of parabolic quasiminimizers related to the heat equation. We assume the measure to be doubling and the underlying space to be such that a weak Poincaré inequality is supported. We define parabolic quasiminimizers in the general metric measure space context, and prove an energy type estimate. Using the energy estimate and properties of the underlying metric measure space, we prove a reverse Hölder inequality type estimate for minimal p-weak upper gradients of parabolic quasiminimizers. Local higher integrability is then established based on the reverse Hölder inequality, by using a modification of Gehring's lemma.

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1 Introduction

In the Euclidean setting, finding a solution to the p-parabolic partial differential equation in $\Omega \times (0, T)$,

$$-\frac{\partial u}{\partial t} + \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \tag{1.1}$$

can be formulated into an equivalent variational problem, which is to find a function u, such that with $K = 1$ we have

$$p \int\limits_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, dx\, dt + \int\limits_{\{\phi \neq 0\}} |\nabla u|^p\, dx\, dt \leq K \int\limits_{\{\phi \neq 0\}} |\nabla u + \nabla \phi|^p\, dx\, dt, \tag{1.2}$$

for every compactly supported $\phi \in C_0^\infty(\Omega \times (0, T))$. Here $\Omega \subset \mathbb{R}^d$ denotes a domain. A generalization of this variational problem is to consider (1.2) with the weakened assumption $K \geq 1$. The function u is then called a parabolic K-quasiminimizer related to (1.1). When $p = 2$, Eq. (1.1) is the classical heat equation.

Our main result is to show that a parabolic quasiminimizer u in a doubling metric measure space has the following higher integrability property: The upper gradient [18] of u is locally integrable to a slightly higher power than is initially assumed (Theorem 2). Although being local, the estimate we obtain is scale and location invariant.

As a first step for examining higher regularity of quasiminimizers in metric measure spaces, we only treat the simplest case $p = 2$, and so the quasiminimizers in this paper are related to the heat equation. Assuming a weak Poincaré inequality and a doubling measure, starting from the definition of quasiminimizers we prove an energy type estimate and a Caccioppoli type inequality for u. Then using these and a Sobolev–Poincaré inequality we show a reverse Hölder inequality, from which higher integrability follows.

The novelty of this paper is that we prove our results in the general metric measure space setting, using a purely variational approach. No reference is made to the explicit scaling properties of the measure, or on assumptions that the measure is translation invariant. Instead we rely on taking integral averages and on the assumption that the measure is doubling. Also, no reference is made to the equation, as the local notion of a weak solution is replaced by quasiminimizers, which are in general known to not be uniquely determined by the quasiminimizing property. The variational nature of quasiminimizers opens up the possibility to substitute gradients by upper gradients, which do not require the existence of partial derivatives on the underlying space. This way, the theory of parabolic PDEs is extended to metric measure spaces.

Methods established in the general metric space setting can be expected to be robust and of fundamental nature, in the sense that they are largely independent of the geometry of the underlying measure space. In connection with this, it is worth noting that already in the Euclidean case, an open question is to show that a weak solution to the doubly non-linear partial differential equation is locally higher integrable. For $p = 2$, the K-quasiminimizer related to the doubly non-linear partial differential equation coincides with the one used in this paper.

In the elliptic case quasiminimizers have been extensively investigated. Originally, quasiminimizers were introduced in the elliptic setting by Giaquinta and Giusti [11,12] as a tool for a unified treatment of variational integrals, elliptic equations and systems, and quasiregular mappings in $\mathbb{R}^d$. Giaquinta and Giusti realized that De Giorgi's method [8] could be extended to quasiminimizers, and they showed, in particular, that elliptic quasiminimizers are locally Hölder continuous. DiBenedetto and Trudinger [9] proved the Harnack inequality for quasiminimizers. As mentioned, unlike partial differential equations, quasiminimizers are a purely variational notion, and so Kinnunen and Shanmugalingam [22] were able to extend these regularity results for elliptic quasiminimizers into the general metric setting by using upper gradients.

In comparison to the elliptic case, already in the Euclidean case the literature available for parabolic quasiminimizers is relatively limited. Following Giaquinta and Giusti, Wieser [34] generalized the notion of quasiminimizers to the parabolic setting in Euclidean spaces. Parabolic quasiminimizers have also been studied by Zhou [35,36], Gianazza and Vespri [14], Marchi [24] and Wang [32]. Recently, also the notions of parabolic quasiminimizers has been extended and studied in metric spaces [21,25].

Higher integrability results were introduced in the parabolic setting by Giaquinta and Struwe [13], when they proved reverse Hölder inequalities and local higher integrability in the case $p = 2$, for weak solutions of parabolic second order p-growth systems. Kinnunen and Lewis [19] extended this local result to the general degenerate and singular case $p \neq 2$. Recently, several authors have worked in the parabolic setting on questions concerning local and global higher integrability and reverse Hölder inequalities, see [1,3–6,10,26,28,29], and in particular for quasiminimizers see [27].

2 Preliminaries

2.1 Doubling measure

Let $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a positive complete doubling Borel measure μ which supports a weak $(1, 2)$-Poincaré inequality.

The measure μ is called *doubling* if there exists a constant $c_\mu \geq 1$, such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X,

$$\mu(2B) \leq c_\mu \mu(B), \text{ where } \lambda B = B(x_0, \lambda r).$$

By iterating the doubling condition, it follows with $s = \log_2 c_\mu$ and $C = c_\mu^{-2}$ that

$$\frac{\mu(B(z, r))}{\mu(B(y, R))} \geq C\Big(\frac{r}{R}\Big)^s, \tag{2.1}$$

for all balls $B(y, R) \subset X$, $z \in B(y, R)$ and $0 < r \leq R < \infty$. However, here c_μ does not have to be optimal. From now on we fix $c_\mu > 1$ and so $s > 0$.

2.2 Notation

Next we introduce more notation used throughout this paper. Given any $z_0 = (x_0, t_0) \in X \times \mathbb{R}$ and $\rho > 0$, let

$$B_\rho(x_0) = \{x \in X : d(x, x_0) < \rho\},$$

denote an open ball in X, and let

$$\Lambda_\rho(t_0) = \left(t_0 - \frac{1}{2}\rho^2,\ t_0 + \frac{1}{2}\rho^2\right),$$

denote an open interval in $\mathbb{R}$. A space-time cylinder in $X \times \mathbb{R}$ is denoted by

$$Q_\rho(z_0) = B_\rho(x_0) \times \Lambda_\rho(t_0),$$

so that $\nu(Q_\rho(z_0)) = \mu(B_\rho(x_0))\rho^2$. When no confusion arises, we shall omit the reference points and write briefly B_ρ, Λ_ρ and Q_ρ. We denote the product measure by $d\nu = d\mu\, dt$. The integral average of u is denoted by

$$u_{B_\rho}(t) = \fint_{B_\rho} u(x, t)\, d\mu = \frac{1}{\mu(B_\rho)} \int_{B_\rho} u(x, t)\, d\mu \tag{2.2}$$

and

$$\fint_{Q_\rho} u\, d\nu = \frac{1}{\nu(Q_\rho)} \int_{Q_\rho} u\, d\nu.$$

Let $\Omega \subset X$ be a domain, and let $0 < T < \infty$. We denote $\Omega_T = \Omega \times (0, T)$.

2.3 Upper gradients

Following [18], a non-negative Borel measurable function $g : \Omega \to [0, \infty]$ is said to be an upper gradient of a function $u : \Omega \to [-\infty, \infty]$ in Ω, if for all compact rectifiable paths γ joining x and y in Ω we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds. \tag{2.3}$$

In case $u(x) = u(y) = \infty$ or $u(x) = u(y) = -\infty$, the left side is defined to be ∞. Assume $1 \leq p < \infty$. The p-modulus of a family of paths Γ in Ω is defined to be

$$\inf_\rho \int_\Omega \rho^p\, d\mu,$$

where the infimum is taken over all non-negative Borel measurable functions ρ such that for all rectifiable paths γ which belong to Γ, we have

$$\int_\gamma \rho\, ds \geq 1.$$

A property is said to hold for p-almost all paths, if the set of non-constant paths for which the property fails is of zero p-modulus. Following [20,30], if (2.3) holds for p-almost all paths γ in Ω, then g is said to be a p-weak upper gradient of u.

When $1 \leq p < \infty$ and $u \in L^p(\Omega)$, it can be shown [15,31] that there exists a minimal p-weak upper gradient of u, we denote it by g_u, in the sense that g_u is a p-weak upper gradient of u and for every p-weak upper gradient g of u it holds $g_u \leq g$ μ-almost everywhere in Ω. Moreover, if $v = u$ μ-almost everywhere in a Borel set $A \subset \Omega$, then $g_v = g_u$ μ-almost everywhere in A. Also, if $u, v \in L^p(\Omega)$, then μ-almost everywhere in Ω, we have

$$\begin{aligned} g_{u+v} &\leq g_u + g_v, \\ g_{uv} &\leq |u| g_v + |v| g_u. \end{aligned}$$

Proofs for these properties and more on upper gradients in metric spaces can be found for example in [2] and the references therein. See also [7] for a discussion on upper gradients.

