LOCAL PROJECTION STABILIZATION OF EQUAL ORDER INTERPOLATION APPLIED TO THE STOKES PROBLEM

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ABSTRACT. The local projection stabilization allows us to circumvent the Babuška–Brezzi condition and to use equal order interpolation for discretizing the Stokes problem. The projection is usually done in a two-level approach by projecting the pressure gradient onto a discontinuous finite element space living on a patch of elements. We propose a new local projection stabilization method based on (possibly) enriched finite element spaces and discontinuous projection spaces defined on the same mesh. Optimal order of convergence is shown for pairs of approximation and projection spaces satisfying a certain inf-sup condition. Examples are enriched simplicial finite elements and standard quadrilateral/hexahedral elements. The new approach overcomes the problem of an increasing discretization stencil and, thus, is simple to implement in existing computer codes. Numerical tests confirm the theoretical convergence results which are robust with respect to the user-chosen stabilization parameter.

1. INTRODUCTION

The standard Galerkin approach for solving the Stokes problem by finite element discretizations is well established [9, 17]. If the finite element spaces approximating velocity and pressure satisfy an inf-sup condition, stability and convergence of the discretization can be proven. Applying this technique on adaptive meshes with hanging nodes, the additional constraints to guarantee the conformity of the velocity space reduce its dimension and it is not a priori clear whether the inf-sup condition remains valid for the considered pair of finite element spaces. Indeed, it is far from being trivial that the stability property proven on a family of regular meshes remains true when considering adaptive grids with hanging nodes [27].

An alternative approach for solving the Stokes problem consists of using equal order interpolations which are simple to implement since the same finite element space is used for approximating the pressure and the velocity components. However, we have to take into consideration that equal order interpolation does not satisfy the inf-sup stability condition and may produce oscillations in the pressure. Starting with [19], a methology has been developed to modify the discrete problem such that the correponding bilinear form is coercive on the product space. In this way stability is achieved for arbitrary pairs of finite elements including equal order interpolations on meshes with hanging nodes. Stability has been generated by adding weighted residuals of the strong form of the differential equation to the discrete problem. This pressure stabilized Petrov–Galerkin (PSPG) formulation has

been successfully developed in different ways. Combined with the streamline upwind Petrov Galerkin (SUPG) stabilization, it has been used to treat two instability phenomena of the Navier–Stokes equations in a unique framework: the dominating advection for higher Reynolds numbers and the violation of the discrete inf-sup condition. The general idea behind this class of *residual based stabilization technique* is to enhance stability without upsetting consistency [14, 30]. Moreover, it has been shown in [15, 18, 29, 30] that an additional least square term for the incompressibility condition, called grad-div stabilization, improves the robustness. Properties of the grad-div stabilization onto the algebraic solver are discussed in [26].

Another approach is the *projection based stabilization technique.* The stabilization by projecting the pressure gradient onto a continuous finite element space has been analyzed in [13]. This method has been shown to be still consistent in the sense that a smooth exact solution satisfies the discrete problem. However, the method is quite expensive due to the nonlocal behavior of the projection. Alternatively, a two-level approach with a projection onto a discontinuous finite element space living on patches of elements has been analyzed in [4]. This more local method is no longer consistent in a strong manner but the consistency error can be bounded by a rate which does not reduce the optimal order of convergence. The technique of local projection has been investigated to stabilize dominating advection for transport equations in [5] and extended to the Oseen equation in [7]. A drawback of the two-level approach from the implementation point of view is the fact that the added stabilizing term produces a larger stencil which may not fit in the data structure of an available programming code. As pointed out in [23], the key point of stabilization by local projection is the existence of a special interpolant with additional orthogonality properties with respect to the projection space. This general view opens the possibility to construct stabilized methods for enriched equal order finite element spaces by local projection onto a discontinuous finite element space living on the same mesh. In particular, the added stabilizing term does not produce a larger stencil in contrast to the two-level approach.

From the implementational point of view, equal order interpolations allow us to work with only one finite element space, but additional stabilization terms have to be assembled. Note that the PSPG approach includes more stabilizing terms compared to the local projection method.

The main objective of this paper is to adapt this new approach to the Stokes problem, to give examples of solution and projection spaces allowing stability and optimal convergence, and to demonstrate the potential of the method by a number of selected numerical tests. The remaining part of the paper is organized as follows. In Section 2, a weak formulation of the Stokes problem and its standard Galerkin discretization is given. We formulate the local projection method for arbitrary equal order interpolations satisfying a certain compatibility condition with respect to the projection space. Section 3 is devoted to optimal a-priori estimates related to an asymptotically optimal choice of the stabilization parameter. In Section 4, we consider examples of enriched finite element spaces satisfying the compatibility condition. In particular, we show the equivalence of the projection based stabilization method of lowest order enriched finite elements on simplices with the PSPG/grad-div stabilization of the unenriched spaces. Finally, the properties of the method are illustrated by numerical tests in Section 5.

Notation. Throughout this paper, C will denote a generic positive constant which is independent of the mesh. Subscripted constants such as C_1 are also independent of the mesh, but have a fixed value. We will write shortly $\alpha \sim \beta$, if there are positive constants $\underline{C}$ and $\overline{C}$ such that

$$\underline{C}\beta \le \alpha \le \overline{C}\beta$$

holds. The Stokes problem will be considered in the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which is assumed to be a polygonal or polyhedral domain with boundary $\partial\Omega$. For a measurable subset G of Ω, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\| \cdot \|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p = 2$, we have $H^m(G) = W^{m,2}(G)$ and the index p will be omitted. The L^2 inner product on G is denoted by $(\cdot, \cdot)_G$. Note that the index G will be omitted for $G = \Omega$. This notation of norms, semi-norms, and inner products is also used for the vector-valued and tensor-valued case.

2. STOKES PROBLEM AND ITS DISCRETIZATION

2.1. Weak formulation.
Let $\Omega \subset \mathbb{R}^d$ be a polygonal $(d = 2)$ or polyhedral $(d = 3)$ domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We consider the Stokes problem

$$(2.1) \qquad \left.\begin{aligned} -\triangle u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}\right\}$$

where f is a given body force, u and p denote the velocity and pressure field, respectively. Introducing the spaces $V := \left(H_0^1(\Omega)\right)^d$ and $Q := L_0^2(\Omega)$, a weak formulation of (2.1) reads:

Find $(u, p) \in V \times Q$ such that

$$(2.2) \qquad (\nabla u, \nabla v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) = (f, v) \qquad \forall (v, q) \in V \times Q.$$

The Lax–Milgram theorem applied to the subspace of divergence-free functions and the inf-sup condition

$$(2.3) \qquad \inf_{q \in Q} \sup_{v \in V} \frac{(q, \operatorname{div} v)}{\|q\|_0 \, |v|_1} > 0,$$

guarantee that there is a unique solution of (2.2); see [17].

2.2. Discrete problem and stabilized formulation.
We are given a family $\{\mathcal{T}_h\}$ of shape-regular decompositions of Ω into d-simplices, quadrilaterals, or hexahedra. The diameter of a cell K is denoted by h_K. The mesh parameter h describes the maximum diameter of the cells $K \in \mathcal{T}_h$.

Let Y_h be a scalar finite element space of continuous, piecewise polynomials functions over $\mathcal{T}_h$. The spaces for approximating the velocity and the pressure are given by $V_h := Y_h^d \cap V$ and $Q_h := Y_h \cap Q$, respectively.

The standard Galerkin discretization reads:

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$,

$$(2.4) \qquad (\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) + (q_h, \operatorname{div} u_h) = (f, v_h).$$

It is known that, in general, equal order interpolations do not satisfy the discrete counterpart of (2.3), i.e., the Babuška–Brezzi condition [9]

$$(2.5) \qquad \exists \beta_0 > 0, \ \forall h : \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(q_h, \operatorname{div} v_h)}{\|q_h\|_0 \, |v_h|_1} \geq \beta_0$$

is violated. Therefore, we intend to add to (2.4) a stabilizing term based on local projection. To this end, we need further notation. Let $P_s(K)$, $s = 0, 1, \ldots$, denote the set of all polynomials of degree less than or equal to s and let $D_h(K)$ be a finite dimensional function space on the cell $K \in \mathcal{T}_h$ with $P_s(K) \subset D_h(K)$. We extend the definition by allowing $D_h(K) = \{0\}$ together with $P_{-1}(K) = D_h(K)$. We introduce the associated global space of discontinuous finite elements

$$D_h := \bigoplus_{K \in \mathcal{T}_h} D_h(K)$$

and the local $L^2(K)$-projection $\pi_K : L^2(K) \to D_h(K)$ which generates the global projection $\pi_h : L^2(\Omega) \to D_h$ by

$$(\pi_h w)|_K := \pi_K(w|_K) \qquad \forall K \in \mathcal{T}_h, \ \forall w \in L^2(\Omega).$$

The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ is given by $\kappa_h := id - \pi_h$ where $id : L^2(\Omega) \to L^2(\Omega)$ is the identity on $L^2(\Omega)$. We will have a closer look at the case $D_h(K) = \{0\}$. The inclusion $P_s(K) \subset D_h(K)$ is satisfied for $s = -1$. Moreover, since $\pi_K w = 0$ for all $w \in L^2(K)$, the fluctuation operator κ_h degenerates to the identity. For simplicity of notation, we will not distinguish between the scalar operators id, π_K, π_h, κ_h acting on scalar functions and their vector-valued versions. Thus, $\kappa_h \nabla p$ has to be understood as acting on each component of ∇p separately.

