Title
Local modification of Pythagorean-hodograph quintic spline curves using the B-spline form

Journal
Advances in Computational Mathematics, 42(1)

ISSN
1019-7168

Authors
Farouki, RT
Giannelli, C
Sestini, A

Publication Date
2016-02-01

DOI
10.1007/s10444-015-9419-y

Peer reviewed

Local modification of Pythagorean-hodograph quintic spline curves using the B-spline form

Rida T. Farouki
Department of Mechanical and Aerospace Engineering,
University of California, Davis, CA 95616, USA.

Carlotta Giannelli
Istituto Nazionale di Alta Matematica,
Unità di Ricerca di Firenze c/o DiMaI "U. Dini,"
Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, ITALY.

Alessandra Sestini
Dipartimento di Matematica e Informatica "U. Dini,"
Università degli Studi di Firenze,
Viale Morgagni 67a, 50134 Firenze, ITALY.

Abstract

The problems of determining the B–spline form of a C^2 Pythagorean–hodograph (PH) quintic spline curve interpolating given points, and of using this form to make local modifications, are addressed. To achieve the correct order of continuity, a quintic B–spline basis constructed on a knot sequence in which each (interior) knot is of multiplicity 3 is required. C^2 quintic bases on uniform triple knots are constructed for both open and closed C^2 curves, and are used to derive simple explicit formulae for the B–spline control points of C^2 PH quintic spline curves. These B-spline control points are verified, and generalized to the case of non–uniform knots, by applying a knot removal scheme to the Bézier control points of the individual PH quintic spline segments, associated with a set of six–fold knots. Based on the B–spline form, a scheme for the local modification of planar PH quintic splines, in response to a control point displacement, is proposed. Only two contiguous spline segments are modified, but to preserve the PH nature of the modified segments, the continuity between modified and unmodified segments must be relaxed from C^2 to C^1. A number of computed examples are presented, to compare the shape quality of PH quintic and "ordinary" cubic splines subject to control point modifications.

Keywords: Pythagorean–hodograph spline curves; B–spline representation; spline knots; spline bases; control points; local modification; end conditions.

e–mail: farouki@ucdavis.edu, carlotta.giannelli@unifi.it, alessandra.sestini@unifi.it

1 Introduction

By incorporating special algebraic structures, Pythagorean–hodograph (PH) curves offer exact algorithms for many fundamental computations that incur numerical approximations for "ordinary" polynomial curves [8]. In geometric design, computer graphics, robotics, real–time motion control, and related applications, the PH curves furnish basic geometric utilities of significantly enhanced accuracy and efficiency. In the realm of PH curves, the quintics are widely exploited as basic design elements, offering shape flexibility analogous to that of the "ordinary" cubics — for example, PH quintics can satisfy first–order Hermite data as single segments, or interpolate sequences of point data with C^2 continuity as multi–segment PH quintic splines.

The standard approach to constructing PH splines is by interpolation of point data, which entails solving tridiagonal systems of quadratic equations. Although the PH curves form a proper subset of the polynomial curves, they do not admit intuitive characterizations through geometrical constraints on their Bézier/B–spline control points — except for the cubics [14, 15]. For PH splines interpolating a sequence of points, the custom has been to represent them as sets of Bézier segments [1, 11]. The goal of this study is to determine the B–spline representation of PH quintic splines, and to use it to formulate a local modification scheme based on displacements of the B–spline control points, that preserves the PH nature of each spline segment.

In order to import C^2 PH quintic splines into commercial CAD systems, it is necessary to express them in the universally recognized B–spline free–form geometry representation. This problem involves determining the knots and control points that exactly define the B–spline form of a C^2 PH quintic spline, specified as a sequence of Bézier curve segments. Although "ordinary" quintic B–spline curves with only simple (interior) knots exhibit C^4 continuity, a PH quintic spline is only C^2 continuous. To accommodate the reduced continuity, a B–spline basis used to specify C^2 PH quintic splines must be constructed on *triple* knots. Moreover, to exploit the advantageous properties of PH curves, the B–spline knots and control points must be augmented with further data specifying the "internal PH structure" of each spline segment.

Although the methodology makes use of well–known B–spline properties, determining the B–spline form of a C^2 PH quintic spline is not an entirely straightforward task, and the absence of algorithms to address this problem is an impediment to the practical exploitation of PH curves. The availability of closed–form expressions for the B–spline basis functions on a sequence of triple knots (see Section 4), for example, is important to development of the local modification procedure discussed later in the paper.

It must be emphasized that the problem treated herein is logically distinct from that addressed in an earlier study [18] which proposed a control–polygon scheme for designing PH quintic splines, as an alternative to interpolation of point data, as follows — (i) the control points and knots are used to define an "ordinary" C^2 cubic B–spline curve with appropriate end conditions; and (ii) the nodal points of this cubic B–spline curve are then interpreted as points $\mathbf{q}_0, \ldots, \mathbf{q}_n$ to be interpolated by the standard C^2 PH quintic spline algorithm, using the same knots and equivalent end conditions. With this scheme, it was observed that the cubic B–spline and quintic PH spline curves are usually in close agreement. However, the control polygon and associated knots are *not* an exact B–spline definition of the C^2 quintic PH spline, but simply an expedient tool for its construction and shape manipulation.

A key feature of the B–spline form, arising from the compact support of the basis functions, is the ability to make strictly local shape modifications. Such a capability is also desirable for PH splines, but more difficult to achieve because of their non–linear nature. Two considerations are paramount: each segment must remain a PH curve; and the new knots and control points must specify an exact B–spline representation for the modified spline. Because of the difficulties these constraints incur, at present we treat this problem only in the context of planar C^2 PH quintic splines on uniform knots. An algorithm for two–segment spline modification is proposed, invoking relaxation from C^2 to C^1 continuity between the modified and unmodified segments to ensure that the former remain PH quintics. This problem is shown to be reducible to the solution of two complex quadratic equations in two unknowns.

This paper is organized as follows. Some basic features of PH splines and the B–spline form are reviewed in Sections 2 and 3. Quintic B–spline bases on uniform triple knots are derived for open and closed curves in Section 4, and are used to determine simple expressions for the B–spline control points of uniformly–parameterized C^2 PH quintic splines. An alternative algorithmic approach is then presented in Section 5, based on a knot removal scheme that admits easy generalization to non–uniform knots (and also to spline curves of higher degree or different orders of continuity). Using the B–spline form, an algorithm for the local modification of uniformly–parameterized planar PH quintic spline curves is developed in Section 6, that preserves the PH nature of the modified spline segments. Finally, Section 7 summarizes the results of the present study, and identifies problems worthy of further investigation.

2 Pythagorean-hodograph quintic splines

A Pythagorean–hodograph (PH) curve has the distinctive property that the components of its derivative satisfy a polynomial Pythagorean condition. For planar PH curves this structure is achieved through a complex–number model [6], while for spatial PH curves a model based on quaternions or the Hopf map from $\mathbb{R}^4$ to $\mathbb{R}^3$ is employed [3, 9]. The fact that a PH curve $\mathbf{r}(t)$ has a polynomial *parametric speed* $\sigma(t) = |\mathbf{r}'(t)|$, specifying the derivative $\mathrm{d}s/\mathrm{d}t$ of arc length s respect to the curve parameter t, facilitates exact formulations for many basic computations that otherwise require numerical approximation.

Apart from such computational advantages, the PH curves may often be preferred over "ordinary" polynomial curves on the basis of *shape quality* or aesthetic considerations. Figure 1 illustrates the C^2 PH quintic and ordinary C^2 cubic splines interpolating a sequence of points $\mathbf{q}_0, \ldots, \mathbf{q}_{12}$ at the uniform parameter values $t_0, \ldots, t_{12} = 0, \ldots, 12$ under periodic boundary conditions ($\mathbf{q}_{12} = \mathbf{q}_0$). The characteristic propensity of PH quintic spline interpolants to exhibit a "rounder" appearance (i.e., a more even curvature distribution) than ordinary cubic splines [7, 11] is clearly evident in this example.

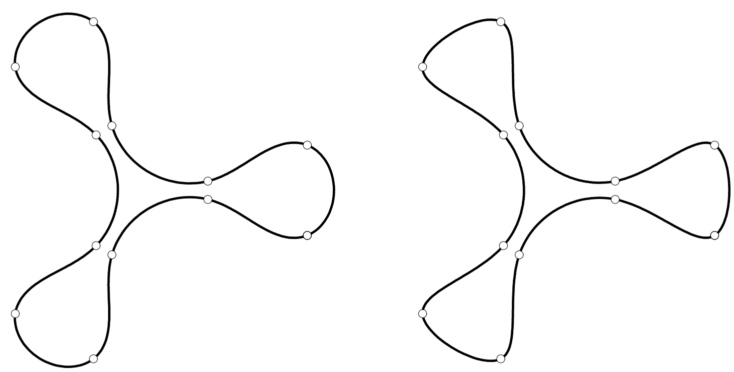

Figure 1: Comparison of C^2 PH quintic (left) and "ordinary" C^2 cubic (right) splines interpolating a given sequence of points with periodic end conditions.

A comprehensive guide to the construction, properties, and applications of PH curves may be found in [8]. The focus of this study is on the B–spline representation of PH quintic splines, a key requirement in importing these curve into CAD software. A B–spline curve is defined by a sequence of knots, that specify a partitioned parameter domain on which a basis for piecewise–polynomial functions of prescribed smoothness may be constructed, and an associated set of control points. However, the knots and control points alone do not allow the unique features of PH splines to be fully exploited: certain "internal structure" variables must also be stored.

The B–spline representations derived in Sections 4 and 5, in terms of knots and control points, apply to both planar and spatial C^2 PH quintic splines. For planar PH splines, the internal structure is completely specified by a set of complex coefficients, but the quaternion formulation used for spatial PH splines incurs a more complicated (and under–determined) internal structure [10, 12, 13]. For brevity, the focus of the local PH spline modification scheme described in Section 6, which requires adjustment of the internal structure, will be on uniformly–parameterized planar PH quintic splines.

The construction of a planar C^2 PH quintic spline interpolating points $\mathbf{q}_0, \ldots, \mathbf{q}_n$ incurs a system of n quadratic equations in n complex unknowns. This non–linear system admits a multiplicity of formal solutions — among which a unique "good" interpolant may be identified, free of undesired loops or curvature extrema. An initial study [1] investigated the complete family of formal solutions to the C^2 PH quintic spline equations, computed by the homotopy method. Since this becomes prohibitively expensive for large n, and in applications only the "good" solution is desired, an iterative scheme to compute *only* this solution was proposed in [11], using an accurate starting approximation to the complex values that identify it. In practice, this scheme has proven to be very efficient and robust, permitting real–time interactive manipulation [4] of planar C^2 PH quintic spline interpolants.

In the complex form [6] the derivative of each segment $\mathbf{r}_k(u)$, $k = 1, \ldots, n$, $u = t-k+1 \in [\,0,1\,]$ of a planar C^2 PH quintic spline is defined as the square

$$\mathbf{r}_k'(u) \;=\; [\,\mathbf{w}_{k,0}(1-u)^2 + \mathbf{w}_{k,1}2(1-u)u + \mathbf{w}_{k,2}u^2\,]^2 \tag{1}$$

of a quadratic complex polynomial in Bernstein form. To interpolate given points $\mathbf{q}_0, \ldots, \mathbf{q}_n$ the coefficients $\mathbf{w}_{k,0}, \mathbf{w}_{k,1}, \mathbf{w}_{k,2}$ must be computed through the spline construction algorithm [11] — they are not independent, but rather satisfy the constraints

$$\mathbf{w}_{k,2} \;=\; \mathbf{w}_{k+1,0} \;=\; \tfrac{1}{2}(\mathbf{w}_{k,1} + \mathbf{w}_{k+1,1}) \tag{2}$$

that enforce the continuity conditions $\mathbf{r}_k'(1) = \mathbf{r}_{k+1}'(0)$ and $\mathbf{r}_k''(1) = \mathbf{r}_{k+1}''(0)$.

To exactly compute PH spline curve properties, such as arc lengths and offset curves, the coefficients $\mathbf{w}_{k,0}, \mathbf{w}_{k,1}, \mathbf{w}_{k,2}$ must be known for each segment $k = 1, \ldots, n$. In view of condition (2), it is only necessary to store the values

$\mathbf{w}_{k,1}$ for each spline segment. Note that, for periodic end conditions, instance $k = n$ of the condition (2) becomes $\mathbf{w}_{n,2} = \mathbf{w}_{1,0} = \frac{1}{2}(\mathbf{w}_{n,1} + \mathbf{w}_{1,1})$, and for cubic end spans we set $\mathbf{w}_{1,0} = 2\,\mathbf{w}_{1,1} - \mathbf{w}_{1,2}$ and $\mathbf{w}_{n,2} = 2\,\mathbf{w}_{n,1} - \mathbf{w}_{n,0}$.

