Local Regularity for Minimizers of non Convex Integrals

E. Acerbi & N. Fusco

1. Introduction

In this paper we study the regularity of minimizers of the functional

$$I(u) = \int_\Omega f\big(x, u(x), Du(x)\big)\, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is open, $u : \Omega \to \mathbb{R}^N$, and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a continuous function satisfying

$$|f(x, s, \xi)| \le L(1 + |\xi|^p)$$

with $p \ge 2$. This problem has been studied under various ellipticity assumptions on f; for the case when f is uniformly strictly convex in ξ, i.e.,

$$f_{\xi\xi}(x_0, s_0, \xi_0)\eta\eta \ge \gamma|\eta|^2 \tag{1.2}$$

for all (x_0, s_0, ξ_0), see e.g. [10], and a comprehensive account in [8].

This work has been supported by the Italian Ministry of Education

If convexity is replaced by uniform strict quasiconvexity, i.e., there is some $\gamma > 0$ such that

$$\begin{aligned}\int_{\operatorname{spt}\varphi} & f\big(x_0, s_0, \xi_0 + D\varphi(y)\big)\,dy \\ & \geq \int_{\operatorname{spt}\varphi} \big[f(x_0, s_0, \xi_0) + \gamma\big(|D\varphi(y)|^2 + |D\varphi(y)|^p\big)\big]\,dy\end{aligned} \tag{1.3}$$

for all $\varphi \in C_0^1$ and all (x_0, s_0, ξ_0), partial regularity of minimizers has been studied in [5],[6] in the case independent of (x, u), in [7],[11] in the case with (x, u), but with second derivatives with respect to ξ bounded by $|\xi|^{p-2}$, and in the general case in [2], see also [9].

These papers are motivated by the fact that in the vector-valued case $(N > 1)$ quasiconvexity, i.e., condition (1.3) with $\gamma = 0$, is essentially equivalent to the semicontinuity of (1.1): see e.g. [15],[14],[1].

Of course the uniform ellipticity conditions (1.2) or (1.3) are not necessary in order for the functional (1.1) to have a minimizer (this happens for example if $f(\xi) = |\xi|^p$ with $p \neq 2$; however, this particular functional may be treated in a special way as far as regularity is concerned, see e.g. [16]).

A new kind of result has been recently proved in [3] which is useful for studying regularity in cases of degenerate ellipticity, by showing that Du is Hölder continuous near points where it is “close” to a value ξ_0 where f is uniformly strictly convex. Precisely, in the case independent of (x, u), if f is convex and with growth $p \geq 1$, and u is a minimizer of I, then if

$$\lim_{r\to 0} ⨍_{B_r(x_0)} |Du - \xi_0|^p\,dx = 0$$

for some x_0 such that (1.2) holds, and f is of class C^2 in a neighbourhood of x_0, then Du is Hölder continuous of any exponent $\alpha < 1$ in a neighbourhood of x_0. A similar result is given when f depends also on (x, u).

In the same spirit, we prove the following result (Theorem 2.1):

Let $p \geq 2$, and let $f : \mathbb{R}^{nN} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying

$$|f(\xi)| \leq L(1 + |\xi|^p), \qquad |Df(\xi)| \leq L(1 + |\xi|^{p-1}).$$

Fix $\xi_0 \in \mathbb{R}^{nN}$ *such that* $f \in C^2$ *in a neighbourhood of* ξ_0 *and*

$$\int_{\operatorname{spt}\varphi} f\big(\xi_0 + D\varphi(y)\big)\,dy \\ \geq \int_{\operatorname{spt}\varphi} \big[f(\xi_0) + \gamma\big(|D\varphi(y)|^2 + |D\varphi(y)|^p\big)\big]\,dy \quad \text{for all } \varphi \in C_0^1.$$

Then if u *is a minimizer of* $\int f(Dv)\,dx$ *and*

$$\lim_{r\to 0} \fint_{B_r(x_0)} |Du - \xi_0|^p\,dx = 0,$$

there is a neighbourhood of x_0 *in which the function* u *is of class* $C^{1,\alpha}$ *for all* $\alpha < 1$.

An extension to the case with (x, u) is also provided (Theorem 3.1).

We remark that in the above theorem we do not require a global quasiconvexity assumption. On the other hand the theorem covers only the case $p \geq 2$; however, it is not clear whether a function which is genuinely quasiconvex at some point ξ_0 and has growth $p < 2$ may exist.

These result allow us to generalize the former partial regularity results of [5],[6],[2]: the strict quasiconvexity need no longer be uniform (Corollaries 4.1 and 4.2).

The last part of the paper is devoted to the study of the set of regular points in the scalar case $N = 1$; as an example, an application to an energy functional of interest in nonlinear elasticity is also provided.

2. The case independent of (x,u)

Let Ω be a bounded open subset of $\mathbb{R}^n$, fix $p \geq 2$ and let $f : \mathbb{R}^{nN} \to \mathbb{R}$ satisfy:

$$f \text{ is locally Lipschitz continuous} \tag{2.1}$$

$$|f(\xi)| \leq L(1+|\xi|^p) \tag{2.2}$$

$$|Df(\xi)| \leq L(1+|\xi|^{p-1}). \tag{2.3}$$

We say that $\xi_0 \in \mathbb{R}^{nN}$ is a regular point for f if there exist $\sigma > 0$, $\gamma > 0$ such that $f \in C^2\big(B_\sigma(\xi_0)\big)$ and

$$\int_{\operatorname{spt}\varphi} f\big(\xi_0 + D\varphi(x)\big)\,dx \geq \int_{\operatorname{spt}\varphi} \big[f(\xi_0) + \gamma\big(|D\varphi(x)|^2 + |D\varphi(x)|^p\big)\big]\,dx$$

for every $\varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N)$.

Set for every $u \in W^{1,p}(\Omega;\mathbb{R}^N)$

$$I(u) = \int_\Omega f\big(Du(x)\big)\,dx;$$

we say that u is a minimizer of I if

$$I(u) \leq I(u+\varphi) \quad \text{for every } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N).$$

Then we have:

Theorem 2.1 . *Let f satisfy* (2.1),(2.2),(2.3), *and let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a minimizer of I. If for some $x_0 \in \Omega$ and some regular point ξ_0*

$$\lim_{r\to 0} ⨍_{B_r(x_0)} |Du(x) - \xi_0|^p\,dx = 0$$

then in a neighbourhood of x_0 the function u is of class $C^{1,\alpha}$ for all $\alpha < 1$.

In the sequel we denote by the same letter c any positive constant, which may vary from line to line; if φ is any vector-valued function, we denote by $(\varphi)_{x_0,r}$ or simply by $(\varphi)_r$ the mean value of φ on $B_r(x_0)$. Finally, we set

$$g_p(t) = |t|^2 + |t|^p.$$

We shall use the following lemmas:

Lemma 2.2 . *Let f satisfy* (2.1),(2.2),(2.3) *and let ξ_0 be a regular point for f, i.e.,*

$$\int_{\operatorname{spt}\varphi} f\big(\xi_0 + D\varphi(y)\big)\,dy \geq \int_{\operatorname{spt}\varphi} \big[f(\xi_0) + \gamma g_p\big(D\varphi(y)\big)\big]\,dy \quad \text{for all } \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N)$$

and $f \in C^2\big(B_{2\sigma}(\xi_0)\big)$. There exists $\varrho > 0$ such that for every $\xi \in B_\varrho(\xi_0)$

$$\int_{\operatorname{spt}\varphi} f\big(\xi + D\varphi(y)\big)\,dy \geq \int_{\operatorname{spt}\varphi} \Big[f(\xi) + \frac{\gamma}{2} g_p\big(D\varphi(y)\big)\Big]\,dy \quad \text{for all } \varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N). \tag{2.4}$$

PROOF . Set $\omega_\varrho = \sup\{|D^2 f(\xi) - D^2 f(\eta)| : \xi, \eta \in B_\sigma(\xi_0), |\xi - \eta| < \varrho\}$, and fix ξ such that $|\xi - \xi_0| < \varrho < \sigma/2$. Then

$$\begin{aligned}\int_\Omega [f(\xi + D\varphi) - f(\xi)]\,dy &= \int_\Omega [f(\xi_0 + D\varphi) - f(\xi_0)]\,dy \\ &\quad + \int_\Omega [G(D\varphi) - G(0) - DG(0)D\varphi]\,dy,\end{aligned} \tag{2.5}$$

where we set

$$G(\eta) = f(\xi + \eta) - f(\xi_0 + \eta).$$

The first integral which appears at the right hand side of (2.5) is greater than $\int_\Omega \gamma g_p(D\varphi)\,dy$. As for the second, we can set

$$S_\sigma = \{y \in \Omega : |D\varphi(y)| \leq \sigma/2\}, \qquad L_\sigma = \{y \in \Omega : |D\varphi(y)| > \sigma/2\};$$

remarking that $G(D\varphi) - G(0) - DG(0)D\varphi = \frac{1}{2}D^2G(\vartheta D\varphi)D\varphi D\varphi$ with $0 < \vartheta < 1$, and that $\xi + D\varphi(y) \in B_\sigma(\xi_0)$ for $y \in S_\sigma$, we have

$$\begin{aligned}\int_\Omega [f(\xi + D\varphi) - f(\xi)]\,dy &\geq \int_\Omega \gamma g_p(D\varphi)\,dy - \frac{\omega_\varrho}{2}\int_{S_\sigma} g_p(D\varphi)\,dy \\ &\quad + \int_{L_\sigma} [G(D\varphi) - G(0) - DG(0)D\varphi]\,dy.\end{aligned} \tag{2.6}$$

Set

$$H(\eta, y) = f\big(\eta + D\varphi(y)\big) - f(\eta) - Df(\eta)D\varphi(y),$$

and

$$M_\sigma = \sup\{|D^2 f(\eta)| : \eta \in B_\sigma(\xi_0)\};$$

we remark that the last integral may be written $\int_{L_\sigma} [H(\xi, y) - H(\xi_0, y)]\, dy$, therefore its absolute value is bounded by

$$\begin{aligned}\int_{L_\sigma} & |\xi - \xi_0||D_\eta H(\tau, y)|\, dy \\ & \le |\xi - \xi_0| \int_{L_\sigma} [|Df(\tau + D\varphi) - Df(\tau)| + M_\sigma |D\varphi|]\, dy \\ & \le c(\sigma, |\xi_0|, L)|\xi - \xi_0| \int_{L_\sigma} (1 + |D\varphi| + |D\varphi|^{p-1})\, dy \\ & \le \tilde{c}\varrho \int_{L_\sigma} g_p(D\varphi)\, dy.\end{aligned}$$

If we choose ϱ such that $\frac{1}{2}\omega_\varrho + \tilde{c}\varrho < \frac{1}{2}\gamma$, the result follows from (2.6). ∎

Lemma 2.3 . *Let f satisfy* (2.1),(2.2),(2.3) *and assume $f \in C^2\big(B_\sigma(\xi_0)\big)$. Set for all $\lambda > 0$ and $\xi, \eta \in \mathbb{R}^{nN}$*

$$f_{\xi,\lambda}(\eta) = \lambda^{-2}[f(\xi + \lambda\eta) - f(\xi) - \lambda Df(\xi)\eta].$$

There exists $c > 0$ such that for every $\xi \in B_{\sigma/3}(\xi_0)$

$$|f_{\xi,\lambda}(\eta)| \le c(|\eta|^2 + \lambda^{p-2}|\eta|^p), \qquad |Df_{\xi,\lambda}(\eta)| \le c(|\eta| + \lambda^{p-2}|\eta|^{p-1}).$$

Proof . Set $K = \max\{|D^2 f(\eta)| : |\eta - \xi_0| < 2\sigma/3\}$; then

$$\begin{aligned} |\lambda\eta| \le \sigma/3 \quad & \Rightarrow \quad |f_{\xi,\lambda}(\eta)| = \frac{1}{2}|D^2 f(\xi + \vartheta\lambda\eta)\eta\eta| \le \frac{1}{2}K|\eta|^2; \\ |\lambda\eta| > \sigma/3 \quad & \Rightarrow \quad |f_{\xi,\lambda}(\eta)| \le \lambda^{-2}c(\sigma)(1 + |\lambda\eta| + |\lambda\eta|^p) \le c(\sigma)\lambda^{p-2}|\eta|^p, \end{aligned}$$

and the first inequality is proven. The second is analogous. ∎

The following result may be easily derived from [8], p. 161.