2.4 Newtonian spaces

Following [30], for $1 < p < \infty$, and $u \in L^p(\Omega)$, we define

$$\|u\|_{1,p,\Omega}^p = \|u\|_{L^p(\Omega)}^p + \|g_u\|_{L^p(\Omega)}^p,$$

and

$$\widetilde{N}^{1,p}(\Omega) = \{u : \|u\|_{1,p,\Omega} < \infty\}.$$

An equivalence relation in $\widetilde{N}^{1,p}(\Omega)$ is defined by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(\Omega)} = 0.$$

The *Newtonian space* $N^{1,p}(\Omega)$ is defined to be the space $\widetilde{N}^{1,p}(\Omega)/\sim$, with the norm

$$\|u\|_{N^{1,p}(\Omega)} = \|u\|_{1,p,\Omega}.$$

A function u belongs to the local Newtonian space $N^{1,p}_{\mathrm{loc}}(\Omega)$ if it belongs to $N^{1,p}(\Omega')$ for every $\Omega' \subset\subset \Omega$. The Newtonian space with zero boundary values is defined as

$$N_0^{1,p}(\Omega) = \{u|_\Omega : u \in N^{1,p}(X),\ u = 0 \text{ in } X \setminus \Omega\}.$$

In practice this means that a function belongs to $N_0^{1,p}(\Omega)$ if and only its zero extension to $X \setminus \Omega$ belongs to $N^{1,p}(X)$. For more properties of Newtonian spaces, see [2,16,30].

2.5 Poincaré's and Sobolev's inequality

For $1 \le q < \infty$, $1 < p < \infty$, the measure μ is said to support a weak (q, p)-Poincaré inequality if there exist constants $c_P > 0$ and $\lambda \ge 1$ such that

$$\left(\fint_{B_\rho(x)} |v - v_{B_\rho(x)}|^q \, d\mu \right)^{1/q} \le c_P \rho \left(\fint_{B_{\lambda\rho}(x)} g_v^p \, d\mu \right)^{1/p}, \tag{2.4}$$

for every $v \in N^{1,p}(X)$ and $B_\rho(x) \subset X$. In case $\lambda = 1$, we say a (q, p)-Poincaré inequality is in force. In a general metric measure space setting, it is of interest to have assumptions which in the special case of Euclidean spaces are invariant under bi-Lipschitz coordinate changing mappings. The weak (q, p)-Poincaré inequality has this quality.

For a metric space X equipped with a doubling measure μ, it is a result by Hajlasz and Koskela [17] that the following Sobolev inequality holds: If X supports a weak $(1, p)$-Poincaré inequality for some $1 < p < \infty$, then X also supports a weak (κ, p)-Poincaré inequality, where

$$\kappa = \begin{cases} \frac{sp}{s-p}, & \text{for } 1 < p < s, \\ 2p, & \text{otherwise,} \end{cases}$$

possibly with different constants $c_P' > 0$ and $\lambda' \ge 1$.

Remark 1 It is a recent result by Keith and Zhong [23], that when $1 < p < \infty$ and (X, d) is a complete metric space with doubling measure μ, the weak $(1, p)$-Poincaré inequality implies a weak $(1, q)$-Poincaré inequality for some $1 < q < p$. Then by the above discussion, X also supports a weak (κ, q)-Poincaré inequality with $\kappa > q$ as above. By Hölder's inequality, we can assume that q is close enough to p, so that $\kappa \geq p$. By Hölder's inequality, the left hand side of the weak (κ, q)-Poincaré inequality can be estimated from below by replacing κ with any positive $\kappa' < \kappa$. Hence we conclude, that if X supports a weak $(1, p)$-Poincaré inequality with $1 < p < \infty$, then X also supports a weak (p, q)-Poincaré and a weak (q, q)-Poincaré inequality with some $1 < q < p$.

2.6 Parabolic Newtonian spaces

For $1 < p < \infty$, we say that

$$u \in L^p(0, T; N^{1,p}(\Omega)),$$

if the function $x \mapsto u(x, t)$ belongs to $N^{1,p}(\Omega)$ for almost every $0 < t < T$, and $u(x, t)$ is measurable as a mapping from $(0, T)$ to $N^{1,p}(\Omega)$, that is, the preimage on $(0, T)$ for any given open set in $N^{1,p}(\Omega)$ is measurable. Furthermore, we require that the norm

$$\|u\|_{L^p(0,T;N^{1,p}(\Omega))} = \left(\int_0^T ||u||^p_{N^{1,p}(\Omega)} \, dt \right)^{1/p}$$

is finite. Analogously, we define $L^p(0, T; N_0^{1,p}(\Omega))$ and $L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$. The space of compactly supported Lipschitz-continuous functions $\mathrm{Lip}_c(\Omega_T)$ consists of functions u, $\operatorname{supp} u \subset \Omega_T$, for which there exists a positive constant $C_{\mathrm{Lip}}(u)$ such that

$$|u(x, t) - u(y, s)| \leq C_{\mathrm{Lip}}(u)(d(x, y) + |t - s|),$$

whenever $(x, t), (y, s) \in \Omega_T$. The parabolic minimal p-weak upper gradient of a function $u \in L^p_{\mathrm{loc}}(t_1, t_2; N^{1,p}_{\mathrm{loc}}(\Omega))$ is defined in a natural way by setting

$$g_u(x, t) = g_{u(\cdot,t)}(x),$$

at ν-almost every $(x, t) \in \Omega \times (0, T)$. When u depends on time, we refer to g_u as the upper gradient of u. The next Lemma on taking limits of upper gradients will be used later in this paper. Here and throughout this paper we denote the time wise mollification of a function by

$$f_\varepsilon(x, t) = \int_{-\varepsilon}^{\varepsilon} \zeta_\varepsilon(s) f(x, t - s) \, ds,$$

where ζ_ε is the standard mollifier with support in $(-\varepsilon, \varepsilon)$.

Lemma 1 *Let $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$. Then the following statements hold:*

(a) *As $s \to 0$, we have $g_{u(x,t-s)-u(x,t)} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$.*
(b) *As $\varepsilon \to 0$, we have $g_{u_\varepsilon - u} \to 0$ pointwise ν-almost everywhere in Ω_T and in $L^p_{\mathrm{loc}}(\Omega_T)$.*

Proof See Lemma 6.8 in [25]. □

The following technical density result will be needed for establishing that the space of compactly supported test functions in $\mathrm{Lip}(\Omega_T)$ is large enough for establishing energy estimates for parabolic quasiminimizers. Since in the following we also want to be able to approximate the sets $\{\phi \neq 0\}$ from outside in the sense of measure, it seems that we cannot avoid a hands on proof.

Lemma 2 *Assume (X,d,μ) is a complete doubling space which supports a weak $(1,p)$-Poincaré inequality, where $1 < p < \infty$. Let $\phi \in L^p(0,T;N^{1,p}_0(\Omega))$. Then for every $\varepsilon > 0$ there exists a function $\varphi \in \mathrm{Lip}(\Omega_T)$ such that $\{\varphi \neq 0\} \subset\subset \Omega_T$ and*

$$\|\phi - \varphi\|_{L^p(0,T;N^{1,p}(\Omega))} < \varepsilon, \quad \text{and} \quad \nu(\{\varphi \neq 0\} \setminus \{\phi \neq 0\}) < \varepsilon.$$

Proof Assume ϕ is as in the claim, and let $\varepsilon > 0$. The measure ν is regular in Ω_T, and so there exists an open set $F \subset \Omega$ such that $\{\phi \neq 0\} \subset F$ and $\nu(F \setminus \{\phi \neq 0\}) < \varepsilon$. For each $t \in (0,T)$, denote the open set $F_t = \{x : (x,t) \in F\}$. Then for almost every $t \in (0,T)$ we have $\phi(\cdot,t) \in N^{1,p}_0(F_t)$. Since by assumption (X,d,μ) is a complete doubling metric space which supports a $(1,p)$-Poincaré inquality, it is a result of Shanmugalingam [31] that the space of compactly supported Lipschitz continuous functions is dense in $N^{1,p}_0$. Hence for almost every $t \in (0,T)$ there exists a function $\widetilde{\psi}(\cdot,t) \in \mathrm{Lip_c}(F_t)$ such that

$$\|\phi(\cdot,t) - \widetilde{\psi}(\cdot,t)\|_{N^{1,p}(\Omega)} < \varepsilon. \tag{2.5}$$

Denote for each $\delta > 0$,

$$\begin{aligned} E_\delta = \{ t \in (\delta, T-\delta) \, : \, d(\{\widetilde{\psi}(\cdot,t) \neq 0\}, \Omega \setminus F_t) \geq \delta \\ C_{\mathrm{Lip}}(\widetilde{\psi}(\cdot,t)) \leq \delta^{-1}, \qquad \|\widetilde{\psi}(\cdot,t)\|_\infty \leq \delta^{-1}\}. \end{aligned}$$

Since each $\widetilde{\psi}(\cdot,t)$ has compact support, is bounded and Lipschitz-continuous, we can see that $|(0,T) \setminus E_\delta| \to 0$ as $\delta \to 0$. Since $\phi \in L^p(0,T;N^{1,p}_0(\Omega))$, this implies that we can fix δ to be such that

$$\|\phi\|_{L^p((0,T)\setminus E_\delta;N^{1,p}(\Omega))} < \varepsilon. \tag{2.6}$$