The control points of the spline segment $\mathbf{r}_k(u)$ in the Bézier representation

$$\mathbf{r}_k(u) \,=\, \sum_{i=0}^{5} \mathbf{c}_{k,i} b_i^5(u)\,, \quad b_i^5(u) \,=\, \binom{5}{i}(1-u)^{5-i}u^i \tag{3}$$

can be determined from the Bernstein coefficients in (1) through the relations

$$\begin{aligned}
\mathbf{c}_{k,1} &= \mathbf{c}_{k,0} + \tfrac{1}{5}\,\mathbf{w}_{k,0}^2\,,\\
\mathbf{c}_{k,2} &= \mathbf{c}_{k,1} + \tfrac{1}{5}\,\mathbf{w}_{k,0}\mathbf{w}_{k,1}\,,\\
\mathbf{c}_{k,3} &= \mathbf{c}_{k,2} + \tfrac{1}{5}\,(\tfrac{2}{3}\,\mathbf{w}_{k,1}^2 + \tfrac{1}{3}\,\mathbf{w}_{k,0}\mathbf{w}_{k,2})\,,\\
\mathbf{c}_{k,4} &= \mathbf{c}_{k,3} + \tfrac{1}{5}\,\mathbf{w}_{k,1}\mathbf{w}_{k,2}\,,\\
\mathbf{c}_{k,5} &= \mathbf{c}_{k,4} + \tfrac{1}{5}\,\mathbf{w}_{k,2}^2\,,
\end{aligned} \tag{4}$$

with the initial control point $\mathbf{c}_{k,0}$ being a free integration constant.

3 B-spline representation

The B–spline basis functions $B_k^d(t)$ of degree d on a given non–decreasing knot sequence[1] $\ldots, t_{k-1}, t_k, t_{k+1}, \ldots$ may be constructed through the recurrence

$$B_k^r(t) \,=\, \omega_{k,r}(t)\, B_k^{r-1}(t) \,+\, [\,1 - \omega_{k+1,r}(t)\,]\, B_{k+1}^{r-1}(t) \tag{5}$$

for $r = 1, \ldots, d$, where we define

$$\omega_{k,r}(t) \,=\, \begin{cases} (t - t_k)/(t_{k+r} - t_k) & \text{if } t_{k+r} > t_k, \\ \quad 0 & \text{otherwise.} \end{cases}$$

The recursion begins by setting $B_k^0(t) = 1$ for $t_k \le t < t_{k+1}$, and 0 otherwise. For $d \ge 1$, the basis function $B_k^d(t)$ has the support interval $t \in [\,t_k, t_{k+d+1}\,]$ — i.e., $B_k^d(t) \equiv 0$ for $t < t_k$ and $t > t_{k+d+1}$ — and is C^{d-1} continuous if the knots are distinct: in particular $B_k^d(t)$ and all its derivatives up to order $d-1$ must vanish at $t = t_k$ and $t = t_{k+d+1}$.

[1]To temporarily avoid complications arising from the end conditions, we consider at present only intermediate knot values, sufficiently far from the initial/final values.

For a prescribed sequence of knots, a B-spline curve $\mathbf{r}(t)$ is determined by associating chosen control points $\mathbf{p}_0, \ldots, \mathbf{p}_N$ with each of the basis functions $B_0^d(t), \ldots, B_N^d(t)$ — the total number of knots required is determined by the number $N+1$ of control points and the degree d of the basis.

If the interior knots are distinct, the B–spline curve $\mathbf{r}(t)$ is of continuity C^{d-1} (i.e., contiguous segments agree in position and derivatives up to order $d-1$ at their junctures). However, the knot sequence may contain repeated values, or *multiple knots*. When $t_k = t_{k+1} = \cdots = t_{k+m-1}$, we have a knot of multiplicity m, and in general the continuity of $\mathbf{r}(t)$ diminishes from C^{d-1} to C^{d-m} at such a knot. With simple knots, each degree–d basis function has a support of $d+1$ consecutive non–zero intervals between knots. However, if a knot of multiplicity $m \geq 2$ lies in the support of a degree–d basis function, its support consists of between $d+2-m$ and $d+1$ non–zero intervals, depending on how many instances of the multiple knot are counted within the support.

4 Control points for uniform triple knots

A (uniformly–parameterized) C^2 PH quintic spline curve interpolating $n+1$ points $\mathbf{q}_0, \ldots, \mathbf{q}_n$ may be constructed as described in [11], the result being a set of n Bézier segments defined on $u \in [\,0,1\,]$ of the form (3) for $k = 1, \ldots, n$. We wish to determine the B–spline representation

$$\mathbf{r}(t) \,=\, \sum_{k=0}^{N} \mathbf{p}_k B_k^5(t)\,, \quad t \in [\,t_5, t_{N+1}\,] \,=\, [\,0, n\,] \tag{6}$$

of such curves, specified by control points $\mathbf{p}_0, \ldots, \mathbf{p}_N$ associated with a quintic B–spline basis $B_0^5(t), \ldots, B_N^5(t)$ constructed on an extended, non–decreasing knot sequence $t_0, \ldots, t_{N+6}$ by means of the recursion formula (5). The initial and final values $t_0, \ldots, t_4$ and $t_{N+2}, \ldots, t_{N+6}$ are auxiliary knots, required to define the B–spline basis, and can be chosen to enforce desired end conditions. Each basis function $B_k^5(t)$ in (6) has support interval $t \in [\,t_k, t_{k+6}\,]$ and is C^4 continuous if $t_k < t_{k+1} < \cdots < t_{k+6}$. To construct a quintic basis that is C^2 only, knots of multiplicity 3 (at least) must be employed.

For an open curve, satisfying $\mathbf{r}(t_5) = \mathbf{p}_0$ and $\mathbf{r}(t_{N+1}) = \mathbf{p}_N$, end knots of multiplicity 6 are used — i.e.,

$$t_0 = t_1 = t_2 = t_3 = t_4 = t_5\,, \quad t_{N+1} = t_{N+2} = t_{N+3} = t_{N+4} = t_{N+5} = t_{N+6}\,.$$

For a smooth closed curve with $\mathbf{r}(t_{N+1}) = \mathbf{r}(t_5)$, a periodic knot sequence is employed, with $t_0, \ldots, t_4$ and $t_{N+2}, \ldots, t_{N+6}$ being obtained from $t_5, \ldots, t_{N+1}$ by requiring for $k = 1, \ldots, 5$ that

$$t_{N+k+1} - t_{N+k} = t_{5+k} - t_{4+k} \quad \text{and} \quad t_k - t_{k-1} = t_{N-4+k} - t_{N-5+k} \, .$$

We focus initially on uniform parameterizations, in which the (distinct) knots are taken as integer values. This is appropriate if the nodal points $\mathbf{q}_0, \ldots, \mathbf{q}_n$ have a reasonably equal spacing. For unevenly spaced points, a non–uniform parameterization [11] should be used — this case is treated in Section 5.

4.1 C^2 quintic B–spline bases

For an open curve, the knot sequence must have initial and final knots of multiplicity 6, with intermediate knots of multiplicity 3. Correspondingly, the number of basis functions is $N+1 = 3n+3$ and the knots are defined by

$$\begin{aligned} t_0 = t_1 = t_2 = t_3 = t_4 = t_5 \;&=\; 0 \, , \\ t_{3k+3} = t_{3k+4} = t_{3k+5} \;&=\; k \, , \quad k = 1, \ldots, n-1 \\ t_{3n+3} = t_{3n+4} = t_{3n+5} = t_{3n+6} = t_{3n+7} = t_{3n+8} \;&=\; n \, . \end{aligned}$$

Consider, for $1 \le k \le n-1$, the functions $B^5_{3k}(t)$, $B^5_{3k+1}(t)$, $B^5_{3k+2}(t)$ with the common support interval $t \in [\, k-1, k+1 \,]$. The knots $t = k-1, k, k+1$ have multiplicities $3, 3, 1$ for $B^5_{3k}(t)$; $2, 3, 2$ for $B^5_{3k+1}(t)$; and $1, 3, 3$ for $B^5_{3k+2}(t)$. Hence, at those points they are of continuity C^2, C^2, C^4 for $B^5_{3k}(t)$; C^3, C^2, C^3 for $B^5_{3k+1}(t)$; and C^4, C^2, C^2 for $B^5_{3k+2}(t)$. When $B^5_{3k}(t), B^5_{3k+1}(t), B^5_{3k+2}(t)$ are constructed for any k between 1 and $n-1$, the basis functions for any other k in this range are translates of them, due to the uniformity of the knots.

The functions $B^5_{3k}(t), B^5_{3k+1}(t), B^5_{3k+2}(t)$ may be uniquely determined from their known orders of continuity at the points $t = k-1, k, k+1$ in conjunction with the normalization (partition of unity) condition

$$\sum_{i=0}^{N} B^5_i(t) \;\equiv\; 1 \, . \tag{7}$$

For $l = 0, 1, 2$ let the components of $B^5_{3k+l}(t)$ on the intervals $t \in [\, k-1, k \,]$ and $t \in [\, k, k+1 \,]$ be expressed in terms of the Bernstein basis $b^5_i(u)$ in the local variables $u = t-(k-1)$ and $u = t-k$, respectively. Then the Bernstein coefficients of these components are determined as

$$\begin{array}{llll} (0,0,0,1,\frac{1}{2},\frac{1}{4}) & \text{and} & (\frac{1}{4},0,0,0,0,0) & \text{for } B^5_{3k}(t)\,, \\ (0,0,0,0,\frac{1}{2},\frac{1}{2}) & \text{and} & (\frac{1}{2},\frac{1}{2},0,0,0,0) & \text{for } B^5_{3k+1}(t)\,, \\ (0,0,0,0,0,\frac{1}{4}) & \text{and} & (\frac{1}{4},\frac{1}{2},1,0,0,0) & \text{for } B^5_{3k+2}(t)\,. \end{array} \tag{8}$$

Figure 2 illustrates these basis functions. $B^5_{3k+1}(t)$ is symmetric about $t = k$, while $B^5_{3k}(t)$ and $B^5_{3k+2}(t)$ are mirror images of each other about that point.

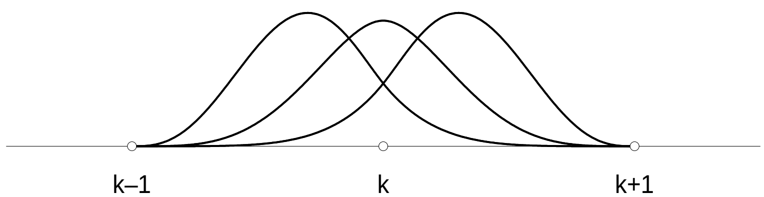

Figure 2: Three quintic B–spline basis functions $B^5_{3k}(t), B^5_{3k+1}(t), B^5_{3k+2}(t)$ on $t \in [\,k-1, k+1\,]$ for a set of uniform knots consisting entirely of triple knots.

However, the initial and final three basis functions $B^5_0(t), B^5_1(t), B^5_2(t)$ and $B^5_{3n}(t), B^5_{3n+1}(t), B^5_{3n+2}(t)$ differ, since for them $t = 0$ and $t = n$ are knots of multiplicity > 3. The functions $B^5_0(t), B^5_1(t), B^5_2(t)$ have support interval $t \in [\,0, 1\,]$ with 0 and 1 being of knot multiplicity 6 and 1 for $B^5_0(t)$; 5 and 2 for $B^5_1(t)$; 4 and 3 for $B^5_2(t)$. Hence, at $t = 0$ and $t = 1$ they have continuity C^{-1} and C^4 for $B^5_0(t)$; C^0 and C^3 for $B^5_1(t)$; C^1 and C^2 for $B^5_2(t)$. Together with (7), these properties determine their Bernstein coefficients on $[\,0, 1\,]$ as

$$(1,0,0,0,0,0)\,, \quad (0,1,0,0,0,0)\,, \quad (0,0,1,0,0,0)\,.$$

By analogous arguments, the basis functions $B^5_{3n}(t), B^5_{3n+1}(t), B^5_{3n+2}(t)$ have support interval $t \in [\,n-1, n\,]$ and corresponding Bernstein coefficients

$$(0,0,0,1,0,0)\,, \quad (0,0,0,0,1,0)\,, \quad (0,0,0,0,0,1)\,.$$

Figure 3 illustrates the quintic C^2 B–spline basis constructed on $t \in [\,0, 6\,]$ as described above — with $t = 0$ and $t = 6$ counted as knots of multiplicity 6, and $t = 1, 2, 3, 4, 5$ as triple knots. Over each interval $t \in [\,k-1, k\,]$ for $k = 1, \ldots, n$ there are six non–zero, linearly–independent basis functions.

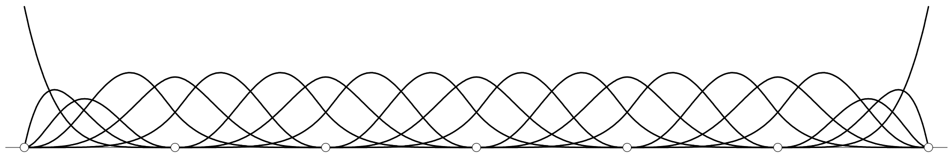

Figure 3: The set of 21 quintic C^2 B–spline basis functions constructed on the sequence of knots $0,0,0,0,0,0,1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6,6,6,6$.

