Local in Time Existence for the Complete Maxwell Equations with Monotone Characteristic in a Bounded Domain (*).

ALBERT MILANI (Torino)

Summary. – *The complete system of the quasi-linear Maxwell equations with monotone characteristic in a bounded domain is studied. Following Kato's theory in* [14] *for quasilinear hyperbolic systems, existence and uniqueness of a local regular solution is established.*

1. – Introduction.

In this paper we are concerned with studying the complete system of Maxwell's equations with nonlinear magnetic characteristic in a bounded region of space: we are in front of a nonlinear hyperbolic evolution problem in a bounded domain of $\boldsymbol{R}^3$. Resorting to the usual vector and scalar potentials for the electric and magnetic fields, Maxwell's equations can be transformed into a second order quasi-linear hyperbolic system. For such systems many results have recently been obtained when the problem is set in the whole space (see for instance KATO [12, 13], HUGHES-KATO-MARSDEN [9], FISCHER-MARSDEN [5], HUGHES-MARSDEN [10]); results on the problem in a bounded domain were first obtained by KATO in [14]. It is the aim of this work to show that under physically reasonable assumptions on the behavior of the magnetic characteristic Kato's method can be applied to Maxwell's equations; more precisely, we consider a homogeneous ferro-magnetic medium and neglect hysteresis phaenomena, so that the nonlinear characteristic can be assumed of monotone type with asymptotically linear behavior; we also assume that eddy currents are everywhere present, that is the conductivity is stricly positive. The hyperbolic evolution equation into which Maxwell's equations are transformed is solved using Kato's theory in [14]: the solution we obtain is local in time, though regular in space; we hope in the future to investigate the problem of global existence (general results at this regard can for instance be found in MATSUMURA, [17]) and study the existence and behavior of weaker solutions. Our motivations in establishing these results lie also in the study of a related singular perturbation problem: since in the study of electromagnetic devices it is usual to neglect the effect of displacement currents, the Maxwell equations describing the electromagnetic field reduce to a parabolic system, which, as it happens in other physical situations (for

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instance with compressible and incompressible fluids, see KLAINERMAN-MAJDA, [16]) may therefore be seen as a "limit system" of the complete system at the vanishing of the dielectric constant. Such convergence is fairly easy to study in the linear case (see [18]); we hope in the future to give some results for the non linear case too. The corresponding quasi-stationary equations have been considered in [20] in a more complex kind of domain under slightly different boundary conditions, and the convergence process may be controlled with tecniques analogous to those used by KLAINERMAN and MAJDA in [15].

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2. – The differential system and the electromagnetic potentials.

Let $\Omega \subseteq \boldsymbol{R}^3$ be a bounded open domain with a sufficiently smooth boundary $\partial\Omega$, and n be the outward normal to $\partial\Omega$. We consider the complete system of Maxwell's equations

$$\left.\begin{aligned}
&\frac{\partial D}{\partial t} + j - \operatorname{curl} H = 0 &&(2.1)\\
&\frac{\partial B}{\partial t} + \operatorname{curl} E = 0 &&(2.2)\\
&\operatorname{div} B = 0 &&(2.3)\\
&\operatorname{div} j = 0 &&(2.4)\\
&j = \sigma E \qquad D = \varepsilon E \qquad H = \zeta(B) &&(2.5)\\
&B(0) = B_0 \qquad E(0) = E_0 &&(2.6)
\end{aligned}\right\} \text{ in } \Omega .$$

$$n \times E = 0 \qquad \text{on } \partial\Omega , \tag{2.7}$$

where σ and ε are positive constants and $\zeta\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is a non linear function; the total charge in Ω is supposed to be null. It is well known that if as usual $\operatorname{div} B_0 = 0$ and $n \cdot B_0 = 0$ then (2.2) and (2.7) imply (2.3) and the condition

$$n \cdot B = 0; \tag{2.8}$$

also, (2.4) is a consequence of (2.1) if $\operatorname{div} E_0 = 0$.

It is well known as well that (2.2) and (2.3) imply the existence of vector and scalar potentials P and φ such that

$$\begin{cases} B = \operatorname{curl} P \\ E = -P' - \nabla\varphi \end{cases} \tag{2.9}$$

(we set $P' = \partial P/\partial t$ and similarly); we notice that the boundary condition

$$n \times P = 0$$

should be imposed in accordance to (2.8), and that P and φ can be so chosen that

$$\operatorname{div} P + \varepsilon\varphi' + \varphi = 0; \quad \operatorname{div} P = 0 \text{ on } \partial\Omega \tag{2.10}$$

(we shall give in Appendix A a more detailed account on how P and φ can be determined in the frame of suitable Sobolev type spaces).

Substitution of (2.9) into (2.1) yields, recalling (2.5) and (2.10), the hyperbolic system of equations

$$\varepsilon P'' + P' + \operatorname{curl} \zeta(\operatorname{curl} P) - \nabla \operatorname{div} P = 0 \tag{2.11}$$

together with the boundary conditions

$$\begin{cases} n \times P = 0 \\ \operatorname{div} P = 0 \end{cases} \quad \text{on } \partial\Omega \tag{2.12}$$

and initial conditions

$$\begin{cases} P(0) = P_0 & \text{where } \operatorname{curl} P_0 = B_0 \\ P'(0) = P_1 & \text{where } P_1 = -E_0 . \end{cases} \tag{2.13}$$

REMARK 1. - Condition (2.10) relating potentials P and φ is rather arbitrary and, as a matter of fact, is not sufficient to characterize either P or φ (see Appendix A). In [20] the condition $\operatorname{div} P = 0$ in all of Ω was imposed; here however such condition would give rise to unnecessary tecnical difficulties.

We shall try to solve Problem (2.11)+(2.12)+(2.13) making use of Kato's theory of integration of quasilinear hyperbolic equations of evolution (see [14]). To this aim we transform (2.11) into a first order equation in time: let T be the non linear operator defined by

$$Tw := \operatorname{curl} \zeta(\operatorname{curl} w) - \nabla \operatorname{div} w$$

and A the matrix

$$A = \begin{Vmatrix} 0 & -1 \\ \tilde{\varepsilon} T & \tilde{\varepsilon}\sigma \end{Vmatrix}$$

where $\tilde{\varepsilon} = 1/\varepsilon$; equation (2.11) is then equivalent to

$$U' + AU = 0 \tag{2.14}$$

where $U = \{\psi, \tilde{\psi}\}$ is a vector unknown. The initial condition

$$U(0) = U_0 \tag{2.15}$$

is added to (2.14), where $U_0 = \{P_0, P_1\}$, while conditions (2.12) will be accounted for in the request that $P(t) \in \mathfrak{D}(A)$.

3. – Functional spaces, assumptions and the main theorem.