Lemma 2.4 . *Let $f : [\frac{r}{2}, r] \to [0, +\infty)$ be a bounded function satisfying*

$$f(t) \le \vartheta f(s) + \frac{A}{(s-t)^2} + \frac{B}{(s-t)^p} + D$$

for some $0 < \vartheta < 1$ and all $\frac{r}{2} \le t < s \le r$. Then there exists a constant $c(\vartheta, p)$ such that

$$f\Big(\frac{r}{2}\Big) \le c\Big(\frac{A}{r^2} + \frac{B}{r^p} + D\Big).$$

The following lemma may be found in [2], Lemma II.4; since we will later refer to the proof, we include it for the readers' convenience.

Lemma 2.5 . *Let $g : \mathbb{R}^{nN} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying*

$$|g(\xi)| \le c_1(|\xi|^2 + \lambda^{p-2}|\xi|^p), \qquad |Dg(\xi)| \le c_1(|\xi| + \lambda^{p-2}|\xi|^{p-1}),$$

$$\int g(D\varphi)\, dx \ge \gamma \int [|D\varphi|^2 + \lambda^{p-2}|D\varphi|^p]\, dx \quad \text{for all } \varphi \in C^1_0(\mathbb{R}^n; \mathbb{R}^N)$$

for suitable constants c_1, λ and γ. Fix $\nu \ge 0$ and let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ satisfy

$$\int_\Omega g(Du)\, dx \le \int_\Omega [g(Du + D\varphi) + \nu|D\varphi|]\, dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega; \mathbb{R}^N).$$

Then there exists $c_2 > 0$, depending only on c_1, γ, such that for every $B_r \subset \Omega$

$$\fint_{B_{r/2}} (|Du|^2 + \lambda^{p-2}|Du|^p)\, dx \le c_2 \fint_{B_r} \Big(\nu^2 + \frac{|u-(u)_r|^2}{r^2} + \lambda^{p-2}\frac{|u-(u)_r|^p}{r^p}\Big)\, dx.$$

PROOF . Fix $B_r \subset \Omega$, let $\frac{r}{2} < t < s < r$ and take a cut-off function $\zeta \in C^1_0(B_s)$ such that $0 \le \zeta \le 1$, $\zeta = 1$ on B_t and $|D\zeta| \le \frac{2}{s-t}$. If we set

$$\varphi_1 = [u - (u)_r]\zeta, \qquad \varphi_2 = [u - (u)_r](1 - \zeta),$$

then $D\varphi_1 + D\varphi_2 = Du$, and

$$\gamma \int_{B_s} [|D\varphi_1|^2 + \lambda^{p-2}|D\varphi_1|^p]\, dx \\ \leq \int_{B_s} g(D\varphi_1)\, dx = \int_{B_s} g(Du - D\varphi_2)\, dx. \tag{2.7}$$

In addition, by the minimality of u,

$$\int_{B_s} g(Du)\, dx \\ \leq \int_{B_s} g(Du - D\varphi_1)\, dx + \nu \int_{B_s} |D\varphi_1|\, dx \\ \leq \int_{B_s \setminus B_t} g(D\varphi_2)\, dx + \frac{\gamma}{2} \int_{B_s} |D\varphi_1|^2\, dx + \frac{\nu^2}{2\gamma} \text{meas}\,(B_r).$$

Then

$$\int_{B_s} g(Du - D\varphi_2)\, dx \\ = \int_{B_s} g(Du)\, dx + \int_{B_s} [g(Du - D\varphi_2) - g(Du)]\, dx \\ \leq \int_{B_s \setminus B_t} g(D\varphi_2)\, dx + \frac{\gamma}{2} \int_{B_s} |D\varphi_1|^2\, dx + \frac{\nu^2}{2\gamma} \text{meas}\,(B_r) \\ + c \int_{B_s \setminus B_t} [|Du| + |D\varphi_2| \\ + \lambda^{p-2}(|Du|^{p-1} + |D\varphi_2|^{p-1})]|D\varphi_2|\, dx. \tag{2.8}$$

By (2.7),(2.8) and the assumptions on g it then follows

$$\int_{B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]\, dx \\ \leq \int_{B_s} [|D\varphi_1|^2 + \lambda^{p-2}|D\varphi_1|^p]\, dx \\ \leq c(\gamma, c_1) \Big[\nu^2 r^n + \int_{B_s \setminus B_t} [|Du|^2 + |D\varphi_2|^2$$

$$
\begin{aligned}
&+ \lambda^{p-2}(|Du|^p + |D\varphi_2|^p)]\,dx\Big] \\
&\le \tilde{c}\Big[\nu^2 r^n + \int_{B_s\setminus B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]\,dx \\
&\quad + \int_{B_s\setminus B_t} \Big(\frac{|u-(u)_r|^2}{(s-t)^2} + \lambda^{p-2}\frac{|u-(u)_r|^p}{(s-t)^p}\Big)\,dx\Big].
\end{aligned}
$$

We fill the hole by adding to both sides the term

$$\tilde{c}\int_{B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]\,dx;$$

then we divide by $\tilde{c}+1$, thus obtaining

$$
\begin{aligned}
&\int_{B_t} [|Du|^2 + \lambda^{p-2}|Du|^p]\,dx \\
&\quad \le \vartheta \int_{B_s} [|Du|^2 + \lambda^{p-2}|Du|^p]\,dx \\
&\quad\quad + c\int_{B_r} \Big[\nu^2 + \frac{|u-(u)_r|^2}{(s-t)^2} + \lambda^{p-2}\frac{|u-(u)_r|^p}{(s-t)^p}\Big]\,dx,
\end{aligned}
$$

with $\vartheta < 1$, and the result follows by Lemma 2.4. ∎

In the sequel we assume that f satisfies (2.1),(2.2),(2.3), and that ξ_0 is a regular point for f, so that (2.4) holds in $B_\varrho(\xi_0)$ and we may assume that $f \in C^2(B_{4\varrho})$. If u is a minimizer of $I(u) = \int_\Omega f(Du)\,dx$, for every $B_r(x_0) \subset \Omega$ we define

$$U(x_0,r) = -\!\!\!\!\!\!\int_{B_r(x_0)} g_p\big(Du - (Du)_r\big)\,dx.$$

The main ingredient to prove Theorem 2.1 is the following decay estimate:

Proposition 2.6 . *There is a constant C, depending only on ξ_0, such that for every $\tau < 1/4$ there exists $\varepsilon(\tau)$ such that if u is a minimizer of $I(u)$ and*

$$|(Du)_{x_0,r} - \xi_0| < \varrho, \qquad |(Du)_{x_0,\tau r} - \xi_0| < \varrho, \qquad U(x_0,r) < \varepsilon$$

then

$$U(x_0,\tau r) < C\tau^2 U(x_0,r).$$

Proof . Fix τ; we shall determine C later. Reasoning by contradiction, we assume that there is a sequence of balls $B_{r_h}(x_h) \subset \Omega$ satisfying

$$|(Du)_{x_h,r_h} - \xi_0| < \varrho, \qquad |(Du)_{x_h,\tau r_h} - \xi_0| < \varrho, \qquad U(x_h,r_h) = \lambda_h^2 \to 0,$$

and

$$U(x_h,\tau r_h) > C\tau^2\lambda_h^2. \tag{2.9}$$

Set $A_h = (Du)_{x_h,r_h}$, $a_h = (u)_{x_h,r_h}$, $\tilde{A}_h = (Du)_{x_h,\tau r_h}$, and

$$v_h(z) = \frac{u(x_h + r_h z) - a_h - r_h A_h z}{\lambda_h r_h},$$

so that

$$(v_h)_{0,1} = 0, \qquad \fint_{B_1} \lambda_h^{-2} g_p(\lambda_h D v_h)\,dz = \fint_{B_1} (|Dv_h|^2 + \lambda_h^{p-2}|Dv_h|^p)\,dz = 1.$$

Then we may assume

$$v_h \rightharpoonup v \quad \text{weakly in } W^{1,2}(B_1;\mathbb{R}^N),$$

$$\lambda_h^{1-2/p} v_h \rightharpoonup 0 \quad \text{weakly in } W^{1,p}(B_1;\mathbb{R}^N)$$

and also $A_h \to A$. Set $L_h = \{z \in B_1 : \lambda_h |Dv_h(z)| \geq \varrho\}$, $S_h = B_1 \setminus L_h$; then

$$\operatorname{meas} L_h \leq \lambda_h^2 \varrho^{-2}, \qquad \int_{L_h} |Dv_h|^{p-1}\,dz \leq \lambda_h^{3-p}\varrho^{-2/p}. \tag{2.10}$$

Now fix $\varphi \in C_0^1(B_1;\mathbb{R}^N)$: by the minimality of u

$$\int_{B_1} [f(\lambda_h Dv_h + A_h + tD\varphi) - f(\lambda_h Dv_h + A_h)]\,dz \geq 0.$$