For a smooth closed curve, a knot sequence of the form

$$t_{3k+3} = t_{3k+4} = t_{3k+5} = k\,, \quad k = -1,\ldots,n+1 \tag{9}$$

is employed. On these knots, a total of $N+1 = 3n+3$ B–spline basis functions $B_0(t),\ldots,B_N(t)$ may be constructed. Since the knots are all multiplicity 3, the basis functions are all defined by the Bernstein coefficients (8) for $k = 0,\ldots,n$. Note that the domain of the curve is $t \in [\,0,n\,]$ although the initial and final three knots are $t_0 = t_1 = t_2 = -1$ and $t_{3n+6} = t_{3n+7} = t_{3n+8} = n+1$.

4.2 B–spline control points

To ensure that expression (6) over $t \in [\,k-1,k\,]$ agrees with (3), we equate Bernstein coefficients. On the interior segments ($2 \le k \le n-1$) this gives

$$\tfrac{1}{4}(\mathbf{p}_{3k-3} + 2\,\mathbf{p}_{3k-2} + \mathbf{p}_{3k-1}) = \mathbf{c}_{k,0}\,,\ \ \tfrac{1}{2}(\mathbf{p}_{3k-2} + \mathbf{p}_{3k-1}) = \mathbf{c}_{k,1}\,,\ \ \mathbf{p}_{3k-1} = \mathbf{c}_{k,2}\,,$$

$$\mathbf{p}_{3k} = \mathbf{c}_{k,3}\,,\ \ \tfrac{1}{2}(\mathbf{p}_{3k} + \mathbf{p}_{3k+1}) = \mathbf{c}_{k,4}\,,\ \ \tfrac{1}{4}(\mathbf{p}_{3k} + 2\,\mathbf{p}_{3k+1} + \mathbf{p}_{3k+2}) = \mathbf{c}_{k,5}\,. \tag{10}$$

For the end segments $k = 1$ and $k = n$ of an open curve, we obtain

$$\begin{gathered} \mathbf{p}_0 = \mathbf{c}_{1,0}\,, \quad \mathbf{p}_1 = \mathbf{c}_{1,1}\,, \quad \mathbf{p}_2 = \mathbf{c}_{1,2}\,, \quad \mathbf{p}_3 = \mathbf{c}_{1,3}\,, \\ \tfrac{1}{2}(\mathbf{p}_3 + \mathbf{p}_4) = \mathbf{c}_{1,4}\,, \quad \tfrac{1}{4}(\mathbf{p}_3 + 2\,\mathbf{p}_4 + \mathbf{p}_5) = \mathbf{c}_{1,5}\,, \end{gathered} \tag{11}$$

$$\begin{gathered} \tfrac{1}{4}(\mathbf{p}_{3n-3} + 2\,\mathbf{p}_{3n-2} + \mathbf{p}_{3n-1}) = \mathbf{c}_{n,0}\,, \quad \tfrac{1}{2}(\mathbf{p}_{3n-2} + \mathbf{p}_{3n-1}) = \mathbf{c}_{n,1}\,, \\ \mathbf{p}_{3n-1} = \mathbf{c}_{n,2}\,, \quad \mathbf{p}_{3n} = \mathbf{c}_{n,3}\,, \quad \mathbf{p}_{3n+1} = \mathbf{c}_{n,4}\,, \quad \mathbf{p}_{3n+2} = \mathbf{c}_{n,5}\,. \end{gathered} \tag{12}$$

This is a system of $6n$ linear equations for the $3n+3$ B–spline control points $\mathbf{p}_0,\ldots,\mathbf{p}_{3n+2}$ — it is not over–determined, since the last three equations from segment k agree with the first three from segment $k+1$, for $k = 1,\ldots,n-1$.

This is a consequence of the fact that segments k and $k+1$ meet with C^2 continuity, so their control points satisfy

$$\mathbf{c}_{k+1,0} \,=\, \mathbf{c}_{k,5} \,=\, \mathbf{q}_k \,, \qquad \mathbf{c}_{k+1,1} - \mathbf{c}_{k+1,0} \,=\, \mathbf{c}_{k,5} - \mathbf{c}_{k,4} \,,$$

$$\mathbf{c}_{k+1,2} - 2\,\mathbf{c}_{k+1,1} + \mathbf{c}_{k+1,0} \,=\, \mathbf{c}_{k,5} - 2\,\mathbf{c}_{k,4} + \mathbf{c}_{k,3} \quad (= \mathbf{a}_k, \text{ say}) \,. \tag{13}$$

To avoid redundancy, we take all six equations from segment 1, but only the last three equations from segments $2, \ldots, n$. Based on these considerations, the B–spline control points for an open curve are defined by assigning $\mathbf{p}_0 = \mathbf{c}_{1,0} = \mathbf{q}_0$, $\mathbf{p}_1 = \mathbf{c}_{1,1}$, $\mathbf{p}_2 = \mathbf{c}_{1,2}$ and $\mathbf{p}_{3n} = \mathbf{c}_{n,3}$, $\mathbf{p}_{3n+1} = \mathbf{c}_{n,4}$, $\mathbf{p}_{3n+2} = \mathbf{c}_{n,5} = \mathbf{q}_n$ as the first and last three points, and defining all intermediate points by

$$\mathbf{p}_{3k} \,=\, \mathbf{c}_{k,3} \,, \quad \mathbf{p}_{3k+1} \,=\, \mathbf{q}_k - \mathbf{a}_k \,, \quad \mathbf{p}_{3k+2} \,=\, \mathbf{c}_{k+1,2} \,,$$

for $k = 1, \ldots, n-1$. Thus, the B–spline control points for open curves may be obtained from the individual Bézier segment control points as follows.

We keep the first two control points of the initial segment, and the last two of the final segment. For each segment $k = 1, \ldots, n$ we also keep the middle two control points $\mathbf{c}_{k,2}$ and $\mathbf{c}_{k,3}$, but the juncture points $\mathbf{q}_k = \mathbf{c}_{k,5} = \mathbf{c}_{k+1,0}$ and preceding and following points $\mathbf{c}_{k,4}$ and $\mathbf{c}_{k+1,1}$ are replaced by the single points $\mathbf{q}_k - \mathbf{a}_k$ for $k = 1, \ldots, n-1$, the vectors $\mathbf{a}_k$ being defined in (13).

Example 1. Consider a uniformly–parameterized open C^2 PH quintic spline interpolating the points $\mathbf{q}_0 = (-2.1, 1.8)$, $\mathbf{q}_1 = (-3.1, 0.0)$, $\mathbf{q}_2 = (-0.3, -0.8)$, $\mathbf{q}_3 = (0.7, 2.2)$, $\mathbf{q}_4 = (3.4, 0.5)$, $\mathbf{q}_5 = (1.1, -0.6)$, $\mathbf{q}_6 = (2.3, -2.4)$ with cubic end spans. This curve is shown in Figure 4 with its B–spline control polygon, connecting the computed control points $\mathbf{p}_0, \ldots, \mathbf{p}_{20}$ (the knot sequence and B–spline basis are as in Figure 3). Note that the B-spline form employs only 21 control points, but 36 control points are required (see Figure 6) when the curve is described as a set of Bézier segments.

For a smooth closed curve, the basis functions are translates of each other, so the relations (10) hold for $k = 1, \ldots, n$ and the B–spline control points $\mathbf{p}_{3k-3}, \ldots, \mathbf{p}_{3k+2}$ can be written in terms of the Bézier points $\mathbf{c}_{k,0}, \ldots, \mathbf{c}_{k,5}$ as

$$\mathbf{p}_{3k-3} = 4\,\mathbf{c}_{k,0} - 4\,\mathbf{c}_{k,1} + \mathbf{c}_{k,2} \,, \quad \mathbf{p}_{3k-2} = 2\,\mathbf{c}_{k,1} - \mathbf{c}_{k,2} \,, \quad \mathbf{p}_{3k-1} = \mathbf{c}_{k,2} \,,$$

$$\mathbf{p}_{3k} = \mathbf{c}_{k,3} \,, \quad \mathbf{p}_{3k+1} = 2\,\mathbf{c}_{k,4} - \mathbf{c}_{k,3} \,, \quad \mathbf{p}_{3k+2} = 4\,\mathbf{c}_{k,5} - 4\,\mathbf{c}_{k,4} + \mathbf{c}_{k,3} \,.$$

Now because of the C^2 continuity conditions (13), the last three expressions in instance k of these formulations coincide with the first three in instance

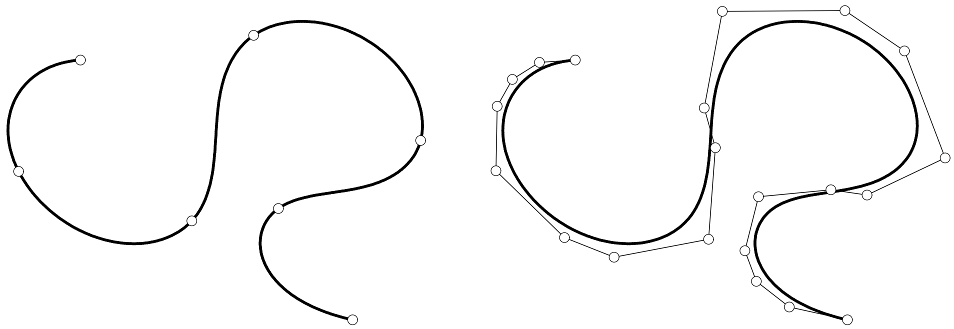

Figure 4: The C^2 PH quintic spline curve in Example 1 (left), together with its B–spline control polygon (right). The end knots $t = 0$ and $t = 5$ are of multiplicity 6, and the interior knots $t = 1, 2, 3, 4$ are all of multiplicity 3.

$k + 1$. To avoid replication, all six expressions are used in the first instance, but only the last three in all subsequent instances. This generates B–spline control points $\mathbf{p}_0, \ldots, \mathbf{p}_{3n+2}$ — i.e., $N = 3n + 2$ in (6).

Note also that, because of the periodic boundary conditions, case $k = n$ of the C^2 continuity conditions (13) becomes $\mathbf{c}_{0,0} = \mathbf{c}_{n,5}$, $\mathbf{c}_{0,1} - \mathbf{c}_{0,0} = \mathbf{c}_{n,5} - \mathbf{c}_{n,4}$, $\mathbf{c}_{0,2} - 2\,\mathbf{c}_{0,1} + \mathbf{c}_{0,0} = \mathbf{c}_{n,5} - 2\,\mathbf{c}_{n,4} + \mathbf{c}_{n,3}$, and hence the last three control points coincide with the first three — i.e., $\mathbf{p}_{3n} = \mathbf{p}_0$, $\mathbf{p}_{3n+1} = \mathbf{p}_1$, $\mathbf{p}_{3n+2} = \mathbf{p}_2$. A periodic quintic B–spline normally requires coincidence of the first and last five control points, but three suffice in the present context because of the reduced support of the basis functions when all knots have multiplicity 3.

Hence, the B–spline control points for closed curves may be determined as follows. For each segment $k = 1, \ldots, n$ we keep the middle two Bézier control points $\mathbf{c}_{k,2}, \mathbf{c}_{k,3}$ but the three points $\mathbf{c}_{k,4}, \mathbf{c}_{k+1,0} = \mathbf{c}_{k,5}, \mathbf{c}_{k+1,1}$ are replaced by the single point $\mathbf{q}_k - \mathbf{a}_k$, with $\mathbf{a}_k$ defined by (13). Note that the index k is interpreted cyclically, e.g., $\mathbf{a}_n = \mathbf{c}_{n,5} - 2\,\mathbf{c}_{n,4} + \mathbf{c}_{n,3} = \mathbf{c}_{1,2} - 2\,\mathbf{c}_{1,1} + \mathbf{c}_{1,0}$.

Example 2. Consider the uniform C^2 PH quintic spline that interpolates the points $\mathbf{q}_0 = (-4.1, -0.8)$, $\mathbf{q}_1 = (-1.5, -1.5)$, $\mathbf{q}_2 = (-0.6, -3.6)$, $\mathbf{q}_3 = (1.2, -1.5)$, $\mathbf{q}_4 = (4.1, 0.4)$, $\mathbf{q}_5 = (1.2, 3.3)$, $\mathbf{q}_6 = (0.9, 0.4)$, $\mathbf{q}_7 = (-1.4, -0.2)$, $\mathbf{q}_8 = (-2.3, 1.7)$, $\mathbf{q}_9 = (-4.1, -0.8)$ with periodic boundary conditions — see Figure 5. For the B–spline representation, the knot sequence consists entirely of triple knots: $-1, -1, -1, 0, 0, 0, 1, 1, 1, \ldots, 9, 9, 9, 10, 10, 10$. Figure 5 shows the B–spline control polygon, computed as described above — the B–spline form requires only 30 control points, but 54 control points are needed when

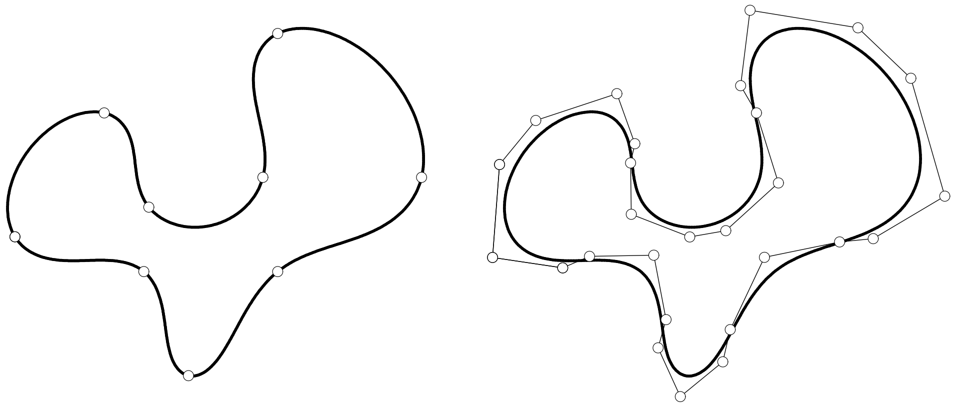

Figure 5: The C^2 PH quintic spline curve interpolating ten points $\mathbf{p}_0, \ldots, \mathbf{p}_9$ under periodic boundary conditions with $\mathbf{p}_9 = \mathbf{p}_0$ in Example 2 (left) and, for the knot sequence (9), the B–spline control polygon of this curve (right).

the spline is defined as a sequence of Bézier segments (see Figure 6).