We shall consider the following spaces:

$$H_0 = \boldsymbol{L}^2(\Omega) = \big(L^2(\Omega)\big)^3; \qquad \boldsymbol{H}^m(\Omega) = \big(H^m(\Omega)\big)^3;$$

$$H_1 = \{u \in \boldsymbol{H}^1(\Omega) | n \times u = 0\}$$

$$H_m = \{u \in \big(H^m(\Omega)\big)^3 | n \times u = 0, \ \operatorname{div} u = 0 \ \text{on} \ \partial\Omega\}, \qquad m = 2, 3, 4 .$$

We have then (DUVAUT-LIONS, [3]) that

$$\{u \in \big(\mathfrak{D}(\bar{\Omega})\big)^3 | n \times u = 0\} \ \text{is dense in} \ H_1;$$

$$H_1 = \{u \in \boldsymbol{L}^2(\Omega) | \operatorname{curl} u \in \boldsymbol{L}^2(\Omega), \ \operatorname{div} u \in L^2(\Omega); \ n \times u = 0\}$$

(we recall that if $u \in \boldsymbol{L}^2(\Omega)$ is such that $\operatorname{curl} u \in \boldsymbol{L}^2(\Omega)$ too, then $n \times u$ is well defined as an element of $\big(H^{-\frac{1}{2}}(\partial\Omega)\big)^3\big)$, and by Friedrichs' inequality (FRIEDRICHS, [7])

$$|u|_1^2 = \|\operatorname{curl} u\|^2_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)}$$

is a norm in H_1 equivalent to the H^1 one; noting then for $m > 1 |\cdot|_m$ the usual H^m norm in H_m, we shall assume that

$$|u|_j \leqslant |u|_{j+1}, \qquad j = 0, 1, 2, 3 .$$

We define then for $j = 0, 1, 2, 3$

$$X_j = H_{j+1} \times \boldsymbol{H}^j(\Omega) \qquad \|u\|_j = |\psi|_{j+1} + |\tilde{\psi}|_j, \qquad u = \{\psi, \tilde{\psi}\} \in X_j$$

$$Y_j = H_{j+1} \times H_j, \qquad j = 1, 2, 3$$

(we shall occasionally write $X = X_0$, $Y = Y_1$).

REMARK 2. – We have chosen the scale of Hilbert spaces $H_0, \ldots, H_4$ in accordance with Kato's theory, in which the highest index $s + 1$ (here $s + 1 = 4$) is related to the space dimension n (here $n = 3$) so that $s > n/2 + 1$. In particular this means that $H^{s-1}(\Omega)$ is an algebra under pointwise multiplication.

We make the following assumptions on the function $\zeta\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$:

a) ζ is a strongly monotone, asymptotically linear function, and the derivative of a convex function $F\colon \boldsymbol{R}^3 \to \boldsymbol{R}$, that is

$$\forall x \in \boldsymbol{R}^3, \quad \zeta(x) = (\partial F)(x) \tag{3.1}$$

$$\forall x, \forall y \in \boldsymbol{R}^3, \quad (\zeta(x) - \zeta(y), x - y) \geqslant l|x - y|^2, \quad l > 0; \tag{3.2}$$

b) ζ has continuous uniformly bounded derivatives up to the 4th order at least, and ζ' is uniformley positive definite, that is

$$\forall x \in \boldsymbol{R}^3, \quad \|\zeta^{(k)}(x)\| \leqslant \delta_k, \quad k = 1, 2, 3, 4 \tag{3.3}$$

$$\forall x, \forall y \in \boldsymbol{R}^3, \quad (\zeta'(x)y, y) \geqslant \gamma |y|^2, \quad \gamma > 0, \tag{3.4}$$

(so that actually (3.2) is a consequence of (3.4)). As an example of such ζ, in FASSINO, [4], the following was considered:

$$\zeta(x) = \alpha(|x|)\frac{x}{|x|}$$

where for $t \in \boldsymbol{R}$, $\alpha(t) = t + \arctan t$.

As for the initial conditions (2.13) we shall assume the following:

$$U_0 = \{P_0, P_1\} \in Y_3; \tag{3.5}$$

also, defining

$$P_2 = \tilde{\varepsilon}[\operatorname{curl} \zeta(\operatorname{curl} P_0) - \nabla \operatorname{div} P_0 + \sigma P_1]$$

$$P_3 = \tilde{\varepsilon}[\operatorname{curl} \zeta'(\operatorname{curl} P_0) \operatorname{curl} P_1 - \nabla \operatorname{div} P_1 + \sigma P_2]$$

$$P_4 = \tilde{\varepsilon} \operatorname{curl}[\zeta''(\operatorname{curl} P_0)(\operatorname{curl} P_1) \operatorname{curl} P_1 + \zeta'(\operatorname{curl} P_0) \operatorname{curl} P_2] + \tilde{\varepsilon}[-\nabla \operatorname{div} P_2 + \sigma P_3]$$

we require that

$$n \times P_j = 0, \quad j \leqslant 4 \tag{3.6}$$

(so that by (3.5) $P_j \in H_{4-j}$, $0 \leqslant j \leqslant 4$).

REMARK 3. – Conditions (3.6) are the so called « compatibility conditions »; though rather awkward, they are natural, in the sense that if P were a regular solution of (2.11)+(2.12)+(2.13), then $P_j = [-d^{(j)}P/dt^j](0)$, $0 \leqslant j \leqslant 4$, so that (3.6) are a consequence of the first of (2.12).

We are now ready to show that under the above made assumptions the system of Maxwell's equations described in section 2 admits a regular solution at least for a short time (local existence); more precisely we state the following

THEOREM 1. – Under the hypotheses (3.1), ..., (3.6) there exists a positive number τ such that problem (2.11)+(2.12)+(2.13) has a unique solution $P \in C^0(0, \tau; H_4) \cap C^1(0, \tau; H_3)$.

4. – Proof of Theorem 1: general outline.

The operator T defined in section 2 has the explicit expression

$$Tu = \sum_{i,j=1}^{3} a_{ij}\, \partial_i \partial_j u$$

where $\partial_i = \partial/\partial x_i$ and $\partial_i \partial_j u = \{\partial_i \partial_j u_h\}$, $h = 1, 2, 3$; the a_{ij} are 3×3 real valued matrices, depending on curl u, which can be explicitly written by a straightforward calculation: setting $f_{ij} = f_{ij}(\text{curl } u) := \partial\zeta_i/\partial y_j|(y = \text{curl } u)$, we have

$$a_{11} = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -f_{33} & f_{32} \\ 0 & f_{23} & -f_{22} \end{Vmatrix}, \quad a_{12} = \begin{Vmatrix} 0 & -1 & 0 \\ f_{33} & 0 & -f_{31} \\ -f_{23} & 0 & f_{21} \end{Vmatrix}, \quad a_{13} = \begin{Vmatrix} 0 & 0 & -1 \\ -f_{32} & f_{31} & 0 \\ f_{22} & -f_{21} & 0 \end{Vmatrix}$$

$$a_{21} = \begin{Vmatrix} 0 & f_{33} & -f_{32} \\ -1 & 0 & 0 \\ 0 & -f_{13} & f_{12} \end{Vmatrix}, \quad a_{22} = \begin{Vmatrix} -f_{33} & 0 & f_{31} \\ 0 & -1 & 0 \\ f_{13} & 0 & -f_{11} \end{Vmatrix}, \quad a_{23} = \begin{Vmatrix} f_{32} & -f_{31} & 0 \\ 0 & 0 & -1 \\ -f_{12} & f_{11} & 0 \end{Vmatrix}$$

$$a_{31} = \begin{Vmatrix} 0 & -f_{23} & f_{22} \\ 0 & f_{13} & -f_{12} \\ -1 & 0 & 0 \end{Vmatrix}, \quad a_{32} = \begin{Vmatrix} f_{23} & 0 & -f_{21} \\ -f_{13} & 0 & f_{11} \\ 0 & -1 & 0 \end{Vmatrix}, \quad a_{33} = \begin{Vmatrix} -f_{22} & f_{21} & 0 \\ f_{12} & -f_{11} & 0 \\ 0 & 0 & -1 \end{Vmatrix}$$

We notice that by (3.1) $f_{ij} = f_{ji}$, so that $a_{ij} = {}^t a_{ji}$.
T is therefore a quasi-linear operator, which we can write

$$T(\text{curl } u)u = \sum_{i,j=1}^{3} a_{ij}(\text{curl } u)\, \partial_i \partial_j u \, .$$