Dividing by $t > 0$ we have

$$\begin{aligned}
\frac{1}{t}\int_{S_h} &[f(\lambda_h Dv_h + A_h + tD\varphi) - f(\lambda_h Dv_h + A_h)]\,dz \\
&\geq -\int_{L_h} \frac{f(\lambda_h Dv_h + A_h + tD\varphi) - f(\lambda_h Dv_h + A_h)}{t}\,dz \\
&\geq -c_\varphi \int_{L_h} (1 + \lambda_h^{p-1}|Dv_h|^{p-1})\,dz \\
&\geq -c_{\varphi,\varrho}\,\lambda_h^2
\end{aligned}$$

by (2.10). If t is smaller than $\varrho/\|\varphi\|_{C^1}$, in the first integral above the argument of f is always in $B_{3\varrho}(\xi_0)$, and the integrand is bounded, therefore as $t \to 0$ we have

$$\int_{S_h} Df(\lambda_h Dv_h + A_h) D\varphi \, dz \geq -c\lambda_h^2.$$

Again by (2.10)

$$\frac{1}{\lambda_h} \int_{S_h} [Df(\lambda_h Dv_h + A_h) - Df(A_h)] D\varphi \, dz \\ \geq -c\lambda_h + \frac{1}{\lambda_h} \int_{L_h} Df(A_h) D\varphi \, dz \geq -c\lambda_h.$$

This may be written also as

$$\int_{B_1} \int_0^1 D^2 f(A_h + s\lambda_h Dv_h) \mathbf{1}_{S_h} Dv_h D\varphi \, ds \, dz \geq -c\lambda_h, \tag{2.11}$$

and remarking that $\lambda_h Dv_h \to 0$ a.e. we have as $h \to \infty$

$$\int_{B_1} D^2 f(A) Dv D\varphi \, dz \geq 0,$$

which yields $\int_{B_1} D^2 f(A) Dv D\varphi \, dz = 0$ for every $\varphi \in C_0^1(B_1; \mathbb{R}^N)$. Then v solves a linear system with constant coefficients; remarking that (2.4) implies $D^2 f(A)\lambda_i \lambda_j \eta_\alpha \eta_\beta \geq \frac{\gamma}{2} |\lambda|^2 |\eta|^2$, by the standard regularity theory we have for every $\tau < 1/4$

$$\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \leq c\tau^2. \tag{2.12}$$

Set

$$w_h = v_h - \frac{\tilde{A}_h - A_h}{\lambda_h} z$$

and remark that

$$\frac{\tilde{A}_h - A_h}{\lambda_h} = (Dv_h)_\tau, \qquad (w_h)_s = (v_h)_s \quad \text{for all } s < 1,$$

and w_h minimizes $\int_{B_1} f_{\tilde{A}_h,\lambda_h}(D\psi)\,dz$; therefore Lemma 2.5 holds with $\nu = 0$, and

$$\begin{aligned}
U(x_h,\tau r_h) &= \fint_{B_{\tau r_h}(x_h)} g_p(Du)\,dx \\
&= \fint_{B_\tau} g_p\big[\lambda_h\big(Dv_h - (Dv_h)_\tau\big)\big]\,dz \\
&= \fint_{B_\tau} g_p(\lambda_h Dw_h)\,dz \\
&\le \tilde{c}\fint_{B_{2\tau}} g_p\left(\lambda_h \frac{w_h - (w_h)_{2\tau}}{\tau}\right)dz \\
&= \tilde{c}\fint_{B_{2\tau}} g_p\left(\lambda_h \frac{v_h - (v_h)_{2\tau} - (Dv_h)_\tau z}{\tau}\right)dz \\
&= \tilde{c}\lambda_h^2\bigg[\fint_{B_{2\tau}} \frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau z|^2}{\tau^2}\,dz \\
&\qquad\qquad + \lambda_h^{p-2}\fint_{B_{2\tau}} \frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau z|^p}{\tau^p}\,dz\bigg] \\
&= \tilde{c}\lambda_h^2(I_h^1 + I_h^2).
\end{aligned} \tag{2.13}$$

By the Sobolev-Poincaré inequality and (2.12)

$$\begin{aligned}
\lim_{h\to\infty} I_h^1 &= \tau^{-2}\fint_{B_{2\tau}} |v - (v)_{2\tau} - (Dv)_\tau z|^2\,dz \\
&\le c\fint_{B_{2\tau}} |Dv - (Dv)_\tau|^2\,dz \\
&\le c\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2\,dz + c|(Dv)_\tau - (Dv)_{2\tau}|^2 \\
&\le c\tau^2 + c\fint_{B_\tau} |Dv - (Dv)_{2\tau}|^2\,dz \\
&\le c\tau^2 + c\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2\,dz \\
&\le \hat{c}\tau^2,
\end{aligned} \tag{2.14}$$

whereas if $\frac{\vartheta}{2}+\frac{1-\vartheta}{p^*}=\frac{1}{p}$ we have

$$
\begin{aligned}
I_h^2 &\leq \lambda_h^{p-2}\left(\fint_{B_{2\tau}} \tau^{-2}|v_h-(v_h)_{2\tau}-(Dv_h)_\tau z|^2\,dz\right)^{p\vartheta/2}\cdot \\
&\quad\cdot\left(\fint_{B_{2\tau}} \tau^{-p^*}|v_h-(v_h)_{2\tau}-(Dv_h)_\tau z|^{p^*}\,dz\right)^{p(1-\vartheta)/p^*} \\
&\leq c\lambda_h^{p-2}(I_h^1)^{p\vartheta/2}\left(\fint_{B_{2\tau}} |Dv_h-(Dv_h)_\tau|^p\,dz\right)^{1-\vartheta} \\
&\leq c\tau^{p\vartheta}\lambda_h^{(p-2)\vartheta}\left[\fint_{B_{2\tau}} \lambda_h^{p-2}|Dv_h|^p\,dz+\fint_{B_\tau} \lambda_h^{p-2}|Dv_h|^p\,dz\right] \\
&\leq c_\tau\lambda_h^{(p-2)\vartheta},
\end{aligned}
$$

so that

$$\lim_{h\to\infty} I_h^2=0. \tag{2.15}$$

By (2.13),(2.14),(2.15) we get

$$\limsup_{h\to\infty}\lambda_h^{-2}U(x_h,\tau r_h)\leq\tilde{c}\hat{c}\tau^2,$$

which contradicts (2.9) if we chose $C>\tilde{c}\hat{c}$.

The fact (which we do not need in the sequel) that C does not depend on the particular minimizer u, could have been proven by taking a different minimizer u_h in each $B_{r_h}(x_h)$. ∎

Proposition 2.7 . *Let ξ_0 be a regular point for f, and take $\alpha<1$; if C is as in Proposition* 2.6, *fix $\tau<1/4$ such that $C\tau^2<\tau^{2\alpha}$. Let u be a minimizer of I and assume that for some $B_r(x_0)\subset\Omega$*

$$|(Du)_{x_0,r}-\xi_0|<\varrho/2, \qquad |(Du)_{x_0,\tau r}-\xi_0|<\varrho/2, \qquad U(x_0,r)<\delta(\tau),$$

where $\delta(\tau)<\min\{\varepsilon(\tau),\varrho^2\tau^{2n}(1-\tau^\alpha)^2/4\}$. Then for every k

$$|(Du)_{x_0,\tau^{k+1}r}-\xi_0|<\varrho, \qquad U(x_0,\tau^k r)<\tau^{2k\alpha}U(x_0,r). \tag{2.16}$$

PROOF . The result is true for $k=0$; we proceed by induction, assuming (2.16) holds for $0\leq k\leq m-1$. Then $U(x_0,\tau^{m-1}r)<\varepsilon(\tau)$, and by Proposition 2.6 we have

$$U(x_0,\tau^m r)\leq C\tau^2U(x_0,\tau^{m-1}r)\leq\tau^{2\alpha}\tau^{2(m-1)\alpha}U(x_0,r)=\tau^{2m\alpha}U(x_0,r).$$

Now,

$$\begin{aligned}|(Du)_{\tau^{m+1}r}-\xi_0| &\le |(Du)_r-\xi_0|+\sum_{k=0}^{m}|(Du)_{\tau^{k+1}r}-(Du)_{\tau^k r}|\\ &\le \frac{\varrho}{2}+\sum_{k=0}^{m}\fint_{B_{\tau^{k+1}r}}|Du-(Du)_{\tau^k r}|\,dx\\ &\le \frac{\varrho}{2}+\tau^{-n}\sum_{k=0}^{m}\Big[\fint_{B_{\tau^k r}}|Du-(Du)_{\tau^k r}|^2\,dx\Big]^{1/2}\\ &\le \frac{\varrho}{2}+\tau^{-n}\sum_{k=0}^{m}[U(x_0,\tau^k r)]^{1/2}\\ &\le \frac{\varrho}{2}+\tau^{-n}\sum_{k=0}^{m}\tau^{k\alpha}[U(x_0,r)]^{1/2}\\ &\le \frac{\varrho}{2}+\frac{1}{\tau^n(1-\tau^\alpha)}[\delta(\tau)]^{1/2}<\varrho,\end{aligned}$$

thus concluding the proof. ∎

Proof of Theorem 2.1 . Suppose ξ_0 is a regular point and

$$\lim_{r\to 0}\fint_{B_r(x_0)}|Du-\xi_0|^p\,dx=0;$$

fix a particular $\hat\alpha<1$: for a suitable $r>0$ the assumptions of Proposition 2.7 are verified uniformly in a neighbourhood of x_0, i.e.,

$$|(Du)_{x,r}-\xi_0|<\varrho,\qquad |(Du)_{x,\tau r}-\xi_0|<\varrho,\qquad U(x,r)<\varepsilon$$

for all $x\in B_s(x_0)$. Then (2.16) implies

$$U(x,\varrho)\le c\Big(\frac{\varrho}{r}\Big)^{2\hat\alpha}U(x,r)\quad\text{for all } x\in B_s(x_0),$$

and $u\in C^{1,\hat\alpha}\big(B_s(x_0)\big)$ by a standard argument — see e.g. [8], Chapter 3.

Since Du is now continuous, by Lemma 2.2 we may suppose that s is so small that $Du(x)$ is a regular point for f for all $x\in B_s(x_0)$; moreover, clearly

$$\lim_{r\to 0}\fint_{B_r(x)}|Du(y)-Du(x)|^p\,dy=0\quad\text{for all } x\in B_s(x_0).$$

Now fix any $\alpha<1$: the same argument employed above shows that $u\in C^{1,\alpha}$ in a neighbourhood of x for all $x\in B_s(x_0)$, therefore $u\in C^{1,\alpha}\big(B_s(x_0)\big)$ for all $\alpha<1$. ∎

3 . The case with (x,u)

Let Ω be a bounded open subset of $\mathbb{R}^n$, fix $p \geq 2$ and assume that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfies:

$$f(x,s,\xi) \text{ is locally Lipschitz continuous with respect to } \xi; \tag{3.1}$$

$$|f(x,s,\xi)| \leq L(1+|\xi|^p); \tag{3.2}$$

$$|f_\xi(x,s,\xi)| \leq L(1+|\xi|^{p-1}); \tag{3.3}$$

$$|f(x,s,\xi) - f(y,t,\xi)| \leq L(1+|\xi|^p)\,\omega(|x-y|^p + |s-t|^p), \tag{3.4}$$

where $\omega(t) \leq t^\delta$, $0 < \delta < 1/p$, and ω is bounded, concave and increasing;

$$f(x,s,\xi) \geq \psi(\xi) \tag{3.5}$$

for a suitable continuous function ψ satisfying

$$\int_\Omega \psi\big(D\varphi(y)\big)\,dy \geq \int_\Omega [\psi(0) + \mu|D\varphi(y)|^p]\,dy \quad \text{for all } \varphi \in C_0^1(\Omega;\mathbb{R}^N),$$

with $\mu > 0$; finally, we assume that

$$\text{either } f \geq 0 \text{ or } f \text{ is quasiconvex.} \tag{3.6}$$

We remark that if f is quasiconvex then (3.3) is implied by (3.2); assumption (3.5) was introduced in [12].

