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LOCAL LIMIT THEOREMS AND RECURRENCE CONDITIONS FOR SUMS OF INDEPENDENT INTEGER-VALUED RANDOM VARIABLES

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Conditions are given which imply that the partial sums of a sequence of independent integer-valued variables which satisfy the classical Lindeberg conditions for the central limit theorem also obey a strong version of the local limit theorem. Application is made to the problem of establishing the interval recurrence of the partial sums.

1. Introduction. A sequence $\{X_k\}_{k=1}^{\infty}$ of independent integer-valued random variables with finite variances, where $EX_k = e_k$, $E(X_k - e_k)^2 = b_k^2$, $\sum_{k=1}^{n} e_k = E_n$, $\sum_{k=1}^{n} b_k^2 = B_n^2$ and $\sum_{k=1}^{n} X_k = S_n$, is said to satisfy a local limit theorem if

$$\lim_{n\to\infty} B_n P\{S_n = x\} - (2\pi)^{-\frac{1}{2}} \exp\{-(x - A_n)^2/2B_n^2\} = 0 \,, \tag{1}$$

uniformly for all integer x. Such theorems have proven by Gnedenko [1] in the case that the random variables are identically distributed, and by Rozanov [4] in the non-identically distributed case. We will say that a strong local limit theorem holds if, for any fixed m, $\{X_{k+m}\}_{k=1}^{\infty}$ satisfies a local limit theorem, i.e., when (1) holds with S_n replaced by $S_n - S_m$, etc.

In Section 2 a generalization of Rozanov's theorem is proven, together with two simple corollaries. A further corollary, useful in applications, gives conditions from which it follows that

$$\lim_{d\to\infty} \frac{1}{d} \limsup_{n\to\infty} |B_n P\{x \leqq S_n < x + d\} - d(2\pi)^{-\frac{1}{2}} \exp\{-x^2/2B_n^2\}| = 0 \,, \tag{2}$$

uniformly in integer x. We will refer to this as an interval limit theorem.

In Section 3 these limit theorems are applied, together with a result of Orey [3] to obtain sufficient conditions for the recurrence of the random walk generated by a sequence $\{X_k\}$. We will say that the random walk is d-recurrent for a given integer d if

$$P\{x \leqq S_n < x + d, \text{ for infinitely many } n\} = 1 \,, \tag{3}$$

for all integer x. This is equivalent to

$$P\{x \leqq S_{n+k} - S_k < x + d, \text{ for some } n > 0\} = 1 \,, \tag{3'}$$

for all integer k and x.

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In the sequel we employ the notation

$$P_k(x) = P(X_k = x)\,, \qquad \varphi_k(t) = \sum e^{itx} P_k(x)\,, \qquad \Psi_n(t) = \prod_{k=1}^n \varphi_k(t)\,.$$

2. Local limit theorems. Throughout this paper we assume

$$B_n \to \infty\,; \tag{4}$$

otherwise entirely different methods are appropriate to discussion of the limiting behavior of the distribution of S_n. We also assume, without loss of generality, that $e_k = 0$, all k.

Rozanov has pointed out that if (4) is satisfied, a necessary condition for $\{X_k\}$ to satisfy (1) is that

$$\prod_{k=1}^\infty [\max_{0 \leqq x < h} P\{X_x \equiv x (\text{mod}\, h)\}] = 0\,, \qquad \text{for all } h \geqq 2\,. \tag{5}$$

Let $\{X_k'\}$ be the symmetrization of the sequence $\{X_k\}$, i.e., let $\{Y_k\}$ be a sequence of independent random variables, independent of $\{X_k\}$, Y_k having the same distribution as X_k. Set $X_k' = X_k - Y_k$ and $P\{X_k' = x\} = P_k'(x)$.