Let now $W \subseteq X_3$ be a bounded open set and for $w = \{\sigma, \tilde{\sigma}\} \in W$ consider the linear operators (depending on w)

$$Lu = L(\mathrm{curl}\,\sigma)\,u := \sum_{i,j=1}^{3} a_{ij}(\mathrm{curl}\,\sigma)\,\partial_i\partial_j u\,.$$

$$A = A(w) := \left\| \begin{matrix} 0 & -1 \\ \tilde{\varepsilon}L & \tilde{\varepsilon}\sigma \end{matrix} \right\|.$$

The theory of integration of the quasi-linear equation

$$U' + A(U)\,U = 0$$

is based upon a fixed point tecnique: more precisely one first solves for each fixed $w \in W$ the linear equation

$$U' + A(w)\,U = 0$$

therefore constructing a correspondence $u = \phi(w)$; then sufficient conditions are seeked to assure that ϕ have a fixed point (in a suitable metric space) which is the solution of (2.14). To achieve this, KATO has shown in [14] that it is sufficient to prove the following properties of the operators $A(w)$:

THEOREM 2. – $A(w) \in \mathcal{G}(X_w, 1, \beta)$, where $\beta > 0$ is a suitable constant and X_w is the space X with an equivalent norm depending Lipschitz continuously on w in the X_2 norm.
(We recall that $\mathcal{G}(X, 1, \beta)$ is the family of generators of C_0 quasi-contractive semigroups on X, see [11]).

THEOREM 3. – For $k = 1, 2, 3$, there exist strongly continuous and locally bounded maps

$$F^k\colon\ W \times \prod_{j=2}^{3-k} X_j \to \bigcup_{j=0}^{3-k} \mathcal{L}(Y_{k+j}, X_j)$$

such that whenever $w \in E^k_X(0, T)$ and $w(t) = \{\sigma(t), \tilde{\sigma}(t)\} \in W$,

$$\frac{d^k}{dt^k} A\big(w(t)\big) = F^k\big(v(t), \dots, v^{(k)}(t)\big)\,. \tag{4.3}$$

Moreover the following estimate holds:

$$\forall w,\ \forall w' \in W\,, \qquad \|A(w) - A(w')\|_{\mathcal{L}(Y_3, X_0)} \leq \mu_1 \|w - w'\|_0\,. \tag{4.4}$$

Here we have set, given a positive number T:

$$E^k_X(0, T) = \{v \mid v^{(j)} \in L^\infty(0, T; X_{3-j})\}\,, \qquad j = 0, \dots, k\,.$$

THEOREM 4. – $\forall w \in W$, $A(w)$ is elliptic in the sense that $\mathcal{D}(A(w)) = Y$ and $\forall j = 0, 1, 2$, if $\phi \in Y$ and $A(w)\phi \in X_j$, then $\phi \in X_{j+1}$ and

$$\|\phi\|_{j+1} \leqslant \nu\{\|A(w)\phi\|_j + \|\phi\|_0\}, \quad \nu > 0 . \tag{4.5$_j$}$$

We shall therefore prove Theorem 1 by proving Theorems 4, 2 and 3 subsequently.

5. – Proof of Theorem 4.

Let $w \in W$ and $L = L(\operatorname{curl}\sigma)$ be defined as in the previous section (from now on we shall not write the correspondence on $\operatorname{curl}\sigma$ explicitly). We claim that

LEMMA 1. – $\forall w \in W$, L is strongly elliptic in the sense that

$$\forall \lambda \in \boldsymbol{R}^3, \quad -\sum_{i,j=1}^{3} a_{ij}\lambda_i\lambda_j \geqslant N|\lambda|^3 \tag{5.1}$$

(matrix inequality). Moreover, the boundary conditions (2.12), that is $n \times u = 0$ and $\operatorname{div} u = 0$ on $\partial\Omega$, are complementing.
(See MORREY, [21], or AGMON-DOUGLIS-NIREMBERG, [2], for the terminology.)

PROOF. – We write

$$Lu = \begin{cases} \displaystyle\sum_{i,j=1}^{3} l_{ij}(x, \partial)u_j \\ i = 1, 2, 3 \end{cases}$$

and recall that (5.1) is equivalent to

$$\alpha = -\sum_{i,j=1}^{3} l_{ij}(x, \xi)\lambda_i\lambda_j \geqslant N_1|\lambda|^2|\xi|^2, \qquad \forall\lambda, \forall\xi \in \boldsymbol{R}^3 . \tag{5.2}$$

A straightforward calculation shows that

$$\alpha = \big(\zeta'(\operatorname{curl}\sigma)(\lambda\times\xi), \lambda\times\xi\big) + |\lambda\cdot\xi|^2$$

so that by (3.4) we get (5.2) with $N_1 = \min(\gamma, 1)$.

Consider now the matrices $B(x, \zeta)$ of the boundary conditions and $\|l^{ij}(x, \xi)\|$ adjoint to $\|l_{ij}(x, \xi)\|$. We have (for instance)

$$B = \begin{Vmatrix} 0 & -n_3 & +n_2 \\ n_3 & 0 & -n_1 \\ \xi_1 & \xi_2 & \xi_3 \end{Vmatrix}$$

($n_j = j$-th component of the normal vector n to $\partial\Omega$). If now $\theta = (\theta_1, \theta_2, 0)$ and $n = (0, 0, 1)$ are vectors tangent and normal to $\partial\Omega$ at a point x_0, to show that the boundary conditions are complementary it is sufficient to show that the matrix $\|Q_{rk}(x_0; \theta; z)\|$, considered as a polinomial in z, has rank 3, where

$$Q_{rk}(x_0; \theta; z) = \sum_{j=1}^{3} B_{rj}(x_0; \theta + zn) l^{jk}(x_0; \theta + zn) , \qquad z \in \boldsymbol{C} .$$

It is immediate now to check that

$$\det \|Q_{rk}\| = z \det \|l^{ij}\|$$

which for $z \neq 0$ cannot be zero because of (5.2).

Consider now the elliptic problem

$$\left\{ \begin{array}{ll} Lu = f & \text{in } \Omega \\ \left. \begin{array}{l} n \times u = 0 \\ \operatorname{div} u = 0 \end{array} \right\} & \text{on } \partial\Omega \end{array} \right. \tag{5.3}$$

since $w \in W$, $\operatorname{curl} \sigma \in \boldsymbol{H}^3(\Omega)$ which is continuously imbedded in $C^1(\bar{\Omega})$ by Sobolev's imbedding theorem (see ADAMS, [1]); the coefficients of L are therefore at least $C^1(\bar{\Omega})$, so that by classical results ([21], [2]) we have that

$$\mathcal{D}(L) = H_2 \tag{5.4}$$

$$|u|_{j+2} \leqslant c_j\{|Lu|_j + |u|_0\} , \qquad j = 0, 1 . \tag{5.5}$$

We would now also need the estimate

$$|u|_4 \leqslant c_2\{|Lu|_2 + |u|_0\} \tag{5.6}$$

for which, to our knowledge, there exists no proof yet. In [19] we give a general result on elliptic systems which would yield (5.6); the boundary condition $u|_{\partial\Omega} = 0$ is considered there, but it is not difficult to adapt the proof to the present situation as well. In appendix B we give a brief sketch of such proof.

The conclusions of Theorem 4 are now an immediate consequence of (5.4), (5.5) and (5.6).

6. – Proof of Theorem 2.