Now we are able to introduce the stabilizing term

$$(2.6) \qquad S_h(p_h, q_h) := \sum_{K \in \mathcal{T}_h} \alpha_K \big(\kappa_h \nabla p_h, \kappa_h \nabla q_h \big)_K$$

where α_K are parameters to be chosen. Our stabilized discrete problem reads:

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$,

$$(2.7) \qquad (\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) + (q_h, \operatorname{div} u_h) + S_h(p_h, q_h) = (f, v_h).$$

In order to study the properties of (2.7) on the product space $V_h \times Q_h$, we introduce the bilinear form

$$(2.8) \qquad A_h\big((u, p); (v, q)\big) := (\nabla u, \nabla v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) + S_h(p, q)$$

and the mesh-dependent norm

$$(2.9) \qquad |||(v, q)||| := \left(|v|_1^2 + \|q\|_0^2 + \sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla q\|_{0, K}^2 \right)^{1/2}.$$

3. Stability and convergence

We are interested in methods of a fixed order $r \in \mathbb{N}$. In order to guarantee stability and convergence of our method (2.7), we suppose that following assumptions are satisfied.

Assumption A1. For the fluctuation operator κ_h, the approximation property

$$\|\kappa_h w\|_{0,K} \le C h_K^{\ell} |w|_{\ell,K} \qquad \forall w \in H^{\ell}(K), \forall K \in \mathcal{T}_h, \, 0 \le \ell \le r - 1$$

holds.

Note that Assumption A1 is fulfilled in the case $P_{r-2}(K) \subset D_h(K)$ for any $r \in \mathbb{N}$. We have for $D_h(K) = \{0\}$ by definition $P_{-1}(K) \subset D_h(K)$ and

$$\|\kappa_h w\|_{0,K} = \|\text{id } w\|_{0,K} = \|w\|_{0,K} \quad \forall w \in L^2(K), \forall K \in \mathcal{T}_h.$$

In the case $\dim(D_h(K)) \ge 1$, Assumption A1 follows from the Bramble–Hilbert lemma.

Assumption A2. The finite element space Y_h approximates $H^1(\Omega)$ of order r, i.e., there exists an interpolation operator $i_h : H^1(\Omega) \to Y_h$ such that $i_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$ and the error estimate

$$\|v - i_h v\|_{0,K} + h_K |v - i_h v|_{1,K} \le C h_K^{\ell} \|v\|_{\ell, \omega(K)}$$

holds for all $v \in H^{\ell}\big(\omega(K)\big)$, for all $K \in \mathcal{T}_h$, and $1 \le \ell \le r + 1$ where $\omega(K)$ denotes a certain local neighborhood of K which appears in the definition of these interpolation operators for non-smooth functions; see [12, 28] for more details.

The existence theory of interpolation operators for non-smooth functions satisfying Assumption A2 is well established in the literature; see [1, 12, 28].

Assumption A3. There exists an interpolation operator $j_h : H^1(\Omega) \to Y_h$ with $j_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$, the orthogonality property

$$(3.1) \qquad (w - j_h w, q_h) = 0 \qquad \forall q_h \in D_h, \forall w \in H^1(\Omega),$$

and the approximation property

$$(3.2) \qquad \|w - j_h w\|_{0,K} + h|w - j_h w|_{1,K} \le C h_K^{\ell} \|w\|_{\ell, \omega(K)},$$

for all $w \in H^{\ell}(\omega(K))$, $1 \le \ell \le r + 1$, where $\omega(K)$ denotes again a certain local neighborhood of K.

Note that (3.1) holds for $D_h(K) = \{0\}$. For $\dim D_h(K) \ge 1$, the existence of an interpolation operator satisfying the additional orthogonality property (3.1) has been investigated in [23]. It turns out that the Assumption A2 together with the local inf-sup condition

$$(3.3) \quad \exists \beta_1 > 0 \quad \forall h > 0, \forall K \in \mathcal{T}_h : \quad \inf_{q_h \in D_h(K)} \sup_{v_h \in Y_h(K)} \frac{(v_h, q_h)_K}{\|v_h\|_{0,K} \|q_h\|_{0,K}} \ge \beta_1$$

for the spaces $D_h(K)$ and $Y_h(K)$ are sufficient for A3; see [23, Theorem 2.2]. Here, $Y_h(K)$ is the restriction of functions from Y_h with support in $\overline{K}$, i.e.,

$$Y_h(K) := \{w_h|_K : w_h \in Y_h, \, w_h = 0 \text{ on } \Omega \backslash K\}.$$

The local inf-sup condition will be fulfilled if $Y_h(K)$ is rich enough compared to $D_h(K)$. Special pairs of spaces $Y_h(K)$ and $D_h(K)$ satisfying the local inf-sup condition will be studied in Section 4.

Lemma 3.1. *Let the Assumptions* A1, A3, *and* $h_K^2/\alpha_K \le C$ *be fulfilled. Then, there exists a positive constant* β *independent of* h *such that*

$$(3.4) \qquad \inf_{(v_h, q_h) \in V_h \times Q_h} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h\big((v_h, q_h); (w_h, r_h)\big)}{|||(v_h, q_h)||| \, |||(w_h, r_h)|||} \ge \beta$$

holds.

Proof. Let (v_h, q_h) be an arbitrary element of $V_h \times Q_h$. We obtain

$$(3.5) \qquad A_h\big((v_h, q_h); (v_h, q_h)\big) = |v_h|_1^2 + \sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2.$$

Furthermore, due to the continuous inf-sup condition (2.3), there exists for any $q_h \in Q_h$ an element $v_{q_h} \in V$ satisfying

$$-(q_h, \operatorname{div} v_{q_h}) = \|q_h\|_0^2, \qquad \|v_{q_h}\|_1 \le C\|q_h\|_0.$$

Consequently, we have for the interpolant $j_h v_{q_h}$,

$$
\begin{aligned}
A_h\big((v_h, q_h); (j_h v_{q_h}, 0)\big) &= -(q_h, \operatorname{div} j_h v_{q_h}) + (\nabla v_h, \nabla j_h v_{q_h}) \\
(3.6) \qquad &= \|q_h\|_0^2 + (q_h, \operatorname{div} (v_{q_h} - j_h v_{q_h})) + (\nabla v_h, \nabla j_h v_{q_h}).
\end{aligned}
$$

Using integration by parts, the orthogonality property (3.1), and the assumption $h_K^2/\alpha_K \le C$, we estimate the second term in (3.6) as follows:

$$
\begin{aligned}
\big|(q_h, \operatorname{div} (v_{q_h} - j_h v_{q_h}))\big| &= |(\nabla q_h, v_{q_h} - j_h v_{q_h})| = |(\kappa_h \nabla q_h, v_{q_h} - j_h v_{q_h})| \\
&\le C \left(\sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2\right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^2}{\alpha_K} |v_{q_h}|_{1,K}^2\right)^{1/2} \\
&\le C_1 \left(\sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2\right)^{1/2} \|q_h\|_0 \\
&\le \frac{\|q_h\|_0^2}{4} + C_1^2 \sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2.
\end{aligned}
$$

The last term in (3.6) is estimated by

$$|(\nabla v_h, \nabla j_h v_{q_h})| \le |v_h|_1 \, |j_h v_{q_h}|_1 \le C_2 |v_h|_1 \, \|q_h\|_0 \le \frac{\|q_h\|_0^2}{4} + C_2^2 \, |v_h|_1^2.$$

Summarizing the estimates of the individual terms, we end up with

$$A_h\big((v_h, q_h); (j_h v_{q_h}, 0)\big) \ge \frac{\|q_h\|_0^2}{2} - C_3 \left(|v_h|_1^2 + \sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla q_h\|_{0,K}^2\right)$$

where $C_3 = \max(C_1^2, C_2^2)$. Multiplying this inequality by $2/(1 + 2C_3)$ and adding it to (3.5) we conclude that for any $(v_h, q_h) \in V_h \times Q_h$ there is $(w_h, r_h) := (v_h, q_h) + 2/(1 + 2C_3)(j_h v_{q_h}, 0)$ such that

$$A_h\big((v_h, q_h); (w_h, r_h)\big) \ge \frac{1}{1 + 2C_3} |||(v_h, q_h)|||^2.$$

Furthermore, the H^1-stability of the interpolation j_h and the definition of v_{q_h} imply

$$
\begin{aligned}
|||(w_h, r_h)||| &\le |||(v_h, q_h)||| + \frac{2}{1 + 2C_3} \, |j_h v_{q_h}|_1 \le |||(v_h, q_h)||| + C_4 \|q_h\|_0 \\
&\le (1 + C_4)|||(v_h, q_h)|||.
\end{aligned}
$$

Thus, the lemma holds with $\beta = ((1 + 2C_3)(1 + C_4))^{-1}$. $\qquad\qquad\square$

Note, that Lemma 3.1 implies the unique solvability of the stabilized discrete problem.