We say that (x_0, s_0, ξ_0) is a regular point for f if there exist $\sigma > 0$, $\gamma > 0$ such that for every $x \in B_\sigma(x_0)$ and $s \in B_\sigma(s_0)$ the function $f(x,s,\cdot)$ is of class C^2 in $B_\sigma(\xi_0)$, and

$$\int_{\operatorname{spt}\varphi} f\big(x,s,\xi + D\varphi(y)\big)\,dy \geq \int_{\operatorname{spt}\varphi} [f(x,s,\xi) + \gamma(|D\varphi(y)|^2 + |D\varphi(y)|^p)]\,dy$$

for every $x \in B_\sigma(x_0)$, $s \in B_\sigma(s_0)$, $\xi \in B_\sigma(\xi_0)$ and $\varphi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N)$.

Set for every $u \in W^{1,p}(\Omega;\mathbb{R}^N)$

$$I(u) = \int_\Omega f\big(x, u(x), Du(x)\big)\,dx;$$

then we have:

Theorem 3.1 . *Let f satisfy (3.1),...,(3.6) and let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a minimizer of I. Then there exists $\alpha \in (0,1)$ such that if for some regular point (x_0, s_0, ξ_0) of f we have*

$$\lim_{r\to 0} \fint_{B_r(x_0)} [|u(x) - s_0|^p + |Du(x) - \xi_0|^p]\, dx = 0$$

then u is of class $C^{1,\alpha}$ in a neighbourhood of x_0.

In the proof we shall use the following results:

Lemma 3.2 . *Let (X, d) be a metric space, and $J : X \to [0, +\infty]$ a lower semicontinuous functional not identically $+\infty$. If*

$$J(u) \leq \alpha + \inf J,$$

there is a $v \in X$ such that

$$d(u, v) \leq 1$$

and

$$J(v) \leq J(w) + \alpha d(v, w) \quad \text{for all } w \in X.$$

The result above may be found in [4], the next one in [8], p. 122.

Lemma 3.3 . *Let Q be a cube in $\mathbb{R}^n$, and suppose that for every ball $B_r(x_0) \subset Q$ such that $2r < \min\{r_0, \operatorname{dist}(x_0, \partial Q)\}$*

$$\fint_{B_{r/2}} g^q\, dx \leq a\Big(\fint_{B_r} g\, dx\Big)^q + \fint_{B_r} f^q\, dx$$

with $f \in L^k(Q)$, $k > q$. Then $g \in L^{q+\epsilon}_{\text{loc}}(Q)$ for some positive $\epsilon(a, q, k)$ and

$$\Big(\fint_{B_{r/2}} g^{q+\epsilon}\, dx\Big)^{1/(q+\epsilon)} \leq c\bigg[\Big(\fint_{B_r} g^q\, dx\Big)^{1/q} + \Big(\fint_{B_r} f^{q+\epsilon}\, dx\Big)^{1/(q+\epsilon)}\bigg].$$

Lemma 3.4 . *Let f satisfy (3.1),(3.2),(3.3),(3.5); there are $q_0 > p$ and $c_0 > 0$, depending only on μ, L, p, such that if $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizer of I, then $u \in W^{1,q_0}_{\text{loc}}(\Omega;\mathbb{R}^N)$ and for every $B_r \subset \Omega$*

$$\Big(\fint_{B_{r/2}} |Du|^{q_0}\, dx\Big)^{1/q_0} \leq c_0\Big(\fint_{B_r} (1 + |Du|^p)\, dx\Big)^{1/p}.$$

PROOF . The argument is similar to Lemma 2.5; fix $B_r \subset \Omega$, let $\frac{r}{2} < t < s < r$, take the cut-off function ζ of 2.5, and again set

$$\varphi_1 = [u - (u)_r]\zeta, \qquad \varphi_2 = [u - (u)_r](1 - \zeta);$$

then $\varphi_1 + \varphi_2 = u - (u)_r$ and $D\varphi_1 + D\varphi_2 = Du$. Now, by (3.5)

$$\begin{aligned} \int_{B_s} [\mu |D\varphi_1|^p + \psi(0)]\, dx \\ &\leq \int_{B_s} \psi(D\varphi_1)\, dx \\ &\leq \int_{B_s} f(x, u, D\varphi_1)\, dx \\ &= \int_{B_s} f(x, u, Du - D\varphi_2)\, dx. \end{aligned} \tag{3.7}$$

By the minimality of u we have

$$\begin{aligned} \int_{B_s} f(x, u, Du)\, dx \\ &\leq \int_{B_s} f(x, u - \varphi_1, Du - D\varphi_1)\, dx \\ &= \int_{B_s} f(x, \varphi_2 + (u)_r, D\varphi_2)\, dx \\ &= \int_{B_s \setminus B_t} f(x, \varphi_2 + (u)_r, D\varphi_2)\, dx + \int_{B_t} f(x, (u)_r, 0)\, dx, \end{aligned}$$

so that by (3.2)

$$\int_{B_s} f(x, u, Du)\, dx \leq L \int_{B_s \setminus B_t} |D\varphi_2|^p\, dx + cr^n,$$

and by (3.3)

$$\begin{aligned}\int_{B_s} & f(x,u,Du-D\varphi_2)\,dx \\ &= \int_{B_s} f(x,u,Du)\,dx \\ &\quad + \int_{B_s} [f(x,u,Du-D\varphi_2)-f(x,u,Du)]\,dx \\ &\le \int_{B_s} f(x,u,Du)\,dx \\ &\quad + c\int_{B_s\setminus B_t} (1+|Du|^{p-1}+|D\varphi_2|^{p-1})|D\varphi_2|\,dx \\ &\le cr^n + c\int_{B_s\setminus B_t} (|D\varphi_2|^p+|Du|^p)\,dx \\ &\le cr^n + c\int_{B_s\setminus B_t} \left(|Du|^p + \frac{|u-(u)_r|^p}{(s-t)^p}\right) dx.\end{aligned}$$

Then by (3.7) we obtain

$$\mu\int_{B_s} |D\varphi_1|^p\,dx \le c\int_{B_s\setminus B_t} |Du|^p\,dx + c\int_{B_r}\left(1+\frac{|u-(u)_r|^p}{(s-t)^p}\right) dx, \quad (3.8)$$

therefore

$$\int_{B_t} |Du|^p\,dx \le c\int_{B_s\setminus B_t} |Du|^p\,dx + c\int_{B_r}\left(1+\frac{|u-(u)_r|^p}{(s-t)^p}\right) dx;$$

we fill the hole, and by Lemma 2.4 we obtain

$$⨍_{B_{r/2}} |Du|^p\,dx \le c⨍_{B_r}\left(1+\frac{|u-(u)_r|^p}{r^p}\right) dx \le c\left(⨍_{B_r}(1+|Du|^{p_*})\,dx\right)^{p/p_*},$$

where $p_* = np/(n+p)$. The conclusion then follows by Lemma 3.3. ■

Lemma 3.5 . *Let f satisfy (3.1),(3.2),(3.3),(3.5) and fix any $(\hat{x},\hat{s})$. Let $B\subset\Omega$ be a ball, and let $u\in W^{1,q}(B;\mathbb{R}^N)$ with $q>p$. There exist $q_0\in$*

(p,q) and $c_0>0$, depending only on μ, L, p, q, such that if $0\le\eta<2^{-p}\mu$ and $v\in u+W_0^{1,p}(B;\mathbb{R}^N)$ satisfies

$$\int_B f(\hat{x},\hat{s},Dv)\,dx \le \int_B [f(\hat{x},\hat{s},Dv+D\varphi)+\eta|D\varphi|]\,dx \quad \text{for all } \varphi\in W_0^{1,p}(B;\mathbb{R}^N)$$

then $v\in W^{1,q_0}(B;\mathbb{R}^N)$ and

$$\left(⨍_B |Dv|^{q_0}\,dx\right)^{1/q_0} \le c_0\left[\left(⨍_B (1+|Dv|^p)\,dx\right)^{1/p}+\left(⨍_B (1+|Du|^q)\,dx\right)^{1/q}\right].$$

PROOF . We begin with the interior estimate; fix any ball $B_r(x_0)\subset B$, and let t, s and ζ be as in Lemma 3.4; define

$$\varphi_1=[v-(v)_r]\zeta, \qquad \varphi_2=[v-(v)_r](1-\zeta).$$

Following the proof of Lemma 3.4, an additional term $\eta\int_{B_s}|D\varphi_1|\,dx$ appears, and instead of (3.8) we are led to

$$\mu\int_{B_s}|D\varphi_1|^p\,dx \le c\int_{B_s\setminus B_t}|Dv|^p\,dx + c\int_{B_r}\left(1+\frac{|v-(v)_r|^p}{(s-t)^p}\right)dx + \eta\int_{B_s}|D\varphi_1|\,dx;$$

by the bounds on η we have

$$\eta\int_{B_s}|D\varphi_1|\,dx \le \frac{\mu}{2}\int_{B_s}(1+|D\varphi_1|^p)\,dx,$$

and we may conclude, as in Lemma 3.4, that if $B_r(x_0)\subset B$ then

$$⨍_{B_{r/2}}|Dv|^p\,dx \le c\left(⨍_{B_r}(1+|Dv|^{p_*})\,dx\right)^{p/p_*}. \tag{3.9}$$

Now we estimate v near the boundary: assume $B_r(x_0)\cap B\ne\emptyset$, $B_r(x_0)\cap(\mathbb{R}^n\setminus B)\ne\emptyset$, and fix t, s, ζ as before; define

$$\varphi_1=(v-u)\zeta, \qquad \varphi_2=(v-u)(1-\zeta),$$

so that $\varphi_1 \in W_0^{1,p}(B_s \cap B)$. Following again the proof of Lemma 3.4 we find

$$\begin{aligned}
\mu \int_{B_s \cap B} |D\varphi_1|^p \, dx& \\
\le \int_{B_s \cap B} &\big\{ -\psi(0) + f(\hat{x}, \hat{s}, Du - D\varphi_2) \\
&\quad + \big[f\big(\hat{x}, \hat{s}, Dv - (Du + D\varphi_2)\big) - f(\hat{x}, \hat{s}, Dv) \big] \big\} \, dx \\
+ \eta \int_{B_s \cap B} &|D\varphi_1| \, dx.
\end{aligned}$$