We shall show that the strong ellipticity of L is a sufficient condition (in fact, it is also necessary) for the existence of an equivalent norm in H_1 with respect to which $-A$ generates a quasi-contractive semi-group on X. As a first step, a direct computation shows that

LEMMA 2 (on integration by parts). – For all $p \in C^1(\bar{\Omega})$, $\forall u \in \boldsymbol{H}^2(\Omega)$, $\forall v \in H_1$ we have

$$\int_\Omega L(p)u\cdot v = \int_\Omega \zeta'(p) \operatorname{curl} u \cdot \operatorname{curl} v + \xi(p;\partial p;\operatorname{curl} u, v) + \int_\Omega \operatorname{div} u \cdot \operatorname{div} v \tag{6.1}$$

where

$$\xi(p;\partial p;\operatorname{curl} u, v) := \sum_{j\in\pi(1,2,3)} \int_\Omega \big(\zeta_j''(p, \partial_{j+1}p) - \zeta_{j+1}''(p, \partial_j p)\big) \operatorname{curl} u \cdot v_{j+2}$$

and $\pi(1, 2, 3)$ means the set $\{1, 2, 3\}$ modulo 3, that is $3+1=2+2=1$, $3+2=2$ and so on.

Let now $w = \{\sigma, \tilde{\sigma}\} \in W$ be fixed, and set $p = \operatorname{curl} \sigma$. Define on H_1 the bilinear form

$$B(p;\psi_1, \psi_2) := \int_\Omega \zeta'(p) \operatorname{curl} \psi_1 \cdot \operatorname{curl} \psi_2 + \int_\Omega \operatorname{div} \psi_1 \cdot \operatorname{div} \psi_2 :$$

because of (3.5) and (3.4) B is a continuous coercive form on H_1, that is there exist positive constants a and b such that

$$|B(p;\psi_1, \psi_2)| \leqslant a|\psi_1|_1|\psi_2|_1 \tag{6.2}$$

$$B(p;\psi,\psi) \geqslant b|\psi|_1^2 \tag{6.3}$$

((6.3) is « Garding's inequality », which is equivalent to the strong ellipticity of L). We define then a new norm in X, depending on w, by

$$\|(\psi, \tilde{\psi})\|_w^2 := B(p;\psi,\psi) + \varepsilon|\tilde{\psi}|_0^2 : \tag{6.4}$$

because of (6.2) and (6.3), (6.4) defines an equivalent norm in X, that is we can find a positive constant ν (depending on a, b, ε but not on w) such that

$$\nu\|\cdot\|_0 \leqslant \|\cdot\|_w \leqslant \nu^{-1}\|\cdot\|_0 . \tag{6.5}$$

We give now two lemmas concerning the regularity of the dependence of the norm on w; for their proof we follow closely [9].

LEMMA 3. – $\forall \psi \in H_1$, $\forall w = \{\sigma, \tilde{\sigma}\} \in W$, $\forall v = \{\varrho, \tilde{\varrho}\} \in W$ we have

$$|B(p;\psi,\psi) - B(q;\psi,\psi)| \leqslant k_1\|w - v\|_2|\psi|_1^2 \qquad (q = \operatorname{curl} \varrho) \tag{6.6}$$

so that $\forall u = \{\psi, \bar{\psi}\} \in X$

$$|\|u\|_w - \|u\|_v| \leqslant k_2 \|w - v\|_2 \|u\|_0 \,. \tag{6.7}$$

PROOF. – *a*) $|B(p; \psi, \psi) - B(q; \psi, \psi)| \leqslant \int\limits_\Omega |[\zeta'(p) - \zeta'(q)]| |\text{curl}\, \psi|^2 + \int\limits_\Omega |\text{div}\, \psi|^2 \leqslant$

$$\leqslant c_1 \int\limits_\Omega |p - q| |\text{curl}\, \psi|^2 + \int\limits_\Omega |\text{div}\, \psi|^2$$

(we recall that ζ' is Lipschitz continuous). Now $|p - q| \leqslant \|p - q\|_{C^0(\bar{\Omega})} \leqslant c_2 |p - q|_2 \leqslant \leqslant c_3 |\sigma - \varrho|_3$, whence (6.6) with $k_1 = \max(1, c_2 c_3)$.

b) $|\|u\|_w^2 - \|u\|_v^2| = |B(p; \psi, \psi) - B(q; \psi, \psi)| \leqslant k_1 \|w - v\|_2 |\psi|_1^2 \leqslant k_1 \|w - v\|_2 \|u\|_0^2$ and therefore, recalling (6.5):

$$|\|u\|_w - \|u\|_v| \leqslant k_1 \|w - v\|_2 \|u\|_0^2 (\|u\|_w + \|u\|_v)^{-1} \leqslant \frac{k_1}{2\nu} \|w - v\|_2 \|u\|_0^2 \|u\|_0^{-1}\,,$$

whence (6.7) with $k_2 = k_1/2\nu$.

LEMMA 4 (on the Lipschitz continuity of $\|\cdot\|_w$). – There exist positive constants λ_N, μ_N such that

$$d(\|\cdot\|_w, \|\cdot\|_0) \leqslant \lambda_N \tag{6.8}$$

$$d(\|\cdot\|_v, \|\cdot\|_w) \leqslant \mu_N \|v - w\|_2 \tag{6.9}$$

where

$$d(\|\cdot\|_v, \|\cdot\|_w) := \ln \max \left\{ \sup_{\substack{u \in X \\ u \neq 0}} \frac{\|u\|_w}{\|u\|_v}, \quad \sup \frac{\|u\|_v}{\|u\|_w} \right\}.$$

PROOF. – *a*) by (6.5), both $\|u\|_w / \|u\|_0 \leqslant 1/\nu$ and $\|u\|_0 / \|u\|_w \leqslant 1/\nu$ whence (6.8) with for instance $\lambda_N = \ln(1/\nu)$.

b) from (6.7) we get

$$\frac{\|u\|_w}{\|u\|_v} = \frac{\|u\|_w - \|u\|_v}{\|u\|_v} + 1 \leqslant \frac{k_2 \|w - v\|_2 \|u\|_0}{\nu \|u\|_0} + 1 = k_3 \|w - v\|_2 + 1 \qquad (k_3 = k_2/\nu)$$

whence (6.9) with $\mu_N = k_3$, noticing that for $y > 0$, $\ln(1 + y) \leqslant y$.

We are now ready to prove Theorem 2 using Hille-Yosida's Theorem.

Since $\mathfrak{D}(A(w)) = Y$ is dense in X, it is sufficient to prove that

(6.10) $\quad \exists \beta > 0, \quad \forall \lambda > \beta,\ A(w) + \lambda \quad$ has a continuous inverse satisfying

$$\|(A(w) + \lambda)^{-1}\|_{\mathfrak{L}(X_w, X_w)} \leqslant \frac{1}{\lambda - \beta}\,. \tag{6.11}$$

To prove the invertibility of $A(w)+\lambda$ we consider the equation

$$A(w)u+\lambda u=\phi\,,\qquad \phi=\{\varphi,\tilde\varphi\}\in X,\ u=\{\psi,\tilde\psi\}\in\mathfrak{D}(A(w))\,,$$

that is the system

$$\left\{\begin{array}{l}-\tilde\psi+\lambda\psi=\varphi\\ \tilde\varepsilon L\psi+\tilde\varepsilon\tilde\psi+\lambda\tilde\psi=\tilde\varphi\end{array}\right.\tag{6.12}$$

