Matematika, 2002, Jilid 18, bil. 1, hlm. 21–31

Local Nonsimilarity Solution for Vertical Free Convection Boundary Layers

Lok Yian Yian & Norsarahaida Amin
Department of Mathematics
Universiti Teknologi Malaysia,
81310 UTM Skudai, Johor, Malaysia

Abstract Laminar free convection boundary layer over a vertical flat plate with an exponential variation of surface temperature in viscous fluids is analyzed using the local nonsimilarity method. The present approach takes into consideration the nonsimilarity terms appearing in the momentum and energy equations, which have been unaccounted for previously, in for example, the similarity and the local similarity methods. The governing equations are solved numerically using the Keller-box method, an efficient implicit finite difference scheme. Numerical results are presented in the form of heat transfer rates, local wall shear stress, velocity and temperature profiles. The heat transfer rates and local wall shear stress obtained show good agreement with available nonsimilar solutions. The effects of various values of transformed stream-wise coordinate ξ and Prandtl numbers on velocity and temperature profiles are also presented.

Keywords Free convection heat transfer, boundary layers, local nonsimilarity method, Keller-box method.

Abstrak *Suatu analisis dalam mengkaji ciri-ciri aliran dan pemindahan haba bagi masalah lapisan sempadan olakan bebas berlamina terhadap plat menegak di dalam bendalir likat dibincangkan dengan menggunakan kaedah ketakserupaan setempat. Kes apabila suhu permukaan berubah secara eksponen dipertimbangkan dalam mengilustrasikan kaedah tersebut. Dalam kaedah ini, sebutan ketakserupaan yang wujud pada persamaan momentum dan tenaga dikekalkan dan pemansuhan sebutan hanya dilakukan pada persamaan bantu yang diterbitkan. Sistem persaman yang diperolehi diselesaikan secara berangka menggunakan kaedah Keller-box. Kadar pemindahan haba dan tegasan ricih dinding yang diperolehi menunjukkan keputusan yang memuaskan. Kesan parameter* ξ *dan nombor Prandtl terhadap taburan halaju dan suhu juga ditunjukkan.*

Katakunci *Olakan bebas, lapisan sempadan, kaedah ketakserupaan setempat, kaedah Keller-box.*

1 Introduction

In many cases, when free convection takes place from a vertical flat plate, the temperature is not isothermal. Most of these problems do not admit similarity solutions, for example, the case of an exponential variation in surface temperature. In 1969, Gebhart and Mollendorf [3] had pointed out that the similarity solution obtained by Sparrow and Gregg [14], was, in fact, an asymptotic solution for x approaches infinity. Gryzagoridis [7] vindicated this in his experimental study. He showed that there was no unique solution, but rather a family of curves for the dimensionless temperature distribution if the data were evaluated in the manner suggested by Sparrow and Gregg [14]. Therefore, it appears that the theoretical solutions in [14] are doubtful. Consequently, Kao, et al. [11] and Yang, et al. [17] solved the problem using the method of strained coordinate and Merk-type series respectively.

Sparrow and his coworkers [15,16] then developed the local nonsimilarity method. This method has been widely used and found to be very efficient to deal with free convection problems. Among the problems solved using the method included that of Kao [10] who considered free convection from vertical plates with sinusoidal temperature variation and constant transpiration. Muntasser and Mulligan [13] solved the problem of free convection from the surface of a horizontal cylindrical with constant wall temperature. Hasan and Eichhorn [8] analyzed the effect of the angle of inclination on free convection flow from an isothermal surface. Chen and Ho [2] studied the effect of flow inertia on vertical free convection in saturated porous media while Gorla [5,6] considered the problem of steady state flow of laminar plane and cylindrical wall jet with nonisothermal wall.

In this paper, the local nonsimilarity method is used to solve free convection boundary layer flow over a vertical plate with an exponential variation in surface temperature. The numerical computation for the resulting nonsimilar system of equation is by using the Keller-box method as outlined in [1]. The present analysis shows that the solution is nonsimilar, as suggested by Yang, et al.[17] in contrast to an earlier suggestion by Sparrow and Gregg [14]. The surface heat transfer rate increases as the distance of the vertical plate from the leading edge is increased. On the other hand, the local wall shear stress decreases as the axial coordinate x increases. It is also shown that the velocity boundary layer is much thicker than the thermal boundary layer for large values of ξ and Prandtl numbers, Pr.