For each $t \in E_\delta$, define the compact set

$$K_t = \{x \in F_t \, : \, d(x, \Omega \setminus F_t) \geq \delta\}.$$

The fact that K_t is a compact subset of F_t implies that for each $t \in E_\delta$ the set $K_t \times \{t\}$ is a compact subset of F. This implies that for each $t \in E_\delta$ there exists a positive constant $0 < \delta_t < \delta$ such that $\overline{B(x,\delta_t)} \times [t-\delta_t, t+\delta_t] \subset F$ for every $x \in K_t$. Denote then for every $0 < \rho < \delta$,

$$\widetilde{E}_\rho = \{t \in E_\delta \,:\, \overline{B(x,\rho)} \times [t-\rho, t+\rho] \subset F \;\text{ for }\; \text{every}\, x \in K_t\}.$$

Then $|E_\delta \setminus \widetilde{E}_\rho| \to 0$ as $\rho \to 0$, and so we can fix $\rho < \delta$ to be such that

$$\|\phi\|_{L^p(E_\delta \setminus \widetilde{E}_\rho; N^{1,p}(\Omega))} < \varepsilon. \tag{2.7}$$

Set

$$\psi(x,t) = \begin{cases} \widetilde{\psi}(x,t), & t \in \widetilde{E}_\rho, \\ 0, & \text{otherwise.} \end{cases}$$

By (2.5), (2.6) and (2.7), we have

$$\begin{aligned} \|\phi - \psi\|_{L^p(0,T;N^{1,p}(\Omega)} &\le \|\phi - \psi\|_{L^p(E_\delta;N^{1,p}(\Omega))} + \|\phi\|_{L^p((0,T)\setminus E_\delta;N^{1,p}(\Omega))} \\ &\le \|\phi - \psi\|_{L^p(\widetilde{E}_\rho;N^{1,p}(\Omega))} + \|\phi\|_{L^p(E_\delta\setminus\widetilde{E}_\rho;N^{1,p}(\Omega))} + \|\phi\|_{L^p(E_\delta;N^{1,p}(\Omega))} \\ &\le \left(\int\limits_{\widetilde{E}_\rho} \|\phi - \widetilde{\psi}\|^p_{N^{1,p}(\Omega)}\, dt \right)^{\frac{1}{p}} + 2\varepsilon \le T^{\frac{1}{p}}\varepsilon + 2\varepsilon. \end{aligned}$$

Now, from the way we constructed ψ, it follows that for every $(x,t) \in \{\psi \neq 0\}$ we have $\overline{B(x,\rho)} \times [t-\rho, t+\rho] \subset F$, and so $\{\psi \neq 0\}$ is compactly contained in F. Hence there exists a compact set K such that for any $0 < \sigma < \rho$, we also have $\{\psi_\sigma \neq 0\} \subset K \subset\subset F$. Since for each $t \in (0,T)$ the function $\psi(\cdot,t)$ is Lipschitz-continuous with Lipschitz constant δ^{-1}, we have for every $x, y \in \Omega$ and $t \in (0,T)$

$$|\psi_\sigma(x,t) - \psi_\sigma(y,t)| \le \int\limits_{-\sigma}^{\sigma} \zeta_\sigma(s)|\psi(x,t-s) - \psi(y,t-s)|\, ds \le \delta^{-1} d(x,y).$$

On the other hand, it is straightforward to show using the theory of mollifiers, that since $|\psi| \le \delta^{-1}$ uniformly in Ω_T, for a fixed $0 < \sigma < \rho$ the time derivative of ψ_σ is uniformly bounded in Ω_T. Hence we have for every $x \in \Omega$ and $t_1, t_2 \in (0,T)$,

$$|\psi_\sigma(x,t_1) - \psi_\sigma(x,t_2)| \le \left\| \frac{\partial \psi_\sigma}{\partial t} \right\|_\infty |t_1 - t_2|.$$

This means that for every $0 < \sigma < \rho$ we have $\psi_\sigma \in Lip(\Omega_T)$ and $\{\psi_\sigma \neq 0\} \subset K$. Now,

$$
\begin{aligned}
&\|\phi - \psi_\sigma\|_{L^p(0,T;N^{1,p}(\Omega))} \\
&\quad \le \|\phi - \psi\|_{L^p(0,T;N^{1,p}(\Omega))} + \|\psi - \psi_\sigma\|_{L^p(0,T;N^{1,p}(\Omega))} \\
&\quad \le T^{\frac{1}{p}}\varepsilon + 2\varepsilon + \|\psi - \psi_\sigma\|_{L^p(K)} + \|g_{\psi-\psi_\sigma}\|_{L^p(K)}. \qquad (2.8)
\end{aligned}
$$

Treating $\zeta_\sigma(s)\,ds$ as a unit measure, we have by Jensen's inequality, and Fubini's theorem

$$
\int_K |\psi - \psi_\sigma|^p\,d\nu \le \int_{-\sigma}^{\sigma} \zeta_\sigma(s) \int_K |\psi(x,t) - \psi(x,t-s)|^p\,d\nu\,ds.
$$

Therefore, by continuity of translation for functions in $L^p(\Omega_T)$, we see that the third term on the right hand side of (2.8) tends to zero as $\sigma \to 0$. Since for every $t \in (0,T)$ we have $\psi(\cdot,t) \in \mathrm{Lip}_c(\Omega)$ uniformly with the same Lipschitz constant δ^{-1}, by the proof of Lemma 6.8 in [25] (note that in that proof, for Lipshitz-functions no density results are used), also the last term on the right hand side of (2.8) tends to zero as $\sigma \to 0$. Moreover, for every $0 < \sigma < \rho$ we have

$$
\nu(\{\psi_\sigma \neq 0\} \setminus \{\phi \neq 0\}) \le \nu(F \setminus \{\phi \neq 0\}) \le \varepsilon.
$$

Setting $\varphi = \psi_\sigma$ with a small enough σ completes the proof. □

For the case $1 < p < 2$, it is not obvious that convergence of the dense Lipshitz functions in the parabolic Newtonian space implies convergence also in $L^2(\Omega_T)$. In the following corollary we check that if the limit function is in $L^2(\Omega_T)$, then this is the case.

Corollary 1 *Assume (X,d,μ) is a complete doubling space supporting a weak $(1,p)$-Poincaré inequality, where $1 < p < \infty$. Let $\phi \in L^p(0,T;N_0^{1,p}(\Omega)) \cap L^2(\Omega_T)$. For every $\varepsilon > 0$ there exists a function $\varphi \in \mathrm{Lip}(\Omega_T)$ such that $\{\varphi \neq 0\} \subset\subset \Omega_T$ and*

$$
\begin{aligned}
&\|\phi - \varphi\|_{L^p(0,T;N^{1,p}(\Omega))} < \varepsilon, \qquad \|\phi - \varphi\|_{L^2(\Omega_T)} < \varepsilon, \\
&\text{and} \quad \nu(\{\varphi \neq 0\} \setminus \{\phi \neq 0\}) < \varepsilon.
\end{aligned}
$$

Proof If $p \ge 2$, the claim is clearly true by Lemma 2 and Hölder's inequality. So let $1 < p < 2$, and Let $\phi \in L^p(0,T;N_0^{1,p}(\Omega)) \cap L^2(\Omega_T)$. Let $\varepsilon > 0$. For each k, denote the cutoff function

$$
\phi_k(x,t) = \begin{cases} k, & \text{when } \phi(x,t) \ge k \\ \phi(x,t), & \text{when } -k < \phi(x,t) < k \\ -k, & \text{when} \phi(x,t) \le -k. \end{cases}
$$

Then there exists a positive constant k such that

$$\|\phi - \phi_k\|_{L^p(0,T;N^{1,p}(\Omega))} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\phi - \phi_k\|_{L^2(\Omega_T)} < \frac{\varepsilon}{2}.$$

By Lemma 2, there exists a sequence of compactly supported functions $(\varphi_i)_i \subset \mathrm{Lip}(\Omega_T)$ such that

$$\|\phi_k - \varphi_i\|_{L^p(0,T;N^{1,p}(\Omega_T))} \to 0 \quad \text{and} \quad \nu(\{\varphi_i \neq 0\} \setminus \{\phi_k \neq 0\}) \to 0$$

as $i \to \infty$. Moreover, since ϕ_k is bounded, we may assume the sequence φ_i to be uniformly bounded by a positive constant m. We can now write

$$\begin{aligned}
&\|\phi_k - \varphi_i\|^2_{L^2(\Omega_T)} \\
&\quad = \int_{\Omega_T} |\phi_k - \varphi_i|^2 \chi_{\{|\phi_k - \varphi_i| > 1\}}\, d\nu + \int_{\Omega_T} |\phi_k - \varphi_i|^2 \chi_{\{|\phi_k - \varphi_i| \leq 1\}}\, d\nu \\
&\quad \leq (k+m)^2 \nu(\{\, \Omega_T \,:\, |\phi_k - \varphi_i| > 1\,\}) + \int_{\Omega_T} |\phi_k - \varphi_i|^p\, d\nu.
\end{aligned}$$

