ISRAEL JOURNAL OF MATHEMATICS 75 (1991), 203–241

LOCAL RIGIDITY FOR CERTAIN GROUPS OF TORAL AUTOMORPHISMS

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ABSTRACT

Let $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ or any subgroup of finite index, $n \geq 4$. We show that the standard action of Γ on $\mathbb{T}^n$ is locally rigid, i.e., every action of Γ on $\mathbb{T}^n$ by C^∞ diffeomorphisms which is sufficiently close to the standard action is conjugate to the standard action by a C^∞ diffeomorphism. In the course of the proof, we obtain a global rigidity result (Theorem 4.12) for actions of free abelian subgroups of maximal rank in $\mathbf{SL}(n, \mathbb{Z})$.

1. Introduction

Let G be a connected semi-simple Lie group with trivial center and without compact factors. Then if Γ is an irreducible lattice in G and G is not isomorphic to $\mathbf{PSL}(2, \mathbb{R})$, there are a number of well-known results which reflect the "rigidity" of Γ in G in the context of finite-dimensional representations, culminating in the celebrated "superrigidity" theorem of Margulis [Ma1] (see also Mostow [Mo] and Prasad [P]). The present work is part of a more recent program, initiated by R. Zimmer, of understanding a special class of non-linear representations of such groups, namely the realizations of Γ and G as smooth transformation groups on compact manifolds. The basic idea, which first appeared in the work of S. Hurder [H2], [H3], is to combine the purely algebraic, finite-dimensional rigidity

*Partially supported by NSF grant DMS9011749.
Received May 21, 1991

properties of Γ with results about hyperbolic dynamical systems, and especially structural stability.

In particular, we consider the following basic example. $\Gamma = \mathbf{SL}(n, \mathbf{Z})$, the lattice of integer matrices in $\mathbf{SL}(n, \mathbf{R})$, or more generally, any subgroup of finite index in $\mathbf{SL}(n, \mathbf{Z})$. Γ acts naturally on $M = \mathbf{T}^n$ as the group of orientation-preserving automorphisms of M as a compact abelian Lie group. Before describing our main result and recent related work which preceded it, we need to establish some general terminology.

If Γ is any finitely-generated discrete group and G is any topological group whatsoever, we let $R(\Gamma, G)$ denote the set of homomorphisms of Γ into G with the compact/open topology. We can describe the topology on $R(\Gamma, G)$ more concretely as follows. Fix generators $\gamma_1, \ldots, \gamma_k$ for Γ and identify $R(\Gamma, G)$ with a closed subset of G^k via $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_k))$; then the topology on $R(\Gamma, G)$ is simply the subspace topology it inherits from G^k. Note that G acts naturally on $R(\Gamma, G)$ by conjugation. A homomorphism $\rho_0 \in R(\Gamma, G)$ is said to be **locally rigid** if its orbit in $R(\Gamma, G)$ is open. Equivalently, ρ_0 is locally rigid if and only if there exists a neighborhood U of ρ_0 in $R(\Gamma, G)$ such that for every $\rho \in U$ there exists $g \in G$ such that $\rho(\gamma) = g\rho_0(\gamma)g^{-1}$ for every $\gamma \in \Gamma$.

In case M and N are (finite-dimensional) C^k manifolds, $0 \le k \le \infty$, we write $C^k(M, N)$ for the set of C^k mappings $M \to N$ and $\mathrm{Diff}^k(M)$ (or simply $\mathrm{Diff}(M)$ in case $k = \infty$) for the set of C^k diffeomorphisms of M. We shall endow $C^k(M, N)$ with the "weak" or "standard" C^k-topology of uniform convergence of k-jets on compact subsets of M and topologize $\mathrm{Diff}^k(M)$ as an open subset of $C^k(M, M)$. A **C^k-action of Γ on M** is an element of $R(\Gamma, \mathrm{Diff}^k(M))$.

In case G is a (finite-dimensional) Lie group, Weil observed [W] that if $\rho \in R(\Gamma, G)$ such that $H^1(\Gamma, \mathrm{Ad}_G \circ \rho) = 0$, then ρ is locally rigid. Now suppose M is a compact C^∞ manifold, and let $\mathrm{Vec}(M)$ denote the space of C^∞ vector fields on M. Any action $\rho \in R(\Gamma, \mathrm{Diff}(M))$ induces a natural linear action of Γ on $\mathrm{Vec}(M)$. Following Zimmer [Zi2], we say that ρ is **infinitesimally rigid** if $H^1(\Gamma, \mathrm{Vec}(M)) = 0$. The terminology is meant to suggest an analogy with Weil's theorem, although in the present context, the connection between infinitesimal and local rigidity has not been established in either direction.

Zimmer raised the question of infinitesimal and local rigidity for the action of $\mathbf{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n$, $n \ge 3$, during the 1984 M.S.R.I. workshop on ergodic theory, Lie groups, and geometry (see [H1]), and again in his 1986 address to the I.C.M. [Zi1]. The first result in this direction was obtained by the second author [L]:

THEOREM 1.1: *Let $\Gamma = \mathbf{SL}(n, \mathbf{Z})$ or any subgroup of finite index. Then the* *action of* Γ *on* $\mathbf{T}^n$ *by automorphisms is infinitesimally rigid for* $n \geq 7$.

Recently, Hurder [H2], [H3] established the following property, which he terms "deformation rigidity," under more general conditions.*

THEOREM 1.2: *Let* $\Gamma = \mathbf{SL}(n, \mathbf{Z})$ *or any subgroup of finite index,* $n \geq 3$. *Let* $t \mapsto \rho_t \in R(\Gamma, \mathrm{Diff}(\mathbf{T}^n))$ *be a continuous path based at* $\rho_0 =$ *the standard action by automorphisms. Then there exists a continuous path* $t \mapsto g_t \in \mathrm{Diff}(\mathbf{T}^n)$ *and* $\epsilon > 0$ *such that* $\rho_t(\gamma) = g_t \rho_0(\gamma) g_t^{-1}$ *for all* $t < \epsilon$ *and* $\gamma \in \Gamma$.

In this paper, we answer, for $n \geq 4$, the principal question posed by Zimmer.

THEOREM 1.3: *Let* $\Gamma = \mathbf{SL}(n, \mathbf{Z})$ *or any subgroup of finite index,* $n \geq 4$. *Then the standard action of* Γ *on* $\mathbf{T}^n$ *is locally rigid.*

Observe that this result, when applicable, is stronger than 1.2, since it has not been established, *a priori*, that the space $R(\Gamma, \mathrm{Diff}(\mathbf{T}^n))$ is locally path-connected. (In fact, the technique can be extended to yield global results under appropriate hypotheses; these will appear in [K-L].)

The proof of (1.3), like that of (1.2), divides naturally into two steps. First, we show that any action which is sufficiently close to the standard one is conjugate to the standard action via a homeomorphism of $\mathbf{T}^n$. This is the principal part of the argument, which is carried out in Section 3. Then in Section 4 we show that, perhaps in a smaller neighborhood, the conjugating homeomorphism is smooth.

We would like to acknowledge at the outset that several important ideas in the proof were first introduced by S. Hurder [H2]. Specifically, two key ingredients in the proof of topological conjugacy, the use of Stowe's theorem (2.3) and the idea of building up the conjugating homeomorphism on rational points, are borrowed from that paper. On the other hand, our argument requires more detailed information about the structure of the group Γ in order to avoid making infinitely many appeals to Stowe's theorem (see Section 3 for more details).

Furthermore, once the existence of a conjugating homeomorphism has been established, the regularity argument in [H3] is applicable under the hypotheses of Theorem 1.3 and could be used to complete the proof. However, we provide a

*Unfortunately, the notation in [H2], [H3] differs from our own. In particular, Hurder uses the term "local rigidity" as well as "deformation rigidity" to refer to the property described in (1.2). He refers to our "local rigidity" as "stability"; in particular, he refers to the property in (3.1) below as "topological stability" and that in (1.3) as "differential stability."

different proof in Section 4, both because our argument is more elementary, and because it provides a more general result (cf. Theorem 4.2).

We shall have more to say about the relationship between Hurder's work and our own at appropriate points below.

We are of course deeply indebted to R. Zimmer, who posed the problem and with whom we have had many helpful conversations. We would also like to acknowledge our debt to G. Stuck and D. Witte, who directed our attention to the paper of Tits [T], and to D. DeLatte, M. Policott, L. Vaserstein, and H. Weiss, all of whom made helpful remarks. Finally, the second author in particular would like to thank the mathematics department at Pennsylvania State University for their generous hospitality during the final preparation of this paper.

2. Preliminaries

In this section, we gather together some more or less well-known results from dynamical systems, stated in a form most suitable for our present purposes. The first of these is the structural stability theorem for Anosov diffeomorphisms [A]. The assertion in the second paragraph follows easily from uniqueness (cf. [P-Y]).

PROPOSITION 2.1: *Suppose $\gamma \in \mathrm{SL}(n,\mathbb{Z})$ is a hyperbolic matrix, i.e., γ has no eigenvalues on the unit circle. Denote by γ as well the corresponding Anosov diffeomorphism of $\mathbb{T}^n$. Then if γ' is another diffeomorphism of $\mathbb{T}^n$ which is sufficiently close to γ in the C^1-topology, γ' is Anosov and there exists $h \in \mathrm{homeo}_1(\mathbb{T}^n)$ such that $\gamma' = h\gamma h^{-1}$. More precisely, there exists a homotopy $\gamma_t = h_t\gamma h_t^{-1}$ with $h_0 = 1$, $h_1 = h$, and $h_t \in \mathrm{homeo}(\mathbb{T}^n)$ for each $t \in [0,1]$. The homotopy h_t "varies continuously" with respect to γ', in the sense that by taking γ' sufficiently close to γ in the C^1-topology, we can arrange that h_t remains arbitrarily close to 1 in the C^0-topology.*

Let $p = h(0)$, which is one of finitely many fixed points for γ'. Then h is the unique homeomorphism of $\mathbb{T}^n$ such that $h\gamma h^{-1} = \gamma'$ and $h(0) = p$, and if φ is any homeomorphism of $\mathbb{T}^n$ which commutes with γ', then $h^{-1}\varphi h$ is a linear map which commutes with γ.

The following statement summarizes all that we need of the "normal hyperbolicity" theory in [H-P-S].

PROPOSITION 2.2: *Let V be a smooth compact submanifold of a smooth Riemannian manifold M, $f \in \mathrm{Diff}(M)$ leaving V invariant. We say that f is* **normally hyperbolic at** *V if the tangent bundle of M, restricted to V, splits into* *three continuous subbundles $T_V M = TV \oplus N^U \oplus N^S$, invariant under Df, such that Df expands N^U and contracts N^S more sharply than TV. Then if f' is another diffeomorphism of M which is sufficiently close to f in the C^1-topology, there is a unique, normally hyperbolic, invariant C^1 submanifold V' for f' near V.*

Finally, the (finite-dimensional) cohomology-vanishing results for Γ will enter the argument via the following theorem due to Stowe [Sto]. Suppose Γ is any compactly-generated Lie group (i.e., Γ^0 is a Lie group and Γ/Γ^0 is a finitely-generated discrete group) and M is a (finite-dimensional) C^1 manifold. Suppose $\alpha \in R(\Gamma, \mathrm{Diff}^1(M))$. Set $S(\alpha) = \{p \in M \mid \alpha(\gamma)p = p \text{ for every } \gamma \in \Gamma\}$, the fixed-point set for the action of Γ under α, and fix $p \in S(\alpha)$. Then α induces a natural (finite-dimensional) linear representation of Γ on T_pM, the tangent space to M at p. We denote by $H^1(\Gamma, \mathrm{T}_pM)$ the ordinary group cohomology with coefficients in this representation.

PROPOSITION 2.3: *If $H^1(\Gamma, \mathrm{T}_pM) = 0$, then p is stable under perturbation of α; i.e., given any neighborhood U of p in M, there is a neighborhood V of α in $R(\Gamma, \mathrm{Diff}^1(M))$ such that each $\beta \in V$ has a fixed point in U.*

Moreover, near p, $S(\alpha)$ is a stable C^1 submanifold of M. More precisely, there is a neighborhood U of p in M, a neighborhood V of α in $R(\Gamma, \mathrm{Diff}^1(M))$, a disk D diffeomorphic to some Euclidean space, and for each $\beta \in V$ a closed embedding $q_\beta \in C^1(D, U)$ such that
(i) *$q_\beta(D) = U \cap S(\beta)$, and*
(ii) *$V \to C^1(D, U)$, $\beta \mapsto q_\beta$ is continuous.*

Stowe's theorem has two obvious corollaries which we state separately for future reference.