2 Governing Equations

We consider the steady laminar incompressible free convection boundary layer flows along a vertical flat plate. The coordinate system is shown in Figure 1. The x coordinate is measured from the leading edge of the plate and the y coordinate is measured normally from the plate to the fluid. The gravitational acceleration g acts downward. In the analysis to follow, the fluid properties are assumed to be constant except for the density variation that induces the buoyancy force. By employing the laminar boundary layer assumptions and the application of the Boussinesq approximations, the governing equations for the free convection from a vertical flat plate with exponential variation in surface temperature can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

with the boundary conditions

$$\begin{aligned} u = v = 0 \quad &\text{and} \quad T = T_w(x) = Ae^{mx} + T_\infty \quad \text{as} \quad y = 0, \\ u = 0 \quad &\text{and} \quad T = T_\infty \quad \text{as} \quad y \to \infty. \end{aligned} \tag{4}$$

In the above equations, u and v are the velocity components in the x and y direction respectively, T is the temperature, T_w is the wall surface temperature, T_∞ is the fluid temperature at a far distance from the surface, β is the thermal expansion coefficient, ν is the kinematic viscosity, α is the thermal diffusivity, A and m are constants.

In this paper, we consider the problem where the surface temperature $T_w(x)$ is greater than the ambient temperature T_∞ . Under this circumstance, the free convective motion of the fluid is upward along the plate, as shown in Figure 1. If the surface is colder than the ambient $T_w(x) < T_\infty$, the boundary layer profile remains the same but the direction is reversed, i.e. the fluid flow is downward [17].

Figure 1: Physical coordinat system

Following [10,11,17], equations (1) to (3) are transformed from the (x, y) coordinates to the dimensionless coordinates (ξ, η) by introducing the variables

$$\xi(x) = \int_0^x (T_w(x) - T_\infty)\, dx \tag{5}$$

$$\eta(x, y) = \frac{C_1(T_w(x) - T_\infty)^{1/2} y}{\xi^{1/4}} \tag{6}$$

$$f(\xi, \eta) = \frac{(T_w(x) - T_\infty)^{1/2}\psi}{4C_1\nu\xi^{3/4}} \tag{7}$$

$$\theta(\xi, \eta) = \frac{T - T_\infty}{T_w(x) - T_\infty} \tag{8}$$

where

$$C_1 = \left(\frac{g\beta}{4\nu^2}\right)^{1/4} \tag{9}$$

and ψ is the stream function defined in the usual way as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{10}$$

In the above equations, $\xi(x)$ is the transformed stream-wise coordinate, $\eta(x, y)$ is the transformed normal coordinate, $f(\xi, \eta)$ is the non-dimensional stream function and $\theta(\xi, \eta)$ is the nondimensional temperature.

Substituting equations (5) to (10) into equations (1) to (3) with the boundary conditions $T_w(x) - T_\infty = Ae^{mx}$, we obtain the following system of equations:

$$f''' + (3 - 2B)ff'' - 2f'^2 + \theta = 4\xi\left(f'\partial f'/\partial\xi - f''\partial f/\partial\xi\right) \tag{11}$$

$$\theta''/\mathrm{Pr} + (3 - 2B)f\theta' - 4Bf'\theta = 4\xi\left(f'\partial\theta/\partial\xi - \theta'\partial f/\partial\xi\right) \tag{12}$$

where primes denote derivatives with respect to η, Pr is the Prandtl number, and

$$B = 1 - e^{-mx}. \tag{13}$$

The boundary conditions (4) now become:

$$f'(\xi, 0) = 0, \quad \theta(\xi, 0) = 1, \quad f'(\xi, \infty) = 0, \quad \theta(\xi, \infty) = 0.$$