Lemma 1. *If $\{X_k\}$ satisfies (5), then $\{X_k'\}$ satisfies (5).*

Proof.

$$\begin{aligned} P\{X_k' \equiv x (\text{mod}\, h)\} &\leqq \textstyle\sum_{y=0}^{h-1} P\{X_k \equiv y (\text{mod}\, h)\} P\{X_k \equiv y - x (\text{mod}\, h)\} \\ &\leqq \max_{0 \leqq y < h} P\{X_k \equiv y (\text{mod}\, h)\}\,. \end{aligned}$$

Theorem 1. *If $\{X_k\}$ is such that*

(A) $\prod_{k=1}^\infty [\max_{0 \leqq x < h} P\{X_k \equiv x (\text{mod}\, h)\}] = 0\,,$ *for all* $h \geqq 2\,,$

(B) $\lim_{n\to\infty} \dfrac{1}{B_n^2} \sum_{k=1}^n \sum_{|x| \leqq \varepsilon B_n} x^2 P_k(x) = 1\,,$ *for any* $\varepsilon > 0\,,$

there exists a sequence $\{M_n\}$, $G > 0$ and $L > 0$ such that

$$\inf_n \frac{1}{B_n^2} \sum_{k=1}^n \sum_{|x| \leqq M_n} x^2 P_k(x) \geqq 2G \tag{C}$$

and, setting

$$Q_n = \sum_{k=1}^n P\{0 < X_k' \leqq L\}\,, \qquad B_n M_n / Q_n \to 0\,, \tag{D}$$

then $\{X_k\}$ satisfies a strong local limit theorem.

Remark. Condition (B) is the classical Lindeberg condition for the central limit theorem. Despite a superficial connection (B) and (C) are independent conditions; because of the restriction (D) places on $\{M_n\}$, neither implies the other. They are both implied by Rozanov's condition (B), if $\{M_n\}$ is replaced by a constant M.

Proof. By the Fourier inversion formula

$$
\begin{aligned}
&2\pi B_n P\{S_n = x\} - (2\pi)^{\frac{1}{2}} \exp\left\{-\frac{x^2}{2B_n{}^2}\right\} \\
&= \int_{|t|\leqq A} \left(\Psi_n\left(\frac{t}{B_n}\right) - e^{-(t^2/2)}\right) e^{-itx/B_n}\, dt \\
&- \int_{|t|>A} \exp\left\{-\frac{t^2}{2} - \frac{itx}{B_n}\right\} dt + B_n \int_{A/B_n<|t|\leqq B/M_n} \Psi_n(t) e^{-itx}\, dt \qquad (6)\\
&+ B_n \int_{B/M_n<|t|\leqq C} \Psi_n(t) e^{-itx}\, dt + B_n \int_{C<|t|\leqq \pi} \Psi_n(t) e^{-itx}\, dt \\
&= I_1 + I_2 + I_3 + I_4 + I_5 .
\end{aligned}
$$

Since (B) is the classical Lindeberg condition, $\Psi_n(t/B_n) \to e^{-(t^2/2)}$ uniformly on compact sets, hence $I_1 \to 0$ for any fixed A. By choosing A sufficiently large, I_2 can be made arbitrarily small, uniformly in x.

Let $A_n = \{k/k \leqq n \text{ and } Gb_k{}^2 \leqq \sum_{|x|\leqq M_n} x^2 P_k(x)\}$. Clearly $\sum_{k\in A_n} \sum_{|x|\leqq M_n} x^2 P_k(x) \geqq GB_n{}^2$. In the remainder of the proof, whenever k appears, we assume $k \in A_n$.

Let $h_{kn} = P\{|X_k| \leqq M_n\}$, $\varphi_{kn}(t) = \sum_{|x|\leqq M_n} e^{itx} P_k(x)$, $e_{kn} = \sum_{|x|\leqq M_n} x P_x(x)$, and $b_{kn}^2 = \sum_{|x|\leqq M_n} x^2 P_k(x)$. We have, manipulating the Taylor expansion for e^{itx},

$$
|\varphi_{kn}(t)|^2 = h_{kn}^2 - t^2\{h_{kn} b_{kn}^2 - (e_{kn})^2\} + \varepsilon(t) \qquad (7)
$$