The last integral is dealt with as above, and using (3.2),(3.3) we have

$$\begin{aligned}
\frac{\mu}{2} \int_{B_t \cap B} &|Dv - Du|^p \, dx \\
&\le c \int_{B_s \cap B} [1 + |Du|^p + |D\varphi_2|^p] \, dx \\
&\quad + c \int_{B_s \cap B} (|Dv - Du|^{p-1} + |D\varphi_2|^{p-1} + |Dv|^{p-1}) \cdot \\
&\qquad \cdot |Du + D\varphi_2| \, dx \\
&\le c_\mu \int_{B_s \cap B} (1 + |Du|^p + |D\varphi_2|^p) \, dx \\
&\quad + \frac{\mu}{4} \int_{B_s \cap B} |Dv - Du|^p \, dx,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{B_t \cap B} &|Dv - Du|^p \, dx \\
&\le c \int_{(B_s \setminus B_t) \cap B} |Dv - Du|^p \, dx \\
&\quad + c \int_{B_r \cap B} \left(1 + |Du|^p + \frac{|u - v|^p}{(s - t)^p} \right) dx.
\end{aligned}$$

The usual hole-filling argument and Lemma 2.4 yield

$$\int_{B_{r/2} \cap B} |Dv - Du|^p \, dx \le c \int_{B_r \cap B} (1 + |Du|^p) \, dx + c r^{-p} \int_{B_r \cap B} |v - u|^p \, dx.$$

Since $v-u$ can be extended as zero outside B, and since the measure of $B_{2r} \setminus B$ is greater than $c_n r^n$, we may apply a modification of Sobolev-Poincaré inequality, and we have

$$\int_{B_r \cap B} |v-u|^p \, dx \le c \Big(\int_{B_{2r} \cap B} |Dv - Du|^{p_*} \, dx \Big)^{p/p_*},$$

so that

$$\begin{aligned} &\int_{B_{r/2} \cap B} |Dv|^p \, dx \\ &\qquad \le c \int_{B_{2r} \cap B} (1 + |Du|^p) \, dx + c r^{-p} \Big(\int_{B_{2r} \cap B} |Dv|^{p_*} \, dx \Big)^{p/p_*}. \end{aligned} \tag{3.10}$$

Then if we set

$$V(x) = \begin{cases} |Dv|^{p_*} & \text{in } B \\ 0 & \text{outside } B \end{cases} \qquad U(x) = \begin{cases} 1 + |Du|^p & \text{in } B \\ 0 & \text{outside } B \end{cases}$$

by (3.9),(3.10) we have for any ball B_r in $\mathbb{R}^n$

$$\fint_{B_{r/2}} V^{p/p_*} \, dx \le c \Big(\fint_{B_{2r}} V \, dx \Big)^{p/p_*} + c \fint_{B_{2r}} U \, dx,$$

with $U \in L^{q/p}(\mathbb{R}^n)$. Aplying Lemma 3.3 the result follows. ∎

Lemma 3.6 . *Let f satisfy* (3.1),(3.2),(3.5) *and fix any* $(\hat{x}, \hat{s})$. *If B is any ball in $\mathbb{R}^n$, and $u \in W^{1,p}(B; \mathbb{R}^N)$, then the functional $\int_B f\big(\hat{x}, \hat{s}, Dw(x)\big) \, dx$ satisfies*

$$\int_B f\big(\hat{x}, \hat{s}, Dw(x)\big) \, dx \ge \mu \int_B |Dw|^p \, dx - c \int_B (1 + |Du|^p) \, dx$$

for every $w \in u + W_0^{1,p}(B; \mathbb{R}^N)$; moreover, if f is also quasiconvex with respect to ξ, then the functional $\int_B f\big(\hat{x}, \hat{s}, Dw(x)\big) \, dx$ is sequentially weakly semicontinuous on $u + W_0^{1,p}(B; \mathbb{R}^N)$.

PROOF . The semicontinuity on the Dirichlet classes follows from [14], Theorem 5.

Let B' be the ball with same center as B, and twice the radius, and let $\tilde{u} \in (u)_B + W_0^{1,p}(B';\mathbb{R}^N)$ be an extension of u such that $\int_{B'} |D\tilde{u}|^p \, dx \le c \int_B |Du|^p \, dx$; if we set for every $w \in u + W_0^{1,p}(B;\mathbb{R}^N)$

$$\tilde{w} = \begin{cases} w & \text{in } B \\ \tilde{u} & \text{in } B' \setminus B, \end{cases}$$

then by (3.5)

$$\begin{aligned} &\int_{B'} [\mu |D\tilde{w}|^p + \psi(0)] \, dx \\ &\qquad \le \int_{B'} f(\hat{x}, \hat{s}, D\tilde{w}) \, dx \\ &\qquad \le \int_B f(\hat{x}, \hat{s}, Dw) \, dx + c \int_{B' \setminus B} (1 + |D\tilde{u}|^p) \, dx, \end{aligned}$$

and the result follows. ■

Lemma 3.7 . *Let f satisfy* (3.1), …,(3.6). *There exist two constants, $0 < \beta_1 < \beta_2 < 1$, a radius $r_0 < 1$, and for every $K > 0$ a constant c_K, such that if u is a minimizer of I, $r < r_0$, $B_{2r}(x_0) \subset \Omega$ and $(|Du|^p)_{x_0,2r} \le K$ then there is a $v \in u + W_0^{1,p}\big(B_r(x_0);\mathbb{R}^N\big)$ such that*

$$\left(\fint_{B_r} |Dv - Du|^p \, dx \right)^{1/p} \le c_K r^{\beta_1}$$

and

$$\begin{aligned} &\int_{B_r} f\big(x_0, (u)_{x_0}, r), Dv(x)\big) \, dx \\ &\qquad \le \int_{B_r} f\big(x_0, (u)_{x_0}, r), Dv(x) + D\varphi(x)\big) \, dx + r^{\beta_2} \int_{B_r} |D\varphi(x)| \, dx \end{aligned}$$

for every $\varphi \in C_0^1\big(B_r(x_0);\mathbb{R}^N\big)$.

Proof . By Lemma 3.4 and the minimality of u follows the existence of $q_0 > p$ and $c_0 > 0$ such that $u \in W^{1,q_0}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and

$$\left(\fint_{B_r} |Du|^{q_0} \, dx \right)^{1/q_0} \le c_0 \left(\fint_{B_{2r}} (1 + |Du|^p) \, dx \right)^{1/p} \tag{3.11}$$

for every $B_{2r} \subset \Omega$. Set

$$I_r^0(w) = \fint_{B_r(x_0)} f\big(x_0, (u)_{x_0,r}, Dw(x)\big)\, dx$$

and

$$J(w) = \begin{cases} I_r^0(w) & \text{if } w \in u + W_0^{1,p}(B_r; \mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

We claim that

$$I_r^0(u) - \inf J \le \tilde{c}(K) r^\beta, \tag{3.12}$$

where $\beta < 1$ depends only on δ, L, p, μ.

First case . We assume that in (3.6) the condition $f \ge 0$ holds.

Denote by (u_h) a sequence in $u + W_0^{1,p}(B_r; \mathbb{R}^N)$ such that

$$I_r^0(u_h) - \inf J \le 1/h,$$

and consider the functional J on the space $u + W_0^{1,1}(B_r; \mathbb{R}^N)$ endowed with the metric

$$d_1(v, w) = \fint_{B_r} |Dv - Dw|\, dx.$$

Since $f \ge 0$, by Fatou's lemma J is semicontinuous in this space; we may then apply Lemma 3.2, so that there exists a sequence (v_h) in $u + W_0^{1,1}(B_r; \mathbb{R}^N)$ such that

$$\fint_{B_r} |Dv_h - Du_h|\, dx \le 1$$

and

$$J(v_h) \le J(w) + \frac{1}{h} \fint_{B_r} |Dv_h - Dw|\, dx \quad \text{for all } w \in W_0^{1,p}(B_r; \mathbb{R}^N). \tag{3.13}$$

In particular $J(v_h) \le J(u_h) + 1/h$, hence

$$\lim_{h\to\infty} J(v_h) = \inf J \tag{3.14}$$

and $J(v_h)$ is finite. Therefore $v_h \in u + W_0^{1,p}(B_r; \mathbb{R}^N)$ and by Lemma 3.6

$$\fint_{B_r} |Dv_h|^p\, dx \le c \fint_{B_r} (1 + |Du|^p)\, dx. \tag{3.15}$$

Moreover by (3.13) we may apply Lemma 3.5 for h large enough, and there exist c_1 and $q_1 \in (p, q_0)$ such that $v_h \in W^{1,q_1}(B_r;\mathbb{R}^N)$ and

$$\begin{aligned}\biggl(\fint_{B_r} |Dv_h|^{q_1}\,dx\biggr)^{1/q_1} & \\ &\le c_1\biggl(\fint_{B_r}(1+|Dv_h|^p)\,dx\biggr)^{1/p} + \biggl(\fint_{B_r}(1+|Du|^{q_0})\,dx\biggr)^{1/q_0} \\ &\le c\biggl(\fint_{B_{2r}}(1+|Du|^p)\,dx\biggr)^{1/p},\end{aligned} \tag{3.16}$$

where we used (3.15) and (3.11). Now

$$\begin{aligned}I_r^0(u) - I_r^0(v_h) &\le \fint_{B_r}\bigl[f\bigl(x_0,(u)_r,Du\bigr) - f(x,u,Du)\bigr]\,dx \\ &\quad + \fint_{B_r}[f(x,u,Du) - f(x,v_h,Dv_h)]\,dx \\ &\quad + \fint_{B_r}\bigl[f(x,v_h,Dv_h) - f\bigl(x_0,(u)_r,Dv_h\bigr)\bigr]\,dx \\ &= a_h + b_h + c_h.\end{aligned}$$

Then by (3.4)

$$\begin{aligned}|a_h| &\le \fint_{B_r} L(1+|Du|^p)\,\omega(r^p + |u-(u)_r|^p)\,dx \\ &\le L\biggl(\fint_{B_r}(1+|Du|^{q_0})\,dx\biggr)^{p/q_0}\cdot \\ &\qquad\cdot\biggl(\fint_{B_r}\omega^{q_0/(q_0-p)}(r^p+|u-(u)_r|^p)\,dx\biggr)^{1-p/q_0}.\end{aligned}$$

Since ω is bounded and concave,

$$\begin{aligned}|a_h| &\le c\biggl(\fint_{B_r}(1+|Du|^{q_0})\,dx\biggr)^{p/q_0}\omega^{1-p/q_0}\biggl(cr^p\fint_{B_r}(1+|Du|^p)\,dx\biggr) \\ &\le c\biggl(\fint_{B_{2r}}(1+|Du|^p)\,dx\biggr)^{1+\delta(q_0-p)/q_0} r^{\delta p(q_0-p)/q_0} \\ &\le c_1(K)r^{\delta p(q_0-p)/q_0}\end{aligned}$$

by (3.11). Analogously

$$|c_h| \le c\left(\fint_{B_r} (1+|Dv_h|^{q_1})\,dx\right)^{p/q_1} \cdot \\ \cdot\, \omega^{1-p/q_1}\left(cr^p \fint_{B_r} (1+|Du|^p + |D(v_h-u)|^p)\,dx\right) \\ \le c_2(K) r^{\delta p(q_1-p)/q_1}$$

by (3.15),(3.16). Since $b_h \le 0$ by the minimality of u, we deduce from the estimates above

$$I_r^0(u) - I_r^0(v_h) \le \tilde{c}(K) r^\beta,$$

with $\beta = \delta p(q_1-p)/q_1$, which together with (3.14) proves (3.12) in the first case; the idea of passing to the sequence (v_h) was first used in [13].