Now we will consider the consistency error.

Lemma 3.2. *Assume that the fluctuation operator κ_h satisfies Assumption* A1. *Let $(u,p) \in V \times (Q \cap H^r(\Omega))$ be the solution of the Stokes problem* (2.2) *and $(u_h, p_h) \in V_h \times Q_h$ the solution of the stabilized problem* (2.7). *Then, the consistency error can be estimated by*

$$\left| A_h\big((u-u_h, p-p_h); (v_h, q_h)\big) \right| \leq C \left(\sum_{K \in \mathcal{T}_h} \alpha_K h_K^{2r-2} |p|_{r,K}^2 \right)^{1/2} |||(v_h, q_h)|||$$

for all $(v_h, q_h) \in V_h \times Q_h$.

Proof. Using (2.8), (2.2), and (2.7), we obtain

$$A_h\big((u-u_h, p-p_h); (v_h, q_h)\big) = S_h(p, q_h).$$

Using the approximation property of the fluctuation operator κ_h,

$$\|\kappa_h \nabla p\|_{0,K} \leq C\, h_K^{r-1} |\nabla p|_{r-1,K} \leq C\, h_K^{r-1} |p|_{r,K}$$

and Cauchy–Schwarz inequality, we end up with the statement of the lemma. $\qquad\square$

Theorem 3.3. *Assume* A1, A2, A3, *and $\alpha_K \sim h_K^2$. Let the solution of* (2.2) *satisfy $(u,p) \in \left(V \cap H^{r+1}(\Omega)^d\right) \times \left(Q \cap H^r(\Omega)\right)$ and let $(u_h, p_h) \in V_h \times Q_h$ be the solution of* (2.7). *Then, there exists a positive constant C independent of h such that the error estimate*

$$|||(u-u_h, p-p_h)||| \leq C\, h^r \left(\|u\|_{r+1} + \|p\|_r \right)$$

holds true. Moreover, if the Stokes problem is $H^2(\Omega)^d \times H^1(\Omega)$-regular, there exists a positive constant C independent of h such that

$$\|u - u_h\|_0 \leq C\, h^{r+1} \left(\|u\|_{r+1} + \|p\|_r \right)$$

holds.

Proof. Using Lemma 3.1, we can estimate

$$|||(j_h u - u_h, i_h p - p_h)||| \leq \frac{1}{\beta} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h\big((j_h u - u_h, i_h p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||}$$

$$\leq \frac{1}{\beta} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h\big((u - u_h, p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||}$$

$$+ \frac{1}{\beta} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h\big((j_h u - u, i_h p - p); (w_h, r_h)\big)}{|||(w_h, r_h)|||}.$$

The consistency estimate of Lemma 3.2 and the choice of α_K yield

$$\sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h\big((u - u_h, p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||} \leq C\, h^r \|p\|_r.$$

The terms in $A_h\big((j_h u - u, i_h p - p); (w_h, r_h)\big)$ will be estimated individually. Employing the approximation properties of j_h and i_h, we get

$$\left| \big(\nabla(j_h u - u), \nabla w_h\big) \right| \leq |j_h u - u|_1 \, |w_h|_1 \leq C\, h^r \|u\|_{r+1} \, |||(w_h, r_h)|||,$$

$$\left| \big(i_h p - p, \mathrm{div}\, w_h\big) \right| \leq C\, \|i_h p - p\|_0 \, |w_h|_1 \leq C\, h^r \|p\|_r \, |||(w_h, r_h)|||.$$

For estimating the next term, we integrate by parts, apply the orthogonality property (3.1) of j_h, use $\alpha_K \sim h_K^2$, and obtain

$$\left| \left(r_h, \text{div } (j_h u - u) \right) \right| = \left| \left(\nabla r_h, j_h u - u \right) \right| = \left| \left(\kappa_h \nabla r_h, j_h u - u \right) \right|$$

$$\leq C \left(\sum_{K \in \mathcal{T}_h} \alpha_K \| \kappa_h \nabla r_h \|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^2}{\alpha_K} h_K^{2r} \| u \|_{r+1, \omega(K)}^2 \right)^{1/2}$$

$$\leq C \, h^r \| u \|_{r+1} \, |||(w_h, r_h)|||.$$

The stabilizing term is estimated by taking into consideration Cauchy–Schwarz inequality, the L^2-stability of the fluctuation operator κ_h, the approximation properties of i_h, and $\alpha_K \sim h_K^2$:

$$S_h(i_h p - p, r_h) \leq \left(\sum_{K \in \mathcal{T}_h} \alpha_K \| \kappa_h \nabla(i_h p - p) \|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \alpha_K \| \kappa_h \nabla r_h \|_{0,K}^2 \right)^{1/2}$$

$$\leq C \left(\sum_{K \in \mathcal{T}_h} \alpha_K h_K^{2r-2} \| p \|_{r, \omega(K)}^2 \right)^{1/2} |||(w_h, r_h)|||$$

$$\leq C \, h^r \| p \|_r \, |||(w_h, r_h)|||.$$

Putting the above estimates together, we obtain

$$|||(j_h u - u_h, i_h p - p_h)||| \leq C \, h^r \left(\| u \|_{r+1} + \| p \|_r \right).$$

Now, the triangle inequality and the estimate of the interpolation error with respect to $||| \cdot |||$ prove the first error estimate.

Under the assumption that the Stokes problem is $H^2(\Omega)^d \times H^1(\Omega)$-regular, we know that for any $g \in L^2(\Omega)^d$ there is a unique solution $(u_g, p_g) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$ of the problem.

Find $(u_g, p_g) \in V \times Q$ such that

$$(\nabla v, \nabla u_g) - (p_g, \text{div } v) + (q, \text{div } u_g) = (g, v) \qquad \forall (v, q) \in V \times Q,$$

satisfying

$$\| u_g \|_2 + \| p_g \|_1 \leq C \, \| g \|_0.$$

Setting $v = u - u_h$ and $q = 0$, we get the representation

$$(3.7) \qquad (g, u - u_h) = (\nabla(u - u_h), \nabla u_g) - (p_g, \text{div } (u - u_h)).$$

It follows from (2.2) and (2.7) that

$$(\nabla(u - u_h), \nabla w_h) = (p - p_h, \text{div } w_h) = -(p - p_h, \text{div } (u_g - w_h)) \quad \forall w_h \in V_h$$

and

$$(r_h, \text{div } (u - u_h)) = S_h(p, r_h) - S_h(p - p_h, r_h) \quad \forall r_h \in Q_h,$$

thus, (3.7) can be written in the form

$$(g, u - u_h) = (\nabla(u - u_h), \nabla(u_g - w_h)) - (p - p_h, \text{div } (u_g - w_h))$$
$$- (p_g - r_h, \text{div } (u - u_h)) + S_h(p - p_h, r_h) - S_h(p, r_h).$$

Choosing $(w_h, r_h) \in V_h \times Q_h$ as an interpolant of (u_g, p_g), we obtain

$$|(\nabla(u - u_h), \nabla(u_g - w_h))| \leq C \, |u - u_h|_1 \, h \|u_g\|_2,$$
$$|(p - p_h, \mathrm{div} \, (u_g - w_h))| \leq C \, \|p - p_h\|_0 \, h \|u_g\|_2,$$
$$|(p_g - r_h, \mathrm{div} \, (u - u_h))| \leq C \, |u - u_h|_1 \, h \|p_g\|_1,$$
$$|S_h(p - p_h, r_h)| \leq (S_h(p - p_h, p - p_h))^{1/2} \, (S_h(r_h, r_h))^{1/2}$$
$$\leq C \, |||(u - u_h, p - p_h)||| \, h \|p_g\|_1,$$
$$|S_h(p, r_h)| \leq \left(\sum_{K \in \mathcal{T}_h} \alpha_K \|\kappa_h \nabla p\|_{0,K}^2 \right)^{1/2} (S_h(r_h, r_h))^{1/2}$$
$$\leq C \, h^r \|p\|_r \, h \|p_g\|_1.$$