where $|\varepsilon(t)| \leqq \frac{1}{6} |t|^3 M_n b_{kn}^2$. Note also that $b_k{}^2 \leqq M_n{}^2/G$. Since increasing the members of the sequence $\{M_n\}$ by a constant factor, such as $4/G$, does not affect conditions (C) and (D), we may assume $b_k{}^2/(GM_n{}^2) < 1/16$, so that $h_{kn} \geqq 1 - b_k{}^2/M_n{}^2 > \frac{1}{2}$. It then follows that

$$
\begin{aligned}
(e_{kn})^2 &= \{\textstyle\sum_{|x|>M_n} x P_k(x)\}^2 \leqq \left\{\dfrac{1}{M_n} \textstyle\sum_{|x|>M_n} x^2 P_k(x)\right\}^2 \qquad (8)\\
&\leqq \left(\dfrac{b_k{}^2}{GM_n{}^2}\right) b_{kn}^2 < \dfrac{b_{kn}^2}{16} .
\end{aligned}
$$

If now $|t| \leqq 6/(16M_n)$,

$$
|\varphi_{kn}(t)|^2 \leqq h_{kn}^2\{1 - t^2(1 - \tfrac{1}{4} - \tfrac{1}{4}) b_{kn}^2\} \qquad (9)
$$

so that

$$
|\varphi_k(t)| \leqq 1 - h_{kn} + h_{kn}\{1 - t^2 G b_k{}^2/4\} \leqq \exp\{-t^2 G b_k{}^2/8\} . \qquad (10)
$$

Therefore, if $B \leqq 6/16$,

$$
|I_3| \leqq B_n \int_{A/B_n}^{B/M_n} \exp\{-t^2 G^2 B_n{}^2/8\}\, dt \leqq \frac{8}{AG^2} \exp\{-A^2 G^2/8\} \qquad (11)
$$

which can be made arbitrarily small by choosing A sufficiently large.

Since $\cos u \leqq 1 - u^2/3$ when $|u| \leqq \pi/2$, if $|t| \leqq \frac{1}{2}\pi/L$, then

$$
|\varphi_k(t)|^2 \leqq 1 - P\{0 < X' \leqq L\} + P\{0 < X' \leqq L\}(1 - t^2/3) . \qquad (12)
$$

Therefore, by (D), if $C = \frac{1}{2}\pi/L$,

$$|I_4| \leqq 2B_n \int_{B/M_n}^{C} \exp\{-Q_n t^2/6\}\, dt < B_n/Q_n^{\frac{1}{2}} \int_{BQ_n^{\frac{1}{2}}/M_n}^{\infty} e^{-u^2/6}\, du$$
$$\leqq \frac{6B_nM_n}{BQ_n} \exp\left\{-\frac{B^2Q_n}{M_n^2}\right\} \to 0 .$$

Let $\{t_i\}$ be the set of points of the interval $[C, \pi]$ which are of the form $2\pi h/j$ where h and j are relatively prime and $2 \leqq j \leqq L$. Indexing the t_i in increasing order with $t_m = \pi$, define

$$\Delta_1 = [C, \tfrac{1}{2}(t_1 + t_2)]$$
$$\Delta_i = [\tfrac{1}{2}(t_{i-1} + t_i)\,,\, \tfrac{1}{2}(t_i + t_{i+1})] \qquad (1 < i < m)$$
$$\Delta_m = [\tfrac{1}{2}(t_{m-1} + t_m)\,,\, \pi]\,.$$

Fixing a value of i, $t_i = 2\pi h_0/j_0$, and $u = t - t_i$, let

$$B_n \int_{\Delta_i} \Psi_n(t)\, dt = \int_{|u| \leq D/B_n} B_n\Psi_n(t)\, du + \int_{D/B_n < |u| \leq E/M_n} B_n\Psi_n(t)\, du$$
$$+ \int_{E/M_n < |u|,\, u+t_i \in \Delta_i} B_n\Psi_n(t)\, du$$
$$= I' + I'' + I'''\,.$$