Since φ_i converges to ϕ_k in $L^p(\Omega_T)$, the above tends to zero as $i \to \infty$. We have

$$\|\phi - \varphi_i\|_{L^p(0,T;N^{1,p}(\Omega))} \leq \frac{\varepsilon}{2} + \|\phi_k - \varphi_i\|_{L^p(0,T;N^{1,p}(\Omega))},$$

and on the other hand

$$\|\phi - \varphi_i\|_{L^2(\Omega_T)} \leq \frac{\varepsilon}{2} + \|\phi_k - \varphi_i\|_{L^2(\Omega_T)}.$$

Moreover, we have

$$\nu(\{\varphi_i \neq 0\} \setminus \{\phi \neq 0\}) = \nu(\{\varphi_i \neq 0\} \setminus \{\phi_k \neq 0\}),$$

and so taking φ_i with a large enough i completes the proof. □

Remark 2 Notice that if ϕ is compactly supported in Ω_T, then Lemma 2 and Corollary 1 are also true with $\operatorname{supp}\phi$ in place of $\{\phi \neq 0\}$, and respectively for the approximating functions. The proofs just have to be repeated after having replaced $\{\phi \neq 0\}$ with $\operatorname{supp}\phi$, respectively for each function. Hence we also have that for $\phi \in L^p(0,T;N_0^{1,p}(\Omega)) \cap L^2(\Omega_T)$ such that $\operatorname{supp}\phi \subset\subset \Omega_T$ and $\varepsilon > 0$, there exists a function $\varphi \in \mathrm{Lip}(\Omega_T)$ such that $\operatorname{supp}\varphi \subset\subset \Omega_T$,

$$\begin{aligned}
&\|\phi - \varphi\|_{L^p(0,T;N^{1,p}(\Omega))} < \varepsilon, \qquad \|\phi - \varphi\|_{L^2(\Omega_T)} < \varepsilon, \\
&\quad \text{and} \quad \nu(\operatorname{supp}\varphi \setminus \operatorname{supp}\phi) < \varepsilon.
\end{aligned}$$

Next we define parabolic quasiminimizers in metric spaces.

2.7 Parabolic quasiminimizers

Definition 1 Let Ω be an open subset of X, $u : \Omega \times (0,T) \to \mathbb{R}$ and $K' \geq 1$. Let $1 < p < \infty$. A function u belonging to the parabolic Newtonian space $L^p_{\mathrm{loc}}(0,T; N^{1,p}_{\mathrm{loc}}(\Omega))$ is a *parabolic quasiminimizer* if

$$\int\limits_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, d\nu + E(u, \{\phi \neq 0\}) \leq K' E(u+\phi, \{\phi \neq 0\}),$$

for every $\phi \in \mathrm{Lip}(\Omega_T)$ such that $\{\phi \neq 0\} \subset\subset \Omega_T$, where we denote

$$E(u, A) = \int\limits_A F(x, t, g_u)\, d\nu,$$

and $F : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

1. $(x,t) \mapsto F(x,t,\xi)$ is measurable for every ξ,
2. $\xi \mapsto F(x,t,\xi)$ is continuous for almost every (x,t),
3. there exist $0 < c_1 \leq c_2 < \infty$ such that for every ξ and almost every (x,t), we have

$$c_1 |\xi|^p \leq F(x,t,\xi) \leq c_2 |\xi|^p .$$

As a consequence of the above, a parabolic quasiminimizer u satisfies

$$\alpha \int\limits_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, d\nu + \int\limits_{\{\phi \neq 0\}} g_u^p\, d\nu \leq K \int\limits_{\{\phi \neq 0\}} g_{u+\phi}^p\, d\nu, \tag{2.9}$$

with $K = c_2 c_1^{-1} K' \geq 1$ and $\alpha = c_1^{-1}$, for every $\phi \in \mathrm{Lip}(\Omega_T)$ such that $\{\phi \neq 0\} \subset\subset \Omega$. There is a subtle difficulty in proving an energy estimate for parabolic quasiminimizers: The proof is based on using a test function which depends on u itself, but u is a priori not necessarily in $\mathrm{Lip}(\Omega_T)$ nor has compact support. We treat this difficulty in the following manner. Consider a test function $\phi \in \mathrm{Lip}(\Omega_T)$ with compact support. Plugging in the test function $\phi(x, t+s)$ and conducting a change of variable in (2.9), we see that there exists a constant $\varepsilon > 0$ such that for every $-\varepsilon < s < \varepsilon$,

$$\alpha \int\limits_{\{\phi \neq 0\}} u(x, t-s) \frac{\partial \phi}{\partial t}\, d\nu + \int\limits_{\{\phi \neq 0\}} g_{u(x,t-s)}^p\, d\nu \leq K \int\limits_{\{\phi \neq 0\}} g_{u(x,t-s)+\phi}^p\, d\nu.$$

Let now $\zeta_\varepsilon(s)$ be a standard mollifier whose support is contained in $(-\varepsilon, \varepsilon)$. We multiply the above inequality with $\zeta_\varepsilon(s)$ and integrate on both sides with respect to s, use Fubini's theorem to change the order of integration, and lastly use partial integration for the first term on the left hand side, to obtain

$$-\alpha \int\limits_{\{\phi \neq 0\}} \frac{\partial u_\varepsilon}{\partial t} \phi\, d\nu + \int\limits_{\{\phi \neq 0\}} \left(g_u^p\right)_\varepsilon d\nu \leq K \int\limits_{\{\phi \neq 0\}} \left(g_{u(x,t-s)+\phi}^p\right)_\varepsilon d\nu, \tag{2.10}$$

for every compactly supported $\phi \in \mathrm{Lip}(\Omega_T)$. Now we use the density results we proved earlier to establish that indeed the space of compactly supported functions in $Lip(\Omega_T)$ is large enough, so that below when proving the energy estimate we may use u in the test function.

Lemma 3 *Assume (X, d, μ) is a complete doubling space supporting a weak $(1, p)$-Poincaré inequality, where $1 < p < \infty$. Assume $u \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$ is a parabolic quasiminimizer. Then inequality (2.10) holds for every $\phi \in \mathrm{Lip}(\Omega_T)$, such that $\{\phi \neq 0\} \subset\subset \Omega_T$, if and only if it holds for every $\phi \in L^p(0, T; N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ such that $\{\phi \neq 0\} \subset\subset \Omega_T$.*

Proof Only the other direction needs a proof. Assume (2.10) holds for every compactly supported $\phi \in \mathrm{Lip}(\Omega_T)$. Let $\psi \in L^p(0, T; N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ be such that $\{\psi \neq 0\} \subset\subset \Omega_T$. Let $\varepsilon > 0$. Let ϕ be a function in $\mathrm{Lip}(\Omega_T)$. By adding and substracting, and using Hölder's inequality, we can write the left hand side of (2.10) as

$$-\alpha \int\limits_{\{\psi\neq 0\}} \frac{\partial u_\varepsilon}{\partial t}\psi\, d\nu + \int\limits_{\{\psi\neq 0\}} \left(g_u^p\right)_\varepsilon d\nu \le \alpha \left(\int\limits_{\{\phi\neq 0\}\cup\{\psi\neq 0\}} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 d\nu \right)^{\frac{1}{2}}$$
$$\cdot \left(\int\limits_{\{\phi\neq 0\}\cup\{\psi\neq 0\}} |\psi - \phi|^2\, d\nu \right)^{\frac{1}{2}} - \left(\int\limits_{\{\phi\neq 0\}\setminus\{\psi\neq 0\}} g_u^p\, d\nu \right)_\varepsilon$$
$$-\alpha \int\limits_{\{\phi\neq 0\}} \frac{\partial u_\varepsilon}{\partial t}\phi\, d\nu + \int\limits_{\{\phi\neq 0\}} \left(g_u^p\right)_\varepsilon d\nu.$$

On the other hand, using Minkowski's inequality we can write

$$\int\limits_{\{\phi\neq 0\}} \left(g^p_{u(x,t-s)+\phi}\right)_\varepsilon d\nu \le \left(\left(\left(\int\limits_{\{\phi\neq 0\}} (g_{u(x,t-s)+\psi} + g_{\psi-\phi})^p\, d\nu\right)^{\frac{1}{p}}\right)^p\right)_\varepsilon$$
$$\le \left(\left(\left(\int\limits_{\{\phi\neq 0\}\setminus\{\psi\neq 0\}} g^p_{u(x,t-s)+\psi}\right)^{\frac{1}{p}} + \left(\int\limits_{\{\psi\neq 0\}} g^p_{u(x,t-s)+\psi}\right)^{\frac{1}{p}}\right.\right.$$
$$\left.\left. + \left(\int\limits_{\{\phi\neq 0\}} g^p_{\psi-\phi}\, d\nu\right)^{\frac{1}{p}}\right)^p\right)_\varepsilon.$$

Since the above two expressions and inequality (2.10) hold for every compactly supported $\phi \in \mathrm{Lip}(\Omega_T)$, by Corollary 1 we can take a sequence $(\phi_i)_i \subset \mathrm{Lip}(\Omega_T)$ such that

$$\|\psi - \phi_i\|_{L^p(0,T;N^{1,p}(\Omega))} \to 0, \quad \|\psi - \phi_i\|_{L^2(\Omega_T)} \to 0$$
$$\text{and} \quad \nu(\{\varphi_i \neq 0\} \setminus \{\psi \neq 0\}) \to 0,$$

as $i \to \infty$. Hence we see that inequality (2.10) holds for ψ, with the same constant K. $\square$

3 Estimates for parabolic quasiminimizers

From now on we assume that $p = 2$, and that $u \in L^2_{\text{loc}}(0, T; N^{1,2}_{\text{loc}}(\Omega))$ is a parabolic K-quasiminimizer.