Here, we used $\alpha_K \sim h_K^2$, Assumption A1, and the H^1 stability of the interpolation $r_h \in Q_h$ of $p_g \in Q$ to estimate $S_h(r_h, r_h)$. Summarizing, we have

$$|(g, u - u_h)| \leq C \, h^r (\|u\|_{r+1} + \|p\|_r) \, h \, (\|u_g\|_2 + \|p_g\|_1)$$
$$\leq C \, h^{r+1}[(\|u\|_{r+1} + \|p\|_r) \, \|g\|_0,$$

and

$$\|u - u_h\|_0 = \sup_{g \in L^2(\Omega)} \frac{(g, u - u_h)}{\|g\|_0}$$

gives the desired L^2 error estimate. $\qquad\square$

Remark 3.4. The classical PSPG-stabilization is based on adding a stabilizing term of the form

$$\sum_{K \in \mathcal{T}_h} \tau_K(-\Delta u + \nabla p - f, \nabla q)_K.$$

For getting coercivity of the associated bilinear form, an upper bound $\tau_K \leq \tau_0 h_K^2$ has assumed to be fulfilled where τ_0 depends on the constant μ in the inverse inequality

$$\|\Delta v_h\|_{0,K} \leq \mu h_K^{-1} |v_h|_{1,K} \quad \forall v_h \in V_h.$$

This results in a conditionally stable method in the sense of [6] unless linear elements are used. The local projection schemes considered here are unconditionally stable in the sense that the corresponding bilinear form satisfies the inf-sup condition (3.4) for $\alpha_K = \alpha_0 h_K^2$ with $0 < \alpha_0 < \infty$.

4. ENRICHED EQUAL ORDER INTERPOLATION

We consider in this section combinations of approximation and projection spaces satisfying the Assumptions A1–A3 leading to methods of order $r \geq 1$. In [4, 7], two-level approaches have been developed where the approximation space Y_h consists of standard finite elements of order $r = 1, 2$ on triangular and quadrilateral meshes $\mathcal{T}_h$. A discontinuous finite element space of order $r - 1 = 0, 1$ on a coarser grid level $\mathcal{T}_{2h}$ has been used as projection space D_h. Drawbacks of this approach are that its stabilizing term increases the size of the stencil and it requires data structures which might not be available in an existing computer code. The general framework developed in [23] opens the way to consider continuous approximation and discontinuous projection spaces which are defined on the same mesh. All examples given for the Oseen problem in [23, Section 4] could also be used for the

Stokes problem. However, due to the missing advection term, the range of possible spaces Y_h and D_h is much larger, in particular, the difference of orders can be two (instead of one for the Oseen problem).

4.1. **Simplicial elements.** Let us briefly discuss first order methods. We start with continuous, piecewise linear finite elements ($r = 1$) for approximating velocity and pressure. Choosing $D_h = \{0\}$, the fluctuation operator κ_h becomes the identity and we end up with the stabilized method.

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$,

$$(4.1) \quad (\nabla u_h, \nabla v_h) - (p_h, \mathrm{div}\ v_h) + (q_h, \mathrm{div}\ u_h) + \sum_{K \in \mathcal{T}_h} \alpha_K (\nabla p_h, \nabla q_h)_K = (f, v_h).$$

This nonconsistent method has already been studied in [10]. Let us enrich only the velocity space elementwise by the bubble functions

$$b_K = (d+1)^{d+1} \prod_{i=1}^{d+1} \lambda_i$$

where λ_i, $i = 1, \ldots, d+1$, denote the barycentric coordinates associated with the cell K. Then, we get the Mini-element, shortly denoted by P_1^+/P_1, which satisfies the Babuška–Brezzi condition (2.5) and has not been stabilized [2]. However, enriching both the velocity and the pressure space, we obtain the equal order interpolation P_1^+/P_1^+ which needs extra stabilization. We use a local projection onto the space of piecewise constant functions, i.e. $D_h = P_0$. These first order element pairs are illustrated in Figure 1:
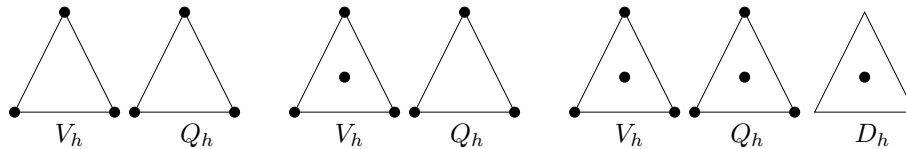

FIGURE 1. Examples of first order methods. Brezzi–Pitkäranta stabilization [10] P_1/P_1 (left), inf-sup stable Mini-element [2] P_1^+/P_1, local projection stabilization $P_1^+/P_1^+/P_0$ (right).

Lemma 4.1. *Assume $Y_h/D_h = P_1^+/P_0$. Then, the assumptions* A1–A3 *are satisfied and the method* (2.7) *converges of first order for $\alpha_K = \alpha_0 h_K^2$, $\alpha_0 > 0$.*

Proof. A1 and A2 hold by construction. For showing A3, it suffices to prove the local inf-sup condition (3.3). Since $\dim (D_h(K)) = \dim (Y_h(K)) = 1$, this is a simple task. □

Now we will consider higher order methods on simplicial meshes, i.e., $r \geq 2$.

We enrich the space of polynomials of degree less than or equal to r by bubble functions

$$P_r^b(K) := P_r(K) + b_K \cdot P_{r-2}(K),$$

and choose the approximation and projection spaces

$$Y_h = \left\{ v \in C(\overline{\Omega})\ :\ v|_K \in P_r^b(K),\ \forall K \in \mathcal{T}_h \right\},$$
$$D_h = \left\{ v \in L^2(\Omega)\ :\ v|_K \in P_{r-2}(K)\ \forall K \in \mathcal{T}_h \right\}.$$

At first glance, the enriched space seems to be large, but a more careful look shows that

$$P_r^b(K) = P_r(K) \oplus \left(b_K \cdot \sum_{i=2}^{d} \widetilde{P}_{r-i}(K) \right)$$

where $\widetilde{P}_m(K) = \text{span} \left(\left\{ \prod_{i=1}^{d} x_i^{\alpha_i}, \ \sum_{i=1}^{d} \alpha_i = m \right\} \right).$

Lemma 4.2. *Assume $r \geq 2$ and $Y_h/D_h = P_r^b/P_{r-2}^{disc}$. Then, the Assumptions A1–A3 are satisfied and the method (2.7) converges of order r for $\alpha_K = \alpha_0 h_K^2$, $\alpha_0 > 0$.*

Proof. A1 and A2 hold by construction. For showing A3, we prove the local inf-sup condition (3.3). Let $q_h \in D_h(K) = P_{r-2}(K)$ be arbitrary. We set $v_h := b_K q_h$ and conclude $v_h \in P_r^b(K)$ where $v_h = 0$ on ∂K. Thus, using the transformation onto a reference domain and the equivalence of norms on finite dimensional spaces, we have

$$(q_h, v_h)_K = (b_K q_h, q_h)_K \geq C_4 \|q_h\|_{0,K}^2.$$

Moreover,

$$\|v_h\|_{0,K} = \|b_K q_h\|_{0,K} \leq \|b_K\|_{\infty,K} \|q_h\|_{0,K} = \|q_h\|_{0,K},$$

consequently, there exists for each $0 \neq q_h \in D_h(K)$ a $v_h \in Y_h(K)$ such that

$$\frac{(q_h, v_h)_K}{\|q\|_{0,K} \|v_h\|_{0,K}} \geq C_4$$

and the local inf-sup condition (3.3) is proven. $\square$

Remark 4.3. To fulfill the local inf-sup condition for the projection space $D_h = P_{r-1}^{\text{disc}}$, we have to enrich the approximation space further:

$$P_r^{b+}(K) := P_r^b(K) \oplus \left(b_K \cdot \widetilde{P}_{r-1}(K) \right);$$

see [23]. For $r = 2$ and $d = 2$, the two variants are indicated in Figure 2 using the notation P_2^b and P_2^{b+} for the enriched spaces with one and three additional bubbles, respectively.