PROPOSITION 2.4: *If p is an isolated fixed point and $H^1(\Gamma, \mathrm{T}_pM) = 0$, then for every neighborhood U of p there is a neighborhood V of α in $R(\Gamma, \mathrm{Diff}^1(M))$ such that every $\beta \in V$ has an isolated fixed point in U.*

PROPOSITION 2.5: *If $N \subset S(\alpha)$ is a compact connected component of $S(\alpha)$ such that $H^1(\Gamma, \mathrm{T}_pM) = 0$ for every $p \in N$, then N is a stable C^1 submanifold of M. More precisely, there is a tubular neighborhood U of N in M, a neighborhood V of α in $R(\Gamma, \mathrm{Diff}^1(M))$, and for each $\beta \in V$ a closed embedding $q_\beta \in C^1(N, U)$ such that*
(i) *$q_\beta(N) = U \cap S(\beta)$, and*

(ii) $V \to C^1(N,U)$, $\beta \mapsto q_\beta$ *is continuous.*

3. Topological Conjugacy

Let $\mathrm{Diff}^1(M)$ denote the set of C^1 diffeomorphisms of M with the C^1 topology.

THEOREM 3.1: *Suppose* $\Gamma = \mathbf{SL}(n,\mathbf{Z})$, *or any subgroup of finite index*, $n \geq 4$. *Then there exists a neighborhood U of the standard action in* $R(\Gamma, \mathrm{Diff}^1(M))$ *such that each* $\rho \in U$ *is* C^0*-conjugate to the standard action, i.e., there exists a unique homeomorphism h of* $\mathbf{T}^n$ *near the identity such that* $\rho(\gamma) = h\gamma h^{-1}$ *for every* $\gamma \in \Gamma$.

In this section, we give the proof of (3.1) in the special case in which Γ equals the full integer lattice $\mathbf{SL}(n,\mathbf{Z})$; the additional argument needed to extend the theorem to subgroups of finite index appears in Section 5, below. We adopt this approach not only to simplify the presentation, but because there are several points in the proof which can be made both more elementary and more explicit for the full group (cf. (3.2) versus (5.2) and (3.8) versus (5.7)).

Before presenting all the details, it will be convenient to give a brief outline of the argument. We begin by observing that Γ is generated by a finite set of hyperbolic matrices, say $\gamma_1, \ldots, \gamma_k$. Then by structural stability (2.1), for ρ close enough to the standard action there exist homeomorphisms h_i such that $\rho(\gamma_i) = h_i\gamma_i h_i^{-1}$, $1 \leq i \leq k$. To complete the proof, we must show that the h_i's coincide.

The first step is to apply (2.4) to obtain a fixed point, p, for the perturbed action, so that $h_i(0) = p$, $1 \leq i \leq k$.

Fix a partition $\{1,\ldots,n\} = \{i_1,\ldots,i_\ell\} \cup \{j_1,\ldots,j_m\}$ of the indices into disjoint subsets with $\ell, m \geq 2$. Corresponding to each such partition, we obtain a pair of complementary subgroups, copies of $\mathbf{SL}(\ell,\mathbf{Z})$ and $\mathbf{SL}(m,\mathbf{Z})$, respectively, sitting inside Γ. For example, if $n = 5$ and the partition is $\{1,\ldots,5\} = \{1,2\} \cup \{3,4,5\}$, we obtain a copy of $\mathbf{SL}(2,\mathbf{Z})$ in the upper left-hand corner and $\mathbf{SL}(3,\mathbf{Z})$ in the lower right.

Note that under the standard action, the set of fixed points for $\mathbf{SL}(\ell,\mathbf{Z})$ is an m-torus through 0, likewise, $\mathbf{SL}(m,\mathbf{Z})$ fixes an ℓ-torus, and the two tori intersect in a single point, 0. Also, both tori are invariant under the action of the product $\mathbf{SL}(\ell,\mathbf{Z}) \times \mathbf{SL}(m,\mathbf{Z}) \subset \Gamma$. The next step is to show that this structure persists for the perturbed action. That is, the fixed-point sets for $\mathbf{SL}(\ell,\mathbf{Z})$ and $\mathbf{SL}(m,\mathbf{Z})$ under the perturbed action are topological tori of the appropriate dimension intersecting in p, and both are invariant under the perturbed action of the product.

Next we observe that the intersections of the various tori corresponding to different partitions are topologically the same as in the unperturbed case. In particular, we show that the two 2-tori fixed by two distinct copies of $\mathbf{SL}(n-2, \mathbf{Z})$ sitting inside a copy of $\mathbf{SL}(n-1, \mathbf{Z}) \subset \Gamma$ intersect in a circle through p, and that this circle is precisely the set of fixed points for the perturbed action of $\mathbf{SL}(n-1, \mathbf{Z})$.

Now fix a copy, Λ, of $\mathbf{SL}(n-1, \mathbf{Z})$ in Γ, and let S denote the corresponding circle of fixed points for the standard action, and $\tilde{S}$ for the perturbed action, of Λ on $\mathbb{T}^n$. We exhibit a sequence of elements $\lambda_N \in \Gamma$ such that each subgroup $\Xi_N = \langle \Lambda, \lambda_N \Lambda \lambda_N^{-1} \rangle$ has finite index in Γ, $S \cap \lambda_N S$ is a set of N points, and $\cup_{N=1}^{\infty}(S \cap \lambda_N S)$ is dense in S. Now since Ξ_N has finite index in Γ and each generator γ_i has infinite order, some non-zero power of γ_i lies in Ξ_N for each i. Also, for each $\gamma \in \Gamma$, $\rho(\gamma)$ induces the same action on homology as does γ. Combining these observations, we are able to conclude that $\tilde{S} \cap \rho(\lambda_N)\tilde{S}$ (which is the fixed-point set for the action of Ξ_N under ρ) is also a set of N points, and that each of the conjugating homeomorphisms h_i restricts to give the same identification $h_i \colon S \cap \lambda_N S \xrightarrow{\sim} \tilde{S} \cap \rho(\lambda_N)\tilde{S}$. Since $\cup_{N=1}^{\infty}(S \cap \lambda_N S)$ is dense in S, this implies that the h_i's agree on S. Finally, since the orbit of S under the action of Γ in $\mathbb{T}^n$ is dense, we conclude that the h_i's coincide, which completes the proof of (3.1).

Finally, we shall comment briefly on the relationship between our proof of (3.1) and the argument by which Hurder [H3] establishes the corresponding result for deformations. In rough outline, the two arguments are similar: Both begin by fixing hyperbolic generators and conjugating homeomorphisms, obtained via structural stability, then show that the conjugating homeomorphisms agree on a dense set of rational points, hence coincide. Also, both arguments use Stowe's theorem (2.3) together with classical cohomology-vanishing results.

The essential difference is that Hurder applies (2.4) at each point in a countably infinite set, while we make one appeal to (2.4), to fix the origin, and another to (2.5), to obtain the circle $\tilde{S}$ of fixed points for the perturbed action of Λ. Since there is no uniform estimate on the size of the neighborhood in (2.4), Hurder's approach requires the existence of a path joining ρ to the standard action (together with local rigidity of the standard linear representation of Γ) in order to obtain stability for this infinite collection of periodic points simultaneously. On the other hand, our approach requires more detailed information about the structure of Γ. In particular, this accounts for the exclusion of the case $n = 3$ in

(1.3) as opposed to (1.2).

We now begin the detailed proof of Theorem (3.1).

LEMMA 3.2: *For $n \geq 2$, $\mathbf{SL}(n, \mathbf{Z})$ is generated by a finite collection of hyperbolic matrices.*

Proof: It is well-known that $\mathbf{SL}(n, \mathbf{Z})$ is generated by the elementary matrices of the form $I + e_{ij}$, $i \neq j$, where e_{ij} is the matrix with a 1 in the i, jth entry and 0's elsewhere. (This follows by the Gaussian elimination algorithm.) The lemma follows by observing that each such elementary matrix is itself the product of a pair of hyperbolic matrices, viz.,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix},$$

and for $n \geq 4$

$$\begin{pmatrix} 1 & 1 & \\ 0 & 1 & \\ & & I_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & -1 & \\ -1 & 2 & \\ & & A \end{pmatrix} \begin{pmatrix} 2 & 3 & \\ 1 & 2 & \\ & & A^{-1} \end{pmatrix}$$

with A any hyperbolic matrix in $\mathbf{SL}(n-2, \mathbf{Z})$. □

Fix generators $\gamma_1, \ldots, \gamma_k$ for Γ as in (3.2). By (2.1), for ρ sufficiently C^1-close to the standard action, there exist unique homeomorphisms $h_1, \ldots, h_k$ of $\mathbf{T}^n$, isotopic to the identity, such that $\rho(\gamma_i) = h_i \gamma_i h_i^{-1}$, $1 \leq i \leq k$.

LEMMA 3.3: *For ρ sufficiently C^1-close to the standard action, $\rho(\Gamma)$ has an isolated fixed point p near the origin in $\mathbf{T}^n$.*

Proof: The linear representation of Γ on the tangent space to $\mathbf{T}^n$ at the origin under the standard action is simply the standard linear representation of Γ on $\mathbf{R}^n$. The cohomology H^1 of Γ with coefficients in this linear representation is known to vanish (cf. [R], [B] for the case $n \geq 5$, [Zu] for the case $n \geq 4$, and [Ma2] for the case $n \geq 3$). Thus (3.3) follows via (2.3). □

LEMMA 3.4: *Fix a partition $\{1,\ldots,n\} = \{i_1,\ldots,i_\ell\} \cup \{j_1,\ldots,j_m\}$ of the indices into disjoint subsets with $\ell, m \geq 2$. Define $V_1, V_2 \subset \mathbb{R}^n$ via*

$$V_1 = \mathbf{R}_{e_i} \oplus \cdots \oplus \mathbf{R}_{e_\ell}, \quad V_2 = \mathbf{R}e_{j_1} \oplus \cdots \oplus \mathbf{R}e_{j_m},$$

where e_i denotes the ith standard basis vector. Set

$$\Lambda_1 \simeq \mathbf{SL}(m,\mathbf{Z}) = \{\gamma \in \Gamma \mid \gamma V_1 = V_1,\ \gamma V_2 = V_2,\ \text{and}\ \gamma|_{V_1} = I_{V_1}\},$$
$$\Lambda_2 \simeq \mathbf{SL}(\ell,\mathbf{Z}) = \{\gamma \in \Gamma \mid \gamma V_1 = V_1,\ \gamma V_2 = V_2,\ \text{and}\ \gamma|_{V_2} = I_{V_2}\}.$$

Let $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ denote the natural projection and set $T_1 = \pi(V_1)$, $T_2 = \pi(V_2)$, so that T_1 is the fixed-point set for the linear action of Λ_1, T_2 for Λ_2. Then for ρ sufficiently C^1-close to the standard action, there exists a homeomorphism h of $\mathbf{T}^n$ such that
(i) $\tilde{T}_i = hT_i = \{x \in \mathbf{T}^n \mid \rho(\gamma)x = x \text{ for every } \gamma \in \Lambda_i\}$, $i = 1$ or 2,
(ii) $h(0) = p$ (so that $\tilde{T}_1 \cap \tilde{T}_2 = \{p\}$), and
(iii) $\rho(\gamma)\tilde{T}_i = \tilde{T}_i$ for every $\gamma \in \Lambda_1 \times \Lambda_2$.

Moreover, the map $\rho \mapsto h$, which maps a neighborhood of the standard action in $R(\Gamma, \mathrm{Diff}^1(\mathbf{T}^n))$ to a neighborhood of I in Homeo $(\mathbf{T}^n)$ is continuous (uniform topology on Homeo $(\mathbf{T}^n)$).