5 Control points by knot removal

For general non–uniform parameterizations, the B–spline representation of a C^2 PH quintic spline that interpolates points $\mathbf{q}_0, \ldots, \mathbf{q}_n$ at parameter values $t_0, \ldots, t_n$ can be obtained by a knot removal process. This begins by taking each value $t_0, \ldots, t_n$ as a six–fold knot, and assigning the Bézier control points of the PH quintic segments as B–spline control points (on an interval between two knots of multiplicity 6, the quintic B–spline basis reduces to the quintic Bernstein basis). Thus, if spline segment $\mathbf{r}_k(t)$ defined on $t \in [\, t_{k-1}, t_k \,]$ has the Bézier form (3) in the local variable $u = (t - t_{k-1})/(t_k - t_{k-1}) \in [\, 0, 1 \,]$, the B–spline knots and control points are initially specified as

$$\begin{aligned} t_{6k+i} &= t_k \qquad \text{for } \; i = 0, \ldots, 5\,, \quad k = 0, \ldots, n\,, \\ \mathbf{p}_{6k+i} &= \mathbf{c}_{k+1,i} \qquad \text{for } \; i = 0, \ldots, 5\,, \quad k = 0, \ldots, n-1\,. \end{aligned}$$

Figure 6 shows the Bézier control polygons for the C^2 PH quintic splines in Figures 4 and 5. For a spline with n segments, the B–spline form defined above requires $6(n+1)$ knots and $6n$ control points.[2] Clearly, this offers no

[2] Since $\mathbf{c}_{k,5} = \mathbf{c}_{k+1,0} = \mathbf{q}_k$, the final and initial control points of consecutive segments

advantage over simply defining a spline as a sequence of Bézier segments, but it serves as the starting point of an algorithm for computing more compact representations, applicable in the general case of non–uniform knots.

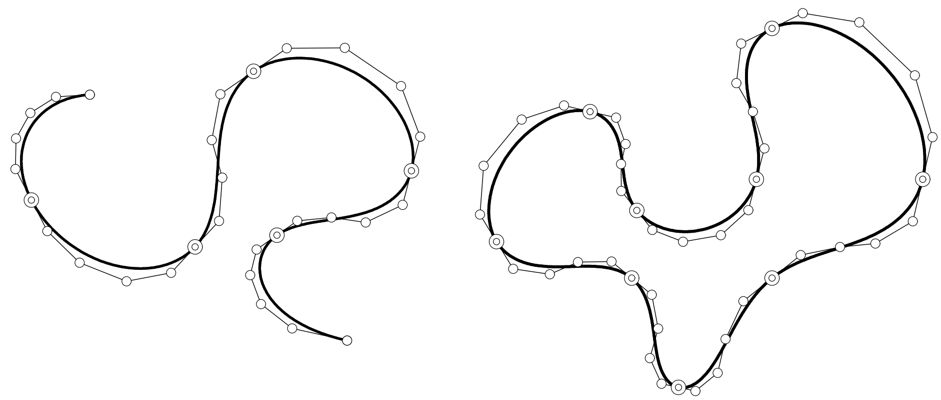

Figure 6: In conjunction with six–fold knots, the Bezier points for each spline segment determine exact B–spline representations for C^2 PH quintic splines.

The process of finding a more compact representation (or, more generally, an approximation) of a B–spline curve by using a smaller number of knots is known [16] as *knot removal.* Since the new knot sequence is a subset of the original sequence, the corresponding spline spaces are nested. Specifically, the spline space on the reduced knot vector is contained within the spline space on the original knot vector, and in the most general case knot removal can only be achieved if the new curve is allowed to deviate from the original curve within some tolerance. In the present context, knot removal is used to reduce knot multiplicities from 6 to 3, without altering the PH spline.

For a general B–spline curve of degree d on a given non–decreasing knot sequence $\ldots, t_{k-1}, t_k, t_{k+1}, \ldots$ let t be a knot of multiplicity $m \le d$ we wish to remove, and let the index s be such that $t_s = t \neq t_{s+1}$. The basic iteration can also be extended to remove the knot t more than once (if possible), i.e., up to m times — see [17, 19] for complete details. Here, we consider removing only one knot at a time. Initially setting $i = s - d$, $j = s - m$, and $\hat{\mathbf{p}}_k = \mathbf{p}_k$ for all k, new control points are iteratively computed using the expressions

$$\hat{\mathbf{p}}_i \,=\, \frac{\mathbf{p}_i - (1-\alpha_i)\,\hat{\mathbf{p}}_{i-1}}{\alpha_i}\,, \qquad \hat{\mathbf{p}}_j \,=\, \frac{\mathbf{p}_j - \alpha_j\,\hat{\mathbf{p}}_{j+1}}{1-\alpha_j}\,, \tag{14}$$

are *double* control points in this B–spline representation.

by increasing i to $i+1$ and decreasing j to $j-1$ while $j-i>0$, where $\alpha_k = (t-t_k)/(t_{i+d+1}-t_k)$ for $k=i,j$. The procedure can be briefly summarized as follows. First, the new control points (if any) are computed through (14). Taking into account the set of original and updated control points, the new control polygon is then checked to see if it defines the same B-spline curve to a specified tolerance. If so, the selected knot can be removed. With each knot removal, the number of knots and control points is reduced by one.

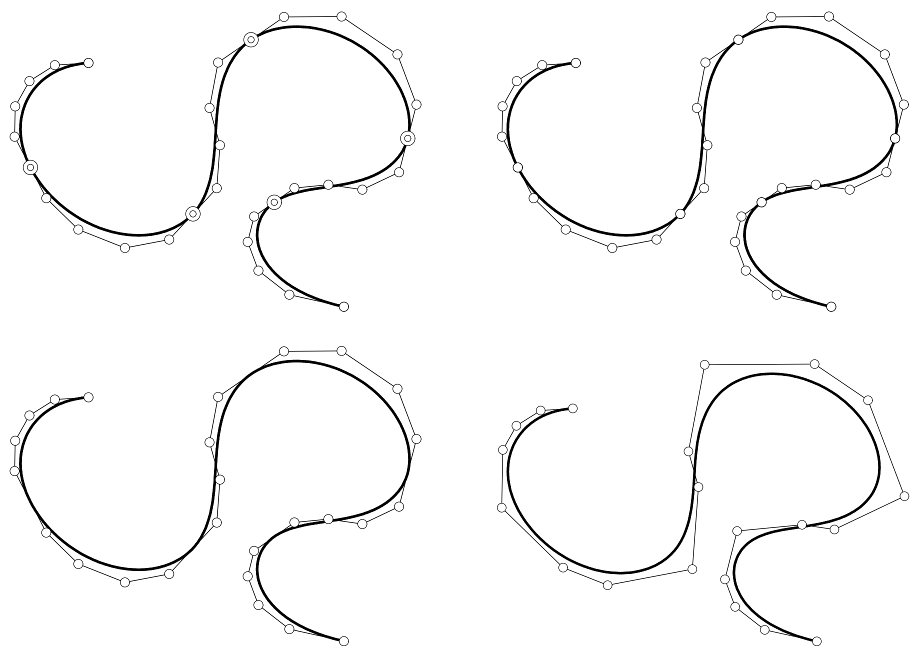

Figure 7: Computation of the minimal B–spline representation for the C^2 PH quintic spline in Figure 4 by knot removal. Starting with the Bézier control points (upper left), the B–spline control polygon is shown with interior knot multiplicities reduced to 5 (upper right), 4 (lower left), and 3 (lower right).

In the present context, the B–spline representation of a C^2 PH quintic spline is intrinsically redundant when all the knots are of multiplicity 6, since internal knots of multiplicity 3 suffice to characterize the C^2 continuity at the junctures of successive spline segments. By using the knot removal algorithm to reduce the multiplicity of each internal knot from 6 to 3, we obtain exactly the same results as in Section 4 for the case of uniform knots. Moreover, the knot removal algorithm can be directly applied without any modification to the case of C^2 PH quintic splines defined on general non–uniform knots.

Example 3. Figure 7 shows the knot removal process for the C^2 PH quintic spline in Example 1, commencing with the Bézier control points as B–spline control points (corresponding to knots of multiplicity 6, with interior segment end points counted as double control points). On reducing the multiplicity of the interior knots from 6 to 5, the segment end points become simple control points. Further reduction of the interior knot multiplicity to 4 then eliminates the segment end points as B–spline control points. Finally, reduction to knot multiplicity 3 replaces the pairs of control points spanning the segment end points by single control points. The three stages of knot removal reduce the total number of B–spline control points from 36 to 31, 26, and finally 21, but at each stage an exact representation of the PH spline curve is maintained.

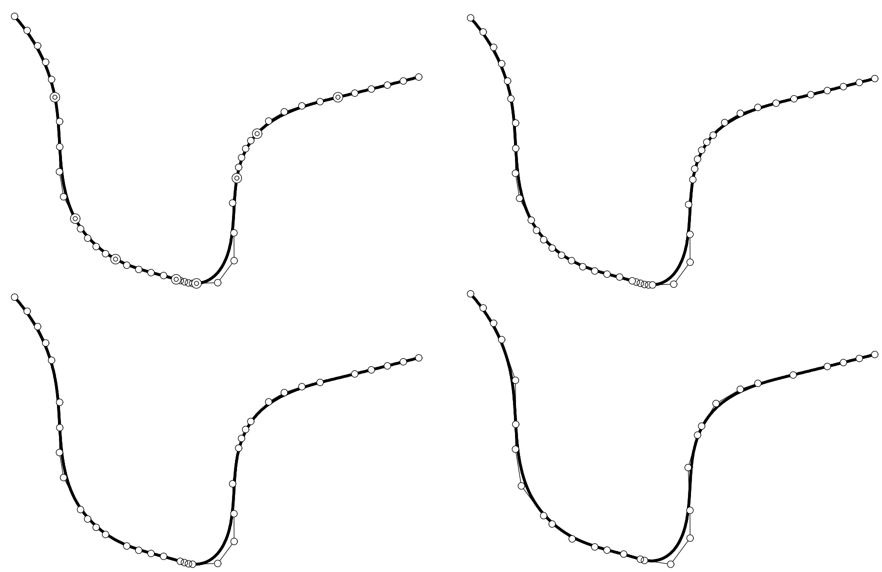

Figure 8: Determination of the B–spline representation for a C^2 PH quintic spline with non–uniform parameterization by the knot removal procedure.

Example 4. Figure 8 illustrates the determination of the minimal B–spline form for a non–uniformly parameterized C^2 PH quintic spline, computed as described in Section 6 of [11]. The procedure starts with the Bézier control points, and successively reduces the interior knot multiplicities from 6 to 3.

The nodal points for this curve, taken from Table 1 in [2], are $\mathbf{q}_0 = (0.0, 10.0)$, $\mathbf{q}_1 = (1.0, 8.0)$, $\mathbf{q}_2 = (1.5, 5.0)$, $\mathbf{q}_3 = (2.5, 4.0)$, $\mathbf{q}_4 = (4.0, 3.5)$, $\mathbf{q}_5 = (4.5, 3.4)$, $\mathbf{q}_6 = (5.5, 6.0)$, $\mathbf{q}_7 = (6.0, 7.1)$, $\mathbf{q}_9 = (8.0, 8.0)$, $\mathbf{q}_9 = (10.0, 8.5)$ while the knots $t_0, \ldots, t_9$ are 0.0, 0.1313, 0.3099, 0.3929, 0.4857, 0.5177, 0.6792, 0.7502, 0.8790, 1.0 (corresponding to a chordal parameterization). Through the three stages of knot removal used to reduce the interior knot multiplicities from 6 to 3, the number of control points decreases from 54 to 30.

6 Local PH quintic spline modification

The *local modification* property is a key feature of the B–spline form, arising from the compact support of the basis functions. For an "ordinary" C^2 cubic B–spline with simple knots, the displacement of a single control point alters only four contiguous spline segments, corresponding to the support interval for the basis function associated with that control point, and C^2 continuity between the modified and unmodified segments is maintained [5].