3 The Local Nonsimilarity Solution

Before proceeding to the local nonsimilarity method, it is useful to examine equations (11) and (12) from the standpoint of local similarity. The local similarity solutions give first order estimates of the heat and mass transfer rates in boundary layers [9]. Therefore, the local similarity solution is also known as the first level truncation [4]. By this concept, the right-hand sides of these equations are assumed to be small and thus can be neglected. The equations on the left sides are treated as ordinary differential equations and solved with B as a parameter. Thus,

$$f''' + (3 - 2B)ff'' - 2f'^2 + \theta = 0 \tag{14}$$

$$\theta''/\mathrm{Pr} + (3 - 2B)f\theta' - 4Bf'\theta = 0 \tag{15}$$

Turning to the rationale of the local similarity model, the reduction of equations (11) and (12) is clearly justifiable for values of ξ that are very close to zero. On the other hand, when ξ is not small, local similarity is based on the postulate that derivatives involving ξ are very small [15]. Uncertainty on whether to neglect the right-hand side of equation or not is the weakness of this local similarity concept.

To correct the drawbacks of the local similarity method, Sparrow and co-workers [15,16] presented a local non-similarity method in obtaining the solutions for the non-similar boundary layer equations. Accordingly, the terms on the right-hand sides of equations (11) and (12) are all retained. Auxiliary differential equations are introduced to approximate them

[10]. These auxiliary equations are obtained simply by differentiating equations (11) and (12) with respect to ξ and defining the new dependent variables as

$$g = \frac{\partial f}{\partial \xi}, \quad \varphi = \frac{\partial \theta}{\partial \xi}, \quad g' = \frac{\partial f'}{\partial \xi} \quad \text{and so on.}$$

To close the system of equations at this second level, the terms involving $\frac{\partial g}{\partial \xi}$ and $\frac{\partial \varphi}{\partial \xi}$ or higher order are neglected. Hence we obtain

$$g''' + (3-2B)fg'' + (7-2B)f''g - 8f'g' + \varphi - 2\frac{dB}{d\xi}ff'' = 4\xi\left(g'^2 - gg''\right), \quad (16)$$

$$\begin{aligned} \varphi''/\mathrm{Pr} + (3-2B)f\varphi' &- 4(1+B)f'\varphi + (7-2B)g\theta' - 4B\theta g' \\ &- \frac{dB}{d\xi}(4f'\theta + 2f\theta') = 4\xi(\varphi g' - g\varphi'), \end{aligned} \quad (17)$$

with the following boundary conditions:

$$g'(\xi,0) = 0, \quad \varphi(\xi,0) = 1, \quad g'(\xi,\infty) = 0, \quad \varphi(\xi,\infty) = 0.$$

4 Numerical Method

The systems of governing equations for the local similarity models (14) and (15), and the local nonsimilarity two-equation models (11), (12), (16) and (17) can be solved simultaneously using an implicit finite difference scheme due to Keller. This method is described by Cebeci and Bradshaw [1] and is known as the Keller-box method.

In this approach, the partial differential equations are first reduced to a system of first-order equations which are then expressed in finite difference forms using simple 'centered-difference' derivatives and averages at the midpoints of net rectangle (ξ, η). This system of equations is nonlinear algebraic equations and it can be linearized using Newton's method. The resulting system of equations is solved along with their boundary conditions by the block-tridiagonal factorization scheme.

The solutions start with $\xi = 0$, with a proper step size $\triangle\eta$, for the interval $0 \le \eta \le \eta_\infty$ by iteration and then proceed to the $\xi > 0$ location with a proper step size $\triangle\xi$. A solution was considered to converge when the difference between the input and output values of the $f''(\xi, 0)$ came within 10^{-5}, respectively. After obtaining a converged solution, the computation was continued by marching in the ξ direction. In the numerical computation, the effects of the step size $\triangle\eta$ and the boundary layer thickness η_∞ on the numerical results as well as on the convergence of the solutions were examined. It was found that sufficiently accurate numerical results were obtained with a step size of $\triangle\eta = 0.04$, $\triangle\xi = 0.02$ and a value of η_∞ between 6 and 8. For further discussions of this method, please refer to [1].