Proof: Fix hyperbolic generators $\alpha_1,\ldots,\alpha_r$ for $\mathbf{SL}(m,\mathbf{Z}) \simeq \Lambda_1$ and $\beta_1,\ldots,\beta_s$ for $\mathbf{SL}(\ell,\mathbf{Z}) \simeq \Lambda_2$, and so that (α_i,β_j) generate $\mathbf{SL}(m,\mathbf{Z}) \times \mathbf{SL}(\ell,\mathbf{Z})$. Note that the matrices in $\Lambda_1 \times \Lambda_2 \subset \Gamma$ corresponding to (α_i,β_j) are hyperbolic; henceforth we abuse notation by writing (α_i,β_j) for the matrix as well. Then by (2.1) there are unique homeomorphisms h_{ij} of $\mathbf{T}^n$ isotopic to the identity with $h_{ij}(0) = p$, $\rho(\alpha_i,\beta_j) = h_{ij}(\alpha_i,\beta_j)h_{ij}^{-1}$, and each h_{ij} varies continuously with respect to ρ.

For fixed j, (I,β_j) and (α_i,β_j) commute for every i, hence

$$\rho(I,\beta_j) = h_{ij}(I,\beta_j)h_{ij}^{-1},$$

by the second part of (2.1). The set of fixed points for the linear action of (I,β_j) on $\mathbf{T}^n$ is a finite union of parallel ℓ-tori; call this set X_j. The leaf of X_j through 0 is T_2. Also, T_1, may be characterized dynamically as the closure of the stable manifold for (I,β_j) at 0. It follows that $h_{ij}X_j$ is the set of fixed points for $\rho(I,\beta_j)$, $h_{ij}T_2$ is the leaf of fixed points through p, and $h_{ij}T_1$ is the closure of the stable manifold at p. (This is the critical application of (3.3).)

Similarly, for fixed i, (α_i, I) and (α_i,β_j) commute for every j, T_2 is the closure of the stable manifold for (α_i, I) at 0, and T_1 is the leaf of fixed points for (α_i, I)

through 0. Thus $h_{ij}T_2$ is the closure of the stable manifold for $\rho(\alpha_i, I)$ at p, and $h_{ij}T_1$ is the leaf of fixed points through p. Thus $\tilde{T}_2 = h_{ij}T_2$ and $\tilde{T}_1 = h_{ij}T_1$ are defined independently of i and j.

By construction, each point of $\tilde{T}_2$ is fixed by the generators for Λ_2 under ρ, hence

$$\tilde{T}_2 \subset \{x | \rho(\gamma)x = x \text{ for every } \gamma \in \Lambda_2\}.$$

To obtain the reverse inclusion, observe that $T_2 = \cap_{j=1}^{s} X_j$. We need to show that $\tilde{T}_2 = \cap_{j=1}^{s} h_{ij}X_j$ for any i.

Fix j_0, and suppose L is one of the other leaves of X_{j_0}, distinct from T_2. Then for some j', $L \cap X_{j'} = \phi$. For fixed i, $h_{ij_0}L \cap h_{ij'}X_{j'} = \phi$ is an open condition on $(h_{ij_0}, h_{ij'})$, hence on ρ. Thus for ρ near the standard action,

$$\bigcap_{j=1}^{s} h_{ij}X_j \subset h_{ij_0}T_2 = \tilde{T}_2.$$

The equality

$$\tilde{T}_1 = \{x \mid \rho(\gamma)x = x \text{ for every } \gamma \in \Lambda_1\}$$

is obtained similarly. □

Establish notation as follows. Suppose $\mathcal{I} = \{i_1, \ldots, i_m\}$, $2 \le m \le n-2$ is a subset of the indices $\{1, \ldots, n\}$. Denote by $\Lambda_{\mathcal{I}} \subset \Gamma$ the corresponding copy of $\mathbf{SL}\ (n-m, \mathbf{Z})$, and $T_{\mathcal{I}}, \tilde{T}_{\mathcal{I}}$ the m-tori of fixed points for $\Lambda_{\mathcal{I}}$ under the standard and perturbed actions, respectively, as in (3.4).

LEMMA 3.5: $\mathcal{I}_1 \subset \mathcal{I}_2 \Rightarrow \tilde{T}_{\mathcal{I}_1} \subset \tilde{T}_{\mathcal{I}_2}$.

Proof: Obvious since $\Lambda_{\mathcal{I}_2} \subset \Lambda_{\mathcal{I}_1}$. □

For $i \in \{1, \ldots, n\}$, let $e_i^{\perp}$ denote the subspace of $\mathbb{R}^n$ orthogonal to e_i, and denote by $\Lambda_i \subset \Gamma$ the subgroup

$$\Lambda_i = \{\gamma \in \Gamma \mid \gamma e_i = e_i,\ \gamma(e_i^{\perp}) \subset e_i^{\perp}\} \simeq \mathbf{SL}(n-1, \mathbf{Z}).$$

LEMMA 3.6: *Suppose $\{i, j, k\} \subset \{1, \ldots, n\}$ is any 3-element subset. Then $\Lambda_{\{i,j\}}$ and $\Lambda_{\{i,k\}}$ together generate Λ_i.*

Proof: This amounts to showing that the two copies of $\mathbf{SL}(n-2, \mathbf{Z})$ in $\mathbf{SL}(n-1, \mathbf{Z})$ obtained by imbedding $\mathbf{SL}(n-2, \mathbf{Z})$ in the upper left and lower right corners, respectively, generate $\mathbf{SL}(n-1, \mathbf{Z})$. This is easy to see, for example, by exhibiting the two "missing" elementary matrices in the subgroup generated. □

LEMMA 3.7: *For $\{i,j,k\}$ as in (3.6) and ρ in a suitable neighborhood,*

$$\tilde{S}_i = \tilde{T}_{\{i,j\}} \cap \tilde{T}_{\{i,k\}} = \{x \in \mathbb{T}^n \mid \rho(\gamma)x = x \text{ for every } \gamma \in \Lambda_i\}$$

is a C^1 circle through p.

Proof: The equality follows immediately from (3.6).

Set $S_i = \pi(\mathbb{R}e_i)$, which is a circle through 0, the fixed-point set for Λ_i under the standard action. For each point $x \in S_i$, the linear representation of Λ_i on $\mathrm{T}_x\mathbb{T}^n$ is equivalent to the direct sum of the standard representation of $\mathbf{SL}(n-1,\mathbb{Z})$ on $\mathbb{R}^{n-1}$ with the trivial representation (on $\mathbb{R}$).

Since $n-1 \geq 3$, we can conclude that $H^1(\Lambda_i, \mathrm{T}_x\mathbb{T}^n) = 0$: We have already observed (under 3.3) that $H^1(\mathbf{SL}(k,\mathbb{Z}),\mathbb{R}^k) = 0$ when $k \geq 3$, and in this case $\mathbf{SL}(k,\mathbb{Z})$ is Kazhdan, hence $H^1(\mathbf{SL}(k,\mathbb{Z}),\mathbb{R}) = 0$ as well. (Cf. [Zi3]. Observe that $H^1(\cdot,\mathbb{R})$ with coefficients in the trivial representation may be identified with the space of additive homomorphisms $\mathrm{Hom}(\cdot,\mathbb{R})$.)

Thus we can apply (2.5): There is a tubular neighborhood U of S_i in M such that for suitable ρ, $\tilde{S}_i \cap U$ is a C^1 circle through p, and since $\tilde{T}_{\{i,j\}}$ and $\tilde{T}_{\{i,k\}}$ vary continuously with ρ near the transverse tori $T_{\{i,j\}}$ and $T_{\{i,k\}}$, respectively, $\tilde{T}_{\{i,j\}} \cap \tilde{T}_{\{i,k\}} \subset U$ is an open condition on ρ near the standard action. □

LEMMA 3.8: *For each $N \geq 2$ set*

$$\lambda_N = \begin{pmatrix} 1 & 1 & \\ N & N+1 & \\ & & I_{n-2} \end{pmatrix} \in \Gamma.$$

Then the subgroup Ξ_N generated by Λ_1 together with $\lambda_N \Lambda_1 \lambda_N^{-1}$ is a subgroup of finite index in Γ.

Proof: In fact, Ξ_N is precisely the subgroup consisting of matrices whose first column is congruent to

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mod N.$$

We provide the following elementary proof since we know of no reference in the literature. To simplify notation, we carry out the argument for the case $n = 3$; the generalization to the case $n \geq 3$ is obvious. So for the time being, write $\Gamma = \mathbf{SL}(3,\mathbb{Z})$, $\Lambda_1 = \{\begin{pmatrix} 1 & \\ & A \end{pmatrix} \mid A \in \mathbf{SL}(2,\mathbb{Z}) \subset \Gamma\}$, and

$$\lambda_N = \begin{pmatrix} 1 & 1 & \\ N & N+1 & \\ & & 1 \end{pmatrix} \in \Gamma.$$

The first step is to exhibit the four elementary matrices

$$\begin{pmatrix}1 & & 1\\ & 1 & \\ & & 1\end{pmatrix},\ \begin{pmatrix}1 & 1 & \\ & 1 & \\ & & 1\end{pmatrix},\ \begin{pmatrix}1 & & \\ & 1 & \\ N & & 1\end{pmatrix},\quad \text{and}\ \begin{pmatrix}1 & & \\ N & 1 & \\ & & 1\end{pmatrix}$$

in Ξ_N.

$$\begin{pmatrix}1 & 1 & \\ N & N+1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & 1\\ & & 1\end{pmatrix}\begin{pmatrix}N+1 & -1 & \\ -N & 1 & \\ & & 1\end{pmatrix}=\begin{pmatrix}1 & & 1\\ & 1 & N+1\\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & 1\\ & 1 & N+1\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & -(N+1)\\ & & 1\end{pmatrix}=\begin{pmatrix}1 & & 1\\ & 1 & \\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & 1\\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ & 1 & \\ & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ & 1 & \\ & 1 & 1\end{pmatrix}\begin{pmatrix}1 & & -1\\ & 1 & \\ & & 1\end{pmatrix}=\begin{pmatrix}1 & 1 & \\ & 1 & \\ & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & \\ & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & \\ & 1 & \\ & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 1 & \\ & 1 & \\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & \\ N & N+1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & 1 & 1\end{pmatrix}\begin{pmatrix}N+1 & -1 & \\ -N & 1 & \\ & & 1\end{pmatrix}=\begin{pmatrix}1 & & \\ & 1 & \\ -N & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & \\ -N & 1 & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -1 & 1\end{pmatrix}=\begin{pmatrix}1 & & \\ & 1 & \\ -N & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & 1\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ -N & & 1\end{pmatrix}=\begin{pmatrix}1 & & \\ -N & 1 & 1\\ -N & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & \\ N & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ -N & 1 & 1\\ -N & & 1\end{pmatrix}=\begin{pmatrix}1 & & \\ -N & 1 & 1\\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & -1\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ -N & 1 & 1\\ & & 1\end{pmatrix}=\begin{pmatrix}1 & & \\ -N & 1 & \\ & & 1\end{pmatrix}$$

Now suppose

$$\gamma = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \Gamma$$

with $a \equiv 1 \bmod N$ and $d, g \equiv 0 \bmod N$. We must show $\gamma \in \Xi_N$.

(i) There exists $A \in \mathbf{SL}(2, \mathbf{Z})$ with $(b, c)A = (b', 0)$. Then

$$\gamma \begin{pmatrix} 1 & \\ & A \end{pmatrix} = \begin{pmatrix} a & b' & 0 \\ & \cdots & \end{pmatrix},$$

so it will suffice to check the case $c = 0$.

(ii) Then f and i are relatively prime, so there exists $A \in \mathbf{SL}(2, \mathbf{Z})$ with

$$A \begin{pmatrix} f \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & \\ & A \end{pmatrix} \gamma = \begin{pmatrix} a & b & 0 \\ \cdots & & 0 \\ & & 1 \end{pmatrix},$$

so suppose $c = f = 0$, $i = 1$.

(iii) $\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix} \gamma = \begin{pmatrix} a' & b' & 1 \\ \cdots & & 0 \\ & & 1 \end{pmatrix}$, so in particular, we may assume $c = 1$.

(iv) Write $a = kN + 1$. Then

$$\begin{pmatrix} 1 & & \\ & 1 & \\ -kN & & 1 \end{pmatrix}, \begin{pmatrix} 1 & -b & -1 \\ & 1 & \\ & & 1 \end{pmatrix} \in \Xi_N$$

and

$$\gamma \begin{pmatrix} 1 & & \\ & 1 & \\ -kN & & 1 \end{pmatrix} \begin{pmatrix} 1 & -b & -1 \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & \cdots & \end{pmatrix}.$$

So suppose $a = 1$, $b = c = 0$.