Now by the ellipticity of L the problems

$$\left\{\begin{array}{l}Lu_1+\eta u_1=\varphi\\ \left.\begin{array}{l}n\times u_1=0\\ \operatorname{div} u_1=0\end{array}\right\}\ \text{on }\partial\Omega\end{array}\right.\qquad\qquad \left\{\begin{array}{l}Lu_2+\eta u_2=\tilde\varphi\\ \left.\begin{array}{l}n\times u_2=0\\ \operatorname{div} u_2=0\end{array}\right\}\ \text{on }\partial\Omega\end{array}\right.$$

have unique solutions u_1 and u_2 if η is sufficiently large, say $\eta>\eta_0$.
Taking $\eta=\lambda(1+\varepsilon\lambda)$, the pair

$$\left\{\begin{array}{l}\psi=(1+\varepsilon\lambda)u_1+\varepsilon u_2\\ \tilde\psi=-Lu_1+\varepsilon\lambda u_2\end{array}\right.$$

solve (6.12) and moreover $\{\psi,\tilde\psi\}\in\mathfrak{D}(A(w))$. We can therefore conclude that $A(w)+\lambda$ is invertible if $\lambda>\beta_1=(-1+\sqrt{1+4\varepsilon\eta_0})/2\varepsilon$ (for this tecnique see YOSIDA, [22], ch. XIV).
To prove (6.11) we show that there exists $\beta_2>0$ such that

$$\forall w\in W,\ \forall u\in\mathfrak{D}(A(w)),\ \forall\lambda>\beta_2\,,\qquad \|(A(w)+\lambda)u\|_w\geqslant(\lambda-\beta_2)\|u\|_w\,.\tag{6.13}$$

By Schwartz's inequality we have:

$$\begin{aligned}\|(A(w)+\lambda)u\|_w\|u\|_w&\geqslant\big((A(w)+\lambda)u,u\big)_w=B(p;(-\tilde\psi+\lambda\psi),\psi)+\\ &+\varepsilon\int_\Omega(\tilde\varepsilon L\psi+\tilde\varepsilon\tilde\psi+\lambda\tilde\psi)\cdot\tilde\psi=-\int_\Omega\zeta'(p)\operatorname{curl}\tilde\psi\cdot\operatorname{curl}\psi-\int_\Omega\operatorname{div}\tilde\psi\cdot\operatorname{div}\psi+\\ &\qquad+\lambda B(p;\psi,\psi)+\int_\Omega L\psi\cdot\tilde\psi+|\tilde\psi|_0^2+\varepsilon\lambda|\tilde\psi|_0^2\,.\end{aligned}$$

By (6.1) we have therefore

$$\begin{aligned}\|(A(w)+\lambda)u\|_w\|u\|_w&\geqslant-\int_\Omega\zeta'(p)\operatorname{curl}\tilde\psi\cdot\operatorname{curl}\psi+\lambda B(p;\psi,\psi)+\\ &\qquad+\int_\Omega\zeta'(p)\operatorname{curl}\psi\cdot\operatorname{curl}\tilde\psi+\xi(p;\partial p;\operatorname{curl}\psi,\tilde\psi)+\lambda\varepsilon|\tilde\psi_0^2|;\end{aligned}$$

now by (3.1) $\zeta'(p)$ is symmetric; moreover

$$|\xi(p;\, \partial p;\, \operatorname{curl}\psi,\, \tilde{\psi})| \leqslant \delta_2 c |\operatorname{curl}\psi|_0 |\tilde{\psi}|_0 \leqslant \frac{1}{2}\, \delta_2 c (|\psi|_1^2 + |\tilde{\psi}|_0^2) \leqslant \frac{1}{2}\, \delta_2 c \|u\|_0^2 \leqslant \frac{\delta_2 c}{2\nu^2} \|u\|_w^2$$

and therefore, setting $\beta_2 = \delta_2 c/2\nu^2$ we get

$$\|(A(w) + \lambda)\, u\|_w \|u\|_w \geqslant (\lambda - \beta_2)\|u\|_w^2$$

whence (6.13). (6.11) is now an easy consequence of (6.10) and (6.13), taking $\beta = \max(\beta_1, \beta_2)$.

REMARK 4. – We emphasize the fact that hypothesis (3.1), that is $\zeta' = \partial F$, is essential in proving Theorem 2. Such hypothesis is naturally related to the conservativeness of the physical system, so that some kind of energy inequality holds (see section 9).

7. – Proof of Theorem 3.

a) for $k = 1, 2, 3$ let $D^{(k)} a_{ij} := a_{ij}^{(k)}(\xi) \in \mathcal{L}(\boldsymbol{R}^3; \boldsymbol{R}^3)$, $\forall \xi \in (\boldsymbol{R}^3)^{1+k}$; for $v = \{\sigma, \tilde{\sigma}\} \in W$, $x = \{\varrho, \tilde{\varrho}\} \in X_2$, $y = \{\tau, \tilde{\tau}\} \in X_1$, $z = \{\psi, \tilde{\psi}\} \in X_0$, let $s = \operatorname{curl}\sigma$, $r = \operatorname{curl}\varrho$, $t = \operatorname{curl}\tau$, $u = \operatorname{curl}\psi$; define then

$$\begin{aligned} f_1(s, r) &= \sum a_{ij}^{(1)}(s, r)\, \partial_i \partial_j \\ f_2(s, r, t) &= \sum [a_{ij}^{(2)}(s, r, r) + a_{ij}^{(1)}(s, t)]\, \partial_i \partial_j \\ f_3(s, r, t, u) &= \sum [a_{ij}^{(3)}(s, r, r, r) + a_{ij}^{(2)}(s, r, t) + a_{ij}^{(1)}(s, u)]\, \partial_i \partial_j \end{aligned}$$

and consequently for $k = 1, 2, 3$

$$F^k = \begin{Vmatrix} 0 & 0 \\ \tilde{\varepsilon} f_k & 0 \end{Vmatrix}.$$

Proof of (4.3) is immediate from the definition of the F^k; to prove that the image of F^k is actually in $\bigcap_{j=0}^{3-k} \mathcal{L}(Y_{j+1}; X_j)$ requires only a lenghty but straightforward application of the chain rule of differentiation and the Sobolev imbedding theorems, recalling assumptions (3.3).

b) let as usual $w = \{\sigma, \tilde{\sigma}\}$, $w' = \{\varrho, \varrho'\}$, $p = \operatorname{curl}\sigma$, $q = \operatorname{curl}\varrho$. For $u = \{\psi, \tilde{\psi}\} \in Y_3$ we have

$$\begin{aligned} \|A(w)u - A(w')u\|_0 = \tilde{\varepsilon}\big| \sum [a_{ij}(p) - a_{ij}(q)]\, \partial_i \partial_j \psi\big|_0 &\leqslant c_1 \delta_2 \|D^2\psi\|_{C^0(\bar{\Omega})} |p - q|_0 \leqslant \\ &\leqslant c_2 |\psi|_4 |\sigma - \varrho|_1 \leqslant \mu_1 \|u\|_3 \|w - w'\|_0 , \end{aligned}$$

whence (4.4) (the first of the above inequalities follows because ζ' is Lipschitz).

8. – Proof of Theorem 1: conclusion.

Theorems 2, 3 and 4 above show that assumptions (Q.1), (Q.2) and (Q.3) of KATO in [14], section 4.1 are met. Hypotheses (3.5) and (3.6) assure that the initial value $U_0 = \{P_0, P_1\}$ satisfies the compatibility conditions ($U_0 \in M$ in Kato's terminology). To apply his Theorem 4.4 and conclude the proof of Theorem 1 we still have to show that the set E defined below is not empty (E is the actual space in which the fixed point argument is carried out). Therefore we prove

LEMMA 5. – The set

$$E = \{v \in E^3_X(0, T) | \|v^{(k)}(t) - u_k\|_{3-k} \leqslant c_k\,, \quad k = 0, \ldots, 3,\ v^{(k)}(0) = u_k,\ k = 0, 1, 2\}$$

where $u_k = \{P_k, P_{k+1}\}$, P_k defined as in section 3, $c_k \in \boldsymbol{R}^+$, $c_0 = \operatorname{dist}(U_0, X_3 \setminus W)$, is not empty.