Having determined the B–spline representation for a C^2 PH quintic spline, it is natural to seek a local modification capability for it. Any displacement of a B–spline control point for a C^2 PH quintic spline will define a modified spline curve, but in general it will also compromise the PH structure of the two spline segments corresponding to the support interval for the associated basis function. To circumvent this, the displacement should be treated in a manner that inherently preserves the PH structure of the modified segments.

An algorithm satisfying this requirement is described below, that incurs the solution of two quadratic equations in two complex variables. To obtain sufficient freedoms to ensure preservation of the PH nature of the modified segments under arbitrary control–point displacements, it is necessary to relax from C^2 to C^1 continuity between modified and unmodified segments.

For an open spline, the first and last three control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_{3n}, \mathbf{p}_{3n+1}, \mathbf{p}_{3n+2}$ only influence the first and last spline segments. Any other control point $\mathbf{p}_i$ with $3 \le i \le 3n-1$ influences two contiguous segments. Setting $k = \lfloor i/3 \rfloor$, these two segments correspond to the parameter intervals $t \in [\,k-1, k\,]$ and $t \in [\,k, k+1\,]$. For a closed spline with $\mathbf{p}_{3n} = \mathbf{p}_0$, $\mathbf{p}_{3n+1} = \mathbf{p}_1$, $\mathbf{p}_{3n+2} = \mathbf{p}_2$, any control point $\mathbf{p}_i$ influences two contiguous spline segments $t \in [\,k-1, k\,]$ and $t \in [\,k, k+1\,]$ with $k = \lfloor i/3 \rfloor$ if $3 \le i \le 3n-1$, and $t \in [\,n-1, n\,]$ and $t \in [\,0, 1\,]$ if $i = 0, 1, 2$ or $i = 3n, 3n+1, 3n+2$.

Consider, in the B–spline form (6) of a C^2 PH quintic spline defined on

$t \in [\,0, n]$ with $N = 3n + 2$, the displacement of a control point $\mathbf{p}_i$ to a new position, and suppose that $3 \le i \le 3n - 1$, so $1 \le k = \lfloor i/3 \rfloor \le n - 1$ (the end conditions are treated later). Then $\mathbf{p}_i$ influences segments k and $k + 1$, defined on $t \in [\,k - 1, k\,]$ and $t \in [\,k, k + 1\,]$, with hodographs corresponding to instances k and $k + 1$ of (1), i.e.,

$$
\begin{aligned}
\mathbf{r}'_k(u) &= [\,\mathbf{w}_{k,0}(1-u)^2 + \mathbf{w}_{k,1}2(1-u)u + \mathbf{w}_{k,2}u^2\,]^2\,, \\
\mathbf{r}'_{k+1}(u) &= [\,\mathbf{w}_{k+1,0}(1-u)^2 + \mathbf{w}_{k+1,1}2(1-u)u + \mathbf{w}_{k+1,2}u^2\,]^2\,,
\end{aligned} \tag{15}
$$

for $u = t-(k-1)$ and $u = t-k$ respectively. The coefficients in (15) satisfy, by construction, the first and second derivative continuity condition (2) and their values are such that integration of (15) yields

$$
\int_0^1 \mathbf{r}'_k(u)\,\mathrm{d}u \,=\, \mathbf{q}_k - \mathbf{q}_{k-1} \quad \text{and} \quad \int_0^1 \mathbf{r}'_{k+1}(u)\,\mathrm{d}u \,=\, \mathbf{q}_{k+1} - \mathbf{q}_k\,, \tag{16}
$$

where, using instances k and $k+1$ of (10) with $\mathbf{c}_{k,0} = \mathbf{q}_{k-1}$, $\mathbf{c}_{k,5} = \mathbf{c}_{k+1,0} = \mathbf{q}_k$, $\mathbf{c}_{k+1,5} = \mathbf{q}_{k+1}$, we have

$$
\begin{aligned}
\mathbf{q}_{k-1} &= \tfrac{1}{4}\,(\mathbf{p}_{3k-3} + 2\,\mathbf{p}_{3k-2} + \mathbf{p}_{3k-1})\,, \\
\mathbf{q}_k &= \tfrac{1}{4}\,(\mathbf{p}_{3k} + 2\,\mathbf{p}_{3k+1} + \mathbf{p}_{3k+2})\,, \\
\mathbf{q}_{k+1} &= \tfrac{1}{4}\,(\mathbf{p}_{3k+3} + 2\,\mathbf{p}_{3k+4} + \mathbf{p}_{3k+5})\,.
\end{aligned}
$$

Hence, by substituting from (15) into (16) and performing the integrations, the coefficients satisfy

$$
\begin{aligned}
&\mathbf{w}_{k,0}^2 + \mathbf{w}_{k,0}\mathbf{w}_{k,1} + \tfrac{2}{3}\,\mathbf{w}_{k,1}^2 + \tfrac{1}{3}\,\mathbf{w}_{k,0}\mathbf{w}_{k,2} + \mathbf{w}_{k,1}\mathbf{w}_{k,2} + \mathbf{w}_{k,2}^2 \\
&\qquad = \tfrac{5}{4}\,(\mathbf{p}_{3k} + 2\,\mathbf{p}_{3k+1} + \mathbf{p}_{3k+2} - \mathbf{p}_{3k-3} - 2\,\mathbf{p}_{3k-2} - \mathbf{p}_{3k-1})\,, \\
&\mathbf{w}_{k+1,0}^2 + \mathbf{w}_{k+1,0}\mathbf{w}_{k+1,1} + \tfrac{2}{3}\,\mathbf{w}_{k+1,1}^2 + \tfrac{1}{3}\,\mathbf{w}_{k+1,0}\mathbf{w}_{k+1,2} + \mathbf{w}_{k+1,1}\mathbf{w}_{k+1,2} + \mathbf{w}_{k+1,2}^2 \\
&\qquad = \tfrac{5}{4}\,(\mathbf{p}_{3k+3} + 2\,\mathbf{p}_{3k+4} + \mathbf{p}_{3k+5} - \mathbf{p}_{3k} - 2\,\mathbf{p}_{3k+1} - \mathbf{p}_{3k+2})\,.
\end{aligned} \tag{17}
$$

These equations are satisfied *prior to* displacement of one of the control points $\mathbf{p}_{3k}, \mathbf{p}_{3k+1}, \mathbf{p}_{3k+2}$ influencing segments k and $k + 1$. After displacement, they are no longer satisfied, and we seek to restore their satisfaction by modifying some of the values $\mathbf{w}_{k,0}, \mathbf{w}_{k,1}, \mathbf{w}_{k,2}$ and $\mathbf{w}_{k+1,0}, \mathbf{w}_{k+1,1}, \mathbf{w}_{k+1,2}$.

Now modification of $\mathbf{w}_{k,0}, \mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}, \mathbf{w}_{k+1,2}$ will compromise first and second derivative continuity with the preceding and succeeding segments, $\mathbf{r}_{k-1}(u)$ and $\mathbf{r}_{k+2}(u)$. If they are unchanged, only the single complex variable

$\mathbf{w}_{k,2} = \mathbf{w}_{k+1,0}$ remains free, which does not suffice to achieve satisfaction of the two complex equations (17) when one of $\mathbf{p}_{3k}, \mathbf{p}_{3k+1}, \mathbf{p}_{3k+2}$ is changed. A two–segment local modification retaining C^2 continuity between the modified and unmodified segments is therefore not, in general, feasible.

However, such a modification becomes possible if we relax to C^1 continuity between the modified and unmodified segments. In that case, assuming the conditions (2) hold (i.e., C^2 continuity is maintained at the juncture of the modified segments) we may interpret $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$ as free parameters, with $\mathbf{w}_{k,0}$ and $\mathbf{w}_{k+1,2}$ unchanged for C^1 continuity with the unmodified segments, and $\mathbf{w}_{k,2} = \mathbf{w}_{k+1,0}$ obtained from (2). To accommodate the C^1 continuity between segments $\mathbf{r}_{k-1}(u)$, $\mathbf{r}_k(u)$ and $\mathbf{r}_{k+1}(u)$, $\mathbf{r}_{k+2}(u)$ one must increase the knot multiplicity of the values $k-1$ and $k+1$ from three to four, by a knot insertion procedure [17], before imposing the control point displacement.

When $k-1$ and $k+1$ are quadruple (rather than triple) knots, let the unmodified C^2 PH quintic spline have the B–spline form

$$\mathbf{r}(t) \,=\, \sum_{k=0}^{N+2} \hat{\mathbf{p}}_k \hat{B}_k^5(t) \tag{18}$$

in terms of control points $\hat{\mathbf{p}}_k$ and basis functions $\hat{B}_k^5(t)$ defined with respect to the new knot sequence — since the number of knots has increased by two, the number of basis functions and control points must also increase by two. By knot insertion, one can verify that the control points $\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{N+2}$ in (18) agree with the original control points $\mathbf{p}_0, \ldots, \mathbf{p}_N$ except that $\mathbf{p}_{3k-2}$ and $\mathbf{p}_{3k+4}$ are replaced by the *pairs* of points $\frac{1}{2}(\mathbf{p}_{3k-3} + \mathbf{p}_{3k-2})$, $\frac{1}{2}(\mathbf{p}_{3k-2} + \mathbf{p}_{3k-1})$ and $\frac{1}{2}(\mathbf{p}_{3k+3} + \mathbf{p}_{3k+4})$, $\frac{1}{2}(\mathbf{p}_{3k+4} + \mathbf{p}_{3k+5})$ — i.e., we have

$$\begin{aligned}
\hat{\mathbf{p}}_i &= \mathbf{p}_i \,, && i = 0, \ldots, 3k-3 \,, \\
\hat{\mathbf{p}}_i &= \tfrac{1}{2}(\mathbf{p}_{i-1} + \mathbf{p}_i) \,, && i = 3k-2, 3k-1 \,, \\
\hat{\mathbf{p}}_i &= \mathbf{p}_{i-1} \,, && i = 3k, \ldots, 3k+4 \,, \\
\hat{\mathbf{p}}_i &= \tfrac{1}{2}(\mathbf{p}_{i-2} + \mathbf{p}_{i-1}) \,, && i = 3k+5, 3k+6 \,, \\
\hat{\mathbf{p}}_i &= \mathbf{p}_{i-2} \,, && i = 3k+7, \ldots, 3n+4 \,.
\end{aligned} \tag{19}$$

Then expressions (6) and (18) define exactly the same C^2 PH quintic spline, but the latter is amenable to local modifications incurring a relaxation to C^1 continuity between the modified and unmodified segments.

Figure 9 illustrates the determination of (18) by knot insertion for a PH quintic spline interpolating points $\mathbf{q}_0, \ldots, \mathbf{q}_6$ at $t = 0, \ldots, 6$ ($t = 1, 2, 3, 4, 5$

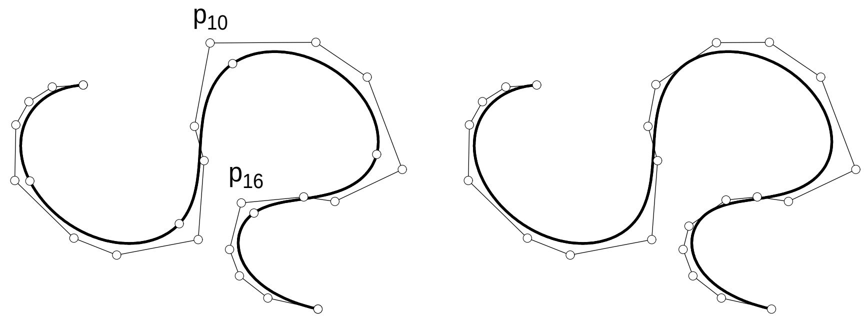

Figure 9: Left: B–spline control polygon for a C^2 PH quintic spline. Right: on changing $t = 3$ and $t = 5$ from triple to quadruple knots, $\mathbf{p}_{10}$ and $\mathbf{p}_{16}$ are replaced by their averages with the preceding and following control points.

are triple knots, while $t = 0$ and $t = 6$ have multiplicity 6). To accommodate modifications to the fourth and fifth segments, while relaxing to C^1 continuity with the unmodified segments, the knot multiplicity of $t = 3$ and $t = 5$ must be increased to 4 by knot insertion. The control polygon is unchanged, except that $\mathbf{p}_{10}$ and $\mathbf{p}_{16}$ are replaced by the pairs of points $\frac{1}{2}(\mathbf{p}_9 + \mathbf{p}_{10})$, $\frac{1}{2}(\mathbf{p}_{10} + \mathbf{p}_{11})$ and $\frac{1}{2}(\mathbf{p}_{15} + \mathbf{p}_{16})$, $\frac{1}{2}(\mathbf{p}_{16} + \mathbf{p}_{17})$. The updated control polygon passes through the nodal points $\mathbf{q}_3$ and $\mathbf{q}_5$, and is tangent to the curve at those points.

In conjunction with the new control points (19), the basis functions must be updated when the knot multiplicity of $k - 1$ and $k + 1$ is increased from three to four. As in Section 4, the Bernstein coefficients of the updated basis functions that contain the modified segments $t \in [\,k-1, k\,]$ and $t \in [\,k, k+1\,]$ in their support can be determined from their known orders of continuity at the knots and the partition of unity property (7), as enumerated in Table 6.