(v) Then $\begin{pmatrix} e & f \\ h & i \end{pmatrix} \in \mathbf{SL}(2, \mathbf{Z})$, hence

$$A = \begin{pmatrix} e & f \\ h & i \end{pmatrix}^{-1} \in \mathbf{SL}(2, \mathbf{Z}),$$

and

$$\gamma \begin{pmatrix} 1 & \\ & A \end{pmatrix} = \begin{pmatrix} 1 & & \\ d & 1 & \\ g & & 1 \end{pmatrix} \in \Xi_N,$$

since d and g are multiples of N. □

For each $N \geq 2$, set $S_1^N = \lambda_N S_1$, which is the fixed-point set for $\lambda_N \Lambda_1 \lambda_N^{-1}$. Note that S_1 and S_1^N intersect in N points, evenly distributed along S_1, so that $\cup_N (S_1 \cap S_1^N)$ is dense in S_1. Recall that $\gamma_1, \ldots, \gamma_k$ denote a fixed collection of hyperbolic generators, and $h_1, \ldots, h_k \in \mathrm{Homeo}_1(\mathbb{T}^n)$ such that $\rho(\gamma_i) = h_i \gamma_i h_i^{-1}$.

LEMMA 3.9: *Let $\{0 = q_0, \ldots, q_{N-1}\} = S_1 \cap S_1^N$. Then the h_i's agree on $S_1 \cap S_1^N$, i.e., $h_1(q_j) = \ldots = h_k(q_j)$, $0 \leq j \leq N-1$.*

Proof: Fix $\tilde{p} = \pi^{-1}(p) \in \mathbb{R}^n$. For each $\gamma \in \Gamma$, $\rho(\gamma)$ fixes p, so $\rho(\gamma)$ lifts uniquely to a diffeomorphism $\tilde{\rho}(\gamma) \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{\rho}(\gamma)$ fixes $\tilde{p}$. Note that for each generator γ_i, $\rho(\gamma_i)$ is homotopic to γ_i, hence induces the same map on $H_1(\mathbb{T}^n)$. Thus $\rho(\gamma)$ and γ induce the same map on H_1 for every $\gamma \in \Gamma$. Consequently we have $\tilde{\rho}(\gamma)(x+z) = \tilde{\rho}(\gamma)x + \gamma z$ for every $x \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$. Also, each h_i lifts to a homeomorphism $\tilde{h}_i \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{h}_i(0) = \tilde{p}$, $\tilde{\rho}(\gamma_i) = \tilde{h}_i \gamma_i \tilde{h}_i^{-1}$, and $\tilde{h}_i(x+z) = \tilde{h}_i(x) + z$ for every $x \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$.

Consider the pre-image $\pi^{-1}(S_1) \subset \mathbb{R}^n$. It is a countable collection of lines parallel to the first coordinate axis. For $z \in \mathbb{Z}^{n-1}$, let ℓ_z denote the component of $\pi^{-1}(S_1)$ through the point $\begin{pmatrix} 0 \\ z \end{pmatrix} \in \mathbb{Z}^n \subset \mathbb{R}^n$. Then ℓ_z may be characterized as

$$\ell_z = \{x \in \mathbb{R}^n \mid \gamma x = x + \gamma \begin{pmatrix} 0 \\ z \end{pmatrix} \text{ for every } \gamma \in \Lambda_1\}.$$

Similarly,

$$\pi^{-1}(S_1^N) = \bigcup_{z \in \mathbb{Z}^{n-1}} \ell_z^N,$$

where

$$\ell_z^N = \{x \in \mathbb{R}^n \mid \gamma x = x + \gamma \lambda_N \begin{pmatrix} 0 \\ z \end{pmatrix} \text{ for every } \gamma \in \lambda_N \Lambda_1 \lambda_N^{-1}\}.$$

Likewise,

$$\pi^{-1}(\tilde{S}_1) = \bigcup_{z \in \mathbb{Z}^{n-1}} \tilde{\ell}_z,$$

where

$$\tilde{\ell}_z = \{x \in \mathbb{R}^n \mid \tilde{\rho}(\gamma) x = x + \gamma \begin{pmatrix} 0 \\ z \end{pmatrix} \text{ for every } \gamma \in \Lambda_1\}.$$

($\tilde{\ell}_0$ is the component of $\pi^{-1}(\tilde{S}_1)$ through $\tilde{p}$, and $\tilde{\ell}_z = \tilde{\ell}_0 + \begin{pmatrix} 0 \\ z \end{pmatrix}$.)

Set $\tilde{S}_1^N = \rho(\lambda_N)\tilde{S}_1$, the circle of fixed points for $\rho(\lambda_N \Lambda_1 \lambda_N^{-1})$. Then

$$\pi^{-1}(\tilde{S}_1^N) = \bigcup_{z \in \mathbb{Z}^{n-1}} \tilde{\ell}_z^N,$$

where

$$\begin{aligned}\tilde{\ell}_z^N &= \{x \in \mathbb{R}^n \mid \tilde{\rho}(\gamma)x = x + \gamma\lambda_N \begin{pmatrix} 0 \\ z \end{pmatrix} \text{ for every } \gamma \in \lambda_N\Lambda_1\lambda_N^{-1}\} \\ &= \tilde{\rho}(\lambda_N)\tilde{\ell}_z.\end{aligned}$$

Recall that $\tilde{S}_1 \subset \tilde{T}_{\{1,2\}}$, and $\lambda_N \in \Lambda_{\{3,\ldots,n\}}$ implies $\rho(\lambda_N)\tilde{T}_{\{1,2\}} = \tilde{T}_{\{1,2\}}$. Thus $\tilde{S}_1^N \subset \tilde{T}_{\{1,2\}}$ as well. $\pi^{-1}(\tilde{T}_{\{1,2\}})$ is a collection of parallel surfaces indexed by $\mathbb{Z}^{n-2}$; let

$$s = s_0 = \{x \in \mathbb{R}^n \mid \tilde{\rho}(\gamma)x = x \text{ for every } \gamma \in \Lambda_{\{1,2\}}\}.$$

Note that $\tilde{\ell}_z, \tilde{\ell}_z^N \subset s_0$ whenever

$$z = \begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{Z}^{n-1}$$

satisfies $z_3 = \ldots = z_n = 0$. (In general, $\tilde{\ell}_z, \tilde{\ell}_z^N \subset s_{z'}$, where

$$z' = \begin{pmatrix} z_3 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{Z}^{n-2}.)$$

Now the distances $d(\tilde{\ell}_z, \ell_z) = d(\tilde{S}_1, S_1)$ and $d(\tilde{\ell}_z^N, \ell_z^N) = d(\tilde{S}_1^N, S_1^N)$ are small (in particular, finite), and $\tilde{\ell}_z$ and $\tilde{\ell}_0^N$ both lie in s provided $z_3 = \ldots = z_n = 0$. Thus for such z, $\tilde{\ell}_z \cap \tilde{\ell}_0^N \neq \phi$.

For each $j = 0, \ldots, N-1$, set

$$z_j = \begin{pmatrix} j \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{Z}^{n-1},$$

so that $\ell_{z_j} \cap \ell_0^N = \{\tilde{q}_j\}$, where $\tilde{q}_0 = 0$, $\pi(\tilde{q}_j) = q_j$, and set $X_j = \tilde{\ell}_{z_j} \cap \tilde{\ell}_0^N \neq \phi$. We will show that X_j consists of a single point $X_j = \{\tilde{p}_j\}$ and that $\tilde{h}_i(\tilde{q}_j) = \tilde{p}_j$ for each i. Then $h_i(q_j) = h_i(\pi(\tilde{q}_j)) = \pi(\tilde{h}_i(\tilde{q}_j)) = \pi(\tilde{p}_j)$ for each i and the proof of (3.9) will be complete.

So suppose $x \in X_j = \tilde{\ell}_{z_j} \cap \tilde{\ell}_0^N$. Then for each $\gamma \in \Lambda_1$ we have $\tilde{\rho}(\gamma)x = x + \gamma \begin{pmatrix} 0 \\ z_j \end{pmatrix}$ and for each $\gamma \in \lambda_N\Lambda_1\lambda_N^{-1}$ we have $\tilde{\rho}(\gamma)x = x$. Since Λ_1 and

$\lambda_N \Lambda_1 \lambda_N^{-1}$ together generate Ξ_N, it follows that there is a cocycle $\alpha\colon \Xi_N \to \mathbf{Z}^N$ such that $\tilde{\rho}(\gamma)x = x + \alpha(\gamma)$ for every $\gamma \in \Xi_N$.

Fix one of the generators γ_i. Then since γ_i has infinite order and Ξ_N has finite index in Γ, it follows that $\gamma_i^K \in \Xi_N$ for some $K \geq 1$. Now we have already ensured that the diffeomorphism $\rho(\gamma_i)$ is hyperbolic. In particular, the mapping $\rho(\gamma_i)^K - \mathrm{Id}\colon \mathbf{T}^n \to \mathbf{T}^n$ is non-singular at each point of $\mathbf{T}^n$, and is therefore a covering map, and $\tilde{\rho}(\gamma_i)^K - \mathrm{Id}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

Since $\tilde{\rho}(\gamma_i)^K - \mathrm{Id}$ is invertible, there is a unique point

$$\tilde{p}_j = (\tilde{\rho}(\gamma_i)^K - \mathrm{Id})^{-1}\alpha(\gamma_i^K) \in \mathbb{R}^n$$

such that $\tilde{\rho}(\gamma_i)^K \tilde{p}_j = \tilde{p}_j + \alpha(\gamma_i^K)$. Thus $X_j = \{\tilde{p}_j\}$. Moreover, $\tilde{q}_j$ is the unique point of $\mathbb{R}^n$ such that $\gamma_i^K \tilde{q}_j = \tilde{q}_j + \alpha(\gamma_i^K)$, hence

$$\begin{aligned}\tilde{\rho}(\gamma_i)^K \tilde{h}_i(\tilde{q}_j) = \tilde{h}_i(\gamma_i^K(\tilde{q}_j)) &= \tilde{h}_i(\tilde{q}_j + \alpha(\gamma_i^K)) \\ &= \tilde{h}_i(\tilde{q}_j) + \alpha(\gamma_i^K).\end{aligned}$$

Thus $\tilde{h}_i(\tilde{q}_j) = \tilde{p}_j$ and the proof of (3.9) is complete. □

COROLLARY 3.10: *$h_1(x) = \cdots = h_k(x)$ for every $x \in S_1$.*

Proof: $\bigcup_N (S_1 \cap S_1^N)$ is dense in S_1. □

COROLLARY 3.11: *Fix $\gamma_0 \in \Gamma$. Then $h_1(x) = \cdots = h_k(x)$ for every $x \in \gamma_0 S_1$.*

Proof: The subgroups $\gamma_0 \Lambda_1 \gamma_0^{-1}$ and $\gamma_0 \lambda_N \Lambda_1 \lambda_N^{-1} \gamma_0^{-1}$ together generate $\gamma_0 \Xi_N \gamma_0^{-1}$, which is again of finite index in Γ. Then the same argument as in (3.9) shows that $h_i(\gamma_0 q_j) = \rho(\gamma_0) h_i(q_j)$, $q_j \in S_1 \cap S_1^N$. □

Since $\bigcup_{\gamma \in \Gamma} \gamma S_1$ is dense in $\mathbf{T}^n$, this completes the proof of (3.1).

4. Smooth Conjugacy

In this section, we show that for suitable ρ, the conjugating homeomorphism h in (3.1) is in fact a C^∞ diffeomorphism. Since neither (3.1) nor the argument below requires more than C^1 control on the perturbed action ρ, we will actually establish the following

THEOREM 4.1: *Suppose $\Gamma = \mathbf{SL}(n, \mathbf{Z})$, or any subgroup of finite index, $n \geq 4$, and let $\mathrm{Diff}^*(\mathbf{T}^n)$ denote the group of C^∞ diffeomorphisms of $\mathbf{T}^n$ under the C^1 topology. Then there is a neighborhood U of the standard (linear) action of Γ on $\mathbf{T}^n$ in $R(\Gamma, \mathrm{Diff}^*(\mathbf{T}^n))$ such that for every $\rho \in U$ there exists a C^∞ diffeomorphism h of $\mathbf{T}^n$ such that $\rho(\gamma) = h\gamma h^{-1}$ for every $\gamma \in \Gamma$.*

Once again, the basic outline of our argument is the same as Hurder's. We begin by selecting a free abelian subgroup $\mathcal{A}$ of Γ generated by $n-1$ hyperbolic matrices. A simultaneous diagonalization of this subgroup yields n transverse, one-dimensional foliations $\mathcal{F}_i$ of $\mathbf{T}^n$ which are each invariant under the action of $\mathcal{A}$, and the topological conjugacy h carries each $\mathcal{F}_i$ to a foliation $h\mathcal{F}_i$ which is invariant under the perturbed action of $\mathcal{A}$. An application of the "stable manifold theorem" [H-P] shows that each foliation $h\mathcal{F}_i$ has C^∞ leaves. The argument is completed by showing that h is smooth along the leaves.