PROOF. – It is sufficient to show the existence of two functions u and $\tilde{u}$ such that

$$u^{(k)} \in \mathrm{C}(\boldsymbol{R}^+_0; H_{4-k})\,, \quad u^{(k)}(0) = u_k \quad k = 0, 1, 2, 3 \tag{8.1}$$

$$\tilde{u}^{(k)} \in \mathrm{C}(\boldsymbol{R}^+_0; H_{3-k})\,, \quad \tilde{u}^{(k)}(0) = \tilde{u}_k \tag{8.2}$$

with $u_k \in H_{4-k}$ and $\tilde{u}_k \in \boldsymbol{H}^{3-k}(\Omega)$.

a) given u_k, $k = 0, 1, 2, 3$, consider the linear systems

$$\begin{cases} v'' + \operatorname{curl}^2 v - \nabla \operatorname{div} v = 0 \\ \left.\begin{array}{l} \operatorname{div} v = 0 \\ n \times v = 0 \end{array}\right\} \quad \text{on } \partial\Omega \\ v(0) = v_0\,, \quad v'(0) = v_1 \end{cases} \tag{8.3}$$

$$\begin{cases} u'' + \operatorname{curl}^2 u - \nabla \operatorname{div} u = v \\ \left.\begin{array}{l} \operatorname{div} u = 0 \\ n \times u = 0 \end{array}\right\} \quad \text{on } \partial\Omega \\ u(0) = u_0\,, \quad u'(0) = u_1 \end{cases} \tag{8.4}$$

where $v_0 = u_2 + \operatorname{curl}^2 u_0 - \nabla \operatorname{div} u_0 \in H_2$, $v_1 = u_3 + \operatorname{curl}^2 u_1 - \nabla \operatorname{div} u_1 \in H_1$.

System (8.3) admits a unique solution $v \in C(\boldsymbol{R}^+_0; H_2) \cap C^1(\boldsymbol{R}^+_0; H_1)$ so that system (8.4) has a unique solution $u \in C(\boldsymbol{R}^+_0; H_4) \cap \mathrm{C}^1(\boldsymbol{R}^+_0; H_3)$. We have then

$$u'' \in \mathrm{C}(\boldsymbol{R}^+_0; H_2)\,, \quad u''(0) = u_2$$

$$u''' \in \mathrm{C}(\boldsymbol{R}^+_0; H_1)\,, \quad u''(0) = u_3$$

that is (8.1). A similar argument can be used to show the existence of u satisfying (8.1) (see KATO, [14]).

9. – Continuous dependence on the data.

It is not yet known if continuous dependence on the initial data holds from $H_4 \times H_3$ into itself (for a result in this line see HUGHES-MARSDEN, [10]). It is easy however to prove such dependence in the larger space $H_1 \times H_0$: this is to be expected, in relation to the validity of an energy inequality for systems in conservation form such as (2.11). More precisely, noting $(\cdot,\cdot)$ the usual scalar product in $\boldsymbol{L}^2(\Omega)$, we get from (2.11)

$$\varepsilon(P'', P') + \sigma(P', P') + \big(\operatorname{curl} \zeta(\operatorname{curl} P), P'\big) - (\nabla \operatorname{div} P, P') = 0$$

from which, recalling (3.1) and (2.12):

$$\frac{1}{2}\varepsilon \frac{d}{dt}|P'(t)|_0^2 + \sigma|P'(t)|_0^2 + \frac{d}{dt}\int_\Omega F\big(\operatorname{curl} P(t)\big) + \frac{1}{2}\frac{d}{dt}|\operatorname{div} P(t)|_0^2 = 0 ,$$

and integrating

$$\varepsilon|P'(t)|_0^2 + 2\sigma\int_0^t |P'(t)|_0^2 + 2\int_\Omega F\big(\operatorname{curl} P(t)\big) + |\operatorname{div} P(t)|_0^2 = \varepsilon|P_1|_0^2 + 2\int_\Omega F(\operatorname{curl} P_0) + |\operatorname{div} P_0|_0^2 .$$

From (3.2) we can deduce that

$$\int_\Omega F\big(\operatorname{curl} P(t)\big) \geqslant \int_\Omega F(0) + \frac{1}{2}l|\operatorname{curl} P(t)|_0^2 + \big(\zeta(0), \operatorname{curl} P(t)\big)$$

in fact,

$$F(y) - F(0) = \int_0^1 \zeta(\alpha y)\cdot y\, d\alpha = \int_0^1 \frac{1}{\alpha}\big(\zeta(\alpha y) - \zeta(0)\big)\cdot \alpha y\, d\alpha + \int_0^1 \zeta(0)\cdot y\, d\alpha;$$

now $\big(\zeta(0), \operatorname{curl} P(t)\big) = 0$ because $\zeta(0)$ is constant and $n \times P(t) = 0$, so that we get

$$\varepsilon|P'(t)|_0^2 + l|\operatorname{curl} P(t)|_0^2 + |\operatorname{div} P(t)|_0^2 \leqslant \varepsilon|P_1|_0^2 + 2\int_\Omega \big(F(\operatorname{curl} P_0) - F(0)\big) + |\operatorname{div} P_0|_0^2;$$

as above we can then deduce that, since ζ is Lipschitz continuous,

$$2\int_\Omega \big(F(\operatorname{curl} P_0) - F(0)\big) \leqslant L|\operatorname{curl} P_0|_0^2 ,$$

so that we finally come to

$$\varepsilon|P'(t)|_0^2 + l|\text{curl } P(t)|_0^2 + |\text{div } P(t)|_0^2 \leqslant \varepsilon|P_1|^2 + L|\text{curl } P_0|_0^2 + |\text{div } P_0|_0^2\,,$$

which shows the asserted continuity.

REMARK 5. – It would also be possible to consider the case of non homogeneous boundary conditions, or more complex kinds of domains. In such cases equation (2.11) would become non homogeneous, but it could still be treated with the method devised by KATO in [14].

Appendix A.

1. – We recall here the definitions and properties of some Sobolev type spaces in which the differential operators curl and div can appropriately be dealt with. We restrict our attention to the geometrical situation considered in this work, that is a simply connected open set; more complex kinds of domains are considered in [20]. A satisfying survey of the properties of these spaces can be found in FOIAS-TEMAM, [6] or in GIRAULT-RAVIART, [8], to which we refer for details.

Considering the spaces

$$H(\text{curl}, \Omega) = \{u \in \boldsymbol{L}^2(\Omega)|\text{curl } u \in \boldsymbol{L}^2(\Omega)\}$$

$$H(\text{div}, \Omega) = \{u \in \boldsymbol{L}^2(\Omega)|\text{ div } u \in L^2(\Omega)\}$$

$$R = \{u \in \boldsymbol{L}^2(\Omega)|\text{div } u = 0,\ n\cdot u = 0\}$$

$$G = \{u \in \boldsymbol{L}^2(\Omega)|u = \nabla p,\ p \in H^1(\Omega)\}$$

$$V = \{u \in H(\text{curl}, \Omega) \cap H(\text{div}, \Omega)|n \times u = 0\}$$

the following results hold:

(A.1) $\quad R, G, V$ are dense closed subspaces of $\boldsymbol{L}^2(\Omega)$;

(A.2) $\quad R = \text{Im(curl)},\ G = \ker(\text{curl}),\ \boldsymbol{L}^2(\Omega) = G \oplus R$;

(A.3) $\quad V = \{u \in \boldsymbol{H}^1(\Omega)|n \times u = 0\}$

(A.4) $\quad \|u\|^2 = |\text{curl } u|_0^2 + |\text{div } u|_0^2$

is a norm in V equivalent to the H^1 norm (this was used in section 3).

We also have the following

PROPOSITION 1. – $\forall\{h, g\} \in R \times \boldsymbol{L}^2(\Omega)$ there exists a uniquely determined $u \in V$ such that $\text{curl } u = h$, $\text{div } u = g$.