SECOND CASE . We assume that in (3.6) the quasiconvexity condition holds.

In this case, by Lemma 3.6 the functional J is semicontinuous, and has a minimum point $\bar{u} \in u + W_0^{1,p}(B_r;\mathbb{R}^N)$ which satisfies

$$\int_{B_r} |D\bar{u}|^p\,dx \le c \int_{B_r} (1+|Du|^p)\,dx;$$

then, by Lemma 3.5 applied with $\eta = 0$, there exist c_1 and $q_1 \in (p, q_0)$ such that $\bar{u} \in W^{1,q_1}(B_r;\mathbb{R}^N)$ and

$$\left(\fint_{B_r} |D\bar{u}|^{q_1}\,dx\right)^{1/q_1} \\ \le c_1\left[\left(\fint_{B_r} (1+|D\bar{u}|^p)\,dx\right)^{1/p} + \left(\fint_{B_r} (1+|Du|^{q_0})\,dx\right)^{1/q_0}\right] \\ \le c\left(\fint_{B_{2r}} (1+|Du|^p)\,dx\right)^{1/p}.$$

The inequality

$$I_r^0(u) - \inf J = I_r^0(u) - I_r^0(\bar{u}) \le \tilde{c}(K) r^\beta$$

may be proved as above, and also the second case is concluded.

We now consider on the space $u + W_0^{1,1}(B_r;\mathbb{R}^N)$ the metric

$$d_2(v,w) = \big(\tilde{c}(K)r^{\beta/2}\big)^{-1} \fint_{B_r} |Dv - Dw|\, dx.$$

By (3.12), applying again Lemma 3.2 we find $v \in u + W_0^{1,p}(B_r;\mathbb{R}^N)$ such that

$$\fint_{B_r} |Dv - Du|\, dx \le \tilde{c}(K)r^{\beta/2} \tag{3.17}$$

and

$$J(v) \le J(v+\varphi) + r^{\beta/2} \fint_{B_r} |D\varphi|\, dx \quad \text{for all } \varphi \in W_0^{1,1}(B_r;\mathbb{R}^N).$$

This proves the last assertion of the lemma, with $\beta_2 = \beta/2$. In addition, by Lemma 3.6

$$\begin{aligned} \mu \fint_{B_r} |Dv|^p\, dx &\le I_r^0(u) + r^{\beta/2} \fint_{B_r} |Dv - Du|\, dx \\ &\quad + c \fint_{B_r} (1 + |Du|^p)\, dx \\ &\le c \fint_{B_r} (1 + |Du|^p)\, dx \end{aligned} \tag{3.18}$$

since $r < 1$. We now select $r_0 = (2^{-p-1}\mu)^{2/\beta}$, so that we may apply Lemma 3.5 to the functional $w \mapsto I_r^0(w) + r^{\beta/2} \fint_{B_r} |Dv - Dw|\, dx$. Then there exist c and $q_1 \in (p, q_0)$ such that

$$\left(\fint_{B_r} |Dv|^{q_1}\, dx\right)^{1/q_1} \le c\left(\fint_{B_{2r}} (1 + |Du|^p)\, dx\right)^{1/p}, \tag{3.19}$$

where we used also (3.18) and (3.11). Now if $\vartheta = \frac{q_1 - p}{p(q_1 - 1)}$ we have

$$\begin{aligned} &\left(\fint_{B_r} |Dv - Du|^p\, dx\right)^{1/p} \\ &\qquad \le \left(\fint_{B_r} |Dv - Du|\, dx\right)^{\vartheta} \cdot \left(\fint_{B_r} |Dv - Du|^{q_1}\, dx\right)^{(1-\vartheta)/q_1} \\ &\qquad \le c(K) r^{\beta\vartheta/2} \end{aligned}$$

by (3.17),(3.11) and (3.19), and the result is proved with $\beta_1 = \beta\vartheta/2$. ∎

The next result is analogous to Proposition 2.6, and after that only the iteration remains to be made. Fix $d < \beta_1$, and set

$$U(x_0, r) = r^d + \fint_{B_r(x_0)} g_p\big(Du - (Du)_{x_0,r}\big)\, dx.$$

Then we have:

Proposition 3.8 . *Let f satisfy (3.1), …,(3.6) and let (x_0, s_0, ξ_0) be a regular point for f. There exists a constant C such that for every $\tau < 1/4$ there exists $\varepsilon(\tau)$ such that if u is a minimizer of I satisfying*

$$\begin{gathered} |\hat{x}-x_0| < \frac{\sigma}{3}, \qquad |(u)_{\hat{x},r} - s_0| < \frac{\sigma}{3}, \qquad |(Du)_{\hat{x},r} - \xi_0| < \frac{\sigma}{3}, \\ |(Du)_{\hat{x},\tau r} - \xi_0| < \frac{\sigma}{3}, \qquad (|Du|^p)_{\hat{x},2r} \le |\xi_0|^p + 1 \end{gathered} \tag{3.20}$$

and

$$U(\hat{x}, r) < \varepsilon$$

then

$$U(\hat{x}, \tau r) < C\tau^d U(\hat{x}, r).$$

PROOF . Fix τ; we shall determine C later. Reasoning as in Proposition 2.6, assume that (3.20) holds in $B_{r_h}(x_h)$, and that

$$U(x_h, r_h) = \lambda_h^2 \to 0, \tag{3.21}$$

but

$$U(x_h, \tau r_h) > C\tau^d \lambda_h^2. \tag{3.22}$$

Applying Lemma 3.7 in each $B_{r_h}(x_h)$ we find a sequence (u_h) in $u + W_0^{1,p}\big(B_{r_h}(x_h); \mathbb{R}^N\big)$ satisfying

$$\left(\fint_{B_{r_h}(x_h)} |Du_h - Du|^p\, dx\right)^{1/p} \le c r_h^{\beta_1} \tag{3.23}$$

and

$$\begin{aligned} &\int_{B_{r_h}(x_h)} f\big(x_h, (u)_{x_h,r_h}, Du_h\big)\, dx \\ &\qquad \le \int_{B_{r_h}(x_h)} f\big(x_h, (u)_{x_h,r_h}, Du_h + D\varphi\big)\, dx \\ &\qquad\quad + r_h^{\beta_2} \int_{B_{r_h}(x_h)} |D\varphi|\, dx \end{aligned} \tag{3.24}$$

for every $\varphi \in C_0^1$. Set

$$A_h = (Du_h)_{x_h,r_h}, \qquad \tilde{A}_h = (Du_h)_{x_h,\tau r_h}, \qquad a_h = (u_h)_{x_h,r_h}.$$

From (3.21) we deduce that

$$r_h^d \le \lambda_h^2, \tag{3.25}$$

therefore in particular $r_h \to 0$; from (3.23) we then get, if h is sufficiently large,

$$|A_h - \xi_0| < \frac{\sigma}{3}, \qquad |\tilde{A}_h - \xi_0| < \frac{\sigma}{3}, \qquad |a_h - s_0| < \frac{\sigma}{3}.$$

Now $g_p(x+y+z) \le (1+\epsilon)g_p(x) + c_\epsilon[g_p(y)+g_p(z)]$, and by the convexity of g_p

$$\begin{aligned}
\frac{1}{\lambda_h^2} &\fint_{B_{r_h}} g_p\big(Du_h - (Du_h)_{r_h}\big)\,dx \\
&\le \frac{(1+\epsilon)}{\lambda_h^2}\fint_{B_{r_h}} g_p\big(Du - (Du)_{r_h}\big)\,dx \\
&\quad + \frac{c_\epsilon}{\lambda_h^2}\fint_{B_{r_h}} \big[g_p(Du_h - Du) + g_p\big((Du_h)_{r_h} - (Du)_{r_h}\big)\big]\,dx \\
&\le \frac{(1+\epsilon)}{\lambda_h^2}\fint_{B_{r_h}} g_p\big(Du - (Du)_{r_h}\big)\,dx \\
&\quad + \frac{2c_\epsilon}{\lambda_h^2}\fint_{B_{r_h}} g_p(Du_h - Du)\,dx \\
&\le \frac{(1+\epsilon)}{\lambda_h^2}\fint_{B_{r_h}} g_p\big(Du - (Du)_{r_h}\big)\,dx + c(\epsilon)\frac{r_h^{2\beta_1}}{\lambda_h^2}
\end{aligned}$$

by (3.23). Since this holds also with u_h and u interchanged, if we set

$$U_h(x_h, r) = r^d + \fint_{B_r(x_h)} g_p\big(Du_h - (Du_h)_{x_h,r}\big)\,dx$$

using (3.25) we deduce easily

$$\lim_{h\to\infty} \frac{U(x_h,r_h) - U_h(x_h,r_h)}{\lambda_h^2} = 0; \tag{3.26}$$

similarly one has

$$\lim_{h\to\infty} \frac{U(x_h,\tau r_h) - U_h(x_h,\tau r_h)}{\lambda_h^2} = 0. \tag{3.27}$$