The essential difference between our approach and Hurder's is the argument which establishes smoothness along the leaves. Hurder applies finite-dimensional cohomology vanishing to show that the infinitesimal representation of the stabilizer at each periodic (rational) point is stable, then invokes some recently-developed machinery from the theory of smooth Anosov systems. The argument we give below is based on a simple idea which first appeared in Koppel [Ko]. Very briefly, we construct a smooth local linearizing parameter g for the perturbed action of $\mathcal{A}$ along each leaf, then show that the composition $g \circ h$ is smooth, hence h is smooth.

An important advantage of our approach over that of [Hu3] is that it takes place entirely within the group $\mathcal{A}$ itself, without referring to the action of the ambient group Γ at all. In fact, the discussion in this section is logically independent from that in the preceding section, since the action of a hyperbolically-generated abelian group is automatically topologically rigid (in the sense of (3.1)) by structural stability (2.1). Thus we need not assume that the action of $\mathcal{A}$ extends to an action of Γ and we shall actually prove the following

THEOREM 4.2: *Suppose $\mathcal{A} \subset \mathbf{SL}(n,\mathbf{Z})$, $n \geq 3$, is a free abelian group of rank $n-1$, generated by $n-1$ hyperbolic matrices. (Such groups exist; cf. below.) Then the standard action of $\mathcal{A}$ on $\mathbf{T}^n$ is locally rigid in $R(\mathcal{A}, \mathrm{Diff}^\infty(\mathbf{T}^n))$.*

In fact, the proof gives a global rigidity result for smooth actions of $\mathcal{A}$ under appropriate hypotheses; these will be summarized under (4.12) below. Moreover, it is easy to see that the argument extends to hyperbolically-generated abelian group actions of rank $k \geq 2$, provided that they satisfy certain conditions on the eigenvalues of the generators. The precise consequences of these ideas for hyperbolic $\mathbf{Z}^k$-actions are currently under investigation. This work may be viewed as a complement to the recent program of Palis and Yoccoz [P-Y] who show, in part by related arguments, that "generically" Anosov diffeomorphisms commute only with their own powers.

We now begin the detailed proof of (4.1) and (4.2) by establishing the necessary algebraic data.

LEMMA 4.3: *Let $\Gamma = \mathrm{SL}(n, \mathbf{Z})$, $n \geq 2$, or more generally, a subgroup of finite index in $\mathrm{SL}(n, \mathbf{Z})$. Then there exists a Cartan subgroup H of $\mathrm{SL}(n, \mathbb{R})$ such that the quotient $H/(H \cap \Gamma)$ is compact. In particular, there exists a subgroup $\mathcal{A} \subset \Gamma$ such that (i) the elements of $\mathcal{A}$ are simultaneously diagonalizable over $\mathbb{R}$, and (ii) $\mathcal{A}$ is isomorphic to a free abelian group of rank $n-1$.*

This follows from the general results in [P-R]*, although in the special case $\Gamma = SL(n, \mathbf{Z})$ it can be established more directly. For example, $\mathcal{A}$ corresponds to a subgroup of finite index in the group of units of norm 1 in the ring of integers in a suitable algebraic number field.

Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be a basis of simultaneous eigenvectors for the group $\mathcal{A}$, and $\lambda_i \colon \mathcal{A} \to \mathbb{R}^\times$ the character on $\mathcal{A}$ defined via $Av_i = \lambda_i(A)v_i$, $A \in \mathcal{A}$. In order to simplify notation, we pass to a subgroup of finite index and assume that each λ_i takes values in $\mathbb{R}^+$. Then H^0 = the identity component in H, above, is a maximal $\mathbb{R}$-split torus in $\mathrm{SL}(n, \mathbb{R})$ with eigenvectors v_i and λ_i extends to $\lambda_i \colon H^0 \to \mathbb{R}^+$ so that

$$\lambda_1 \times \cdots \times \lambda_n \colon H^0 \to \{(x_1, \ldots, x_n) \in (\mathbb{R}^+)^n | x_1 \ldots x_n = 1\}$$

is an isomorphism of analytic groups and $H/\mathcal{A}$ is compact. As an immediate consequence of the cocompactness of $\mathcal{A}$ in H^0, we have that for each i, $1 \leq i \leq n$, there exists $A_i \in \mathcal{A}$ such that $\lambda_i(A_i) < 1$, $\lambda_j(A_i) > 1$ for each $j \neq i$.

By standard results, the torus H^0 is $\mathbb{Q}$-anisotropic. Equivalently, none of the eigenspaces $\mathbb{R}v_i$ is a rational line in $\mathbb{R}P^n$. In fact, let $\pi \colon \mathbb{R}^n \to \mathbf{T}^n$ denote the natural projection. Then $\pi(\mathbb{R}v_i)$ is the stable manifold through 0 for the hyperbolic diffeomorphism A_i on the torus; in particular, $\pi(\mathbb{R}v_i)$ is dense in $\mathbf{T}^n$ for each i.

LEMMA 4.4: *Suppose $A \in \mathcal{A}$, $A \neq 1$. Then $\lambda_i(A) \neq 1$ for each $1 \leq i \leq n$.*

Proof: Otherwise $Av_i = v_i \Rightarrow A$ fixes each point of $\overline{\pi(\mathbb{R}v_i)} = \mathbf{T}^n \Rightarrow A = 1$. □

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*We are indebted to S. Hurder for directing our attention to this reference.

LEMMA 4.5: *Fix generators $B_1, \dots, B_{n-1}$ for $\mathcal{A} \simeq \mathbf{Z}^{n-1}$. Then there is no nontrivial relation of the form $\lambda_i(B_1)^{p_1} \dots \lambda_i(B_{n-1})^{p_{n-1}} = 1$, with $1 \le i \le n$, $p_j \in \mathbf{Z}$, and at least one $p_j \neq 0$.*

Proof: Otherwise $B_1^{p_1} \dots B_{n-1}^{p_{n-1}} \neq 1$ and $\lambda_i(B_1^{p_1} \dots B_{n-1}^{p_{n-1}}) = 1$, contradicting (4.4). □

The particular consequence of (4.5) which we need is the following:

COROLLARY 4.6: *For each $1 \le i \le n$, $\{\lambda_i(A) | A \in \mathcal{A}\}$ is a dense subgroup in $\mathbb{R}^+$.*

For each $1 \le i \le n$, fix $A_i \in \mathcal{A}$ such that $\lambda_i(A_i) < 1$, $\lambda_j(A_i) > 1$ for $j \neq i$. Let $\mathcal{F}_i^S, \mathcal{F}_i^U$ denote the stable and unstable foliations. respectively, for the hyperbolic diffeomorphism A_i of $\mathbb{T}^n$. I.e., the leaves of $\mathcal{F}_i^S$ are the images under π of lines in $\mathbb{R}^n$ parallel to v_i, those of $\mathcal{F}_i^U$ the images of hyperplanes parallel to the span of the remaining v_j, $j \neq i$.

For suitable ρ, each $\rho(A_i)$ is Anosov; $\tilde{\mathcal{F}}_i^S = h(\mathcal{F}_i^S)$ and $\tilde{\mathcal{F}}_i^U = h(\mathcal{F}_i^U)$ are the stable and unstable foliations for $\rho(A_i)$. The following properties of the foliations $\tilde{\mathcal{F}}_i^S$ and $\tilde{\mathcal{F}}_i^U$ are all described in [H-P]. Recall that a family $\{W_x\}_{x \in M}$ of k-dimensional C^∞ submanifolds of M is said to **vary continuously** if for each $x \in M$ there exists a neighborhood U of x in M and a continuous map $\varphi: U \to C^\infty(D^k, M)$ such that φ_x maps D^k diffeomorphically onto a neighborhood centered at x in W_x, where D^k denotes the unit disk in $\mathbb{R}^k$.

LEMMA 4.7: *("Stable manifold theorem") For each $1 \le i \le n$, the one-dimensional foliation $\tilde{\mathcal{F}}_i^S$ and the $(n-1)$-dimensional foliation $\tilde{\mathcal{F}}_i^U$ are transverse continuous foliations with C^∞ leaves. The families $\{\tilde{\mathcal{F}}_i^S(x)\}$ and $\{\tilde{\mathcal{F}}_i^U(x)\}$ of leaves vary continuously. Moreover, the foliations are Hölder continuous. I.e., let $\tilde{\mathcal{E}}_i^S, \tilde{\mathcal{E}}_i^U$ denote the corresponding tangent distributions. Then $\tilde{\mathcal{E}}_i^S$ and $\tilde{\mathcal{E}}_i^U$ are uniformly Hölder continuous with respect to the standard Riemannian metric on $\mathbb{T}^n$ and corresponding induced metric on the bundle of distributions.*

Fix an index i_0, $1 \le i_0 \le n$. Set $A = A_{i_0}$, $\mathcal{F} = \mathcal{F}_{i_0}^S$, $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{i_0}^S$. The leaves of both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ inherit natural Riemannian metrics as submanifolds of $\mathbb{T}^n$. For each $x \in \mathbb{T}^n$, let $\varphi_x: \mathbb{R} \to \mathcal{F}(x)$ denote the arc-length parameterization based at x, oriented so that v_{i_0} points in the positive direction. I.e., $\varphi_x(0) = x$, the distance along $\mathcal{F}(x)$ between x and $\varphi_x(t) \in \mathcal{F}(x)$ is $|t|$, and $\langle v_i, (\varphi_x)_*(\frac{d}{dt})\rangle > 0$ (standard inner product on $\mathrm{T}_x\mathbb{T}^n \simeq \mathbb{R}^n$). Define $\tilde{\varphi}_x: \mathbb{R} \to \tilde{\mathcal{F}}(x)$ similarly, oriented so that $\tilde{\varphi}_{h(x)}^{-1} \circ h \circ \varphi_x : \mathbf{R} \to \mathbf{R}$ is an orientation-preserving (monotone increasing) homeomorphism. Our next objective is to show that h is smooth along the leaves of $\mathcal{F}$. More precisely, we shall show that $x \mapsto \tilde{\varphi}_{h(x)}^{-1} \circ h \circ \varphi_x$ is a continuous map $M \to C^\infty(\mathbf{R})$

By construction, $\varphi_x : \mathbf{R} \to \mathcal{F}(x)$ and $\tilde{\varphi}_x : \mathbf{R} \to \tilde{\mathcal{F}}(x)$ are diffeomorphisms for every $x \in \mathbf{T}^n$. Let $\mathcal{L} = \mathbf{T}^n \times \mathbf{R}$ denote the trivial line bundle over $\mathbf{T}^n$. It follows easily from (4.7) that $\varphi : \mathcal{L} \to \mathbf{T}^n$, $(x,t) \mapsto \varphi_x(t)$ and $\tilde{\varphi} : \mathcal{L} \to \mathbf{T}^n$, $(x,t) \mapsto \tilde{\varphi}_x(t)$ are continuous, and that $x \mapsto \varphi_x$, $x \mapsto \tilde{\varphi}_x$ are continuous maps $\mathbf{T}^n \to C^\infty(\mathbf{R}, \mathbf{T}^n)$.

Let $f = \rho(A) \in \mathrm{Diff}(\mathbf{T}^n)$. Extend f and h to transformations on $\mathcal{L}$ in the obvious way, namely, define

$$F : \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto \big(f(x), F_x(t)\big) \quad \text{and} \quad H : \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto \big(h(x), H_x(t)\big)$$

so that

$$\tilde{\varphi}(F(x,t)) = f\big(\tilde{\varphi}(x,t)\big) \quad \text{and} \quad \tilde{\varphi}\big(H(x,t)\big) = h\big(\varphi(x,t)\big).$$

Then F and H are continuous, $F_x \in C^\infty(\mathbf{R})$ for each $x \in \mathbf{T}^n$, $0 < F_x'(t) < 1$ for every $x \in \mathbf{T}^n$, $t \in \mathbf{R}$, and $x \mapsto F_x$ is a continuous map $\mathbf{T}^n \to C^\infty(\mathbf{R})$. We must show that $H_x \in C^\infty(\mathbf{R})$ and $x \mapsto H_x$ is continuous.