5 Results and Discussion

The physical quantity of interest includes the local Nusselt number, Nu_x that represents the surface heat transfer rate. For the case of exponential variation in surface temperature,

$$Nu_x = \frac{\theta'(\xi,0)}{2^{1/2}} Gr_x^{1/4} \left[(T_w(x) - T_\infty)x \Big/ \int_0^x (T_w(x) - T_\infty)\, dx \right]^{1/4} \quad (18)$$

where Gr_x is the so-called Grashof number, which may be interpreted physically as a dimensionless group representing the ratio of the buoyancy force to the viscous force acting on the fluid. The local wall shear stress in terms of $f''(\xi,0)$ is

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_{\eta=0} = 4\mu C_1^3 \xi^{1/4}\left(T_w(x) - T_\infty\right)^{1/2} f''(\xi,0) \tag{19}$$

where μ is the dynamic viscosity and C_1 is the constant defined in equation (9). Lok [12] contains details on how to obtain the expressions (18) and (19) from the variables that were introduced in equations (5) to (10).

Numerical solutions are obtained for a Prandtl number 0.7 with A and m having the value unity. The surface heat transfer rates in term of the ratios of the local Nusselt number Nu_x and Grashof number, Gr_x in equation (18) are plotted in Figure 2. Also shown are the results from the Merk-type series [17], local similarity model and from Sparrow and Gregg's similarity solution [14]. In order to verify the accuracy of the present method, the results are also presented in Table 1 and the deviations between the local nonsimilarity method and the Merk-type series are also calculated.

Table 1: The numerical results of the surface heat transfer rates for Pr 0.7 and $A = m = 1$

		$\theta'(\xi,0)$		
B	x	**Merk-type series [17]**	**Present method**	**Deviation (%)**
0.00	0.00	-0.4995	-0.4995	0.0000
0.17	0.186	-0.5382	-0.5372	0.1858
0.33	0.401	-0.5750	-0.5723	0.4696
0.50	0.693	-0.6146	-0.6104	0.6834
0.67	1.109	-0.6551	-0.6502	0.7480
0.75	1.386	-0.6743	-0.6690	0.7860
0.80	1.609	-0.6864	-0.6812	0.7576
0.84	1.833	-0.6960	-0.6911	0.7040
0.88	2.120	-0.7057	-0.7013	0.6235

As seen from Figure 2, all three solutions show close agreement near the leading edge. However, the local similarity model starts to diverge for $x > 0.2$. On the other hand, the results from the local nonsimilarity method and Merk-type series are in good agreement over the entire range of x that are considered in the analysis. The largest deviation between these two models is 0.786 percent as shown in Table 1. From equation (13), we find that x must satisfy the equation $x = -\frac{1}{m}\ln(1-B)$. Therefore, the values of the parameter x presented here are limited because the values of B are limited to $0 \leq B < 1$.

The surface heat transfer rate increases as the distance from the leading edge of the vertical plate is increased. This can be seen from Figure 2, where the local Nusselt number increases with increasing values of x. Figure 3 shows the local wall shear stress in terms of equation (19) with increasing values of x. As can be seen in Figure 3, the local wall shear stress decreases as the values of x increases.

Representative velocity and temperature profiles are shown, respectively, in Figures 4 and 5 for the case $m = 1$ and $\mathrm{Pr} = 0.7$. From the figures, one can see that the dimensionless momentum boundary layer thickness increases as the parameter x (or ξ) increases. It can be explained as follows: the η coordinate in equation (6) is dependent on the values of x and ξ. The values of $e^{x/2}$ may be substantially much larger than $\xi^{\frac{1}{4}}$ and these act to increase the range of η. However, from Figure 4, the values of ξ (or x) are seen to affect any significant change in the thickness of thermal boundary layers.

Typical curves showing the relationship between the velocity profiles for different Prandtl numbers are shown in Figure 6 for and $m = 1$ and $\xi = 0$ while Figure 7 shows the relative orientation of the temperature profiles for Pr = 0.7, 1.0 and 7.0. The maximum values of the dimensionless velocity distributions occur at large values of the argument η as the Prandtl number decreases and the velocities decrease with increasing Pr. For $\mathrm{Pr} \gg 1$ the velocity boundary layer is much thicker than the thermal boundary layer.