We now define

$$v_h(z) = \frac{u_h(x_h + r_h z) - a_h - r_h A_h z}{\lambda_h r_h},$$

and remark that $(v_h)_{0,1} = 0$, and that by (3.26)

$$\limsup_{h\to\infty} \int_{B_1} [|Dv_h|^2 + \lambda_h^{p-2}|Dv_h|^p]\, dz \le 1,$$

so that we may suppose

$$v_h \rightharpoonup v \quad \text{weakly in } W^{1,2}(B_1;\mathbb{R}^N),$$

$$\lambda_h^{1-2/p} v_h \rightharpoonup 0 \quad \text{weakly in } W^{1,p}(B_1;\mathbb{R}^N),$$

and also

$$A_h \to A, \qquad (u)_{x_h,r_h} \to a, \qquad x_h \to \bar{x}.$$

Remark that $(\bar{x}, a, A)$ is a regular point for f. Now define

$$L_h = \{z \in B_1 : \lambda_h |Dv_h(z)| \ge \sigma/3\}, \qquad S_h = B_1 \setminus L_h,$$

and use (3.24) as in Proposition 2.6 to obtain, instead of (2.11),

$$\int_{B_1}\int_0^1 f_{\xi\xi}(x_h, a_h, A_h + s\lambda_h Dv_h)\mathbf{1}_{S_h} Dv_h D\varphi\, ds\, dz$$
$$\ge -\frac{r_h^{\beta_2}}{\lambda_h}\int_{B_1} |D\varphi|\, dz - c\lambda_h,$$

whence again

$$\int_{B_1} f_{\xi\xi}(\bar{x}, a, A) Dv D\varphi\, dz = 0 \quad \text{for all } \varphi \in C_0^1(B_1;\mathbb{R}^N),$$

and for all $\tau < 1/4$

$$\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \, dz \leq c\tau^2.$$

Define

$$w_h = v_h - \frac{\tilde{A}_h - A_h}{\lambda_h} z$$

and remark that

$$\int_{B_1} f_{\tilde{A}_h,\lambda_h}\big(x_h, (u)_{x_h,r_h}, Dw_h\big) \, dz \\ \leq \int_{B_1} f_{\tilde{A}_h,\lambda_h}\big(x_h, (u)_{x_h,r_h}, Dw_h + D\psi\big) \, dz + r_h^{\beta_2} \int_{B_1} |D\psi| \, dz.$$

Then we may apply Lemma 2.5, this time with $\nu = r_h^{\beta_2}$; repeating the argument of Proposition 2.6, and recalling (3.25), we get from (3.27)

$$\limsup_{h\to\infty} \frac{U(x_h, \tau r_h)}{\lambda_h^2} \leq c\tau^2 + \limsup_{h\to\infty} \tau^d r_h^d / \lambda_h^2 \leq c\tau^d,$$

which gives the required contradiction with (3.22). ■

Proposition 3.9 . *Take a regular point (x_0, s_0, ξ_0) and $\alpha < d$; if C is as in Proposition 3.8, fix $\tau < 1/4$ such that $C\tau^d < \tau^\alpha$. Let u be a minimizer of I and assume that for some $B_r(x_0)$*

$$|(u)_{x_0,r} - s_0| < \frac{\sigma}{6}, \qquad |(Du)_{x_0,r} - \xi_0| < \frac{\sigma}{6}, \qquad |(Du)_{x_0,r} - \xi_0| < \frac{\sigma}{6},$$

$$(|Du|^p)_{x_0,r} \leq |\xi|^p + \frac{1}{2}, \qquad (|Du|^p)_{x_0,2r} \leq |\xi|^p + \frac{1}{2}$$

and $U(x_0, r) < \eta$, with $\eta > 0$ sufficiently small. Then for all k

$$|(u)_{x_0,\tau^k r} - s_0| < \frac{\sigma}{3}, \qquad |(Du)_{x_0,\tau^k r} - \xi_0| < \frac{\sigma}{3},$$

$$|(Du)_{x_0,\tau^{k+1} r} - \xi_0| < \frac{\sigma}{3}, \qquad (|Du|^p)_{x_0,2\tau^k r} \leq |\xi|^p + 1$$

and

$$U(x_0, \tau^k r) \leq \tau^{k\alpha} U(x_0, r).$$

PROOF . Reasoning as in Proposition 2.7 we have

$$|(Du)_{\tau^{m+1}r}-\xi_0|\le \frac{\sigma}{6}+\tau^{-n}\sum_{k=0}^{m}[U(x_0,\tau^k r)]^{1/2} \tag{3.28}$$

$$|(u)_{\tau^m r}-s_0|\le \frac{\sigma}{6}+c\tau^{-n}\sum_{k=0}^{m-1}\tau^k r(|Du|^p)^{1/p}_{x_0,2\tau^k r}; \tag{3.29}$$

now, since $\tau<1/4$, for $m\ge 2$ we have

$$\begin{aligned}
&\left(\fint_{B_{2\tau^m r}}|Du|^p\,dx\right)^{1/p}\\
&\qquad\le |(Du)_{2\tau^{m-1}r}|+\left(\fint_{B_{2\tau^m r}}|Du-(Du)_{2\tau^{m-1}r}|^p\,dx\right)^{1/p}\\
&\qquad\le |(Du)_{2\tau^{m-1}r}|+\tau^{-n/p}\left(\fint_{B_{2\tau^{m-1}r}}|Du-(Du)_{2\tau^{m-1}r}|^p\,dx\right)^{1/p}\\
&\qquad\le (|Du|^p)^{1/p}_{2\tau^{m-1}r}+(2\tau^2)^{-n/p}[U(x_0,\tau^{m-2}r)]^{1/p},
\end{aligned}$$

whereas for $m=1$

$$\begin{aligned}
&\left(\fint_{B_{2\tau r}}|Du|^p\,dx\right)^{1/p}\\
&\qquad\le |(Du)_r|+\left(\fint_{B_{2\tau r}}|Du-(Du)_r|^p\,dx\right)^{1/p}\\
&\qquad\le (|Du|^p)^{1/p}_r+(2\tau)^{-n/p}[U(x_0,r)]^{1/p}.
\end{aligned}$$

Thus, combining these two estimates,

$$\begin{aligned}
(|Du|^p)^{1/p}_{2\tau^m r}\le (|\xi_0|^p+\frac{1}{2})^{1/p}&+(2\tau)^{-n/p}[U(x_0,r)]^{1/p}\\
&+(2\tau^2)^{-n/p}\sum_{k=0}^{m-2}[U(x_0,\tau^k r)]^{1/p}.
\end{aligned}$$

From (3.28),(3.29) and this inequality, an induction argument proves the result if η was chosen sufficiently small. ■

PROOF OF THEOREM 3.1 . One may follow the lines of the proof of Theorem 2.1, except that α must be less than d. ■

4. Additional remarks

In this section we state two corollaries which follow from our results, then we apply Theorem 2.1 in a case which is relevant in nonlinear elasticity, and finally we study the scalar-valued case $N = 1$, identifying exactly the set of regular points.

Theorems 2.1 and 3.1 yield two new (global) partial regularity results: precisely, we have

Corollary 4.1 . *Let $f : \mathbb{R}^{nN} \to \mathbb{R}$ be a function of class C^2 satisfying for some $p \geq 2$*

$$|f(\xi)| \leq c(1 + |\xi|^p);$$

assume that for every ξ there exists a positive number $\gamma(\xi)$ such that

$$\int_{\mathrm{spt}\varphi} f(\xi + D\varphi(y))\, dy \geq \int_{\mathrm{spt}\varphi} [f(\xi) + \gamma(\xi)(|D\varphi(y)|^2 + |D\varphi(y)|^p)]\, dy$$

for every $\varphi \in C^1_0(\mathbb{R}^n; \mathbb{R}^N)$. Then if $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\int f(Dv(x))\, dx$ there exists an open subset Ω_0 of Ω with $\mathrm{meas}\,(\Omega \setminus \Omega_0) = 0$ *such that $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ for all $\alpha < 1$.*

Corollary 4.2 . *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfy for some $p \geq 2$*

$$f \text{ is twice differentiable with respect to } \xi;$$

$$f_{\xi\xi}(x, s, \xi) \text{ is continuous;}$$

$$|f(x, s, \xi)| \leq c(1 + |\xi|^p);$$

$$|f(x, s, \xi) - f(y, t, \xi)| \leq L(1 + |\xi|^p)\, \omega(|x - y|^p + |s - t|^p),$$

where $\omega(t) \leq t^\delta$, $0 < \delta < 1/p$, and ω is bounded, concave and increasing;

$$f(x, s, \xi) \geq \psi(\xi)$$

for a suitable continuous function ψ satisfying

$$\int_\Omega \psi\big(D\varphi(y)\big)\, dy \geq \int_\Omega [\psi(0) + \mu|D\varphi(y)|^p]\, dy \quad \text{for all } \varphi \in C^1_0(\Omega; \mathbb{R}^N),$$

with $\mu > 0$; finally, we assume that there exists a positive lower semicontinuous function $\gamma(x, s, \xi)$ such that for every (x_0, s_0, ξ_0)

$$\int_{\operatorname{spt}\varphi} f\big(x, s, \xi + D\varphi(y)\big)\, dy$$
$$\geq \int_{\operatorname{spt}\varphi} [f(x, s, \xi) + \gamma(x, s, \xi)(|D\varphi(y)|^2 + |D\varphi(y)|^p)]\, dy$$

for every $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a local minimizer of $\int_\Omega f\big(x, v(x), Dv(x)\big)\, dx$. Then there exist an open subset Ω_0 of Ω with $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$ *such that $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ for some $\alpha < 1$.*

To prove this second Corollary, it is enough to remark that (see Propositions 3.8 and 3.9) the Hölder exponent α must satisfy only $\alpha < d$, and the number d is independent of γ.

These results improve the former general regularity theorems of [5], [6]: not only, as already in [2], the boundedness of the second derivatives of f is dropped, but also the strict quasiconvexity need no longer be uniform.

Example 4.3 . Let $n = N$, and define f by

$$f(\xi) = |{}^t\xi\, \xi - I|^2,$$

where I is the $n \times n$ identity matrix; if $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is the deformation of an n-dimensional body Ω, the functional $\int_\Omega f(Du(x))\, dx$ is an important model of nonlinear elastic energy associated with u. The "expansion points" of u are the points x at which the n eigenvalues of the matrix $\sqrt{{}^tDu\, Du}$ are greater than 1. A not too hard computation shows that if the eigenvalues of $\sqrt{{}^t\xi\, \xi}$ are all greater than 1, then ξ is a regular point for f; therefore, a deformation u is of class $C^{1,\alpha}$ around each of its expansion points.

We shall henceforth confine ourselves to the scalar-valued case $N = 1$; in this case, it is well known that a function being quasiconvex everywhere is equivalent to its being convex (everywhere). This is not true for quasiconvexity and convexity at a single point, as is shown by the following proposition (for any function f we denote by

$$f^{**} = \max\{g \leq f : g \text{ is convex}\}$$

the convex hull of f).