In this section we deduce from the definition of parabolic quasiminimizers a fundamental energy estimate for u. From this energy estimate we obtain a Caccioppoli inequality by the so called hole filling iteration. By combining the Caccioppoli inequality, a parabolic version of Poincaré's inequality derived from the fundamental energy estimate, and Sobolev's inequality, we obtain a reverse Hölder inequality for g_u. Since $p = 2$, no difficulties arise when combining the estimates together in proving the reverse Hölder inequality. Higher integrability then follows from the reverse Hölder inequality by a modification of Gehring's famous lemma.

We begin by establishing a fundamental energy estimate by testing the parabolic quasiminimizer with a suitable function.

Lemma 4 (Fundamental energy estimate) *There exists a positive constant $c = c(K)$, such that for every $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, $\rho < \sigma$ such that $Q_\sigma \subset \Omega_T$, we have*

$$\operatorname*{ess\,sup}_{t \in \Lambda_\rho} \int_{B_\rho} |u - u_\sigma(t)|^2 \, d\mu + \int_{Q_\rho} g_u^2 \, d\nu \leq c \int_{(Q_\sigma \setminus Q_\rho)} g_u^2 \, d\nu + \frac{c}{(\sigma - \rho)^2} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{Q_\sigma} |u - u_\sigma(t)|^2 \, d\nu.$$

Proof Assume $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$, and $\rho < \sigma$ are such that $B_\sigma(x_0) \subset \Omega$ and $\Lambda_\rho \subset (0, T)$. Let $t' \in \Lambda_\rho(t_0)$. Define

$$\begin{cases} 0, & t > t' \\ \frac{t'-t}{h}, & t' - h \leq t \leq t', \\ 1, & t \leq t' - h. \end{cases}$$

Let $\varphi_1 \in N^{1,2}(B_\sigma)$, $0 \leq \varphi_1 \leq 1$, be such that $\varphi_1 = 1$ in $B_\rho(x_0)$, $\operatorname{spt} \varphi_1$ is a compact subset of $B_\sigma(x_0)$, and

$$g_{\varphi_1}^2 \leq \frac{c}{(\sigma - \rho)^2}.$$

Define then $\varphi_2 \in \operatorname{Lip}(0, T)$ such that $0 \leq \varphi_2 \leq 1$ and $|\partial \varphi_2 / \partial t| \leq c(\sigma - \rho)^{-2}$, by setting

$$\varphi_2(t) = \begin{cases} 1, & t \geq t_0 - \frac{\rho^2}{2}, \\ \frac{2t - 2t_0 + \sigma^2}{\sigma^2 - \rho^2}, & t_0 - \frac{\sigma^2}{2} < t < t_0 - \frac{\rho^2}{2}, \\ 0, & t \leq t_0 - \frac{\rho^2}{2}. \end{cases}$$

Set now $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$. For a function $f(x,t)$, define for every $t \in (0,T)$

$$f_\sigma^\varphi(t) = \frac{\int_{B_\sigma} f(x,t)\varphi(x,t)\,d\mu}{\int_{B_\sigma} \varphi(x,t)\,d\mu} = \frac{\int_{B_\sigma} f(x,t)\varphi_1(x)\,d\mu}{\int_{B_\sigma} \varphi_1(x)\,d\mu}, \tag{3.1}$$

where when $t \notin \Lambda_\sigma(t_0)$ we use the right hand side expression as the definition. Choose $\phi = -\varphi(u_\varepsilon - (u_\varepsilon)_\sigma^\varphi)\chi_h$. Since u is a parabolic quasiminimizer and $\phi \in L^2(0,T;N^{1,2}(\Omega))$ is compactly supported, by Lemma 3 we can insert ϕ as a test function into inequality (2.10) and examine the resulting terms. In the first term on the left hand side, after performing partial integration, we add and subtract $(u_\varepsilon)_\sigma^\varphi(\partial\phi/\partial t)$ to obtain

$$\int\limits_{\Omega_T} u_\varepsilon \frac{\partial \phi}{\partial t}\,d\nu = \int\limits_{\Omega_T} (u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t))\frac{\partial \phi}{\partial t}\,d\nu + \int\limits_{\Omega_T} (u_\varepsilon)_\sigma^\varphi(t)\frac{\partial \phi}{\partial t}\,d\nu. \tag{3.2}$$

Integrating by parts and using the definition of $(u_\varepsilon)_\sigma^\varphi(t)$, we see that the last term on the right hand side vanishes

$$\begin{aligned} &\int\limits_{\Omega_T} (u_\varepsilon)_\sigma^\varphi(t)\frac{\partial \phi}{\partial t}\,d\nu \\ &\quad = -\int\limits_{t_0 - \frac{\sigma^2}{2}}^{t'} \frac{\partial}{\partial t}(u_{\sigma,\varepsilon}^\varphi(t)) \left(\int\limits_{B_\sigma} u_\varepsilon \varphi\,d\mu - \frac{\int_{B_\sigma} \varphi\,d\mu \int_{B_\sigma} u_\varepsilon\varphi\,d\mu}{\int_{B_\sigma}\varphi\,d\mu} \right) \chi_h(t)\,dt = 0. \end{aligned}$$

Then we integrate the first term on the right side of (3.2) by parts, and so we obtain for every h small enough

$$\begin{aligned} \int\limits_{\Omega_T} u_\varepsilon \frac{\partial \phi}{\partial t}\,d\nu &= -\frac{1}{2}\int\limits_{\Omega_T} (u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t))^2 \frac{\partial}{\partial t}(\varphi\chi_h)\,d\nu \\ &= \frac{1}{2h}\int\limits_{t'-h}^{t'}\int\limits_{B_\sigma} \left|u_\varepsilon(x,t) - (u_\varepsilon)_\sigma^\varphi(t)\right|^2 \varphi\,d\mu\,dt \\ &\quad -\frac{1}{2}\int\limits_{\Omega_T} (u_\varepsilon - (u_\varepsilon)_\sigma^\varphi(t))^2 \chi_h \frac{\partial \varphi}{\partial t}\,d\nu. \end{aligned}$$

Now, letting first $\varepsilon \to 0$ and then $h \to 0$, we obtain

$$\begin{aligned}
&\liminf_{\varepsilon,h\to 0} \int_{\Omega_T} u_\varepsilon \frac{\partial\phi}{\partial t}\, d\nu \\
&\geq \frac{1}{2}\int_{B_\sigma} \left|u(x,t') - u_\sigma^\varphi(t')\right|^2 \varphi(x,t')\, d\mu - \frac{1}{2}\int_{\Omega_T} (u - u_\sigma^\varphi(t))^2 \left|\frac{\partial\varphi}{\partial t}\right| d\nu \\
&\geq \frac{1}{2}\int_{B_\sigma} \left|u(x,t') - u_\sigma^\varphi(t')\right|^2 \varphi(x,t')\, d\mu - \frac{1}{(\sigma-\rho)^2}\int_{Q_\sigma} (u - u_\sigma^\varphi(t))^2\, d\nu.
\end{aligned}$$

On the right hand side of inequality (2.10), we note that for every $h, \varepsilon > 0$ small enough, in the set $\{\phi \neq 0\}$ we have

$$\begin{aligned}
(g^2_{u(\cdot,\cdot-s)-\varphi(u_\varepsilon-(u_\varepsilon)_\sigma^\varphi)\chi_h})_\varepsilon &\leq c(g^2_{u(\cdot,\cdot-s)-u})_\varepsilon + cg^2_{u-\varphi(u-u_\sigma^\varphi)} \\
&+ cg^2_{\varphi(u-u_\sigma^\varphi)}(1-\chi_h)^2 + cg^2_\varphi((u_\varepsilon)_\sigma^\varphi - u_\sigma^\varphi)^2\chi_h^2 \\
&+ c(u-u_\varepsilon)^2 g_\varphi^2 \chi_h^2 + c\varphi^2 g^2_{u-u_\varepsilon}\chi_h^2.
\end{aligned}$$