By the "non-stationary Sternberg lemma" described in the appendix, there exists a unique continuous linearization

$$G : \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto \big(x, G_x(t)\big)$$

such that
(i) $G_x \in C^\infty(\mathbf{R})$ with $G_x'(0) = 1$ for every $x \in \mathbf{T}^n$,
(ii) $\mathbf{T}^n \to C^\infty(\mathbf{R})$, $x \mapsto G_x$ is continuous,
(iii) and $GFG^{-1}(x,t) = (f(x), F_x'(0)t)$ for every $x \in \mathbf{T}^n$ and $t \in \mathbf{R}$.

LEMMA 4.8: *Suppose $p \in \mathbf{T}^n$ is rational (i.e., p is a periodic point for the standard action of every matrix in $\mathrm{SL}(n, \mathbf{Z})$). Then $G_{h(p)} \circ H_p|_{\mathbf{R}^+} : \mathbf{R}^+ \to \mathbf{R}^+$ has the form $G_{h(p)} \circ H_p(t) = c_p t^{\nu_p}$ for some $c_p, \nu_p > 0$.*

Proof: Since $\mathcal{A}$ is abelian, it follows from the uniqueness (A.2) that G simultaneously linearizes the transformations on $\mathcal{L}$ corresponding to $\rho(A)$ for each $A \in \mathcal{A}$. By (4.6), we can find $B, C \in \mathcal{A}$ such that $\lambda_{i_0}(B) = \beta$, $\lambda_{i_0}(C) = \gamma$ with $\beta, \gamma > 1$ such that β and γ generate a dense subgroup in $\mathbf{R}^+$. By replacing B and C with appropriate powers, we may assume that p is a fixed point for the action of both B and C.

Let $r = \rho(B)$, $s = \rho(C)$, and define $R, S\colon \mathcal{L} \to \mathcal{L}$ as above, so that $\tilde\varphi R = r\tilde\varphi$ and $\tilde\varphi S = s\tilde\varphi$. Then

$$GRG^{-1}(x,t) = \big(r(x), \tilde\beta_x t\big),\ GSG^{-1}(x,t) = \big(s(x), \tilde\gamma_x t\big),$$

where $\tilde\beta_x = R'_x(0)$, $\tilde\gamma_x = S'_x(0)$. In particular, since $h(p)$ is fixed by r and s,

$$G_{h(p)} \circ R_{h(p)} = \tilde\beta G_{h(p)} \quad \text{and} \quad G_{h(p)} \circ S_{h(p)} = \tilde\gamma G_{h(p)},$$

with $\tilde\beta = \tilde\beta_{h(p)}$, $\tilde\gamma = \tilde\gamma_{h(p)}$. Also, since h intertwines ρ and the standard action,

$$R_{h(p)} \circ H_p(t) = H_p(\beta t) \quad \text{and} \quad S_{h(p)} \circ H_p(t) = H_p(\gamma t).$$

Let

$$\psi = G_{h(p)} \circ H_p|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}^+.$$

Then we have shown that for every $t \in \mathbb{R}^+$,

$$\psi(\beta t) = \tilde\beta\psi(t) \quad \text{and} \quad \psi(\gamma t) = \tilde\gamma\psi(t).$$

By construction, ψ is an orientation-preserving homeomorphism.

Let $c = \psi(1)$. Then $\psi(\beta^k\gamma^\ell) = c\tilde\beta^k\tilde\gamma^\ell$ for every $k, \ell \in \mathbf{Z}$. Hence

$$\{\beta^k\gamma^\ell | k,\ \ell \in \mathbf{Z}\} \to \{\tilde\beta^k\tilde\gamma^\ell | k,\ \ell \in \mathbf{Z}\},\ \beta^k\gamma^\ell \mapsto \tilde\beta^k\tilde\gamma^\ell$$

is an order-preserving map between these two subsets of $\mathbb{R}^+$. It follows easily that

$$\frac{\log\beta}{\log\gamma} = \frac{\log\tilde\beta}{\log\tilde\gamma},$$

hence

$$\psi(t) = ct^\nu \quad \text{for every } t \in \{\beta^k\gamma^\ell\},$$

where

$$\nu = \frac{\log\tilde\beta}{\log\beta} = \frac{\log}{\tilde\gamma}\log\gamma.$$

But this set is dense in $\mathbf{R}^+$ and ψ is continuous, hence $\psi(t) = ct^\nu$ for every $t \in \mathbf{R}^+$. □

Now for each $x \in \mathbb{T}^n$, set $\psi_x = G_{h(x)} \circ H_x|_I \colon I \to \mathbb{R}^+$, $I = [0,1]$. Since I is compact and $G \circ H|_{\mathbb{T}^n \times I}$ is continuous it follows that $\mathbb{T}^n \to C^0(I)$, $x \mapsto \psi_x$ is continuous with respect to the uniform topology on $C^0(I)$. By (4.6), $\psi_p(t) = c_p t^{\nu_p}$ for a dense set of $p \in \mathbb{T}^n$. It follows that $p \mapsto c_p$ and $p \mapsto \nu_p$ must extend to continuous functions $\mathbb{T}^n \to \mathbb{R}^+$ such that $\psi_x(t) = c_x t^{\nu_x}$ for every $x \in \mathbb{T}^n$. An entirely analogous argument works with $-I = [-1,0]$ in place of I and $\mathbb{R}^-$ in place of $\mathbb{R}^+$. Also, we can replace I with any compact interval $[0,T]$.

Thus we have proved the following:

Lemma 4.9: *There exist continuous functions $c^{\pm}, \nu^{\pm} \colon \mathbb{T}^n \to \mathbb{R}^+$ such that for every $x \in \mathbb{T}^n$, $G_{h(x)} \circ H_x \colon \mathbb{R} \to \mathbb{R}$ has the form*

$$G_{h(x)} \circ H_x(t) = \begin{cases} c_x^+ t^{\nu_x^+}, & t \geq 0, \\ -c_x^- |t|^{\nu_x^-}, & t \leq 0. \end{cases}$$

Now for each $x \in \mathbb{T}^n$, $G_{h(x)} \circ H_x$ is smooth away from 0, and $G_{h(x)}$ is a C^∞ diffeomorphism, hence H_x is smooth away from 0. But φ maps $\mathbb{T}^n \times (\mathbb{R} - \{0\})$ onto $\mathbb{T}^n$, so this implies that h is C^∞ along each leaf of $\mathcal{F}$, more precisely, $h|_{\mathcal{F}(x)} \colon \mathcal{F}(x) \to \tilde{\mathcal{F}}(h(x))$ is C^∞ for every $x \in \mathbb{T}^n$. Thus $G_{h(x)} \circ H_x$ must be smooth at 0 as well, hence $c_x^+ = c_x^-$, and $\nu_x^+ = \nu_x^- = 1$ for every $x \in \mathbb{T}^n$. We have shown that $x \mapsto G_{h(x)} \circ H_x$ defines a continuous map $\mathbb{T}^n \to C^\infty(\mathbb{R})$. The same is true for $x \mapsto G_{h(x)}$, and each $G_{h(x)}$ is a diffeomorphism. Since the diffeomorphisms of $\mathbb{R}$ form a topological group with respect to the subspace topology inherited from $C^\infty(\mathbb{R})$, we conclude that $\mathbb{T}^n \to C^\infty(\mathbb{R})$, $x \mapsto H_x = G_{h(x)}^{-1} \circ (G_{h(x)} \circ H_x)$ is continuous.

We will say that a topological foliation of a smooth n-dimensional manifold M by k-dimensional leaves has **uniformly C^∞ leaves** if there exists an atlas of continuous foliation charts of the form $\psi \colon I^{n-k} \times I^k \to M$, $(x,y) \mapsto \psi_x(y)$, with ψ_x bounded in $C^\infty(I^k, M)$, i.e., with uniform bounds on ψ_x and each of its derivatives of all orders independent of $x \in I^{n-k}$. In this case, a function $f \colon M \to N$ is said to be **uniformly C^∞ along leaves** if f_x is bounded in $C^\infty(I^k, N)$, where f_x is defined via $f \circ \psi(x,y) = f_x(y)$, for each of the charts ψ as above. Note that if $x \mapsto f_x$ defines a continuous map $I^{n-k} \to C^\infty(I^k, N)$ then this condition is automatically satisfied since I^{n-k} is compact.

The foliation $\mathcal{F}$ is smooth (in fact, the leaves of $\mathcal{F}_i^S$, $1 \leq i \leq n$, constitute a smooth parallelism on $\mathbb{T}^n$) and the smooth foliation charts determine a uniformly C^∞ structure along the leaves. For each $x \in \mathbb{T}^n$, we can construct a continuous foliation chart for $\tilde{\mathcal{F}}$ centered at x as follows. First fix a small transverse slice with continuous coordinates centered at x, e.g., a smooth coordinate chart centered at x in $\mathcal{F}^U_{i_0}(x)$. Then extend along the leaves of $\tilde{\mathcal{F}}$ via the arc length parameter to obtain a continuous foliation chart centered at x with C^∞ leaves and such that the C^∞ coordinate charts along the leaves vary continuously with respect to the transverse coordinate. In particular, this determines a uniformly C^∞ structure along the leaves of $\tilde{\mathcal{F}}$. We can encapsulate all that we need from the preceding discussion as follows, again making use of the fact that inversion defines a continuous involution on the C^∞ diffeomorphisms of $\mathbb{R}$.

LEMMA 4.10: *For each i, $1 \le i \le n$, the one-dimensional foliations $\mathcal{F}^S_i, \tilde{\mathcal{F}}^S_i$ have uniformly C^∞ leaves, and $h, h^{-1}\colon \mathbb{T}^n \to \mathbb{T}^n$ are uniformly C^∞ along the leaves of $\mathcal{F}^S_i, \tilde{\mathcal{F}}^S_i$, respectively.*

We are now in a position to apply the following theorem of Journé [J]:

LEMMA 4.11: *Let M be a C^∞ manifold and $\mathcal{F}$ and $\mathcal{F}'$ be two Hölder foliations, transverse, and with uniformly C^∞ leaves. If a function f is uniformly C^∞ along the leaves of the two foliations, then it is C^∞ on M.*

For $1 \le j \le n$, define C^∞ j-dimensional foliations $\mathcal{G}_j$ of $\mathbb{T}^n$ as follows. The leaves of $\mathcal{G}_j$ are the images under π of j-planes in $\mathbb{R}^n$ parallel to the span of the first j basis vectors v_i, $1 \le i \le j$, so that $\mathcal{G}_1 = \mathcal{F}^S_1$ and $\mathcal{G}_n$ is the trivial foliation with one leaf. Then $\mathcal{G}_{j-1}$ and $\mathcal{F}^S_j$ restrict to transverse foliations on each leaf of $\mathcal{G}_j$. Let $\tilde{\mathcal{G}}_j = h(\mathcal{G}_j)$. Since $\mathcal{A}$ is co-compact in H^0, there exists $C_j \in \mathcal{A}$, $1 \le j \le n-1$, such that $\lambda_i(C_j) < 1$, $i \le j$, and $\lambda_i(C_j) > 1$, $i > j$, i.e., so that $\mathcal{G}_j$ is the stable foliation for the standard action of C_j on $\mathbb{T}^n$. For suitable ρ, each of the diffeomorphisms $\rho(C_j)$ is Anosov, and $\tilde{\mathcal{G}}_j$ is the stable foliation for the Anosov diffeomorphism $\rho(C_j)$. Thus we can apply the stable manifold theorem to conclude that the foliation $\tilde{\mathcal{G}}_j$ is Hölder continuous with continuously varying C^∞ leaves.