(The proof is easy, considering the elliptic system

$$\begin{cases} \operatorname{curl} u + \nabla p = h \\ \operatorname{div} u \qquad = g \\ n \times u = 0 \end{cases}$$

and noting that $\nabla p = 0$ since p is a solution of

$$\Delta p = 0 \quad \text{in } \Omega\,, \qquad \partial p/\partial n = 0 \quad \text{on } \partial\Omega.)$$

2. – These results allow us to give a more precise formulation to equations (2.9) and (2.10), supposing B and E to be regular enough. Let $h_0 \neq 0$ be a sufficiently regular function null on $\partial\Omega$ (for instance $h_0 \in H^2(\Omega) \cap H_0^1(\Omega)$) and let h and φ be the unique solutions of

$$\text{(A.5)} \qquad \begin{cases} \varepsilon h'' + h' - \Delta h = 0 \\ h|_{\partial\Omega} = 0 \\ h(0) = h_0\,, \quad h'(0) = 0 \end{cases}$$

$$\text{(A.6)} \qquad \begin{cases} \varepsilon \varphi' + \varphi = h \\ \varphi(0) = 0\,. \end{cases}$$

By (2.3) and (2.8) we have that $B \in R$, so that by Proposition 1 there exists a unique vector P such that

$$\operatorname{curl} P = B; \qquad \operatorname{div} P = -h; \qquad n \times P = 0;$$

by (2.2), $P' + E \in \ker(\operatorname{curl}) = G$, so that there exists a unique $\psi \in H^1(\Omega)/\boldsymbol{R}$ such that $P' + E = -\nabla\psi$. We claim that $\alpha = \varphi - \psi$ is constant, so that $\nabla\psi = \nabla\varphi$ and (2.9), (2.10) hold. To achieve this, we first show that φ is also the solution of

$$\text{(A.7)} \qquad \begin{cases} \varepsilon \varphi'' + \varphi' - \Delta\varphi = 0 \\ \varphi|_{\partial\Omega} = 0 \\ \varphi(0) = 0\,, \quad \varphi'(0) = \bar{\varepsilon} h_0; \end{cases}$$

the boundary conditions and the initial ones are immediately checked; if now χ is the solution of (A.7) and $h_1 = \varepsilon\chi' + \chi$ we have that $\varepsilon h_1'' + h_1' - \Delta h_1 = 0$, $h_1|_{\partial\Omega} = 0$, and $h_1(0) = \varepsilon\chi'(0) + \chi(0) = h_0$, $h_1'(0) = \varepsilon\chi''(0) + \chi'(0) = \Delta\chi(0) = 0$, so that h_1 is a solution of (A.5) and hence $h_1 = h$. χ is therefore solution of (A.6) and hence $\chi = \varphi$.

Now we have:

$$-\Delta\psi = \operatorname{div}(P' + E) = -h' = -(\varepsilon\varphi'' + \varphi') = -\Delta\varphi$$

$$n\times\nabla\alpha = n\times\nabla\varphi - n\times\nabla\psi = n\times(P' + E) = 0$$

so that α, being harmonic in Ω and constant on $\partial\Omega$, is constant in all of Ω. We define then P_0 and P_1 in (2.13) by

$$\operatorname{curl} P_0 = B_0; \quad \operatorname{div} P_0 = -h_0; \quad n\times P_0 = 0; \quad P_1 = -E_0 .$$

Conversely, if P is a solution of (2.11)+(2.12)+(2.13), defining $B = \operatorname{curl} P$, $h = -\operatorname{div} P$, φ solution of (A.6), $E = -P' - \nabla\varphi$, it is easy to see, using the same tecnique, that B and E are solutions of Maxwell's equations (2.1), ..., (2.7). In fact, it is sufficient to note that

$$\varepsilon E' + \sigma E - \operatorname{curl} H = -\nabla(\operatorname{div} P + \varepsilon\varphi' + \varphi)$$

$$\operatorname{div} E = -d/dt(\operatorname{div} P + \varepsilon\varphi' + \varphi) .$$

REMARK 6. – The choice $h_0 = 0$ would lead to the characterization $\operatorname{div} P = 0$, $E = -P'$, which was used in [20] (see also Remark 1). The homogeneous boundary condition is arbitrary, and could be replaced by $h|_{\partial\Omega} =$ constant with respect to x.

Appendix B.

We briefly report the method followed in [19], with the modification needed to prove estimate (5.6) for the solution of (5.3). The idea of the method is based on proving that if the coefficients of L are not only of class H^3, but also of class $C^3(\bar{\Omega})$ (as well as f), then it is possible to get estimate (5.6) with c_2 depending at most on the H^3 norm of such coefficients (which is in general not assured by classical results, see [21] or [2]). Then the conclusion follows by a standard approximation method. To achieve this we separately consider the problem of regularity at the boundary and regularity ar the interior of Ω. We give here only an outline of the proof; for all details we refer to [19].

1. – Regularity at the boundary.

We estimate u in $\boldsymbol{H}^4(\Omega \cap U)$ where U is an open set such that $U \cap \partial\Omega \neq \emptyset$. Via changement of local coordinates $y = Tx$ we flatten $U \cap \partial\Omega$ into $\Gamma' \subseteq \{y_3 = 0\}$ ($y = (y_1, y_2, y_3)$ are the new coordinates). System (5.3) becomes

$$\text{(B.1)} \qquad \begin{cases} \tilde{D}v := \tilde{L}v(y) + \tilde{B}v(y) = \tilde{f}(y) \\ \left.\begin{aligned} & v_1(y) = v_2(y) = 0 \\ & Pv(y) := \operatorname{div} v(y) - \{\alpha_1(y)\,\partial_3 v_1(y) + \alpha_2(y)\,\partial_3 v_2(y)\} = 0 \end{aligned}\right\} \text{ on } \Gamma' \end{cases}$$

where $v(y) = u(T^{-1}(y))$, $\tilde{L} = \sum \alpha_{ij}(y)\,\partial_i\partial_j$, $\tilde{B} = \sum \beta_j(y)\,\partial_j$, $\tilde{f}(y) = f(T^{-1}(y))$, $\alpha_i = [\partial g/\partial x_i]$ $(x = T^{-1}(y))$, g is the function describing $U \cap \partial\Omega$; α_{ij} and β_j are 3×3 matrices which are still of class $H^3 \cap C^3$ if g is sufficiently regular.

Let now $B_R^+ \subseteq \{y_3 > 0\}$ be a ball centered at the origin of the new coordinate system, and $R' < R$. We prove

PROPOSITION 2. – $\|\partial_1 v\|_{H^2(B_{R'}^+)} + \|\partial_2 v\|_{H^2(B_{R'}^+)} \leq k_1\{\|\tilde{D}v\|_{H^2} + \|v\|_{L^2}\}$ with k_1 depending at most on the H^3 norm of the coefficients of L.

(We shall state this by writing $k_1 \in \mathcal{O}$).