6 Conclusion

In this paper, the problem of laminar free convection over vertical plates with exponential wall temperature has been analyzed using the local nonsimilarity method. In this approach, the nonsimilarity terms appearing in the conservation equations are retained and they are selectively deleted only in the subsidiary equations. These have been unaccounted for previously, in for example the similarity and local similarity method.

The heat transfer rate and local wall shear stress presented herein demonstrate that the present method of solution can provide a very good approximation to the nonsimilar free convection problems especially for small values of the transformed stream-wise coordinate ξ (or x). The increased values of ξ (or x) will increase the thickness of the velocity boundary layer but give no significant effect on the thickness of the thermal boundary layer. The velocity and temperature profiles are also shown to be strongly dependent on the Prandtl number.

Acknowledgement The authors gratefully acknowledge the financial support from UTM Fellowship Awards (Industrial and Technology Development).

References

[1] T. Cebeci, & P. Bradshaw, *Physical and Computational Aspects of Convective Heat Transfer*, Springer-Verlag, New York, 1988, Chapter 13.

[2] K.S. Chen & J.R. Ho, *Effects of Flow Inertia on Vertical, Natural Convection in Saturated, Porous Media*, International Journal of Heat and Mass Transfer, 29(1986), 753-758.

[3] B. Gebhart & J. Mollendorf, *Viscous Dissipation in External Natural Convection Flows*, Journal of Fluid Mechanics, 38(1969), 97-107.

[4] R.S.R. Gorla, *Non-Newtonian Nonsimilar boundary layers*, Canadian Journal of Chemical Engineering, 53(1975), 563-567.

[5] R.S.R. Gorla, *Heat Transfer in a Laminar Cylindrical Wall Jet with Uniform Surface Heat Flux*, Chemical Engineering Communication, 130(1994), 167-179.

[6] R.S.R. Gorla, *Nonsimilar Solutions for Heat Transfer in Wall Jet Flows*, Chemical Engineering Communication, 140(1996), 139-156.

[7] J. Gryzagoridis, *Temperature Profiles of a Nonisothermal Vertical Flat Plate in Free Convection*, Letters in Heat and Mass Transfer, 3(1976), 343-353.

[8] M.M. Hasan & R. Eichhorn, *Local Nonsimilarity Solution of Free Convection Flow and Heat Transfer from an Inclined Isothermal Plate*, Trans. ASME, Journal of Heat Transfer, 101(1979), 642-647.

[9] M. Hasan & A.S. Mujumdar, *Simultaneous Heat and Mass Transfer in Free Convection from a Horizontal Cylinder*, AIAA Journal, 23(1985), 1602-1608.

[10] T.T. Kao, *Locally Nonsimilar Solution for Laminar Free Convection Adjacent to a Vertical Wall*, Trans, ASME, Journal of Heat Transfer, 98(1976), 321-322.

[11] T.T. Kao, G.A. Domoto & Jr.H.G. Elrod, *Free Convection along a Nonisothermal Vertical Flat Plate*, Trans. ASME, Journal of Heat Transfer, 99(1977), 72-78.

[12] Y.Y. Lok, *Nonisothermal Free Convection Boundary Layers over Vertical Flat Plates*, M. Sc. Thesis, Universiti Teknologi Malaysia (2002).

[13] M.A. Muntasser & J.C. Mulligan, *A Local Nonsimilarity Analysis of Free Convection from a Horizontal Cylindrical Surface*, Trans. ASME, Journal of Heat Transfer, 100(1978), 165-167.

[14] E.M. Sparrow & J.L.Gregg, *Similar Solutions for Laminar Free Convection from a Nonisothermal Vertical Plate*, Trans. ASME, Journal of Heat Transfer, 80(1958), 379-387.

[15] E.M. Sparrow, H. Quack & C.J. Boerner, *Local Nonsimilarity Boundary-Layer Solutions*, AIAA Journal, 8(1970), 1936-1942.

[16] E.M. Sparrow & H.S. Yu, *Local Nonsimilarity Thermal Boundary-Layer Solutions*, Trans. ASME, Journal of Heat Transfer, 93(1971), 328-334.

[17] J. Yang, D.R. Jeng & K.J. DeWitt, *Laminar Free Convection from a Vertical Plate with Nonuniform Surface Conditions*, Numerical Heat Transfer, 5(1982), 165-184.