Thus, by comparing Bernstein coefficients, the B–spline control points associated with the basis functions in Table 6 are related to the Bézier control points for the segments $\mathbf{r}_k(u)$ and $\mathbf{r}_{k+1}(u)$ by

$$
\begin{aligned}
\tfrac{1}{2}\,\hat{\mathbf{p}}_{3k-2} + \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k-1} &= \mathbf{c}_{k,0}\,, & \tfrac{1}{4}\,\hat{\mathbf{p}}_{3k+1} + \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+2} + \tfrac{1}{4}\,\hat{\mathbf{p}}_{3k+3} &= \mathbf{c}_{k+1,0}\,, \\
\hat{\mathbf{p}}_{3k-1} &= \mathbf{c}_{k,1}\,, & \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+2} + \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+3} &= \mathbf{c}_{k+1,1}\,, \\
\hat{\mathbf{p}}_{3k} &= \mathbf{c}_{k,2}\,, & \hat{\mathbf{p}}_{3k+3} &= \mathbf{c}_{k+1,2}\,, \\
\hat{\mathbf{p}}_{3k+1} &= \mathbf{c}_{k,3}\,, & \hat{\mathbf{p}}_{3k+4} &= \mathbf{c}_{k+1,3}\,, \\
\tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+1} + \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+2} &= \mathbf{c}_{k,4}\,, & \hat{\mathbf{p}}_{3k+5} &= \mathbf{c}_{k+1,4}\,, \\
\tfrac{1}{4}\,\hat{\mathbf{p}}_{3k+1} + \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+2} + \tfrac{1}{4}\,\hat{\mathbf{p}}_{3k+3} &= \mathbf{c}_{k,5}\,, & \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+5} + \tfrac{1}{2}\,\hat{\mathbf{p}}_{3k+6} &= \mathbf{c}_{k+1,5}\,,
\end{aligned}
$$

	$[\,k-2,k-1\,]$	$[\,k-1,k\,]$	$[\,k,k+1\,]$	$[\,k+1,k+2\,]$
$\hat{B}_{3k-2}(t)$	$(0,0,0,0,1,\frac{1}{2})$	$(\frac{1}{2},0,0,0,0,0)$		
$\hat{B}_{3k-1}(t)$	$(0,0,0,0,0,\frac{1}{2})$	$(\frac{1}{2},1,0,0,0,0)$		
$\hat{B}_{3k}(t)$		$(0,0,1,0,0,0)$		
$\hat{B}_{3k+1}(t)$		$(0,0,0,1,\frac{1}{2},\frac{1}{4})$	$(\frac{1}{4},0,0,0,0,0)$	
$\hat{B}_{3k+2}(t)$		$(0,0,0,0,\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2},0,0,0,0)$	
$\hat{B}_{3k+3}(t)$		$(0,0,0,0,0,\frac{1}{4})$	$(\frac{1}{4},\frac{1}{2},1,0,0,0)$	
$\hat{B}_{3k+4}(t)$			$(0,0,0,1,0,0)$	
$\hat{B}_{3k+5}(t)$			$(0,0,0,0,1,\frac{1}{2})$	$(\frac{1}{2},0,0,0,0,0)$
$\hat{B}_{3k+6}(t)$			$(0,0,0,0,0,\frac{1}{2})$	$(\frac{1}{2},1,0,0,0,0)$

Table 1: The Bernstein coefficents of the B–spline basis functions in (18) that are non–zero over the two modified segments $t \in [\,k-1,k]$ and $t \in [\,k,k+1\,]$.

where $\mathbf{c}_{k,0} = \mathbf{q}_{k-1}$, $\mathbf{c}_{k,5} = \mathbf{c}_{k+1,0} = \mathbf{q}_k$, $\mathbf{c}_{k+1,5} = \mathbf{q}_{k+1}$. From these relations, one may infer that

$$\begin{aligned}
\hat{\mathbf{p}}_{3k+1} &= \mathbf{c}_{k,3} \;=\; 4\,\mathbf{c}_{k+1,0} - 4\,\mathbf{c}_{k+1,1} + \mathbf{c}_{k+1,2}\,, \\
\hat{\mathbf{p}}_{3k+2} &= 2\,\mathbf{c}_{k,4} - \mathbf{c}_{k,3} \;=\; 2\,\mathbf{c}_{k+1,1} - \mathbf{c}_{k+1,2}\,, \\
\hat{\mathbf{p}}_{3k+3} &= 4\,\mathbf{c}_{k,5} - 4\,\mathbf{c}_{k,4} + \mathbf{c}_{k,3} \;=\; \mathbf{c}_{k+1,2}\,.
\end{aligned}$$

Expressing the control points $\mathbf{c}_{k,0},\ldots,\mathbf{c}_{k,5}$ and $\mathbf{c}_{k+1,0},\ldots,\mathbf{c}_{k+1,5}$ in terms of the coefficients $\mathbf{w}_{k,0},\mathbf{w}_{k,1},\mathbf{w}_{k,2}$ and $\mathbf{w}_{k+1,0},\mathbf{w}_{k+1,1},\mathbf{w}_{k+1,2}$ — with $\mathbf{w}_{k,0},\mathbf{w}_{k+1,2}$ fixed and $\mathbf{w}_{k,2},\mathbf{w}_{k+1,0}$ satisfying (2) — these relations yield two quadratic equations for the $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$ values that accommodate displacement of one of the control points $\hat{\mathbf{p}}_{3k+1} = \mathbf{p}_{3k}$, $\hat{\mathbf{p}}_{3k+2} = \mathbf{p}_{3k+1}$, $\hat{\mathbf{p}}_{3k+3} = \mathbf{p}_{3k+2}$.

For brevity, we illustrate only the case where $\hat{\mathbf{p}}_{3k+2}$ is displaced (similar principles apply for $\hat{\mathbf{p}}_{3k+1}$ and $\hat{\mathbf{p}}_{3k+3}$). In that case, we obtain the equations

$$\begin{aligned}
10\,\mathbf{w}_{k,1}^2 + 6\,\mathbf{w}_{k,1}\mathbf{w}_{k+1,1} + \mathbf{a}_1\mathbf{w}_{k,1} + \mathbf{b}_1\mathbf{w}_{k+1,1} + \mathbf{c}_1 &= 0\,, \\
10\,\mathbf{w}_{k+1,1}^2 + 6\,\mathbf{w}_{k,1}\mathbf{w}_{k+1,1} + \mathbf{a}_2\mathbf{w}_{k,1} + \mathbf{b}_2\mathbf{w}_{k+1,1} + \mathbf{c}_2 &= 0\,,
\end{aligned} \tag{20}$$

for $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$, in terms of the known quantities

$$\begin{aligned} &\mathbf{a}_1 = 7\,\mathbf{w}_{k,0}\,, \quad \mathbf{b}_1 = \mathbf{w}_{k,0}\,, \quad \mathbf{c}_1 = 6\,\mathbf{w}_{k,0}^2 - 30\,(\hat{\mathbf{p}}_{3k+2} - \mathbf{q}_{k-1})\,, \\ &\mathbf{a}_2 = \mathbf{w}_{k+1,2}\,, \quad \mathbf{b}_2 = 7\,\mathbf{w}_{k+1,2}\,, \quad \mathbf{c}_2 = 6\,\mathbf{w}_{k+1,2}^2 - 30\,(\mathbf{q}_{k+1} - \hat{\mathbf{p}}_{3k+2})\,. \end{aligned} \tag{21}$$

In principle, one can eliminate either $\mathbf{w}_{k,1}$ or $\mathbf{w}_{k+1,1}$ among the quadratic equations (20) to obtain a quartic equation in the other, which may be solved in closed form through Ferrari's method [20] — see the Appendix. As usual with PH curve constructions, a unique "good" modified spline is observed among the four formal solutions, the other three having undesired loops or curvature extrema. In practice, it is more efficient to use Newton–Raphson iteration to obtain just the "good" solution of equations (20), employing the $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$ values for the unmodified spline as a starting approximation. Convergence to machine precision typically occurs within a few iterations.

After solving equations (20) for $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$ the neighboring control points $\hat{\mathbf{p}}_{3k} = \mathbf{c}_{k,2}$, $\hat{\mathbf{p}}_{3k+1} = \mathbf{c}_{k,3}$ and $\hat{\mathbf{p}}_{3k+3} = \mathbf{c}_{k+1,2}$, $\hat{\mathbf{p}}_{3k+4} = \mathbf{c}_{k+1,3}$ must also be adjusted, since they depend on $\mathbf{w}_{k,1}, \mathbf{w}_{k+1,1}$ through the expressions

$$\begin{aligned} \hat{\mathbf{p}}_{3k} &= \mathbf{q}_{k-1} + \tfrac{1}{5}\,(\mathbf{w}_{k,0}^2 + \mathbf{w}_{k,0}\mathbf{w}_{k,1})\,, \\ \hat{\mathbf{p}}_{3k+1} &= \mathbf{q}_{k-1} + \tfrac{1}{5}\,(\mathbf{w}_{k,0}^2 + \mathbf{w}_{k,0}\mathbf{w}_{k,1} + \tfrac{2}{3}\,\mathbf{w}_{k,1}^2 + \tfrac{1}{3}\,\mathbf{w}_{k,0}\mathbf{w}_{k,2})\,, \\ \hat{\mathbf{p}}_{3k+3} &= \mathbf{q}_{k+1} - \tfrac{1}{5}\,(\mathbf{w}_{k+1,2}^2 + \mathbf{w}_{k+1,1}\mathbf{w}_{k+1,2} + \tfrac{2}{3}\,\mathbf{w}_{k+1,1}^2 + \tfrac{1}{3}\,\mathbf{w}_{k+1,0}\mathbf{w}_{k+1,2})\,, \\ \hat{\mathbf{p}}_{3k+4} &= \mathbf{q}_{k+1} - \tfrac{1}{5}\,(\mathbf{w}_{k+1,2}^2 + \mathbf{w}_{k+1,1}\mathbf{w}_{k+1,2})\,. \end{aligned} \tag{22}$$

Thus, to preserve the internal PH structure of the modified spline segments, the displacement of $\hat{\mathbf{p}}_{3k+2}$ incurs automatic "sympathetic" displacements of the two preceding and following B–spline control points.

Remark 1. Since the scheme depends on relaxing from C^2 to C^1 continuity between the modified and unmodified segments by altering $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$, instances $k-1$ and $k+1$ of (2) are no longer valid. Hence, together with the altered $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$ values, the values $\mathbf{w}_{k-1,2} = \mathbf{w}_{k,0}$ and $\mathbf{w}_{k+1,2} = \mathbf{w}_{k+2,0}$ must be stored, instead of relying on the relation (2) to compute them.

The PH quintic spline modification scheme may be summarized as follows.

Algorithm

input: knots $t_0, \ldots, t_{N+6}$; control points $\mathbf{p}_0, \ldots, \mathbf{p}_N$;
complex coefficients $\mathbf{w}_{k,0}, \mathbf{w}_{k,1}, \mathbf{w}_{k,2}$ for $k = 1, \ldots, n$;
and desired displacement of $\Delta\mathbf{p}$ of control point $\mathbf{p}_{3k+1}$.

1. increase knot multiplicity of $k-1$ and $k+1$ from 3 to 4, and compute new control points $\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{N+2}$ from (19);

2. apply the displacement $\Delta\mathbf{p}$ to control point $\hat{\mathbf{p}}_{3k+2} = \mathbf{p}_{3k+1}$;

3. solve equations (20), with known coefficients (21), for new values $\mathbf{w}_{k,1}$ and $\mathbf{w}_{k+1,1}$ associated with $\mathbf{r}'_k(u)$ and $\mathbf{r}'_{k+1}(u)$;

4. assign a new value to $\mathbf{w}_{k,2} = \mathbf{w}_{k+1,0}$ using equation (2);

5. update the two control points $\hat{\mathbf{p}}_{3k}, \hat{\mathbf{p}}_{3k+1}$ and $\hat{\mathbf{p}}_{3k+3}, \hat{\mathbf{p}}_{3k+4}$ preceding and following $\hat{\mathbf{p}}_{3k+2}$ through expressions (22);

output: new knots $t_0, \ldots, t_{N+8}$; control points $\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{N+2}$; and complex coefficients $\mathbf{w}_{k,0}, \mathbf{w}_{k,1}, \mathbf{w}_{k,2}$ for $k = 1, \ldots, n$ of the modified PH spline corresponding to the control point displacement $\Delta\mathbf{p}$.