By Lemma 1, we know that $g^2_{u-u_\varepsilon} \to 0$ and $g^2_{u(\cdot,\cdot-s)-u} \to 0$ in $L^1_{\mathrm{loc}}(\Omega_T)$ as $\varepsilon \to 0$. Also, by (3.1) we have for every small enough ε

$$\begin{aligned}
\int_{\{\phi\neq 0\}} (u_\sigma^\varphi - (u_\varepsilon)_\sigma^\varphi)^2\, d\nu &\leq \mu(\Omega) \int_{t_0-\frac{\sigma^2}{2}}^{t'} (u_\sigma^\varphi(t) - (u_\sigma^\varphi)_\varepsilon(t))^2\, dt \\
&\leq \mu(\Omega) \int_{-\varepsilon}^{\varepsilon} \int_{t_0-\frac{\sigma^2}{2}}^{t'} |u_\sigma^\varphi(t) - u_\sigma^\varphi(t-s)|^2\, dt\, \zeta_\varepsilon(s)\, ds,
\end{aligned}$$

and therefore the fact that $u_\sigma^\varphi \in L^2(0,T)$ implies by the continuity of translation, that the above expression tends to zero as $\varepsilon \to 0$. We obtain

$$\limsup_{\varepsilon,h\to 0} \int_{\{\phi\neq 0\}} \left(g^2_{u(\cdot,\cdot-s)-\phi}\right)_\varepsilon d\nu \leq c\int_{Q_\sigma} g^2_{u-\varphi(u-u_\sigma^\varphi)}\, d\mu\, dt.$$

Next we note that since u_σ^φ does not depend on x, and hence its upper gradient vanishes, we can write, after noting that $\phi = 1$ in Q_ρ,

$$\int_{Q_\sigma} g^2_{u-\varphi(u-u^\varphi_\sigma)}\,d\nu \le c\int_{Q_\sigma} |1-\varphi|^2 g_u^2\,d\nu + c\int_{Q_\sigma} |u-u^\varphi_\sigma(t)|^2 g_\varphi^2\,d\nu$$
$$\le c\int_{Q_\sigma\setminus Q_\rho} g_u^2\,d\nu + \frac{c}{(\sigma-\rho)^2}\int_{Q_\sigma} |u-u^\varphi_\sigma(t)|^2\,d\nu.$$

Since t' was assumed to be arbitrary in $\Lambda_\rho(t_0)$ and the constants in the estimates are independent of t', combining the above expressions through (2.10), and remembering the definition of φ, leads us to the estimate

$$\operatorname*{ess\,sup}_{t\in\Lambda_\rho}\int_{B_\rho} |u-u^\varphi_\sigma(t)|^2\,d\mu + \int_{Q_\rho} g_u^2\,d\nu$$
$$\le c\int_{(Q_\sigma\setminus Q_\rho)} g_u^2\,d\nu + \frac{c}{(\sigma-\rho)^2}\int_{Q_\sigma} |u-u^\varphi_\sigma(t)|^2\,d\nu,$$

where $c = c(K)$. We complete the proof by noting that for any $t \in (0, T)$, we have

$$\int_{B_\rho} |u-u_\sigma(t)|^2\,d\mu$$
$$\le 2\int_{B_\rho} |u-u^\varphi_\sigma(t)|^2\,d\mu + 2\int_{B_\rho}\left(\fint_{B_\sigma} |u^\varphi_\sigma(t)-u|^2\,d\mu\right)d\mu$$
$$\le 4\int_{B_\rho} |u-u^\varphi_\sigma(t)|^2\,d\mu.$$

On the other hand, by the triangle inequality and by Jensen's inequality, and since $\varphi_1 = 1$ in B_ρ,

$$\int_{B_\sigma} |u-u^\varphi_\sigma(t)|^2\,d\mu \le 2\int_{B_\sigma} |u-u_\sigma(t)|^2\,d\mu$$
$$+\,2\int_{B_\sigma}\left(\left(\int_{B_\sigma}\varphi_1\,d\mu\right)^{-1}\int_{B_\sigma} |u_\sigma(t)-u|^2\varphi_1\,d\mu\right)d\mu$$
$$\le 4\frac{\mu(B_\sigma)}{\mu(B_\rho)}\int_{B_\sigma} |u-u_\sigma(t)|^2\,d\mu.$$

□

Having obtained the energy estimate, we use the so called hole filling iteration [33] to extract a Caccioppoli inequality from it.

Lemma 5 (Caccioppoli inequality) *A positive constant $c = c(c_\mu, K)$ exists, so that for any $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$ such that $Q_{2\rho} \subset \Omega_T$, we have*

$$\int_{Q_\rho} g_u^2 \, d\nu \leq \frac{c}{\rho^2} \int_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 \, d\nu.$$

Proof By Lemma 4, for any cylinder $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$ such that $Q_{2\rho} \subset \Omega_T$, we have for any $\rho < \sigma \leq 2\rho$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t \in \Lambda_\rho} \int_{B_\rho} |u - u_\sigma(t)|^2 \, d\mu + \int_{Q_\rho} g_u^2 \, d\nu \\
&\leq c \int_{(Q_\sigma \setminus Q_\rho)} g_u^2 \, d\nu + \frac{c}{(\sigma - \rho)^2} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{Q_\sigma} |u - u_\sigma(t)|^2 \, d\nu
\end{aligned}$$

where $c = c(K)$. We add $c \int_{Q_\rho} g_u^2 \, d\nu$ to both sides of the expression, and divide by $1 + c$, to obtain

$$\int_{Q_\rho} g_u^2 \, d\nu \leq \frac{c}{1 + c} \int_{Q_\sigma} g_u^2 \, d\nu + \frac{c}{(1 + c)(\sigma - \rho)^2} \frac{\mu(B_\sigma)}{\mu(B_\rho)} \int_{Q_\sigma} |u - u_\sigma(t)|^2 \, d\nu.$$

Then we choose

$$\rho_0 = \rho, \quad \rho_i - \rho_{i-1} = \frac{1 - \beta}{\beta} \beta^i \rho, \quad i = 1, 2, \ldots, k, \quad \beta^2 = \frac{1}{2} \left(\frac{c}{1 + c} + 1 \right),$$

replace ρ by ρ_{i-1} and σ by ρ_i, and iterate, to have

$$\begin{aligned}
\int_{Q_\rho} g_u^2 \, d\nu &\leq \left(\frac{c}{1 + c} \right)^k \int_{Q_{\rho_k}} g_u^2 \, d\nu \\
&\quad + \sum_{i=1}^{k} \left(\frac{c}{1 + c} \right)^i \frac{\mu(B_{\rho_i})}{\mu(B_{\rho_{i-1}})} \frac{1}{(\rho_i - \rho_{i-1})^2} \int_{Q_{\rho_i}} |u - u_{\rho_i}(t)|^2 \, d\nu.
\end{aligned}$$

Here among other things $\rho_i \leq 2\rho_{i-1}$ for every i, and so by the doubling property of μ, the ratio $\mu(B_{\rho_i})/\mu(B_{\rho_{i-1}})$ is uniformly bounded. Also, for each i we can estimate after using Fubini's theorem,

$$\int\limits_{Q_{\rho_i}} |u - u_{\rho_i}(t)|^2 \, d\nu \le 2 \int\limits_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 \, d\nu + 2 \int\limits_{Q_{2\rho}} \fint_{B_{\rho_i}} |u_{2\rho}(t) - u|^2 \, d\nu$$

$$\le 2c \int\limits_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 \, d\nu,$$

where $c = c(c_\mu)$. Hence, taking the limit $k \to \infty$ yields the estimate

$$\int\limits_{Q_\rho} g_u^2 \, d\nu \le \frac{c}{\rho^2} \int\limits_{Q_{2\rho}} |u - u_{2\rho}(t)|^2 \, d\nu,$$

where $c = c(c_\mu, K)$. □

Now we combine the Caccioppoli inequality, the fundamental energy estimate, Poincaré's inequality and Sobolev's inequality together to prove a reverse Hölder inequality type estimate for the upper gradient of u.

Lemma 6 (Reverse Hölder inequality) *There exists a positive constant* $c = c(c_\mu, c_P, \lambda, K)$*, and a* $1 < q < 2$*, so that for any* $Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0)$*, such that* $Q_{2\lambda\rho} \subset \Omega_T$ *, we have*

$$\fint\limits_{Q_\rho} g_u^2 \, d\nu \le \varepsilon c \fint\limits_{Q_{2\lambda\rho}} g_u^2 \, d\nu + 2\varepsilon^{-1} c \left(\fint\limits_{Q_{2\lambda\rho}} g_u^q \, d\nu \right)^{\frac{2}{q}}.$$