To bound I' we use the fact that $|\varphi_k(t)| \leqq \exp\{\frac{1}{2}(|\varphi_k(t)|^2 - 1)\}$. We have

(13) $$\sum_{k=1}^{n} (|\varphi_k(t)|^2 - 1) \leqq \sum_{k=1}^{n} \sum^{j} P_k'(x)\, (\cos xt - 1) = R_n(t)$$

where the j on the summation sign indicates the summation is taken over those values of x for which $x \not\equiv 0(j)$. For such values of x, when $t = 2\pi h/j$, $\cos xt \leqq \cos 2\pi/j$. Since by condition (A) and Lemma 1, $\sum_{k=1}^{n} \sum^{j} P_k'(x) \to \infty$ as $n \to \infty$, $R_n(2\pi h/j) \to -\infty$ as $n \to \infty$. Since $R_n(t) \geqq R_{n+1}(t)$ and the functions are continuous, for any M there is a symmetric interval of length 2δ around $2\pi h/j$, $\delta = \delta(M)$, such that for n sufficiently large, and t in the interval, $-M \geqq R_n(t)$. Therefore, for any fixed D, if n is sufficiently large, and if $\delta > D/B_n$, then $|I'| \leqq B_n \int_{|u| \leq D/B_n} e^{R_n(t)}\, du \leqq De^{-M}$.

Set $E = \pi/2L$. Then if $|u| \leqq E/M_n$, and $|j| \leqq 2M_n$, we have $|uj| \leqq \pi/L$. Supposing $(2\pi h_0/j_0)j = 2\pi k_0 + 2\pi h_1/j_1$ so $\pi > 2\pi h_1/j_1 > 2\pi/L$, we have $\cos(u + 2\pi h_0/j_0)j \leqq \cos uj$.
Therefore,

$$\sum_{|j| \leq 2M_n} P_k'(j)\Big[\cos\Big(u + \frac{2\pi h_0}{j_0}\Big)j - 1\Big] \leqq \sum_{|j| \leq 2M_n} P_k'(j)[\cos uj - 1]$$
$$\leqq \frac{u^2}{2} \sum_{|j| \leq 2M_n} P_k'(j)j^2\,.$$

Now

(14) $$\begin{aligned} \sum_{|j| \leq 2M_n} P_k'(j)j^2 &= \sum_{|j| \leq 2M_n} \sum_{l=-\infty}^{\infty} P_k(l)P_k(l+j)j^2 \\ &\geqq \sum_{l=-M_n}^{M_n} P_k(l) \sum_{j=-2M_n}^{2M_n} P_k(l+j)j^2 \\ &\geqq \sum_{|l| \leq M_n} P_k(l) \sum_{i=-M_n}^{M_n} P_k(i)(i-l)^2 \\ &= 2h_{kn}b_{kn}^2 - 2e_{kn}^2 \geqq b_{kn}^2 - \tfrac{1}{8} b_{kn}^2\,, \qquad \text{by (8).} \end{aligned}$$

It follows that for $|u| < E/M_n$, since $k \in A_n$

$$|\varphi_k(t)| \leqq \exp \tfrac{1}{2}\{|\varphi_k(t)|^2 - 1\} \leqq -u^2 G b_k^2/8$$

so that

$$|I''| \leqq B_n \int_{D/B_n < |u| \leqq E/M_n} \exp - (u^2/16G) B_n^2 \, du < \int_{D/4G^{\frac{1}{2}}}^{\infty} e^{-v^2} \, dv < \frac{4G^{\frac{1}{2}}}{D} e^{-D^2/16G}$$

which can be made arbitrarily small by a suitable choice of D.