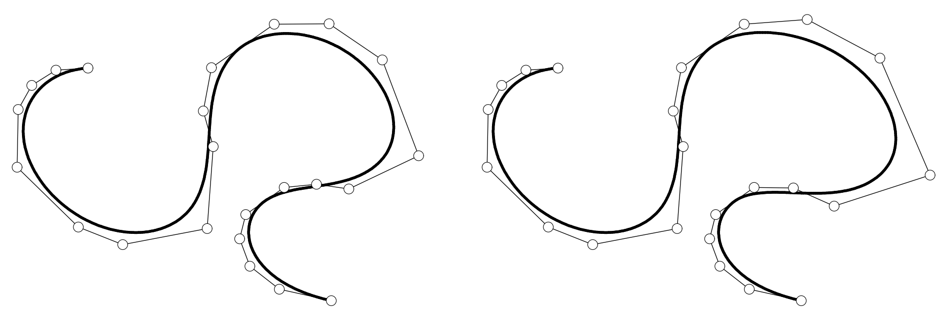

Figure 10: A local modification of the PH quintic spline in Example 1. Left: control polygon after knot multiplicity of $t = 3$ and $t = 5$ is increased from three to four. Right: the control point $\hat{\mathbf{p}}_{14}$ is displaced by $\Delta\mathbf{p} = (0.73, -0.35)$ and the control points $\hat{\mathbf{p}}_{12}, \hat{\mathbf{p}}_{13}$ and $\hat{\mathbf{p}}_{15}, \hat{\mathbf{p}}_{16}$ are correspondingly modified by the algorithm, to preserve the PH quintic nature of the modified segments.

Example 5. Figure 10 illustrates the modification of a PH quintic spline by this process. After increasing the multiplicity of $t = 3$ and $t = 5$ from 3 to 4 by knot insertion, the control point $\hat{\mathbf{p}}_{14} = \mathbf{p}_{13}$ is displaced by $\Delta\mathbf{p} = (0.75, -0.35)$ resulting in modifications to the fourth and fifth spline segments. Note that the modified segments are still PH quintics — they maintain C^1 continuity with the unmodified segments, and are C^2 at their juncture. In Figure 11 we compare curvature profiles for the modified and unmodified splines. Although the modified curve is C^1 at $t = 3$ and $t = 5$, the curvature discontinuities are quite modest compared to the overall range of curvature variation.

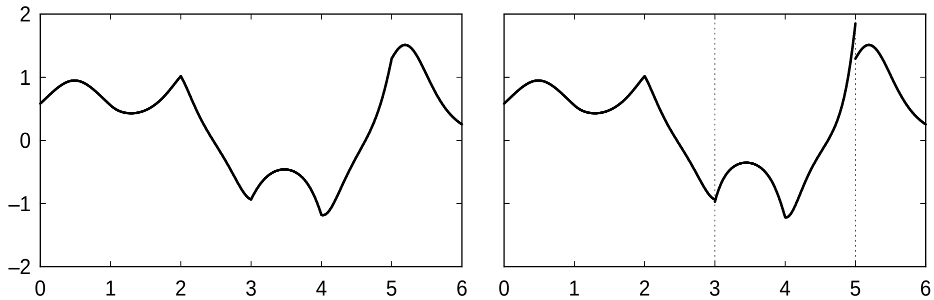

Figure 11: Curvature plots for the unmodified (left) and modified (right) PH quintic splines in Figure 10 (note the curvature discontinuities in the latter).

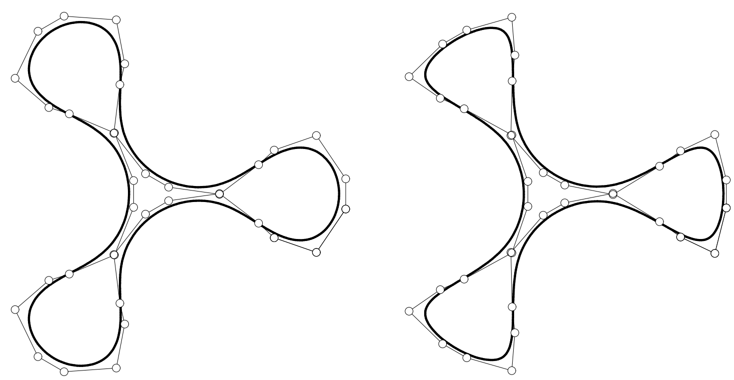

Figure 12: B–spline control polygons for the C^2 PH quintic and "ordinary" C^2 cubic splines shown in Figure 1. The latter has been elevated to degree 5, so as to possess the same number of control points as the PH quintic spline.

Example 6. Figure 12 illustrates the B–spline control polygons for the C^2 PH quintic and "ordinary" C^2 cubic splines shown in Figure 1. This example will be used to compare the local shape–modification behavior of the two spline forms. To ensure a fair comparison, the cubic spline is first elevated to degree 5, so as to have the same number of control points as the PH spline. This entails using triple knots for the cubic spline, so the displacement of a single control point will only influence two contiguous spline segments.

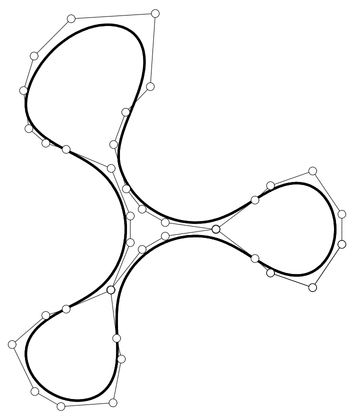

Figure 13: Comparison of *bella* modified PH quintic spline (left) and *brutta* modified "ordinary" degree–elevated cubic spline (right), after applying the same displacement to a corresponding B–spline control point for each curve.

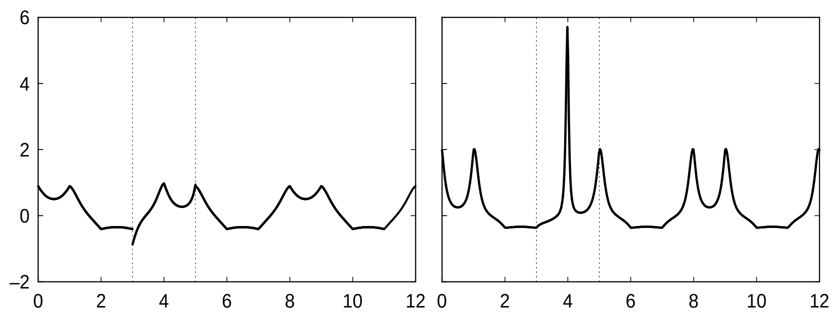

Figure 14: Comparison of curvature profiles for modified PH quintic spline (left) and ordinary cubic spline (right) after displacement of one control point.

Figure 13 illustrates the outcome of applying a displacement $\Delta\mathbf{p}$ to the control point $\mathbf{p}_{13}$ of both splines. After modification, the PH quintic spline preserves its attractive "roundness" while the cubic spline develops a sharp curvature extremum. This is highlighted by the curvature plots shown in Figure 14, in which the dashed lines indicate the domain influenced by $\mathbf{p}_{13}$.

The modified PH spline is nominally just C^1 at the junctures of modified and unmodified segments, but the magnitudes of the curvature discontinuities are modest relative to the overall range of curvature variation. On the other hand, although the modified cubic spline remains C^2, the sharp curvature spike incurred by the control point modification is arguably a worse artifact than the modest curvature discontinuities of the PH quintic spline.

Finally, we consider a displacement of one of the first or last three control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_{3n}, \mathbf{p}_{3n+1}, \mathbf{p}_{3n+2}$ reflecting the end conditions, so that $k = \lfloor i/3 \rfloor = 0$ or n. For a closed C^2 spline, $\mathbf{p}_{3n} = \mathbf{p}_0$, $\mathbf{p}_{3n+1} = \mathbf{p}_1$, $\mathbf{p}_{3n+2} = \mathbf{p}_2$, and these three control points influence the final and initial segments, $\mathbf{r}_n(u)$ and $\mathbf{r}_1(u)$, meeting with C^2 continuity. We first increase the knot multiplicity of $t = 1$ and $t = n - 1$ from three to four, and then the approach described above still holds when the indices are interpreted cyclically — i.e., we replace $\mathbf{w}_{0,i}$ with $\mathbf{w}_{n,i}$ and $\mathbf{w}_{n+1,i}$ with $\mathbf{w}_{1,i}$ to obtain quadratic equations in $\mathbf{w}_{n,1}, \mathbf{w}_{1,1}$ and set $\mathbf{w}_{n,2} = \mathbf{w}_{1,0} = \frac{1}{2}(\mathbf{w}_{n,1} + \mathbf{w}_{1,1})$ upon solving them.

In the case of an open spline, C^2 continuity can be preserved when one of the first or last three control points is displaced: no knot insertion is required. Consider the first segment $\mathbf{r}_1(u)$ with

$$\mathbf{r}_1'(u) = [\, \mathbf{w}_{1,0}(1-u)^2 + \mathbf{w}_{1,1}2(1-u)u + \mathbf{w}_{1,2}u^2 \,]^2 \,, \qquad \int_0^1 \mathbf{r}_1'(u)\,\mathrm{d}u = \mathbf{q}_1 - \mathbf{q}_0 \,.$$

From equations (11), the nodal points $\mathbf{q}_0, \mathbf{q}_1$ can be expressed as

$$\mathbf{q}_0 = \mathbf{p}_0 = \mathbf{p}_1 - \tfrac{1}{5}\,\mathbf{w}_{1,0}^2 = \mathbf{p}_2 - \tfrac{1}{5}\,(\mathbf{w}_{1,0}^2 + \mathbf{w}_{1,0}\mathbf{w}_{1,1}) \,, \quad \mathbf{q}_1 = \tfrac{1}{4}\,\mathbf{p}_3 + \tfrac{1}{2}\,\mathbf{p}_4 + \tfrac{1}{4}\,\mathbf{p}_5$$

since $\mathbf{c}_{1,0} = \mathbf{q}_0$, $\mathbf{c}_{1,1} = \mathbf{q}_0 + \frac{1}{5}\,\mathbf{w}_{1,0}^2$, and $\mathbf{c}_{1,2} = \mathbf{q}_0 + \frac{1}{5}(\mathbf{w}_{1,0}^2 + \mathbf{w}_{1,0}\mathbf{w}_{1,1})$. Hence, using each of the three expressions for $\mathbf{q}_0$ above, we obtain

$$\begin{aligned}
\mathbf{w}_{1,0}^2 + \mathbf{w}_{1,0}\mathbf{w}_{1,1} + \frac{2\,\mathbf{w}_{1,1}^2 + \mathbf{w}_{1,0}\mathbf{w}_{1,2}}{3} + \mathbf{w}_{1,1}\mathbf{w}_{1,2} + \mathbf{w}_{1,2}^2 &= 5\,(\mathbf{q}_1 - \mathbf{p}_0) \,, \\
\mathbf{w}_{1,0}\mathbf{w}_{1,1} + \frac{2\,\mathbf{w}_{1,1}^2 + \mathbf{w}_{1,0}\mathbf{w}_{1,2}}{3} + \mathbf{w}_{1,1}\mathbf{w}_{1,2} + \mathbf{w}_{1,2}^2 &= 5\,(\mathbf{q}_1 - \mathbf{p}_1) \,, \\
\frac{2\,\mathbf{w}_{1,1}^2 + \mathbf{w}_{1,0}\mathbf{w}_{1,2}}{3} + \mathbf{w}_{1,1}\mathbf{w}_{1,2} + \mathbf{w}_{1,2}^2 &= 5\,(\mathbf{q}_1 - \mathbf{p}_2) \,.
\end{aligned}$$

Substituting a modified $\mathbf{p}_0$, $\mathbf{p}_1$, or $\mathbf{p}_2$ into the first, second, or third relation and keeping $\mathbf{w}_{1,1}$ and $\mathbf{w}_{1,2}$ fixed yields a quadratic equation that can be solved for a new $\mathbf{w}_{1,0}$ value (which amounts to changing $\mathbf{r}_1'(0)$) to accommodate the modified control point. Since $\mathbf{w}_{1,1}$ and $\mathbf{w}_{1,2}$ are unaltered, C^2 continuity with the following segment $\mathbf{r}_2(u)$ is maintained. An analogous approach holds for modification of $\mathbf{p}_{3n}$, $\mathbf{p}_{3n+1}$, or $\mathbf{p}_{3n+2}$ — $\mathbf{w}_{n,0}$ and $\mathbf{w}_{n,1}$ are fixed, and $\mathbf{w}_{n,2}$ is adjusted to accommodate the control–point modification using (12).

Remark 2. The PH spline modification scheme is not intended for repeated applications, except when the modified control points influence only disjoint parameter domains. However, by relaxing to C^1 continuity at all the knots, it may be possible to accommodate repeated modification of arbitrary control points. We defer a complete investigation of this possibility to a future study.

7 Closure

Methods to determine the B–spline representation of open or closed C^2 PH quintic spline curves that interpolate given points have been developed, based on triple knot sequences. In the case of uniform distinct knots, the B–spline basis functions were explicitly constructed, and simple expressions for the control points were derived. In the case of non–uniform knots, standard knot–removal algorithms can be used to determine the B–spline control points, starting from the Bézier control points of individual spline segments. These procedures apply to both planar and spatial PH quintic splines, but in the latter case greater care is required in specifying the "internal structure" of the PH quintic segments (required to fully exploit their advantageous properties).

Based on the B–spline form, a scheme for the local modification of planar PH quintic splines with uniform distinct knots was proposed. Since the basis functions have a support of two non–zero intervals, only two spline segments are modified in response to a control point displacement. To ensure that the PH nature of the modified segments is preserved for arbitrary control point displacements, it is necessary to relax from C^2 to C^1 continuity between the modified and unmodified segments, and adjust pairs of neighboring control points. An alternative modification scheme may be based on, for example, an *a priori* elevation of the degree of the modified segments to 7.