Proof By the Caccioppoli Lemma 5, by the doubling property of μ and since $\nu(Q_\rho) = \rho^2 \mu(B_\rho)$, we obtain

$$\fint\limits_{Q_\rho} g_u^2 \, d\nu \le \frac{c}{\rho^4} \int\limits_{\Lambda_\rho} \fint\limits_{B_{2\rho}} |u - u_{2\rho}|^2 \, d\mu \, dt,$$

where $c = c(c_\mu, c_P, K)$. On the other hand we can write

$$\frac{c}{\rho^4} \int\limits_{\Lambda_\rho} \fint\limits_{B_{2\rho}} |u - u_{2\rho}|^2 \, d\mu \, dt \le \left(\frac{c}{\rho^2} \operatorname*{ess\,sup}_{t \in \Lambda_\rho} \fint\limits_{B_\rho} |u - u_{2\rho}|^2 \, d\mu \right)^{1 - \frac{q}{2}}$$

$$\cdot \frac{c}{\rho^2} \int\limits_{\Lambda_\rho} \left(\frac{c}{\rho^2} \fint\limits_{B_{2\rho}} |u - u_{2\rho}|^2 \, d\mu \right)^{\frac{q}{2}} dt$$

$$\le \left\{ \fint\limits_{Q_{2\rho}} g_u^2 \, d\nu + c \fint\limits_{Q_{2\lambda\rho}} g_u^2 \, d\nu \right\}^{1 - \frac{q}{2}} \fint\limits_{Q_{2\lambda\rho}} g_u^q \, d\nu,$$

were we used the fundamental energy estimate Lemma 4 and the (2, 2)-Poincaré inequality (see Remark 1) for the former term and the Sobolev type $(2, q)$-Poincaré inequality for the latter term. By the ε-Young inequality we now obtain for every positive ε

$$\fint_{Q_\rho} g_u^2 \, d\nu \leq \varepsilon c \fint_{Q_{2\lambda\rho}} g_u^2 \, d\nu + \varepsilon^{-1} c \left(\fint_{Q_{2\lambda\rho}} g_u^q \, d\nu \right)^{\frac{2}{q}},$$

where $c = c(c_\mu, c_P, \lambda, K)$. □

4 Establishing higher integrability

Having established a reverse Hölder inequality for u, the remaining part of proving local higher integrability for the upper gradient of u is abstract in nature. It relies only on the reverse Hölder inequality and on properties of the underlying doubling metric measure space. We use the following modification of Gehring's lemma.

Theorem 1 *Let $g \in L^2_{\mathrm{loc}}(\Omega_T)$ be a non negative measurable functions defined in Ω_T. Let s be the constant from (2.1) and let q be such that $2s/(2+s) < q < 2$. Consider a parabolic cylinder $Q_{2R}(z_0) \subset \Omega_T$. Suppose that there exists a positive constant $A > 1$, for which with any $z' = (x', t')$ and ρ such that $Q_{A\rho}(z') \subset Q_{2R}(z_0)$, we have*

$$\fint_{Q_\rho(z')} g^2 \, d\nu \leq \varepsilon \fint_{Q_{A\rho}(z')} g^2 \, d\nu + \gamma \left(\fint_{Q_{A\rho}(z')} g^q \, d\nu \right)^{2/q}, \tag{4.1}$$

for any $\varepsilon > 0$, where γ may depend on ε. Then there exists positive constants $\varepsilon_0 = \varepsilon_0(c_\mu, A, \gamma, q)$ and $c = c(c_\mu, A, \gamma)$ such that

$$\left(\fint_{Q_R(z_0)} g^{2+\varepsilon} \, d\nu \right)^{\frac{1}{2+\varepsilon}} \leq c \left(\fint_{Q_{2R}(z_0)} g^2 \, d\nu \right)^{\frac{1}{2}},$$

for every $0 < \varepsilon \leq \varepsilon_0$.

Proof Assume a parabolic cylinder Q_{2R} with center point $z_0 = (x_0, t_0)$ such that $Q_{2R}(z_0) \subset \Omega_T$. Define for every $z_1 = (x_1, x_2)$, $z_2 = (x_2, t_2) \in X \times \mathbb{R}$ the parabolic distance

$$\mathrm{dist}_p(z_1, z_2) = d(x_1, x_2) + |t_1 - t_2|^{1/2}.$$

Using this, set for every $z \in Q_{2R}$ the functions

$$r(z) = \operatorname{dist}_p(z, (X \times \mathbb{R}) \setminus Q_{2R}),$$
$$\alpha(z) = \frac{\nu(Q_{2R})}{\nu(Q_{\frac{r(z)}{5A}}(z))}.$$

From the definition of $r(z)$ it can readily be checked that $Q_{r(z)}(z) \subset Q_{2R}$ for every $z \in Q_{2R}$. For $z \in Q_{2R}$, define

$$h(z) = \alpha^{-1/2}(z) g(z),$$

and for every $\beta > 0$, set

$$G(\beta) = \{ z \in Q_{2R} \cap \Omega_T \ : \ h(z) > \beta \}.$$

Denote

$$\beta_0 = \left(\fint_{Q_{2R}} g^2 \, d\nu \right)^{1/2}.$$

Assume $\beta > \beta_0$. For ν-almost every $z' \in G(\beta)$, we have for every $r \in [r(z')/(5A), r(z')]$,

$$\fint_{Q_r(z') \cap \Omega_T} g^2 \, d\nu \le \alpha(z') \fint_{Q_{2R}} g^2 \, d\nu \le \alpha(z')\beta^2, \tag{4.2}$$

and by the definition of $G(\beta)$, since μ is a positive Borel measure,

$$\lim_{r \to 0} \fint_{Q_r(z')} g^2 \, d\nu = g^2(z') > \alpha(z')\beta^2. \tag{4.3}$$

Now (4.2) and (4.3) imply that for ν-almost every $z' \in G(\beta)$, there exists a corresponding radius $\rho(z') \in (0, r(z')/(5A))$, for which it holds

$$\fint_{Q_{A\rho(z')}(z')} g^2 \, d\nu \le \alpha(z')\beta^2 \le \fint_{Q_{\rho(z')}(z')} g^2 \, d\nu. \tag{4.4}$$

Thus by choosing $\varepsilon = 1/2$ in (4.1), we can absorb the first term on the right hand side of (4.1) into the left hand side and obtain

$$-\!\!\!\!\!\!\int_{Q_{\rho(z')}(z')} g^2\,d\nu \le \left(c -\!\!\!\!\!\!\int_{Q_{A\rho(z')}(z')} g^q\,d\nu \right)^{2/q},$$

for ν-almost every $z' \in G(\beta)$, where $c = c(\gamma)$. This together with (4.4) yields

$$-\!\!\!\!\!\!\int_{Q_{5A\rho(z')}(z')} g^2\,d\nu \le \left(c -\!\!\!\!\!\!\int_{Q_{A\rho(z')}(z')} g^q\,d\nu \right)^{2/q}, \tag{4.5}$$

where $c = c(A, c_\mu, \gamma)$. From the definitions of a parabolic cylinder and the parabolic distance, it follows that

$$2^{-1/2} r(z') \le r(z) \le 2r(z') \qquad \text{for every } z \in Q_{r(z')}(z'),\ z' \in Q_{2R}.$$

From this it is straightforward to check that

$$\begin{array}{c} Q_{r(z)}(z) \subset Q_{3r(z')}(z'), \\ Q_{r(z')}(z') \subset Q_{4r(z)}(z) \end{array} \quad \text{for every } z \in Q_{r(z')}(z'),\ z' \in Q_{2R},$$

and so by the doubling property of the measure there exists positive constants $c = c(c_\mu)$, $c' = c'(c_\mu)$ such that

$$c\alpha(z') \le \alpha(z) \le c'\alpha(z) \qquad \text{for every } z \in Q_{r(z')}(z'),\ z' \in Q_{2R}. \tag{4.6}$$

Because of this, we see from (4.5) that there exists a positive constant $c = c(A, c_\mu, \gamma)$, such that for ν-almost every $z' \in G(\beta)$, after also using the fact that $\alpha(z) \ge 1$,

$$\fint_{Q_{5A\rho(z')}(z')} h^2\,d\nu \le \left(\fint_{Q_{A\rho(z')}(z')} h^q\,d\nu \right)^{2/q}. \tag{4.7}$$

On the other hand, by Hölder's inequality since $1 < q < 2$, and then by (4.6), we obtain from (4.4),

$$\left(\fint_{Q_{A\rho(z')}(z')} h^q\,d\nu \right)^{(2-q)/q} \le \left(\fint_{Q_{A\rho(z')}(z')} h^2\,d\nu \right)^{(2-q)/2} \le c\beta^{2-q}, \tag{4.8}$$

where $c = c(c_\mu)$. Assume now any $\delta > 0$. By (4.7) and by the definition of $G(\delta\beta)$, we have for ν-almost every $z' \in G(\beta)$,

$$\fint_{Q_{5A\rho(z')}(z')} h^2\,d\nu \le c\delta^2\beta^2 + \left(\frac{c}{\nu(Q_{A\rho(z')}(z'))}\int_{Q_{A\rho(z')}(z')\cap G(\delta\beta)} h^q\,d\nu\right)^{2/q}.$$

By (4.6) and (4.4), we can now choose a small enough positive number $\delta(c_\mu, A, \gamma) < 1$ to absorb the first term on the right hand side into the left hand side. We obtain a positive $c = c(A, c_\mu, \gamma)$, such that for ν-almost every $z' \in G(\beta)$ and any $\beta > \beta_0$, after using (4.8),

$$\fint_{Q_{5A\rho(z')}(z')} h^2\,d\nu \le \beta^{2-q}\frac{c}{\nu(Q_{A\rho(z')}(z'))}\int_{Q_{A\rho(z')}(z')\cap G(\delta\beta)} h^q\,d\nu. \tag{4.9}$$