SKETCH OF PROOF. – We already know that $v \in \boldsymbol{H}^4(\Omega)$, since the coefficients are supposed more regular. Let $\xi \in C_0^\infty(B_R)$, $0 \leq \xi \leq 1$, $\xi \equiv 1$ on $B_{R'}$; let $\bar{v} = \xi\partial_r\partial_s v$, $s \neq 3$. We have that $\bar{v} \in \boldsymbol{H}^2(B_R)$ and since $\tilde{D}$ is uniformely elliptic

$$\|\bar{v}\|_{H^2(B_R^+)} \leq h_0\{\|\tilde{D}\bar{v}\|_{L^2(B_R^+)} + \|\bar{v}\|_{L^2(B_R^+)} + \|n\times\bar{v}\|_{H^{\frac{3}{2}}(\partial B_R^+)} + \|P\bar{v}\|_{H^{\frac{1}{2}}(\partial B_R^+)}\}\,, \qquad h_0 \in \mathcal{O}\,. \tag{B.2}$$

Proposition 2 is proved by careful estimation of the right side of (B.2), for which we shall give brief details.

a) estimate of $\|n\times\bar{v}\|$: on ∂B_R, $\bar{v} \equiv 0$; on $\Gamma = \{y \in \Gamma' \,|\, |y| \leq R\}$, $v_1 = v_2 = 0$ and $n = (0, 0, 1)$ so that $n \times \partial_r\partial_s v = 0$ because $rs \neq 33$.

b) estimate of $\|P\bar{v}\|$: $P\bar{v} = 0$ on ∂B_R, while on Γ' we have (since $v_1 = v_2 = 0$) $P\bar{v} = \partial_3\xi\partial_r\partial_s v_3 + \xi\partial_3\partial_r\partial_s v$. The first term is easily estimated with the H^3 norm of v, and the same is true for the second, since $Pv = 0$ on $\Gamma' \Rightarrow \partial_3 v_3 = \alpha_1\partial_3 v_1 + \alpha_2\partial_3 v_2$.

c) estimate of $\|\bar{v}\|$ and $\|\tilde{B}\bar{v}\|$: there is no problem in giving estimates with the H^3 norm of v, since B is a first order operator.

d) estimate of $\|\tilde{L}\bar{v}\|$: we have

$$\tilde{L}\bar{v} = \xi\tilde{L}(\partial_r\partial_s v) + 2\sum\alpha_{ij}[\partial_i\xi\partial_j\partial_r\partial_s v] + \sum\alpha_{ij}[\partial_i\partial_j\xi\partial_r\partial_s v]:$$

the second and third terms can be estimated with the H^3 norm of v; as for the first we have

$$\tilde{L}(\partial_r\partial_s v) = \partial_r\partial_s\tilde{L}v - \sum(\partial_r\alpha_{ij})[\partial_s\partial_i\partial_j v] - \sum(\partial_s\alpha_{ij})[\partial_r\partial_i\partial_j v] - \sum(\partial_r\partial_s\alpha_{ij})[\partial_i\partial_j v];$$

we estimate the second and third term as before; as for the fourth, we notice that since α_{ij} and v are H^3, $\partial_r\partial_s\alpha_{ij}$ and $\partial_i\partial_j r$ are $H^1 \hookrightarrow L^6$, and therefore their product is in L^2 with

$$\|(\partial_r\partial_s\alpha_{ij})[\partial_i\partial_j v]\|_{L^2} \leq c\|\partial_r\partial_s\alpha_{ij}\|_{L^6}\|\partial_i\partial_j v\|_{L^6} \leq C\|\alpha_{ij}\|_{H^3}\|v\|_{H^3}$$

so that the fourth term also can be estimated with the H^3 norm of v and a constant in $\mathcal{O}$. To estimate $\partial_r\partial_s\tilde{L}v$ we have from (B.1) that

$$\int_{B_R^+}\partial_r\tilde{L}v\cdot\partial_s(\xi^2\partial_r\partial_s\tilde{L}v)=\int_{B_R^+}(\partial_r\tilde{f}-\partial_r\tilde{B}v)\cdot\partial_s(\xi^2\partial_r\partial_s\tilde{L}v);\tag{B.3}$$

now

$$\int_{B_R^+}\partial_r\tilde{L}v\cdot\partial_s(\xi^2\partial_r\partial_s\tilde{L}v)=\int_{B_R^+}\partial_s(\partial_r\tilde{L}v\cdot\xi^2\partial_r\partial_s\tilde{L}v)-\int_{B_R^+}\xi^2|\partial_r\partial_s\tilde{L}v|^2$$

and

$$\int_{B_R^+}\partial_s(\partial_r\tilde{L}v\cdot\xi^2\partial_r\partial_s\tilde{L}v)=\int_{B_R^+}n_s(\partial_r\tilde{L}v\cdot\xi^2\partial_r\partial_s\tilde{L}v)+\int_{\partial B_r\cap\{y_3\geqslant 0\}}n_s(\partial_r\tilde{L}v\cdot\xi^2\partial_r\partial_s\tilde{L}v)=0$$

since $\xi=0$ on ∂B_R and $n_s=0$ on Γ because $s\neq 3$. By a similar argument on the right side of (B.3) we eventually come to the identity

$$\int_{B_R^+}\xi^2|\partial_r\partial_s\tilde{L}v|^2=\int_{B_R^+}\xi^2(\partial_r\partial_s\tilde{f}-\partial_r\partial_s\tilde{B}v)\cdot\partial_r\partial_s\tilde{L}v$$

whence the desired estimate, by Schwartz' inequality.

PROPOSITION 3. – $\|\partial_{3333}v\|_{L^2(B_{R'}^+)}\leqslant k_2\{\|\tilde{D}v\|_{H^2}+\|v\|_{L^2}\}$, $k_2\in\mathcal{O}$.

PROOF. – From (B.1) we get

$$\alpha_{33}\partial_{33}v=\tilde{f}-\tilde{B}v-\sum{}'\alpha_{ij}\partial_i\partial_jv$$

where $\sum'$ means summation over all indexes except $ij=33$. The strong ellipticity of the operator $\tilde{L}$ assures that α_{33} is invertible; moreover, it is easy to check that α_{33}^{-1} is of class H^2 and $\|\alpha_{33}^{-1}\|_{H^2}\in\mathcal{O}$.

Hence, from

$$\|\partial_{33}v\|_{H^2(B_{R'}^+)}=\|\alpha_{33}^{-1}(\tilde{f}-\tilde{B}v-\sum{}'\alpha_{ij}\partial_i\partial_jv)\|_{H^2(B_R^+)}\tag{B.4}$$

we conclude the proof, since by Proposition 2 we already have an estimate of the right side of (B.4) (we recall that H^2 is an algebra).

2. – Interior regularity and conclusion.

We estimate u in $\boldsymbol{H}^4(\Omega')$, where Ω' is an open set such that $\Omega'\subset\subset\Omega$. The procedure is identical with that of Proposition 2, and even simpler, since the values of $\bar{u}=\xi\partial_r\partial_su$ on $\partial\Omega$ are null $(\xi\in C_0^\infty(\Omega),\ 0\leqslant\xi\leqslant 1,\ \xi\equiv 1$ on $\Omega')$. The global estimate (5.6) can now be easily obtained considering a finite open covering of $\bar{\Omega}$ and the related partition of unity. The final result is then proved considering the sequence of problems

$$\text{(B.5)} \qquad \begin{cases} L_m u_m + \lambda u_m = f_m + \lambda w_m \\ \left.\begin{array}{l} n \times u_m = 0 \\ \operatorname{div} u_m = 0 \end{array}\right\} \quad \text{on } \partial\Omega \end{cases}$$

where

$$L_m = \sum l_{ij}^m(x, \partial)$$

and $l_{ij}^m \to l_{ij}$ in H^3, $f_m \to f$ in $\boldsymbol{H}^2$, $w_m \to u$ in $\boldsymbol{H}^2$ and are such that Propositions 2 and 3 hold for u_m solution of (B.5). Then $\{u_m\}$ is uniformely bounded in $\boldsymbol{H}^4(\Omega)$, therefore converging weakly to a vector $\chi \in \boldsymbol{H}^4(\Omega)$ which is solution of

$$\text{(B.6)} \qquad \begin{cases} L\chi + \lambda\chi = f + \lambda u \\ \left.\begin{array}{l} n \times \chi = 0 \\ \operatorname{div} \chi = 0 \end{array}\right\} \quad \text{on } \partial\Omega\,. \end{cases}$$

Since u is obviously a solution of (B.6), we have that $\chi = u$ so that $u \in \boldsymbol{H}^4(\Omega)$ and (5.6) holds.

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