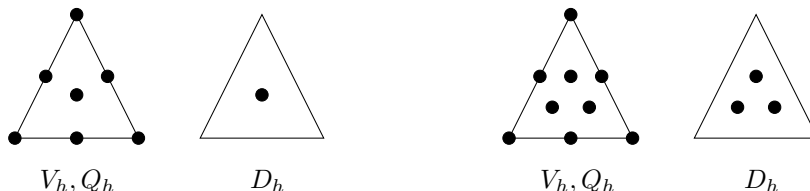V_h, Q_h D_h V_h, Q_h D_h

FIGURE 2. Examples of approximation and projection spaces of second order: P_2^b/P_0 (left) and $P_2^{b+}/P_1^{\text{disc}}$ (right).

4.2. **Quadrilateral and hexahedral elements.** Now we consider the standard finite element spaces on regular families $\mathcal{T}_h$ of quadrilateral or hexahedral meshes [11]. Let $\widehat{K} = (-1, +1)^d$ be the reference cell,

$$
Q_r(\widehat{K}) = \left\{ \hat{v} : \hat{v}(\hat{x}) = \sum_{0 \le \alpha_i \le r} c_{\alpha_1, \ldots, \alpha_d} \hat{x}_1^{\alpha_1} \cdots \hat{x}_d^{\alpha_d}, \right\},
$$

and $\hat{b} = \prod_{i=1}^{d} (1 - \hat{x}_i^2) \in Q_2(\widehat{K})$ the bubble function on $\widehat{K}$. We denote by $F_K : \widehat{K} \to K$ the mapping from the reference cell onto the cell $K \in \mathcal{T}_h$ and set $v = \hat{v} \circ F_K^{-1}$. We introduce the finite element spaces

$$
Q_r := \left\{ v \in C(\overline{\Omega}) : v\big|_K = \hat{v} \circ F_K^{-1}, \ \hat{v} \in Q_r(\widehat{K}), \ \forall K \in \mathcal{T}_h \right\}, \quad r \ge 1,
$$

$$
Q_r^{\mathrm{disc}} := \left\{ v \in L^2(\Omega) : v\big|_K = \hat{v} \circ F_K^{-1}, \ \hat{v} \in Q_r(\widehat{K}), \ \forall K \in \mathcal{T}_h \right\}, \quad r \ge 0,
$$

$$
Q_1^+ := \left\{ v \in C(\overline{\Omega}) : v\big|_K = \hat{v} \circ F_K^{-1}, \ \hat{v} \in Q_1(\widehat{K}) + \mathrm{span}\,(\hat{b}), \ \forall K \in \mathcal{T}_h \right\}.
$$

A first order method can be derived by setting $Y_h = Q_1$ and $D_h = \{0\}$, this results in the stabilized method (4.1). Note that in this case no projection is used.

Examples of enrichments of the velocity space such that the enriched velocity space together with the Q_1 pressure space satisfies the Babuška–Brezzi condition has been given in [3, 25, 16]. See [21] for a more general investigation. It should be mentioned that the local bubble space for enriching the Q_1 quadrilateral space has at least dimension 3, more precisely, one additional degree of freedom for each component of the velocity and one degree of freedom shared by both components. This is true in the conforming [3] as well as in the nonconforming [16] case.

We turn back to equal order interpolations stabilized by local projection. For a local projection onto $D_h = Q_0 = Q_0^{\mathrm{disc}}$, we enrich the approximation space and set $Y_h = Q_1^+$.

Lemma 4.4. *Assume $Y_h / D_h = Q_1^+ / Q_0$. Then, the Assumptions A1–A3 are satisfied and method (2.7) converges of first order for $\alpha_K = \alpha_0 h_K^2$, $\alpha_0 > 0$.*

Proof. A1 and A2 hold by construction. For showing A3, it is enough to prove the local inf-sup condition (3.3). Since $\dim (D_h(K)) = \dim (Y_h(K)) = 1$, this is a simple task.   □

We now turn to higher order methods.

Lemma 4.5. *Assume $r \ge 2$ and $Y_h / D_h = Q_r / Q_{r-2}^{disc}$. Then, the Assumptions A1–A3 are satisfied and method (2.7) converges of order r for $\alpha_K = \alpha_0 h_K^2$, $\alpha_0 > 0$.*

Proof. A1 and A2 hold by construction. For showing A3, we prove the local inf-sup condition (3.3). Let $q_h \in D_h(K)$ be arbitrary. We set $v_h := (\hat{b} \circ F_K^{-1}) \cdot q_h$. Since $v_h \circ F_K = \hat{b} \cdot (q_h \circ F_K)$ and $q_h \circ F_K \in Q_{r-2}(\widehat{K})$, we conclude $v_h \in Y_h(K)$. Moreover, it follows from

$$
\|v_h\|_{0,K} \le \|q_h\|_{0,K},
$$

that

$$
\frac{(q_h, v_h)_K}{\|q_h\|_{0,K}\,\|v_h\|_{0,K}} \ge \frac{\int_{\widehat{K}} \hat{b}\, \hat{q}_h^2 |\det DF_K|\, d\hat{x}}{\int_{\widehat{K}} \hat{q}_h^2 |\det DF_K|\, d\hat{x}}.
$$

Now, using the upper and lower bound

$$C_5 \, h_K^d \leq |\det DF_K| \leq C_6 \, h_K^d$$

shown in [24] and the equivalence of the norms

$$\hat{q} \mapsto \left(\int_{\widehat{K}} \hat{q}^2 \, d\hat{x} \right)^{1/2}, \qquad \hat{q} \mapsto \left(\int_{\widehat{K}} \hat{b} \cdot \hat{q}^2 \, d\hat{x} \right)^{1/2}$$

on the finite dimensional space $Q_{r-2}(\widehat{K})$, we get the assertion of the lemma. $\qquad \square$

Remark 4.6. Note that for $r \geq 2$ and quadrilateral or hexahedral elements we do not need any enrichments. Thus, the standard finite element spaces can be used.

Remark 4.7. The smallest projections space which guarantees Assumption A1 is the space P_{r-2}^{disc}. Note that there are two versions of this space. The unmapped version is defined on the original cell $K \in \mathcal{T}_h$ by

$$P_{r-2}^{\mathrm{disc},u} := \left\{ v : v\big|_K \in P_{r-2}(K) \right\}$$

and satisfies the usual approximation properties on shape regular meshes. Using the mapping $F_K : \widehat{K} \to K$ from the reference cell $(-1,+1)^d$ onto $K \in \mathcal{T}_h$, the mapped version is defined by

$$P_{r-2}^{\mathrm{disc},m} := \left\{ v : v\big|_K \circ F_K \in P_{r-2}(\widehat{K}) \right\}.$$

It satisfies the usual approximation properties only on families of successively refined quadrilateral and hexahedral meshes; see [22] for more details. Since both spaces are subspaces of Q_{r-2}^{disc}, the local inf-sup condition (3.3) is satisfied and Assumption A3 follows.

Remark 4.8. The local projection method with the larger projection space $D_h = P_{r-1}^{\mathrm{disc}}$ was analyzed in [23]. In order to fulfill the local inf-sup condition, the approximation space Y_h has to be enriched to

$$Q_r^b := \left\{ v \in C(\overline{\Omega}) : v\big|_K = \hat{v} \circ F_K^{-1}, \ \hat{v} \in Q_r^b(\widehat{K}), \ \forall K \in \mathcal{T}_h \right\}$$

where

$$Q_r^b(\widehat{K}) := Q_r(\widehat{K}) \oplus \mathrm{span} \left(\hat{b} \, \hat{x}_i^{r-1} \, : \, i = 1, \dots, d \right).$$

Note that $Q_r^b(\widehat{K})$ contains for any $r \geq 2$ exactly d functions more than $Q_r(\widehat{K})$ where d is the space dimension.