The collection $\{Q_{A\rho(z')}(z') : z' \in G(\beta)\}$ is now an open cover of $G(\beta)$. By the Vitali covering lemma, there exists a countable and pairwise disjoint subcollection $\{\,Q_{A\rho(z_i')}(z_i') : z_i' \in G(\beta)\,\}_{i=1}^{\infty}$, such that

$$G(\beta) \subset \bigcup_{i=1}^{\infty} Q_{5A\rho(z_i')}(z_i') \subset Q_{2R}.$$

The last inclusion follows from the fact that $5A\rho(z) \le r(z)$. This property is the reason why we introduced the number 5 into the proof earlier. Now we can write for any $\beta > \beta_0$, after multiplying inequality (4.9) with $\nu(Q_{A\rho(z')}(z'))$ and using the doubling property of μ,

$$\begin{aligned}\int_{G(\beta)} h^2\,d\nu &\le \sum_{i=1}^{\infty}\int_{Q_{5A\rho(z_i')}(z_i')} h^2\,d\nu\\ &\le \sum_{i=1}^{\infty} c\beta^{2-q}\int_{Q_{A\rho(z_i')}(z_i')\cap G(\delta\beta)} h^q\,d\nu \le c\beta^{2-q}\int_{G(\delta\beta)} h^q\,d\nu.\end{aligned} \tag{4.10}$$

From now on the higher integrability result is a consequence of (4.10) and Fubini's theorem. To see this, we integrate over $G(\beta_0)$ and use Fubini's theorem to obtain

$$\begin{aligned}\int_{G(\beta_0)} h^{2+\varepsilon}\,d\nu &= \int_{G(\beta_0)}\left(\int_{\beta_0}^{h}\varepsilon\beta^{\varepsilon-1}\,d\beta + (\beta_0)^\varepsilon\right)h^2\,d\nu\\ &= \int_{\beta_0}^{\infty}\varepsilon\beta^{\varepsilon-1}\int_{G(\beta)} h^2\,d\nu\,d\beta + (\beta_0)^\varepsilon\int_{G(\beta_0)} h^2\,d\nu,\end{aligned}$$

and now by (4.10)

$$\int_{\beta_0}^{\infty} \varepsilon\beta^{\varepsilon-1} \int_{G(\beta)} h^2 \, d\nu \, d\beta \le c \int_{\beta_0}^{\infty} \varepsilon\beta^{\varepsilon+1-q} \int_{G(\delta\beta)} h^q \, d\nu \, d\beta.$$

By Fubini's theorem again, we see that

$$\begin{aligned} &\int_{\beta_0}^{\infty} \varepsilon\beta^{\varepsilon+1-q} \int_{G(\delta\beta)} h^q \, d\nu \, d\beta + \beta_0^{\varepsilon} \int_{G(\beta_0)} h^2 \, d\nu \\ &= \varepsilon \int_{G(\delta\beta_0)} \left(\int_{\beta_0}^{h/\delta} \beta^{\varepsilon-1+2-q} \, d\beta \right) h^q \, d\nu + \beta_0^{\varepsilon} \int_{G(\beta_0)} h^2 \, d\nu \\ &\le \frac{\varepsilon}{\delta^{2+\varepsilon-q}(\varepsilon+2-q)} \int_{G(\beta_0)} h^{\varepsilon+2} \, d\nu + \beta_0^{\varepsilon} \int_{G(\delta\beta_0)} h^2 \, d\nu, \end{aligned}$$

where $c = c(A, c_\mu, \gamma)$. Observe that in the last step we also used the fact that $h^{\varepsilon+2} \le \beta_0^{\varepsilon} h^2$ in $G(\delta\beta_0) \setminus G(\beta_0)$. We can now choose a positive $\varepsilon = \varepsilon(c_\mu, A, \gamma, q)$ small enough to absorb the term containing $h^{2+\varepsilon}$ into the left hand side of (4.10), and conclude that

$$\int_{G(\beta_0)} h^{2+\varepsilon} \, d\nu \le c(\beta_0)^{\varepsilon} \int_{G(\delta\beta_0)} h^2 \, d\nu, \tag{4.11}$$

where $c = (c_\mu, A, \gamma)$. In case the term containing $h^{2+\varepsilon}$ is infinite, we replace h by $h_k = \min\{h, k\}$ where $k > \beta$. Starting from (4.10) we estimate that

$$\int_{\{h_k > \beta\}} h_k^{2-q} \, d\zeta \le c\beta^{2-q} \int_{\{h_k > \delta\beta\}} d\zeta. \tag{4.12}$$

where $d\zeta = h^q \, d\nu$. Performing now as above the calculations involving Fubini's theorem yields

$$\int_{\{h_k > \beta_0\}} h_k^{2+\varepsilon-q} \, d\zeta \le \varepsilon c \int_{\{h_k > \beta_0\}} h_k^{2+\varepsilon-q} \, d\zeta + \beta_0^{\varepsilon} \int_{\{h_k > \delta\beta_0\}} h_k^{2-q} \, d\zeta.$$

Now we can absorb the term containing $h_k^{2+\varepsilon-q}$ into the left hand side side, and finally let $k \to \infty$ to obtain (4.11).

Finally, from the definitions of the parabolic distance and the parabolic cylinder, it is again straighforward to check that $Q_R \subset Q_{4r(z)}(z)$ for every $z \in Q_R$. Hence, by the doubling property of the measure,

$$\alpha(z) \le \frac{\nu(Q_{2R})}{\nu(Q_R)} \frac{\nu(Q_{4r(z)}(z))}{\nu(Q_{\frac{r(z)}{5A}}(z))} \le c_1, \qquad \text{for every } z \in Q_R,$$

where $c_1 = c_1(c_\mu, A) > 0$. On the other hand, clearly $\alpha(z) \ge 1$ for every $z \in Q_{2R}$. Now (4.11) and the definition of β_0 imply that

$$\int_{Q_R} g^{2+\varepsilon}\, d\nu \le c_1^{\frac{2+\varepsilon}{2}} \left((\beta_0)^\varepsilon \int_{Q_R \setminus G(\beta_0)} h^2\, d\nu + \int_{G(\beta_0)} h^{2+\varepsilon}\, d\nu \right)$$
$$\le c \frac{1}{(\nu(Q_{2R}))^{\varepsilon/2}} \left(\int_{Q_{2R}} g^2\, d\nu \right)^{\frac{2+\varepsilon}{2}},$$

where $c = c(c_\mu, A, \gamma) > 0$. From this expression the proof can readily be completed. □

We conclude this article by stating and proving the main result and a corollary.

Theorem 2 (Local higher integrability) *Let $u \in L^2_{\mathrm{loc}}(0, T; N^{1,2}_{\mathrm{loc}}(\Omega))$ be a parabolic K-quasiminimizer. Then there exists positive constants $\varepsilon = \varepsilon(c_\mu, c_P, \lambda)$ and $c = c(c_\mu, c_P, \lambda, K)$, so that for every z_0 and R such that $Q_{2R}(z_0) \subset \Omega_T$, we have*

$$\left(\fint_{Q_R(z)} g_u^{2+\varepsilon}\, d\nu \right)^{\frac{1}{2+\varepsilon}} \le c \left(\fint_{Q_{2R}(z)} g_u^2\, d\nu \right)^{\frac{1}{2}}.$$

Proof Let z and R be such that $Q_{2R}(z) \subset \Omega_T$. By Lemma 6 there exists a constant $c = c(c_\mu, c_P, \lambda, K)$, and a $1 < q < 2$, such that

$$\fint_{Q_\rho} g_u^2\, d\nu \le \varepsilon c \fint_{Q_{2\lambda\rho}} g_u^2\, d\nu + 2\varepsilon^{-1} c \left(\fint_{Q_{2\lambda\rho}} g_u^q\, d\nu \right)^{\frac{2}{q}}.$$

Theorem 1 with $A = 2\lambda$ now completes the proof. □

Corollary 2 *Let $u \in L^2_{\mathrm{loc}}(0, T; N^{1,2}_{\mathrm{loc}}(\Omega))$ be a parabolic K-quasiminimizer. Then there exists a positive constant $\varepsilon = \varepsilon(c_\mu, c_P, \lambda)$, such that for any compact $F \subset \Omega_T$, we have*

$$\left(\int_F g_u^{2+\varepsilon}\, d\nu \right)^{\frac{1}{2+\varepsilon}} < \infty.$$

Proof Since F is compact, there exists a finite collection of points $z_i \in F$ and parabolic cylinders $Q_{R_i}(z_i)$ such that $Q_{2R_i}(z_i) \subset \Omega_T$ and

$$F \subset \bigcup_{i=1}^{n} Q_{R_i}(z_i).$$

By Theorem 2, we have

$$\begin{aligned}\int_F g_u^{2+\varepsilon}\,d\nu &\le \max_{1\le i\le n} \nu(Q_{R_i}(z_i)) \sum_{i=1}^{n} \fint_{Q_{R_i}(z_i)} g_u^{2+\varepsilon}\,d\nu \\ &\le c \max_{1\le i\le n} \nu(Q_{R_i}(z_i)) \sum_{i=1}^{n} \left(\fint_{Q_{2R_i}(z_i)} g_u^2\,d\nu \right)^{\frac{2+\varepsilon}{2}} < \infty.\end{aligned}$$

□

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