Now apply (4.11) inductively. I.e., suppose that we have shown that h^{-1} is uniformly C^∞ along the leaves of $\tilde{\mathcal{G}}_j$. Then the restrictions of $\tilde{\mathcal{G}}_j$, $\tilde{\mathcal{F}}^S_{j+1}$, and h^{-1} to the leaves of $\tilde{\mathcal{G}}_{j+1}$ satisfy the hypotheses of (4.11), and we can conclude that h^{-1} is C^∞ along the leaves of $\tilde{\mathcal{G}}_{j+1}$. Also Journé's argument yields uniform bounds on the derivatives of h^{-1} along the leaves of $\tilde{\mathcal{G}}_{j+1}$ which depend only on the bounds on the derivatives of h^{-1} along the leaves of $\tilde{\mathcal{G}}_j$ and $\tilde{\mathcal{F}}^S_{j+1}$ and the (uniform) Hölder constants associated with the foliations. Thus h^{-1} is uniformly C^∞ along the leaves of $\tilde{\mathcal{G}}_{j+1}$, the induction goes through, and we conclude that h^{-1} is C^∞. A similar argument shows that h is C^∞. Thus h is a C^∞ diffeomorphism, and the proof of (4.2) is complete.

In fact, the proof yields the following global result, which we summarize here for future reference.

THEOREM 4.12: *Suppose $\mathcal{A} \subset \mathbf{SL}(n,\mathbb{Z})$, $n \geq 3$, is a free abelian group of rank $n-1$, and $\rho \in R(\mathcal{A}, \mathrm{Diff}^*(\mathbb{T}^n))$. Then if ρ satisfies*
(i) the induced action ρ_* on $\pi_1(\mathbb{T}^n)$ is conjugate to the standard one, and
(ii) for some collection $\{A_i \mid 1 \leq i \leq n\} \subset \mathcal{A}$ of codimension-1 hyperbolic matrices satisfying the conditions $\lambda_i(A_i) < 1$, $\lambda_j(A_i) > 1$ for each $j \neq i$ (as described under (4.3) above), the diffeomorphisms $\rho(A_i)$ are all Anosov
there exists a subgroup $\mathcal{A}' \subset \mathcal{A}$ of finite index and a C^∞ diffeomorphism h of $\mathbb{T}^n$ such that $\rho(A) = hAh^{-1}$ for every $A \in \mathcal{A}'$.

The only additional observation needed to establish (4.12) is that by a theorem of Franks [F], condition (i) implies that the action of each Anosov generator $\rho(A_i)$ is topologically conjugate to the standard linear action of A_i, and the conjugating homeomorphism h_i is unique in the homotopy class of the identity up to a finite set of rational translations, corresponding to the fixed points of the linear Anosov diffeomorphism A_i. Since $\mathcal{A}$ is abelian, each of these finite sets is invariant under the full group $\mathcal{A}$, and there exists $p \in \mathbb{T}^n$ and a subgroup $\mathcal{A}' \subset \mathcal{A}$ of finite index such that $\rho(A)p = p$ for every $A \in \mathcal{A}'$. As we have remarked above, once we require that $h_i(0) = p$, the conjugacies h_i for the commuting Anosov diffeomorphisms $\rho(A_i)$ must coincide. Thus (i) immediately implies the existence of a topological conjugacy between ρ and the standard action on a subgroup of finite index.

5. Subgroups of Finite Index

In this section we extend the proof of (3.1) to the case in which Γ is a subgroup of finite index in $\mathbf{SL}(n,\mathbb{Z})$. The first step is to obtain a suitable set of hyperbolic generators.

LEMMA 5.1: *Suppose $A, H \in \mathbf{SL}(n,\mathbb{R})$ with H hyperbolic. Let E_S, E_U denote the stable and unstable subspaces of $\mathbb{R}^n$, respectively, for the action of H, so that $\mathbb{R}^n = E_S \oplus E_U$, $HE_S = E_S$, $HE_U = E_U$, and there exists ϵ, $0 < \epsilon < 1$, such that $\|Hx\| \leq (1-\epsilon)\|x\|$ for every $x \in E_S$ and $\|Hx\| \geq (1+\epsilon)\|x\|$ for every $x \in E_U$. Then either $H^N A$ is hyperbolic for some $N \in \mathbb{Z}$ or $AE_U \cap E_S \neq 0$.*

Proof: Suppose $H^N A$ fails to be hyperbolic (i.e., has an eigenvalue on the unit circle) for every $N \geq 0$. Then there exist sequences v_N, $w_N \in \mathbb{R}^n$ such that $H^N A v_N = w_N$, $||v_N|| = ||w_N|| = 1$, and $w_N = \lambda_N v_N$, with $\lambda_N \in \mathbb{C}$, $|\lambda_N| = 1$. Passing to a subsequence, we may assume $v_N \to v$, $w_N \to w$, with $||v|| = ||w|| = 1$, $w = \lambda v$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Denote by $\pi_S \colon \mathbb{R}^n \to E_S$ and $\pi_U \colon \mathbb{R}^n \to E_U$ the complementary projections corresponding to the decomposition $\mathbb{R}^n = E_S \oplus E_U$. Since $||H^N x|| \geq (1+\epsilon)^N ||x||$ for every $x \in E_U$, convergence of $w_N = H^N A v_N$ implies $\pi_U A v_N \to 0$, i.e., $Av \in E_S$.

On the other hand, $||v_N|| = 1$ for every N, hence $||Av_N||$ is bounded, and since $||H^N x|| \leq (1-\epsilon)^N ||x||$ for every $x \in E_S$, $\pi_S w_N = \pi_S H^N A v_N \to 0$, i.e., $w \in E_U$. But $w = \lambda v$. Thus $v \in E_U$ and $Av \neq 0 \in E_S$. □

LEMMA 5.2: *Suppose Γ is a subgroup of finite index in $\mathbf{SL}(n, \mathbb{Z})$. Then Γ is generated by a finite collection of hyperbolic matrices.*

Proof: As a subgroup of finite index in the finitely-generated group $\mathbf{SL}(n,\mathbb{Z})$, Γ is finitely-generated (see, for example, [Ku]), say $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$. Also, as we have observed above, Γ must contain some non-zero power of each hyperbolic matrix in $\mathbf{SL}(n,\mathbb{Z})$, so in particular, there exists a hyperbolic matrix $\gamma_0 \in \Gamma$.

Denote by E_S and E_U, respectively, the stable and unstable subspaces in $\mathbb{R}^n$ for the action of γ_0. Then for $A \in \mathbf{SL}(n,\mathbb{R})$, the matrix $A\gamma_0 A^{-1}$ is hyperbolic with stable and unstable subspaces AE_S and AE_U, respectively. Then by (5.1), the condition that $(A\gamma_0 A^{-1})^N \gamma_i$ is not hyperbolic for any N implies that $\gamma_i A E_U \cap AE_S \neq 0$. For fixed i, this is a non-trivial polynomial condition on $A \in \mathbf{SL}(n,\mathbb{R})$: One can easily construct $A \in \mathbf{SL}(n,\mathbb{R})$ for which $\gamma_i A E_U \cap AE_S = 0$, and the condition $\gamma_i A E_U \cap AE_S \neq 0$ is equivalent to $\det(\pi_U A^{-1} \gamma_i A \mid_{E_U}) = 0$, where $\pi_U \colon \mathbb{R}^n \to E_U$ is the projection with kernel E_S, and this is a polynomial condition on the entries of $A \in \mathbf{SL}(n,\mathbb{R})$. Thus $\{A \mid \gamma_i A E_U \cap AE_S = 0\}$ is a non-empty Zariski-open subset of $\mathbf{SL}(n,\mathbb{R})$.

But Γ is a lattice in $\mathbf{SL}(n,\mathbb{R})$, and is therefore Zariski-dense by the Borel density theorem (see [Zi3]). Thus there exists $\alpha_i \in \Gamma$ such that $\gamma_i \alpha_i E_U \cap \alpha_i E_S = 0$ and $N_i \in \mathbb{Z}$ such that $(\alpha_i \gamma_0 \alpha_i^{-1})^{N_i} \gamma_i$ is hyperbolic.

Then Γ is generated by the $2k$ hyperbolic matrices $\alpha_i \gamma_0 \alpha_i^{-1}$ and $(\alpha_i \gamma_0 \alpha_i^{-1})^{N_i} \gamma_i$, $1 \leq i \leq k$. □

Fix $n \geq 2$, and establish notation as follows: For each $m \in \mathbb{Z}^+$, let $\Gamma(m) \subset \mathbf{SL}(n,\mathbb{Z})$ denote the **principal congruence subgroup** mod m,

$$\Gamma(m) = \{\gamma \in \mathbf{SL}(n,\mathbb{Z}) \mid \gamma \equiv I \bmod m\}.$$

A subgroup in $\mathbf{SL}(n, \mathbf{Z})$ is called a **congruence subgroup** if it contains $\Gamma(m)$ for some m. Obviously any congruence subgroup has finite index in $\mathbf{SL}(n, \mathbf{Z})$. In fact, for $n \geq 3$, the converse is also true, by a celebrated theorem first established independently by Mennicke [Me] and Bass-Lazard-Serre [B-L-S] (see also [B-M-S]).

PROPOSITION 5.3: *For $n \geq 3$, every subgroup of finite index in $\mathbf{SL}(n, \mathbf{Z})$ is a congruence subgroup.*

By combining (5.2) and (5.3), we reduce the proof of (3.1) to the case in which $\Gamma = \Gamma(m)$ is a principal congruence subgroup. For by (5.3), $\Gamma \supset \Gamma(m)$ for some m, and by (5.2), Γ is generated by a finite collection of hyperbolic elements, say $\gamma_1, \ldots, \gamma_k$. Since $\Gamma(m)$ has finite index in Γ, for each i, $1 \leq i \leq k$, there exists $N_i \in \mathbf{Z}^+$ such that $\gamma_i^{N_i} \in \Gamma(m)$.

Now suppose that the theorem has been proved for $\Gamma(m)$. Then for suitable ρ, there exists a unique homeomorphism $h \in \mathrm{Homeo}_1(\mathbf{T}^n)$ such that $\rho(\gamma) = h\gamma h^{-1}$ for every $\gamma \in \Gamma(m)$; in particular, $\rho(\gamma_i^{N_i}) = h\gamma_i^{N_i}h^{-1}$, $1 \leq i \leq k$. But this implies $\rho(\gamma_i) = h\gamma_i h^{-1}$ by (2.1).

So we shall assume henceforth that $\Gamma = \Gamma(m)$.

In place of the single fixed point 0 for the action of $\mathbf{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n$, the subgroup $\Gamma = \Gamma(m)$ fixes m^n points, the m-division points $(m^{-1}\mathbf{Z})^n/\mathbf{Z}^n \subset \mathbf{R}^n/\mathbf{Z}^n = \mathbf{T}^n$. The linear representation of Γ at each of these points is again simply the standard representation, and we again have H^1 vanishing for $n \geq 3$ by [Ma2]. Thus we obtain

LEMMA 5.4: *For ρ sufficiently C^1-close to the standard action, $\rho(\Gamma)$ has an isolated fixed point near each m-division point in $\mathbf{T}^n$.*

We adopt the notation of (3.4), and extend it as follows. Set $\Lambda_i(m) = \Lambda_i \cap \Gamma(m)$, $i = 1$ or 2, and set $T_i(m)$ equal to the fixed-point set for the action of $\Lambda_i(m)$. Then $T_i(m)$ is a finite collection of tori parallel to T_i; T_i is the leaf of $T_i(m)$ through 0. Each m-division point is uniquely determined as the intersection of some leaf of $T_1(m)$ with some leaf of $T_2(m)$. Then the argument which establishes (3.4) yields

LEMMA 5.5: *For ρ sufficiently C^1-close to the standard action, there exists $h \in \mathrm{Homeo}_1(\mathbf{T}^n)$ such that*

(i) $\tilde{T}_i(m) = hT_i(m) = \{x \in \mathbf{T}^n \mid \rho(\gamma)x = x \text{ for every } \gamma \in \Lambda_i(m)\}$, $i = 1$ *or* 2,

(ii) *h maps each m-division point on $\mathbb{T}^n$ to the corresponding nearby fixed point for $\rho(\Gamma)$ (cf. 5.4), and*
(iii) $\rho(\gamma)\tilde{T}_i(m) = \tilde{T}_i(m)$ *for every* $\gamma \in \Lambda_1(m) \times \Lambda_2(m)$.

Moreover, the map $\rho \mapsto h$ is continuous.