4.3. **Elimination of enrichments.** In the examples above, standard finite element spaces have been enriched with bubble functions. These additional degrees of freedom can be eliminated locally by static condensation. We concentrate in this section on the special case $V_h/Q_h/D_h = P_1^+/P_1^+/P_0$ on simplices, i.e., our approximation space consists of continuous, piecewise linear functions enriched by one bubble function b_K per cell:

$$Y_h = \left\{ v \in C(\overline{\Omega}) \, : \, v|_K \in P_1(K) \oplus \mathrm{span} \, b_K, \ \forall K \in \mathcal{T}_h \right\}.$$

We set $V_h := (Y_h \cap H_0^1(\Omega))^d$ and $Q_h := Y_h \cap L_0^2(\Omega)$. The stabilized discrete problem (2.7) can be written in the form:

Find $(u_h, p_h) \in V_h \times Q_h$ such that

(4.2) $\qquad (\nabla u_h, \nabla v_h) - (p_h, \mathrm{div} \, v_h) = (f, v_h) \qquad\qquad \forall v_h \in V_h,$

(4.3) $\qquad (q_h, \mathrm{div} \, u_h) + S_h(p_h, q_h) = 0 \qquad\qquad\qquad \forall q_h \in Q_h.$

Let us denote by $\langle \cdot, \cdot \rangle_{\partial G}$ the inner product in $L^2(\partial G)$ and by n the outer unit normal with respect to Ω. Since

$$(1, \operatorname{div} u_h) + S_h(p_h, 1) = \langle 1, u_h \cdot n \rangle_{\partial \Omega} + \sum_{K \in \mathcal{T}_h} \alpha_K(\kappa_h \nabla p_h, \kappa_h \nabla 1)_K = 0,$$

the equation (4.3) is true for all $q_h \in Y_h$.

According to the splitting of the spaces V_h and Y_h into a linear and a bubble part

$$V_h = V_L \oplus V_B, \qquad Y_h = Y_L \oplus Y_B,$$

we split the functions $v_h \in V_h$ and $q_h \in Y_h$ into

$$v_h = v_L + v_B, \qquad q_h = q_L + q_B.$$

Next, we decompose the stabilized discrete problem by using test functions from the linear and bubble part, respectively. This results in:

Find $(u_L, p_L) \in V_L \times Y_L$ and $(u_B, p_B) \in V_B \times Y_B$ such that $p_L + p_B \in Q_h$ and

$$(\nabla u_L, \nabla v_L) + (\nabla u_B, \nabla v_L) - (p_L, \operatorname{div} v_L) - (p_B, \operatorname{div} v_L) = (f, v_L) \quad \forall v_L \in V_L,$$
$$S_h(p_L + p_B, q_L) + (q_L, \operatorname{div} u_L) + (q_L, \operatorname{div} u_B) = 0 \qquad \forall q_L \in Y_L,$$
$$(\nabla u_L, \nabla v_B) + (\nabla u_B, \nabla v_B) - (p_L, \operatorname{div} v_B) - (p_B, \operatorname{div} v_B) = (f, v_B) \quad \forall v_B \in V_B,$$
$$S_h(p_L + p_B, q_B) + (q_B, \operatorname{div} u_L) + (q_B, \operatorname{div} u_B) = 0 \qquad \forall q_B \in Y_B.$$

A careful look shows that several terms in the discrete formulation vanish,

$$(\nabla u_B, \nabla v_L) = \sum_{K \in \mathcal{T}_h} (\nabla u_B, \nabla v_L)_K = \sum_{K \in \mathcal{T}_h} \left[\left\langle \frac{\partial v_L}{\partial n_K}, u_B \right\rangle_{\partial K} - (\triangle v_L, u_B)_K \right] = 0$$

due to $\triangle v_L|_K = 0$ and $u_B|_{\partial K} = 0$. Analogously, $(\nabla u_L, \nabla v_B) = 0$. It follows from

$$\left(b_K, \frac{\partial b_K}{\partial x_i} \right)_K = \frac{1}{2} \int_K \frac{\partial b_K^2}{\partial x_i} dx = \frac{1}{2} \int_{\partial K} b_K^2 n_i \, d\gamma = 0, \quad i = 1, \ldots, d,$$

that

$$(p_B, \operatorname{div} v_B) = (q_B, \operatorname{div} u_B) = 0.$$

Furthermore, since $\nabla(v_L|_K) \in P_0(K)$, we have $\pi_h \nabla v_L = \nabla v_L$. Consequently, $\kappa_h \nabla v_L = 0$ such that $S_h(p_L + p_B, q_L) = 0$ and $S_h(p_L + p_B, q_B) = S_h(p_B, p_B)$. Now, the bubble part of our discrete system reduces to:

Find $(u_B, p_B) \in V_B \times Y_B$ such that

$$(\nabla u_B, \nabla v_B) - (p_L, \operatorname{div} v_B) = (f, v_B) \qquad \forall v_B \in V_B,$$
$$S_h(p_B, q_B) + (q_B, \operatorname{div} u_L) = 0. \qquad \forall q_B \in Y_B.$$

We extend the bubble function $b_K : K \to \mathbb{R}$ onto $\overline{\Omega}$ by setting $b_K = 0$ outside of $\overline{K}$. Then, using the basis representations

$$u_B = \sum_{K \in \mathcal{T}_h} u_K \, b_K, \qquad p_B = \sum_{K \in \mathcal{T}_h} p_K \, b_K,$$

we can locally solve the bubble part of the algebraic system and get

$$p_K = -\frac{(b_K, \operatorname{div} u_L)}{S_h(b_K, b_K)}, \quad (u_K)_i = \frac{1}{|b_K|_{1,K}^2} \left(f_i - \frac{\partial p_L}{\partial x_i}, b_K \right), \quad i = 1, \ldots, d.$$

Elimination of u_B and p_B in the linear part of the algebraic system yields

Find $(u_L, p_L) \in V_L \times Y_L$ such that for all $(v_L, q_L) \in V_L \times Y_L$,

$$(\nabla u_L, \nabla v_L) - (p_L, \operatorname{div} v_L) + \sum_{K \in \mathcal{T}_h} \gamma_K (\operatorname{div} u_L, \operatorname{div} v_L) = (f, v_L)$$

$$(q_L, \operatorname{div} u_L) + \sum_{K \in \mathcal{T}_h} (-\Delta u_L + \nabla p_L, \tau_K \nabla q_L)_K = \sum_{K \in \mathcal{T}_h} (f, \tau_K \nabla q_L)_K$$

where

$$\gamma_K = \frac{1}{|K|} \frac{\|b_K\|_{0,1,K}^2}{S_h(b_K, b_K)} = \frac{\|b_K\|_{0,1,K}^2}{\alpha_K |K| \|\kappa_h \nabla b_K\|_{0,K}^2}, \qquad \alpha_K(x) = \frac{\|b_K\|_{0,1,K}}{|b_K|_{1,K}^2} b_K(x).$$

Of course the pressure $p_L \in Y_L$ is only determined up to an additive constant. We make the pressure unique by replacing Y_L by $Q_L = Y_L \cap L_0^2(\Omega)$. It is remarkable that the reduced problem corresponds to the PSPG method [19] in combination with the grad-div stabilization [15, 18, 29, 30].

Let us study the dependency of the parameter γ_K and τ_K on h_K. We have

$$\|b_K\|_{0,1,K} = \frac{(d+1)^{d+1} d!}{(2d+1)!} |K|, \quad |b_K|_{1,K}^2 \sim h_K^{-2} |K|, \quad \|\kappa_h \nabla b_K\|_{0,K}^2 \sim h_K^{-2} |K|$$

from which with the standard choice $\alpha_K = \alpha_0 h_K^2$ follows:

$$\gamma_K \sim \alpha_K^{-1} h_K^2 \sim 1, \qquad \tau_K(x) \sim h_K^2 b_K(x).$$

Starting with the solution $(u_L, p_L) \in V_L \times Q_L$, we can reconstruct a velocity field

$$u_h = u_L + u_B = u_L + \sum_{K \in \mathcal{T}_h} \frac{1}{|b_K|_{1,K}^2} (f - \nabla p_L, b_K) b_K$$

satisfying the incompressibility constraint in the sense that

$$(q_L, \operatorname{div} u_h) = 0 \qquad \forall q_L \in Q_L.$$

5. Numerical tests

This section presents numerical results for solving the Stokes problem with equal order spaces where the discretization is stabilized by the local projection method. All calculations were performed with the code MooNMD [20] on a Linux PC (Pentium IV, 2.8 GHz).

Let $\Omega = (0,1)^2$. We consider the Stokes problem

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma,$$

where the right hand side f and the inhomogeneous Dirichlet boundary condition g have been chosen such that

$$u = \big(\sin(x) \sin(y), \cos(x) \cos(y) \big), \quad p = 2 \cos(x) \sin(y) - 2 \sin(1) \big(1 - \cos(1) \big)$$

is the solution. This example was taken from [8].

We have carried out calculations on triangular and quadrilateral meshes which were obtained by successive regular refinement of the initial coarse meshes. The coarsest mesh (level 0) consists of either two triangles or a single quadrilateral. The meshes on level 1 are shown in Figure 3.