It $t = t_i + u \in \Delta_i$, then $|u| \leqq \pi/2L$. If also $|j| \leqq L$, $\cos jt = \cos j(u + 2\pi h_0/j_0) = \cos(uj + 2\pi h_1/j_1) \leqq \cos u$, so that for such t,

$$(15) \qquad \begin{aligned} |\varphi_k(t)|^2 &\leqq 1 - P\{0 < X_k' \leqq L\} + P\{0 < X_k' \leqq L\} \cos u \\ &\leqq 1 - P\{0 < X_k' \leqq L\}(u^2/3), \end{aligned}$$

and hence,

$$|I'''| \leqq B_n \int_{E/M_n \leqq |u|} \exp - \{Q_n u^2/6\} \, du \leqq \frac{6 B_n M_n}{E Q_n} \exp\left\{\frac{-Q_n E^2}{M_n^2}\right\} \to 0 .$$

Since $I_5 = \sum_{i=1}^m B_n \int_{\Delta_i} \Psi_n(t) e^{-itx} \, dt$, $|I_5|$ can be made arbitrarily small by choosing n sufficiently large.

We note the following corollaries of Theorem 1:

COROLLARY 1. *If* $\{X_k\}$ *satisfy* (A), (B),

(C_1) $\liminf_{n\to\infty} 1/B_n^2 \sum_{k=1}^n \sum_{|x| < \varepsilon n^{\frac{1}{4}}} x^2 P_k(x) > 2G$ *for all* $\varepsilon > 0$,

and

(D_1) $\exists L$ *such that*

$$\inf 1/n \sum_{k=1}^n \min_{|x| \leqq L} P\{X_k \neq x \text{ and } |X_k| < L\} > 0$$

then the sequence satisfies a strong local limit theorem.

PROOF. Clearly

$$[\min_{|x| \leqq L} P\{X_k \neq x, \text{ and } |X_k| < L\}]^2 \leqq P\{0 < X_k' \leqq 2L\}$$

so that (D_1) implies $\inf Q_n/n > 0$. By (C_1), for any $\varepsilon > 0$, n sufficiently large, $\varepsilon n^{\frac{3}{4}} > (2G)^{\frac{1}{2}} B_n$. Hence if we set $M_n = \varepsilon n^{\frac{1}{4}}$, then $B_n M_n / Q_n = \varepsilon O(1)$, and by suitable choice of ε, we can make the contributions of I_4 and the integrals I''' arbitrarily small.

COROLLARY 2. *If* $\{X_k\}$ *satisfy* (A), (B), *and*

(C_2) *there exists an* M *and* $G < 0$ *such that*

$$\inf 1/B_n^2 \sum_{k=1}^n \sum_{|x| < M} x^2 P_k(x) \geqq 2G$$

then the sequence satisfies a strong local limit theorem.

PROOF. If $k \in A_n$, then $b_k \leqq M/G^{\frac{1}{2}}$, and for sufficiently large L, $P\{0<X_k'<L\}$ is bounded away from 0. The first corollary can then be applied.

If the hypothesis (A), which is necessary for a strong local limit theorem, does not hold, then a weaker result can be obtained which we refer to as an interval limit theorem.

THEOREM 2. *If* $\{X_k\}$ *satisfy* (B), (C), *and* (D), *then*

(16) $$\lim_{d\to\infty} 1/d \limsup_{n\to\infty} |B_n P\{x \leqq S_n < x+d\} - d(2\pi)^{-\frac{1}{2}} \exp\{-x^2/2B_n^2\}| = 0$$

uniformly in x, *as n and* $d \to \infty$.

PROOF. Let Y be a random variable with characteristic function $\Psi(t) = 1 - |t|/C$ for $|t| < C$ and $\Psi(t) = 0$, for $|t| \geqq C$. Choose C as in Theorem 1. Given ε, we can find δ, such that $P\{|Y| \geqq \delta\} < \varepsilon$. Then,

$$P\{x \leqq S_n < x+d\} \leqq [P\{|Y| < \delta\}]^{-1} P\{x - \delta \leqq S_n + Y < x + d + \delta\} .$$

By the proof of Theorem 1,

$$B_n P\{x - \delta \leqq S_n + Y < x + d + \delta\} - (d + 2\delta)(2\pi)^{-\frac{1}{2}} \exp\{-x^2/2B_n^2\} \to 0$$

uniformly in x. Therefore, if we call the expression inside the absolute value sign in (16) $T_n(d)$

$$1/d \limsup T_n(d) \leqq (1-\varepsilon)^{-1}(1 + 2\delta/d) - 1$$

a similar lower bound can be found for $\liminf T_n(d)$.
Since ε is arbitrary, this proves the corollary.