Proposition 4.4 . *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, and assume there exists some ξ_0 such that*

$$\int_{\mathrm{spt}\varphi} f(\xi_0 + D\varphi(x))\,dx \geq \int_{\mathrm{spt}\varphi} [f(\xi_0) + \gamma(|D\varphi(x)|^2 + |D\varphi(x)|^p)]\,dx$$

for all $\varphi \in C_0^{0,1}(\mathbb{R}^n)$, where $p \geq 2$ and $\gamma \geq 0$ is a constant. Then

$$f(\xi_0) = f^{**}(\xi_0),$$

and, if $\gamma > 0$, we have for some positive constants c, c' depending only on (ξ_0, γ, p)

$$f(\xi) \geq c|\xi|^p - c' \qquad \text{for all } \xi.$$

If in addition f is twice differentiable at ξ_0, then

$$D_{ij} f(\xi_0)\xi_i\xi_j \geq 2\gamma|\xi|^2.$$

Proof . The idea is not new; take any $\tilde{\xi}$, $\tilde{\eta}$ and $\lambda \in (0,1)$ such that

$$\lambda\tilde{\xi} + (1-\lambda)\tilde{\eta} = \xi_0,$$

and set $\xi = \tilde{\xi} - \xi_0$, $\eta = \tilde{\eta} - \xi_0$, so that $\lambda\xi + (1-\lambda)\eta = 0$. Let Q be a unit cube with an edge parallel to ξ, and fix a face F of Q which is orthogonal to ξ; for every positive integer m slice Q into m stripes orthogonal to ξ, and call F_m the union of their faces parallel to F, then divide again each stripe in two, a stripe with thickness λ/m, the other with thickness $(1-\lambda)/m$, and call Q_m^λ, $Q_m^{1-\lambda}$ the union of the λ-stripes and the union of the $(1-\lambda)$-stripes respectively (they are thus intertwined). Then we may define a Lipschitz continuous function v_m on Q by setting

$$v_m = 0 \quad \text{on } F_m$$

$$Dv_m = \begin{cases} \xi & \text{in } Q_m^\lambda \\ \eta & \text{in } Q_m^{1-\lambda}. \end{cases}$$

In addition, $\max|v_m| = \lambda|\xi|/m$: therefore, if Q_δ is the cube concentric with Q and whose side is $1+\delta$, we may extend v_m to a Lipschitz continuous function φ_m vanishing outside Q_δ and such that

$$\sup_{Q_\delta \setminus Q} |D\varphi_m| \leq c(|\xi| + |\eta| + \frac{\lambda|\xi|}{m\delta}).$$

Set $\omega(t) = \sup\{f(\zeta) : |\zeta| \leq t\}$; then the quasiconvexity inequality yields

$$\begin{aligned}\gamma \int_Q (|D\varphi_m|^2 + |D\varphi_m|^p)\,dx + (1+\delta)^n f(\xi_0) \\ \leq \int_Q f(\xi_0 + D\varphi_m)\,dx \\ + \int_{Q_\delta \setminus Q} \omega[c(1 + |\xi_0|^p + |\xi|^p + |\eta|^p + \frac{|\xi|^p}{m^p\,\delta^p})]\,dx \\ \leq \int_Q f(\xi_0 + D\varphi_m)\,dx + c\delta\,\omega[c(1 + |\xi_0|^p + |\xi|^p + |\eta|^p)]\end{aligned}$$

for m large enough, but $D\varphi_m = Dv_m$ in Q, hence

$$\xi_0 + D\varphi_m = \begin{cases} \tilde{\xi} & \text{in } Q_m^\lambda \\ \tilde{\eta} & \text{in } Q_m^{1-\lambda}, \end{cases}$$

thus

$$\begin{aligned}\gamma[\lambda(|\xi|^2 + |\xi|^p) + (1-\lambda)(|\eta|^2 + |\eta|^p)] + (1+\delta)^n f(\xi_0) \\ \leq \lambda f(\tilde{\xi}) + (1-\lambda) f(\tilde{\eta}) \\ + c\delta\omega[c(1 + |\xi_0|^p + |\xi|^p + |\eta|^p)]\end{aligned}$$

for m large enough; by letting $m \to \infty$, then $\delta \to 0$, we get

$$\lambda f(\tilde{\xi}) + (1-\lambda) f(\tilde{\eta}) \geq f(\xi_0) + \gamma[\lambda(|\xi|^2 + |\xi|^p) + (1-\lambda)(|\eta|^2 + |\eta|^p)]. \quad (4.1)$$

From this we deduce in particular

$$\lambda f(\tilde{\xi}) + (1-\lambda) f(\tilde{\eta}) \geq f(\xi_0),$$

and by taking the infimum for $\lambda\tilde{\xi} + (1-\lambda)\tilde{\eta} = \xi_0$

$$f^{**}(\xi_0) \geq f(\xi_0),$$

thus proving the first assertion since the opposite inequality is obvious.

Set $M(\xi_0) = \max\{f(\xi_0 + \eta) : |\eta| \leq 1\}$, and take $|\eta| = 1$ in (4.1), so that $\lambda = 1/(1+|\xi|)$; dropping $|\xi|^2$ and some other terms on the right-hand side, we have

$$\begin{aligned}f(\tilde{\xi}) &\geq -2M(\xi_0)(1 + |\xi|) + \gamma|\xi|^p \\ &\geq -2M(\xi_0)(1 + |\xi_0| + |\tilde{\xi}|) - \gamma|\xi_0|^p + 2^{1-p}\gamma|\tilde{\xi}|^p,\end{aligned}$$

and the second assertion follows easily.

Finally, this time dropping $|\xi|^p$ and $|\eta|^p$, and taking $\eta = -\xi$, so that $\lambda = 1/2$, we have again from (4.1)

$$\frac{1}{2}[f(\xi_0+\xi)+f(\xi_0-\xi)-2f(\xi_0)] \geq \gamma|\xi|^2,$$

and the last assertion follows by taking $\xi = t\zeta$, dividing by t^2 and letting $t \to 0$. ∎

As an example, note that the function $f(x) = x^2 - 3x^4 + \frac{5}{3}x^6$ is convex at 0, but it is not quasiconvex at 0, since $f^{**}(0) = -\frac{1}{3} < f(0)$.

We also remark that in the proof we did not fully use the continuity of f, but almost only the fact that it is bounded on bounded sets.

Proposition 4.5 . *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying for some $p \geq 2$*

$$|f(\xi)| \leq c(1+|\xi|^p), \qquad |Df(\xi)| \leq c(1+|\xi|^{p-1})$$

and such that the set of its regular points is not empty. Then this set is

$$\{\xi_0 : f^{**} \in C^2(B_\sigma(\xi_0)) \text{ for some } \sigma > 0, \quad D^2 f^{**}(\xi_0)\eta\eta > 0 \text{ for all } \eta \neq 0\}.$$

Proof . If there is at least a regular point, then by Proposition 4.4 we have

$$f(\xi) \geq c|\xi|^p - c'; \tag{4.2}$$

take a regular point ξ_0: by Lemma 2.2 a whole ball $B_\sigma(\xi_0)$ is made of regular points, so by Proposition 4.4 $f(\xi) = f^{**}(\xi)$ in $B_\sigma(\xi_0)$: thus $f^{**} \in C^2(B_\sigma(\xi_0))$ and (again by Proposition 4.4) $D^2 f^{**}(\xi_0)$ is positive definite.

To prove the converse, assume f^{**} is of class C^2 around ξ_0, and $D^2 f^{**}(\xi) \geq \gamma I$ for $|\xi - \xi_0| < \sigma$; then necessarily $f(\xi) = f^{**}(\xi)$ in $B_\sigma(\xi_0)$, so $f \in C^2(B_\sigma(\xi_0))$ too. Now for $|\xi - \xi_0| < \sigma/2$

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \frac{\gamma}{2}|\xi - \xi_0|^2, \tag{4.3}$$

and for $|\xi - \xi_0| \geq \sigma/2$

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \frac{\gamma\sigma^2}{8}. \tag{4.4}$$

By (4.2) we have for a suitably large R that if $|\xi - \xi_0| \geq R$ then

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \frac{c}{2}|\xi - \xi_0|^p. \tag{4.5}$$

Now if $\lambda \leq c/2$ satisfies

$$\lambda(t^2 + t^p) \leq \frac{\gamma\sigma^2}{8} \quad \text{for all } t \leq R,$$

we immediately deduce from (4.3),(4.4),(4.5)

$$f(\xi) \geq f(\xi_0) + Df(\xi_0)(\xi - \xi_0) + \lambda(|\xi = \xi_0|^2 + |\xi - \xi_0|^p),$$

and the strict quasiconvexity at ξ_0 follows immediately. ∎

References

[1] Acerbi, E., & N. Fusco: *Semicontinuity problems in the calculus of variations.* Arch. Rational Mech. Anal. **86** (1984), 125–145.

[2] Acerbi, E., & N. Fusco: *A regularity theorem for minimizers of quasiconvex integrals.* Arch. Rational Mech. Anal. **99** (1987), 261–281.

[3] Anzellotti, G., & M. Giaquinta: *Convex functionals and partial regularity.* Arch. Rational Mech. Anal. **102** (1988), 243–272.

[4] Ekeland, I.: *Nonconvex minimization problems.* Bull. Amer. Math. Soc. **1** (1979), 443–474.

[5] Evans, L. C.: *Quasiconvexity and partial regularity in the calculus of variations.* Arch. Rational Mech. Anal. **95** (1986), 227–252.

[6] Evans, L. C., & R. F. Gariepy: *Blow-up, compactness and partial regularity in the calculus of variations.* Indiana Univ. Math. J. **36** (1987), 361–371.

[7] Fusco. N., & J. Hutchinson: $C^{1,\alpha}$ *partial regularity of functions minimising quasiconvex integrals.* Manuscripta Math. **54** (1985), 121–143.

[8] Giaquinta, M.: *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Annals of Mathematics Studies **105**, Princeton University Press, Princeton, 1983.

[9] Giaquinta, M.: *Quasiconvexity, growth conditions, and partial regularity.* Partial differential equations and calculus of variations, 211–237, Lecture Notes in Math., 1357, Springer, Berlin-New York, 1988.

[10] Giaquinta, M., & E. Giusti: *On the regularity of minima of variational integrals.* Acta Math. **148** (1982), 31–46.

[11] Giaquinta, M., & G. Modica: *Partial regularity of minimizers of quasiconvex integrals.* Ann. Inst. H. Poincaré, Analyse non linéaire **3** (1986), 185–208.

[12] Hong, M. C.: *Existence and partial regularity in the calculus of variations.* Ann. Mat. Pura Appl. **149** (1987), 311–328.

[13] Marcellini, P., & C. Sbordone: *On the existence of minima of multiple integrals in the calculus of variations.* J. Math. Pures Appl. **62** (1983), 1–9.

[14] Meyers, N. G.: *Quasi-convexity and lower semicontinuity of multiple variational integrals of any order.* Trans. Amer. Math. Soc. **119** (1965), 1–28.

[15] Morrey, C. B., jr.: *Quasi-convexity and the semicontinuity of multiple integrals.* Pacific J. Math. **2** (1952), 25–53.

[16] Uhlenbeck, K.: *Regularity for a class of nonlinear elliptic systems.* Acta Math. **138** (1977), 219–240.

Authors' addresses

Emilio Acerbi
Dipartimento di Matematica
Politecnico
Torino, Italy

Nicola Fusco
Dipartimento di Matematica
Università
Salerno, Italy