In the following, we evaluate the results of our calculations by considering the L^2 norm and the H^1 semi-norm of the velocity error, the L^2 norm of the pressure error, and the error in the local projection norm $||| \cdot |||$.
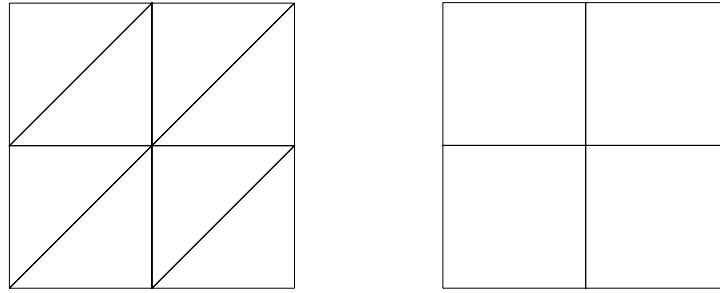
FIGURE 3. Meshes on level 1 consisting of triangles (left) and quadrilaterals (right).

The first group of calculations confirms predicted convergence rates. Furthermore, the results on triangular and quadrilateral meshes are compared. The parameter α_0 is kept fixed for these calculations.

We start with first order elements on triangles and quadrilaterals. We consider the element P_1^+ with projection onto P_0 on triangles, the Mini-element on triangles, and the element Q_1^+ with projection onto P_0. The results of the Mini-element and the element P_1^+ are almost the same although the Mini-element uses a smaller pressure space. Hence, the behavior of the error with respect to different norms is shown in Figure 4 only for the element P_1^+ and the element Q_1^+. We have chosen for these calculations the parameter $\alpha_0 = 1$. The errors for the quadrilateral element are always smaller than the corresponding errors for the triangular elements. The largest difference appears for the errors in the local projection norm and for the L^2 norm of the pressure. The results differ by a factor of roughly 10. Furthermore, the H^1 semi-norm of the velocity error and the error in the local projection norm indicate, as predicted, first order convergence on both triangles and quadrilaterals. Moreover, the L^2 norm of the velocity error converges with second order on both types of meshes. The numerical results show that the convergence order for the L^2 norm of the pressure error is on both triangular and quadrilateral meshes approximately $3/2$ while our theory proves just first order convergence. A reason for this behavior might be some superconvergence phenomena for the first order elements.

Higher order elements are investigated next. Figure 5 presents the results for the third order elements P_3^{b+} with projection onto P_2^{disc} on triangles and Q_3^b with projection onto P_2^{disc} on quadrilaterals. Furthermore, the predicted asymptotic behavior is shown in the graphs of Figure 5. For these calculations, we have again chosen $\alpha_K = h_K^2$, i.e., $\alpha_0 = 1$. The L^2 norm of the pressure error, the H^1 semi-norm of the velocity error, and the error in the local projection norm converge with third order which confirms our theory. For these third order elements, there is no longer a better convergence of the L^2 norm of the pressure error as observed for the first order elements. The results for the L^2 norm of the velocity error indicate fourth order accuracy, in agreement with Theorem 3.3. The calculations on quadrilateral meshes result in a smaller error than the calculations on triangular meshes. For the velocity error, this effect is more pronounced for higher order elements.

We will study the behavior of different third order elements on quadrilaterals. We consider the following four elements: Q_3^b with projection onto P_2^{disc}, Q_3^b with
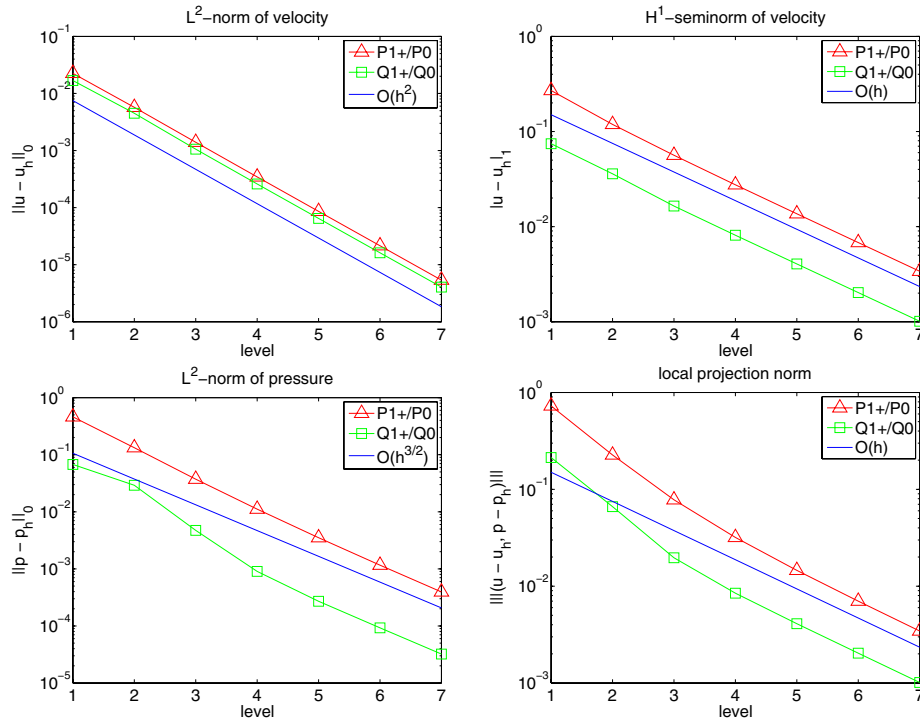
FIGURE 4. First order elements on triangles and quadrilaterals.

projection onto P_1^{disc}, Q_3 with projection onto P_1^{disc}, and Q_3 with projection onto Q_1^{disc}. The presented theory predicts for all four elements the same asymptotic convergence behavior. The parameter α_0 was again chosen to be 1.

The graphs in Figure 6 show that the element Q_3^b with projection onto P_2^{disc} gives the best results which are 1–2 order of magnitude better than the results of the other three elements. The results for the two elements which are projected onto P_1^{disc} differ so slightly that the difference cannot be seen in the graphs. Looking at these results, we find that a larger projection space produces smaller errors. This is also indicated by the fact that the errors for the element which is projected onto Q_1^{disc} are between the errors for the elements with projection space P_1^{disc} and the element with projection space P_2^{disc}.

Next we consider how the results depend on the parameter α_0 in the stabilization term. To this end, we have chosen three different cases: the first order element P_1^+ with projection onto P_0 on triangles, the third order element Q_3^b with projection onto P_2^{disc} on quadrilaterals, and the third order element Q_3 with projection onto P_1^{disc}.

Let us start with the first order element P_1^+ with projection onto P_0 on triangles. The graphs in Figure 7 show the results on level 5 (2,048 triangles, 9,411 degrees of freedom for pressure and both velocity components), level 6 (8,192 triangles, 37,251 degrees of freedom), and level 7 (32,768 triangles, 148,227 degrees
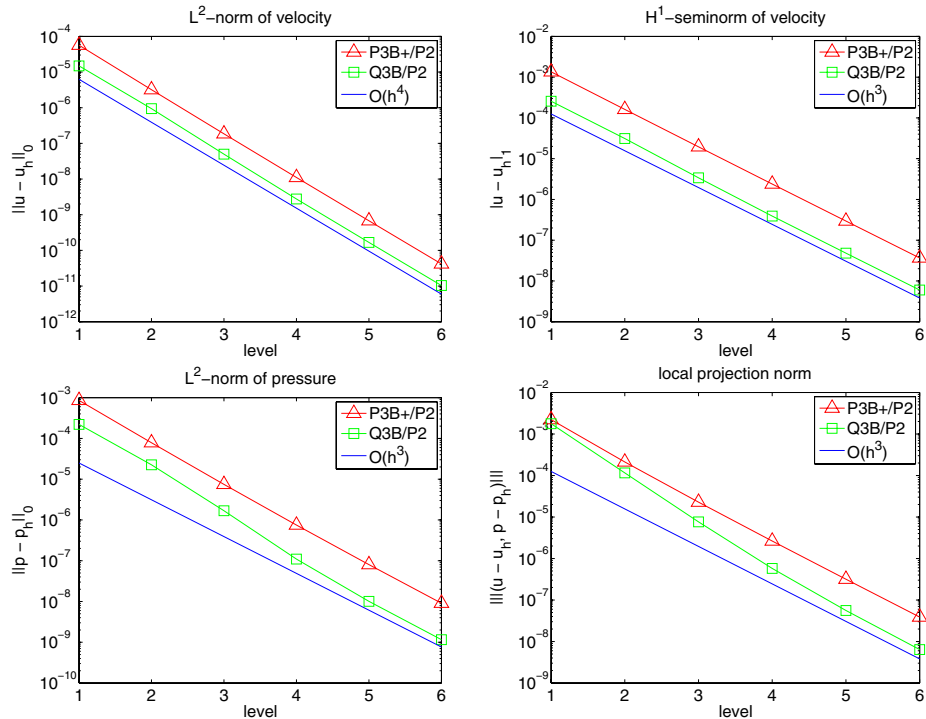
FIGURE 5. Third order elements on triangles and quadrilaterals.

of freedom). These results indicate that the velocity error is not very sensitive with respect to the parameter α_0; see the top row of graphs in Figure 7 and note the scaling of the axes. The behavior of the pressure error and the error in the local projection norm is different. If the parameter α_0 tends to zero then the error increases. This shows the instability of the equal order interpolation for the Stokes problem without stabilization. For each norm, the dependence of the error on the parameter α_0 is very similar on the different refinement levels.