3. Recurrent random walks. In this section we apply the interval limit theorem of the previous section to obtain sufficient conditions for the recurrence, or more precisely, the "d-recurrence" of the random walk generated by a sequence of independent random variables $\{X_k\}$. We employ some results of Orey [3] concerning the equicontinuous solutions $\{h_k(x)\}$ of the sequence of equations

(17) $$h_k(x) = \int h_{k+1}(x+y)\, dF_{k+1}(y) \qquad k = 0, 1, 2, \dots$$

where $F_k(y) = P\{X_k \leqq y\}$. Clearly if we set $L_k(x) = P\{x \leqq S_{n+k} - S_k < x + d,$ for infinitely many $n\}$, then

(18) $$h_k(x) = \int_x^{x+1} L_k(t)\, dt$$

is such a solution to (17).

In Orey's terminology, for a given sequence $\{X_k\}$ define a likely sequence of integers $\{l_i\}$ to be a sequence such that $\inf_i P\{X_{k_i} = l_i\} > 0$, where $\{k_i\}$ is a subsequence of the natural numbers. Let $\Gamma = \{x|$ for some likely sequence

$\{l_i\}$, $\sum_{i=1}^{\infty} P\{X_{k_i} - l_i = x\} = \infty\}$, and $\Gamma^* =$ subgroup of the integers generated by Γ.

THEOREM (Orey). *If* $\Gamma^* = \{nd\}_{n=1}^{\infty}$, *then all equicontinuous solutions of* (17) *have period* d, *i.e.*, $h_k(x) = h_k(x + d)$, *all* x.

From this we easily obtain the following

LEMMA 2. *If* $\Gamma^* = \{nd\}$, *then* $P\{x \leqq S_{n+k} < x + d$, *infinitely many* $n\}$ *is either identically zero or identically* 1 *for all integer* x, *i.e., the random walk generated by* $\{X_k\}$ *is either transient or d-recurrent.*

PROOF. Suppose that for some integer x, and some k, $L_k(x) = 2\delta > 0$. For integer-valued x, $L_k(x) = h_k(x) = h_k(x + d)$. If z is an integer such that $d > z > 0$, since $L_k(x - z) + L_k(x - z + d) \geqq L_k(x)$, we must have $L_k(x) \geqq \delta$, for all integer x, and hence for all $k \geqq 1$.

Choose now N_1 such that

$$P\{x \leqq S_n < x + d, \quad \text{for no} \quad n < N_1\} < 1 - \delta/2 .$$

Choose $N_2 = N_2(S_{N_1})$ such that

$$P\{x - S_{N_1} < S_{n+N_1} - S_{N_1} < x + d - S_{N_1} \quad \text{for no} \quad n < N_2\} < 1 - \delta/2 .$$

Continuing in this manner, we find

$$P\{x \leqq S_n < x + d \quad \text{for some} \quad n < N\} \to 1 , \qquad \text{as} \quad N \to \infty .$$

Since this is also true when S_n is replaced by $S_{n+k} - S_k$, (3′) is verified and the random walk is d-recurrent.

THEOREM 3. *If* $\{X_k\}$ *is a sequence of independent random variables such that*

$$\text{(A}'\text{)} \qquad \Gamma^* = \{nd\} ,$$

and such that the conditions of the interval limit theorem are satisfied uniformly in m *for the sequences* $\{X_{k+m}\}_{k=1}^{\infty}$, *i.e., setting* $B_{nm}^2 = B_{n+m}^2 - B_m{}^2$, *it is true that*