It is hoped that the availability of a B–spline representation for C^2 PH quintic splines, and the ability to interactively make localized modifications, will help facilitate their importation into commercial CAD/CAM systems.

Acknowledgements

This work was supported by the Gruppo Nazionale per il Calcolo Scientifico (GNCS) of the Istituto Nazionale di Alta Matematica Francesco Severi (INdAM) and by the project DREAMS (MIUR Futuro in Ricerca RBFR13FBI3).

Appendix

We consider here the closed–form determination of all solutions to the system of equations (20). Substituting

$$\mathbf{w}_{k,1} \;=\; \mathbf{u}+\mathbf{v}\,, \quad \mathbf{w}_{k+1,1} \;=\; \mathbf{u}-\mathbf{v}\,, \tag{23}$$

in (20), we take the sum and difference of the resulting equations to obtain

$$\begin{aligned} 32\,\mathbf{u}^2 + 8\,\mathbf{v}^2 + \mathbf{d}_1\,\mathbf{u} + \mathbf{e}_1\,\mathbf{v} + \mathbf{f}_1 \;&=\; 0\,, \\ 40\,\mathbf{uv} + \mathbf{d}_2\,\mathbf{u} + \mathbf{e}_2\,\mathbf{v} + \mathbf{f}_2 \;&=\; 0\,, \end{aligned} \tag{24}$$

where $\mathbf{d}_1, \mathbf{e}_1, \mathbf{f}_1$ and $\mathbf{d}_2, \mathbf{e}_2, \mathbf{f}_2$ are defined in terms of the quantities (21) by

$$\begin{aligned} \mathbf{d}_1 \;&=\; \mathbf{a}_1+\mathbf{b}_1+\mathbf{a}_2+\mathbf{b}_2\,, \quad & \mathbf{e}_1 \;&=\; \mathbf{a}_1-\mathbf{b}_1+\mathbf{a}_2-\mathbf{b}_2\,, \quad & \mathbf{f}_1 \;&=\; \mathbf{c}_1+\mathbf{c}_2\,, \\ \mathbf{d}_2 \;&=\; \mathbf{a}_1+\mathbf{b}_1-\mathbf{a}_2-\mathbf{b}_2\,, \quad & \mathbf{e}_2 \;&=\; \mathbf{a}_1-\mathbf{b}_1-\mathbf{a}_2+\mathbf{b}_2\,, \quad & \mathbf{f}_2 \;&=\; \mathbf{c}_1-\mathbf{c}_2\,. \end{aligned}$$

From the second of equations (24), we obtain

$$\mathbf{v} \;=\; -\,\frac{\mathbf{d}_2\mathbf{u}+\mathbf{f}_2}{40\,\mathbf{u}+\mathbf{e}_2}\,, \tag{25}$$

and substituting this into the first equation then gives a quartic in $\mathbf{u}$, namely

$$\mathbf{u}^4 \,+\, \boldsymbol{\alpha}_3\mathbf{u}^3 \,+\, \boldsymbol{\alpha}_2\mathbf{u}^2 \,+\, \boldsymbol{\alpha}_1\mathbf{u} \,+\, \boldsymbol{\alpha}_0 \;=\; 0\,, \tag{26}$$

with coefficients

$$\begin{aligned} \boldsymbol{\alpha}_3 \;&=\; (5\,\mathbf{d}_1 + 8\,\mathbf{e}_2)/160\,, \\ \boldsymbol{\alpha}_2 \;&=\; (\mathbf{d}_2^2 + 4\,\mathbf{e}_2^2 - 5\,\mathbf{d}_2\mathbf{e}_1 + 10\,\mathbf{d}_1\mathbf{e}_2 + 200\,\mathbf{f}_1)/6400\,, \\ \boldsymbol{\alpha}_1 \;&=\; (\mathbf{d}_1\mathbf{e}_2^2 - \mathbf{d}_2\mathbf{e}_1\mathbf{e}_2 + 16\,\mathbf{d}_2\mathbf{f}_2 - 40\,\mathbf{e}_1\mathbf{f}_2 + 80\,\mathbf{e}_2\mathbf{f}_1)/51200\,, \\ \boldsymbol{\alpha}_0 \;&=\; (8\,\mathbf{f}_2^2 + \mathbf{e}_2^2\mathbf{f}_1 - \mathbf{e}_1\mathbf{e}_2\mathbf{f}_2)/51200\,. \end{aligned}$$

All four roots $\mathbf{u}$ of the quartic (26) can be computed by Ferrari's method [20], and the corresponding values of $\mathbf{v}$ can be obtained from (25). From each pair of $(\mathbf{u},\mathbf{v})$ values, the coefficients $\mathbf{w}_{k,1}$, $\mathbf{w}_{k+1,1}$ that define a formal solution to the two–segment C^1 modification problem are then obtained from (23).

Ferrari's method may be formulated as follows. Let $\mathbf{z}$ be any root of the *resolvent cubic equation*

$$\mathbf{z}^3 \,+\, \boldsymbol{\beta}_2\mathbf{z}^2 \,+\, \boldsymbol{\beta}_1\mathbf{z} + \boldsymbol{\beta}_0 \;=\; 0\,, \tag{27}$$

with coefficients $\boldsymbol{\beta}_2 = -\,\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_1\boldsymbol{\alpha}_3 - 4\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0 = 4\boldsymbol{\alpha}_2\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1^2 - \boldsymbol{\alpha}_3^2\boldsymbol{\alpha}_0$. Then the roots of (26) coincide with the roots of the two quadratic equations

$$\mathbf{u}^2 \,+\, (\tfrac{1}{2}\boldsymbol{\alpha}_3 \pm \boldsymbol{\gamma})\,\mathbf{u} \,+\, (\tfrac{1}{2}\mathbf{z} \pm \boldsymbol{\delta}) \,=\, 0\,,$$

where we define $\boldsymbol{\gamma} = \frac{1}{2}\sqrt{\boldsymbol{\alpha}_3^2 + 4(\mathbf{z} - \boldsymbol{\alpha}_2)}$ and $\boldsymbol{\delta} = (\boldsymbol{\alpha}_3\mathbf{z} - 2\boldsymbol{\alpha}_1)/4\boldsymbol{\gamma}$. The roots of the cubic (27) may be obtained by Cardano's method [20]. Set

$$\mathbf{Q} \,=\, \frac{3\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2^2}{9}\,, \qquad \mathbf{R} \,=\, \frac{9\boldsymbol{\beta}_1\boldsymbol{\beta}_2 - 27\boldsymbol{\beta}_0 - 2\boldsymbol{\beta}_2^3}{54}\,, \qquad \boldsymbol{\Delta} \,=\, \mathbf{Q}^3 + \mathbf{R}^2\,,$$

and let $\mathbf{S}$ be any of the three complex values specified by $\mathbf{S}^3 = \mathbf{R} + \sqrt{\boldsymbol{\Delta}}$. Then, writing $\mathbf{A} = \mathbf{S} - \mathbf{Q}/\mathbf{S}$ and $\mathbf{B} = \mathbf{S} + \mathbf{Q}/\mathbf{S}$, the roots $\mathbf{z}$ of (27) are

$$-\frac{\boldsymbol{\beta}_2}{3} + \mathbf{A}\,, \quad -\frac{\boldsymbol{\beta}_2}{3} - \frac{\mathbf{A} - \sqrt{3}\,\mathrm{i}\,\mathbf{B}}{2}\,, \quad -\frac{\boldsymbol{\beta}_2}{3} - \frac{\mathbf{A} + \sqrt{3}\,\mathrm{i}\,\mathbf{B}}{2}\,.$$

Figure 15 illustrates all four formal solutions to the PH spline modification problem shown in Figure 10, computed as described above. As usual with PH curve constructions, there is a unique "good" solution, and the other solutions exhibit extreme curvature variations or undesired "looping" behavior.

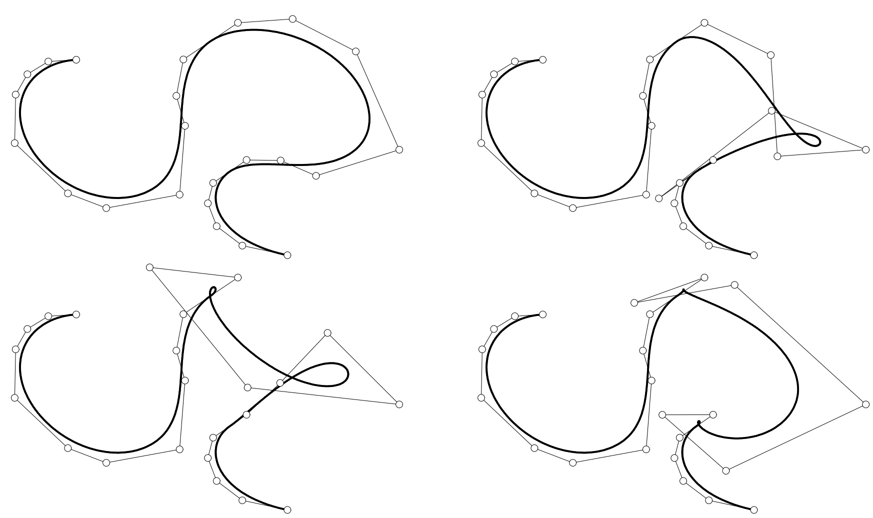

Figure 15: The four formal solutions to the PH quintic spline modification problem in Figure 10, on moving the control point $\hat{\mathbf{p}}_{14}$ by $\Delta\mathbf{p} = (0.75, -0.35)$.

References

[1] G. Albrecht and R. T. Farouki (1996), Construction of C^2 Pythagorean–hodograph interpolating splines by the homotopy method, *Adv. Comp. Math.* **5**, 417–442.

[2] S. Asaturyan, P. Costantini, and C. Manni (1998), G^2 shape–preserving parametric planar curve interpolation, in *Creating Fair and Shape–Preserving Curves and Surfaces* (H. Nowacki and P. D. Kaklis, eds.), Teubner, Stuttgart, 89–98.

[3] H. I. Choi, D. S. Lee, and H. P. Moon (2002), Clifford algebra, spin representation, and rational parameterization of curves and surfaces, *Adv. Comp. Math.* **17**, 5–48.

[4] B. Dong and R. T. Farouki (2014), `PHquintic`: A library of basic functions for the construction and analysis of planar quintic Pythagorean–hodograph curves, preprint.

[5] G. Farin, (1997), *Curves and Surfaces for Computer Aided Geometric Design*, 4th edition, Academic Press, San Diego.

[6] R. T. Farouki (1994), The conformal map $z \to z^2$ of the hodograph plane, *Comput. Aided Geom. Design* **11**, 363–390.

[7] R. T. Farouki (1996), The elastic bending energy of Pythagorean–hodograph curves, *Comput. Aided Geom. Design* **13**, 227–241.

[8] R. T. Farouki (2008), *Pythagorean–Hodograph Curves: Algebra and Geometry Inseparable*, Springer, Berlin.

[9] R. T. Farouki, M. al–Kandari, and T. Sakkalis (2002), Structural invariance of spatial Pythagorean hodographs, *Comput. Aided Geom. Design* **19**, 395–407.

[10] R. T. Farouki, C. Giannelli, C. Manni, and A. Sestini (2008), Identification of spatial PH quintic Hermite interpolants with near–optimal shape measures, *Comput. Aided Geom. Design* **25**, 274–297.

[11] R. T. Farouki, B. K. Kuspa, C. Manni, and A. Sestini (2001), Efficient solution of the complex quadratic tridiagonal system for C^2 PH quintic splines, *Numer. Algor.* **27**, 35–60.

[12] R. T. Farouki, C. Manni, F. Pelosi, and M. L. Sampoli (2012), Design of C^2 spatial Pythagorean–hodograph quintic splines by control polygons, in (J. D. Boissonnat et al., eds.), Lecture Notes in Computer Science Vol. 6920, 253–269, Springer.

[13] R. T. Farouki, C. Manni, and A Sestini (2003), Spatial C^2 PH quintic splines, *Curve and Surface Design: Saint–Malo 2002* (T. Lyche, M.–L. Mazure, and L. L. Schumaker, eds.), Nashboro Press, 147–156.

[14] R. T. Farouki and T. Sakkalis (1990), Pythagorean hodographs, *IBM J. Res. Develop.* **34**, 736–752.

[15] R. T. Farouki and T. Sakkalis (1994), Pythagorean–hodograph space curves, *Adv. Comp. Math.* **2**, 41–66.

[16] T. Lyche and K. Morken (1987), Knot removal for parametric B–spline curves and surfaces, *Comput. Aided Geom. Design* **4**, 217–230.

[17] L. Piegl and W. Tiller (1997), *The NURBS Book*, 2nd edition. Springer, New York.

[18] F. Pelosi, M. L. Sampoli, R. T. Farouki, and C. Manni (2007), A control polygon scheme for design of planar C^2 PH quintic spline curves, *Comput. Aided Geom. Design* **24**, 28–52.

[19] W. Tiller (1992), Knot–removal algorithms for NURBS curves and surfaces, *Comput. Aided Design* **24**, 445–453.

[20] J. V. Uspensky (1948), *Theory of Equations*, McGraw–Hill, New York.