We continue by considering the third order element Q_3^b with projection onto P_2^{disc} on quadrilaterals. While varying the coefficient α_0 in $\alpha_K = \alpha_0 \, h_K^2$ over the wide range from 10^{-4} up to 10^4, the obtained results changed only slightly. Furthermore, the behavior is similar on different refinement levels. If the parameter α_0 tends towards zero in these calculations, then the error in the L^2 norm of the pressure and the error in the local projection norm increase while the velocity error remains bounded. This shows again the instability of equal order interpolation pairs for the Stokes problem.

Finally, we look at the element Q_3 with projection onto P_1^{disc}, i.e., the order difference between approximation space and projection space is 2. The graphs of Figure 8 present the results on level 3 (64 quadrilaterals, 1,875 degrees of freedom for velocity and pressure), level 4 (256 quadrilaterals, 8,739 degrees of freedom), and level 5 (1,024 quadrilaterals, 28,227 degrees of freedom). These results show that this element is on all refinement levels much more sensitive with respect to the parameter α_0 as the previously considered element Q_3^b with projection onto P_2^{disc}. The error changes several orders of magnitude if α_0 is varying.
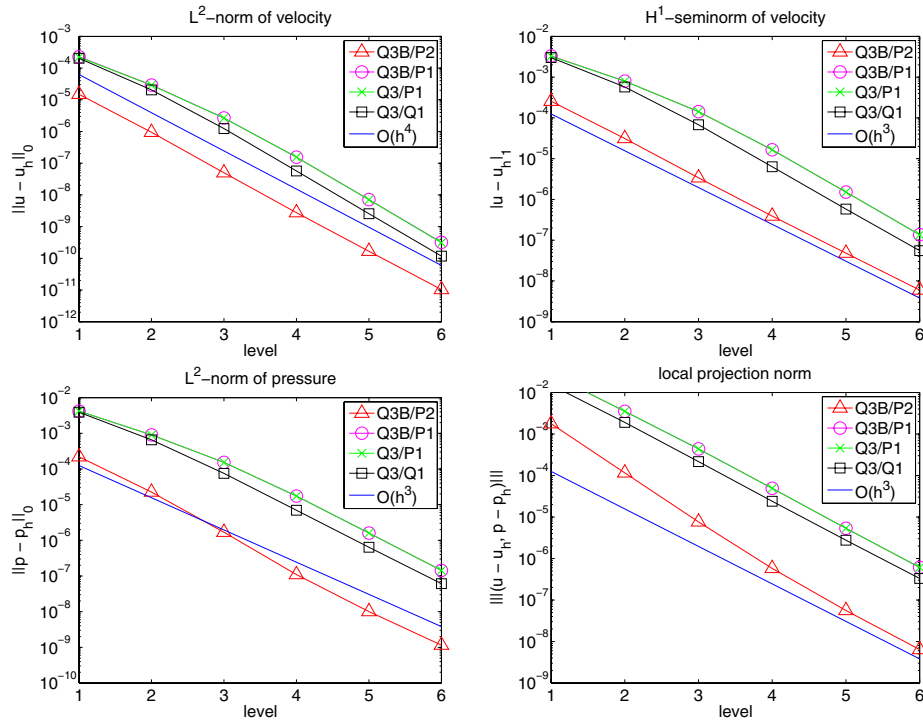
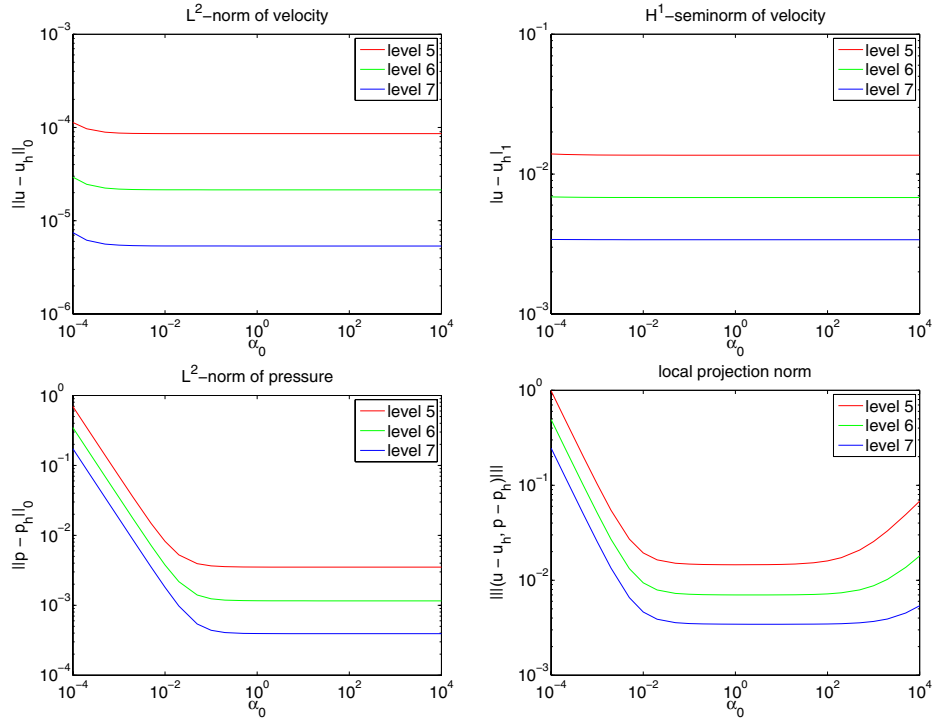FIGURE 6. Comparison of third order elements on quadrilaterals.

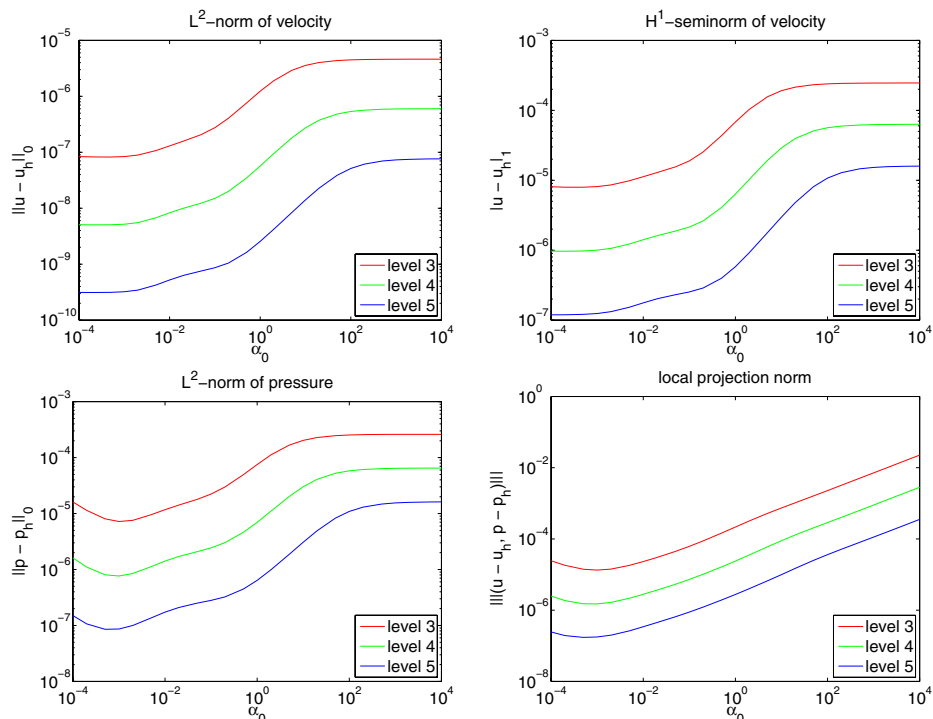FIGURE 7. Dependence of norms on the parameter α_0 for first order elements on triangles.

FIGURE 8. Dependence of norms on the parameter α_0 for third order elements with difference 2 on quadrilaterals.

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