(B′) *for any* $\varepsilon > 0$,

$$1/B_{nm}^2 \sum_{k=m+1}^{n+m} \sum_{|x| \leqq \varepsilon B_{nm}} x^2 P_k(x) \to 1 ,$$

uniformly in m,

(C′) *there exist* $\{M_{nm}\}$ *and* $G < 0$ *such that*

$$\liminf_{n\to\infty} 1/B_{nm}^2 \sum_{k=m+1}^{n+m} \sum_{|x| \leqq M_{nm}} x^2 P_k(x) \geqq 2G , \qquad \textit{uniformly in } m,$$

(D′) *there exists L such that if* $\{M_{nm}\}$ *is as in* (C′), *and* $Q_{nm} = \sum_{k=m+1}^{n+m} P\{0 < X_k' \leqq L\}$, *then* $B_{nm} M_{nm}/Q_{nm} \to 0$, *uniformly in m, and*

(E′) $\liminf_{l\to\infty} \inf_{n+m \leqq l} B_{nm}/B_n = 2H > 0$,

then $\{X_k\}$ *generates a d-recurrent random walk.*

REMARKS. A random walk will certainly be d-recurrent in for some sub-sequence $\{n_k\}$,

$$P\{x \leqq S_{n_k} < x + d, \quad \text{for infinitely many } k\} = 1 .$$

Therefore, the conclusion of the theorem follows if we replace $\{X_k\}$ in the hypotheses by $\{Y_k\}$ where $Y_k = X_{n_k - 1} + \cdots + X_{n_{k+1}}$.

Note that in the case that $d = 1$, (A′) is a somewhat stronger condition than (A), since if no likely sequences exist, then Γ^* is empty.

For (E′) to be satisfied, it is sufficient that b_k be bounded above, since $B_n \to \infty$ and, as above, we can find a sub-sequence $\{n_k\}$ such that $b^2_{n_k - 1} + \cdots + b^2_{n_{k+1}}$ is bounded away from zero.

PROOF. Set $P_{nk}(x, y) = P\{x \leqq S_{n+k} - S_k < y\}$. By the interval limit theorem, for bounded x and y if c is sufficiently large, there are constants l_1 and l_2 such that

$$0 < l_1 < B_{nk} P_{nk}(x, x + c) < l_2 < \infty , \tag{19}$$

and

$$\limsup_{n \to \infty} \frac{|P_{n0}(x, x + c) - P_{nk}(y, y + c)|}{P_{n0}(x, x + c)} \leqq \tfrac{1}{2} . \tag{20}$$

We will show that for any x and k,

$$P\{x \leqq S_{n+k} - S_k < x + c , \quad \text{for some} \quad n \geqq 1\} \geqq \tfrac{2}{3} \tag{21}$$

and hence, since the random walk cannot be transient, by Lemma 2 it must be d-recurrent. Without loss of generality we assume $x = k = 0$.

Let $f_m(y) = P\{S_m = y,\ S_k < 0 \text{ or } S_k \geqq c \text{ for all } k < m\}$. Then by a renewal argument

$$P_{k0}(0, c) = \textstyle\sum_{y=0}^{c-1} \sum_{m=1}^{k} f_m(y) P_{k-m,m}(-y, c - y) .$$

Summing this expression for $1 \leqq k \leqq n$, and dividing by the left-hand side, we have

$$1 = \textstyle\sum_{y=0}^{c-1} \sum_{m=1}^{n} f_m(y) [\sum_{k=1}^{n-m} P_{km}(-y, c - y) \{\sum_{k=1}^{n} P_{k0}(0, c)\}^{-1}] .$$

By (19) and hypothesis (E′), the expression in brackets is uniformly bounded, since $P_{km}(-y, c - y)\{P_{k0}(0, c)\}^{-1} \leqq l_2/l_1 H$ for k sufficiently large. By (20), for any fixed k and y the expression in brackets is bounded above by $1 + \frac{1}{2}$. Therefore by the dominated convergence theorem,

$$\tfrac{2}{3} \leqq \textstyle\sum_{y=0}^{c-1} \sum_{m=1}^{\infty} f_m(y)$$

which is equivalent to (21), proving the theorem.

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