We continue to extend the notation of Section 3 as follows. For each subset $\mathcal{I} = \{i_1, \ldots, i_k\}$, $2 \leq k \leq n-2$, of the indices $\{1, \ldots, n\}$, we set $\Lambda_{\mathcal{I}}(m) = \Lambda_{\mathcal{I}} \cap \Gamma(m)$, and denote by $T_{\mathcal{I}}(m)$ and $\tilde{T}_{\mathcal{I}}(m)$ the fixed-point sets for $\Lambda_{\mathcal{I}}(m)$ under the standard and perturbed actions, respectively, as in (3.5). Obviously $\mathcal{I}_1 \subset \mathcal{I}_2 \Rightarrow \tilde{T}_{\mathcal{I}_1}(m) \subset \tilde{T}_{\mathcal{I}_2}(m)$.

For each $i \in \{1, \ldots, n\}$, define $\Lambda_i(m) = \Lambda_i \cap \Gamma(m)$, with Λ_i as in (3.6). Denote by $S_i(m)$ and $\tilde{S}_i(m)$ the fixed-point sets for $\Lambda_i(m)$ under the standard and perturbed actions, respectively. Note that $S_i(m)$ is a finite union of circles parallel to S_i (S_i as in (3.7)); S_i is the leaf of $S_i(m)$ through 0.

Now suppose $\{i, j, k\} \subset \{1, \ldots, n\}$ is any 3-element subset. Since

$$\Lambda_{\{i,j\}}(m),\ \Lambda_{\{i,k\}}(m) \subset \Lambda_i(m),$$

$$S_i(m) \subset T_{\{i,j\}}(m) \cap T_{\{i,k\}}(m)$$

and

$$\tilde{S}_i(m) \subset \tilde{T}_{\{i,j\}}(m) \cap \tilde{T}_{\{i,k\}}(m).$$

Also, for $n \geq 4$, $\Lambda_i(m)$ is again (isomorphic to) a lattice in the higher-rank group $\mathbf{SL}(n-1, \mathbb{R})$, and we have

$$H^1(\Lambda_i(m), \mathrm{T}_x\mathbb{T}^n) = 0 \quad \text{for every } x \in S_i(m).$$

Thus the argument which establishes (3.7) yields

LEMMA 5.6: *$\tilde{S}_i(m)$ is a finite union of (disjoint) C^1 circles.*

LEMMA 5.7: *For each $N \geq 2$ set*

$$\lambda_N(m) = \begin{pmatrix} 1 & m & \\ Nm & Nm^2+1 & \\ & & I_{n-2} \end{pmatrix} \in \Gamma(m).$$

Then the subgroup $\Xi_N(m)$ generated by $\Lambda_1(m)$ together with

$$\lambda_N(m)\Lambda_1(m)\lambda_N^{-1}(m)$$

is a subgroup of finite index in $\Gamma(m)$.

Proof: In fact, $\Xi_N(m)$ contains the principal congruence subgroup $\Gamma(N^2m^6)$. Again, to simplify notation, we carry out the argument for the case $n = 3$; the generalization to $n \geq 3$ is obvious. The first step is to exhibit the four elementary matrices

$$\begin{pmatrix}1 & & m^2\\ & 1 & \\ & & 1\end{pmatrix}, \begin{pmatrix}1 & m^3 & \\ & 1 & \\ & & 1\end{pmatrix}, \begin{pmatrix}1 & & \\ & 1 & \\ Nm^2 & & 1\end{pmatrix}, \text{ and } \begin{pmatrix}1 & & \\ Nm^3 & 1 & \\ & & 1\end{pmatrix}$$

in $\Xi_N(m)$.

$$\begin{pmatrix}1 & m & \\ Nm & Nm^2+1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & m\\ & & 1\end{pmatrix}\begin{pmatrix}Nm^2+1 & -m & \\ -Nm & 1 & \\ & & 1\end{pmatrix} = \begin{pmatrix}1 & & m^2\\ & 1 & Nm^3+m\\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & m^2\\ & 1 & Nm^3+m\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & -(Nm^3+m)\\ & & 1\end{pmatrix} = \begin{pmatrix}1 & & m^2\\ & 1 & \\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & m^2\\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & m & 1\end{pmatrix} = \begin{pmatrix}1 & m^3 & m^2\\ & 1 & \\ & m & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & m^3 & m^2\\ & 1 & \\ & m & 1\end{pmatrix}\begin{pmatrix}1 & & -m^2\\ & 1 & \\ & & 1\end{pmatrix} = \begin{pmatrix}1 & m^3 & \\ & 1 & \\ & m & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & \\ & -m & 1\end{pmatrix}\begin{pmatrix}1 & m^3 & \\ & 1 & \\ & m & 1\end{pmatrix} = \begin{pmatrix}1 & m^3 & \\ & 1 & \\ & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & m & \\ Nm & Nm^2+1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & m & 1\end{pmatrix}\begin{pmatrix}Nm^2+1 & -m & \\ -Nm & 1 & \\ & & 1\end{pmatrix} = \begin{pmatrix}1 & & \\ & 1 & \\ -Nm^2 & m & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & \\ -Nm^2 & m & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -m & 1\end{pmatrix} = \begin{pmatrix}1 & & \\ & 1 & \\ -Nm^2 & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & & \\ & 1 & m\\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ -Nm^2 & & 1\end{pmatrix} = \begin{pmatrix}1 & & \\ -Nm^3 & 1 & m\\ -Nm^2 & & 1\end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ Nm^2 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -Nm^3 & 1 & m \\ -Nm^2 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -Nm^3 & 1 & m \\ & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & -m \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -Nm^3 & 1 & m \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -Nm^3 & 1 & \\ & & 1 \end{pmatrix}$$

The lemma now follows from the following proposition, first observed by Vaserstein [V], which is a special case of the main result in [T].

PROPOSITION 5.8: *For $n \geq 3$, the subgroup generated by the elementary matrices in the principal congruence subgroup $\Gamma(m) \subset \mathbf{SL}(n, \mathbf{Z})$ contains $\Gamma(m^2)$.*

□

For each $N \geq 2$, set $S_1^N(m) = \lambda_N(m)S_1(m)$, which is the fixed-point set for $\lambda_N(m)\Lambda_1(m)\lambda_N^{-1}(m)$. Then $S_1(m)$ and $S_1^N(m)$ intersect in a finite set, and $\cup_N(S_1(m) \cap S_1^N(m))$ is dense in $S_1(m)$.

Now fix a finite collection $\gamma_1, \ldots, \gamma_k$ of hyperbolic generators for $\Gamma(m)$ and $h_1, \ldots, h_k \in \mathrm{Homeo}_1(\mathbf{T}^n)$ such that $\rho(\gamma_i) = h_i\gamma_i h_i^{-1}$. Then the argument which establishes (3.10) shows that the h_i's agree on $S_1(m) \cap S_1^N(m)$ for each N, and hence on $S_1(m)$. Finally, since $\cup_{\gamma \in \Gamma(m)} \gamma S_1(m)$ is dense in $\mathbf{T}^n$, the theorem follows exactly as in Section 3.

6. Additional Results and Future Directions

The purpose of this section is to indicate, without detailed proofs, some results extending those of the preceding sections, and to discuss very briefly the direction of future research.

Recall that if $\mathbf{G} \subset \mathbf{GL}(n, \mathbf{C})$ is a semi-simple algebraic $\mathbf{Q}$-group, then by a theorem of Borel–Harish-Chandra [B-H-C] the group $\mathbf{G}_\mathbf{Z} = \mathbf{G} \cap \mathbf{GL}(n, \mathbf{Z})$ is a lattice in $G = \mathbf{G}_\mathbf{R} = \mathbf{G} \cap \mathbf{GL}(n, \mathbf{R})$. Then as a subgroup of $\mathbf{GL}(n, \mathbf{Z})$, $\Gamma = \mathbf{G}_\mathbf{Z}$ acts naturally on $\mathbf{T}^n$. We expect that the argument we have given above for $\mathbf{SL}(n, \mathbf{Z})$ can be generalized to establish local rigidity for many of these actions. For example, a fairly straightforward modification of the proof of (1.3) yields the following

THEOREM 6.1: *Let $\Gamma = \mathbf{Sp}(n, \mathbf{Z})$ $(= \{\gamma \in \mathbf{SL}(2n, \mathbf{Z}) \mid \gamma^t J \gamma = J\}$, where*

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and γ^t indicates transpose) or any subgroup of finite index, $n \geq 3$. Then the natural action of Γ on $\mathbf{T}^{2n}$ is locally rigid.

$\mathbf{Sp}(n-1,\mathbf{Z})$ imbeds in $\mathbf{Sp}(n,\mathbf{Z})$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(n-1,\mathbf{Z}) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \in \mathbf{Sp}(n,\mathbf{Z}),$$

and this subgroup has as its set of fixed points a rational 2-torus. The argument which establishes (3.4), above, shows that this structure persists for small perturbations of the action. (In this case no appeal to (2.5) is necessary, and this much of the proof works for $\mathbf{Sp}(2,\mathbf{Z})$ as well.) Conjugating by the matrix

$$\begin{pmatrix} 1+N & -N & & & & \\ N & 1-N & & & & \\ & & I_{n-2} & & & \\ & & & 1-N & -N & \\ & & & N & 1+N & \\ & & & & & I_{n-2} \end{pmatrix},$$

we obtain a second imbedding of $\mathbf{Sp}(n-1,\mathbf{Z})$ in $\mathbf{Sp}(n,\mathbf{Z})$, and the corresponding 2-torus of fixed points intersects the first in N^2 rational points. The hypothesis $n \geq 3$ is necessary to ensure that the two subgroups together generate a subgroup of finite index. The proof of topological conjugacy is now entirely analogous to that for $\mathbf{SL}(n,\mathbf{Z})$, above.

In order to establish smoothness for the conjugacy, we begin by observing that there exists a Cartan subgroup H in $\mathbf{Sp}(n,\mathbb{R})$ such that $H/(H\cap\Gamma)$ is compact ([P-R]). In particular, there exists a subgroup $\mathcal{A}\subset\Gamma$, free abelian of rank n and simultaneously diagonalizable over $\mathbb{R}$. The matrices in $\mathcal{A}$ are again hyperbolic, but in this case the eigenvalues are paired; each eigenvalue appears together with its inverse. Thus the stable and unstable foliations corresponding to each element of $\mathcal{A}$ are n-dimensional, and we cannot hope to recover the 1-dimensional foliations as stable foliations for any element. However, each of the invariant 1-dimensional foliations can be identified as the "fast-contracting" foliation for some element (i.e., corresponding to the unique eigenvalue of minimum absolute value) and is therefore a Hölder foliation with C^∞ leaves. It again follows easily from co-compactness of $\mathcal{A}$ in H that $\mathcal{A}$ acts densely along each leaf, and the remainder of the discussion in Section 4 carries over verbatim.

It is interesting to observe that although the restriction enters at a different point in the argument, we again require real rank 3 in order to obtain topological conjugacy, whereas the appropriate hypothesis for finite-dimensional cohomology vanishing, super-rigidity, and Hurder's proof of deformation rigidity is real rank 2. It appears that this may reflect a fundamental limitation on the scope of our technique.

THEOREM 6.2: *For each $i \in \{1, \ldots, k\}$, let Γ_i be a subgroup of finite index in* $\mathbf{SL}(n_i, \mathbf{Z})$*,* $n_i \geq 4$*, or* $\mathbf{Sp}(n_i, \mathbf{Z})$*,* $n_i \geq 3$*, and set* $m_i = n_i$ *or* $2n_i$*, respectively. Then the natural product action of* $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$ *on* $\mathbf{T}^n = \mathbf{T}^{m_1} \times \cdots \times \mathbf{T}^{m_k}$*,* $n = \sum m_i$*, is locally rigid.*

Again by the argument which establishes (3.4), the product structure for the action persists under small perturbations. More precisely, there is an isotopy h of $\mathbf{T}^n$ near the identity such that $h(\mathbf{T}^{m_i})$ is invariant under the perturbed action of Γ for each i, and is fixed pointwise by the perturbed action of Γ_j for all $j \neq i$. Moreover, it is easy to see that the (perturbed) action of Γ_i on $h(\mathbf{T}^{m_i})$ is C^∞-close to the original action of Γ_i on $\mathbf{T}^{m_i}$. Direct application of (3.1) (and the corresponding result for $\mathbf{Sp}(n, \mathbf{Z})$) now yields topological conjugacy. Finally, to establish smoothness for the conjugacy, we fix an appropriate diagonalizable abelian subgroup in Γ and proceed in the obvious way.

Among other natural examples are the remaining noncompact classical groups, certain rational representations of the exceptional groups, and the $\mathbf{Q}$-rational realizations of semi-simple algebraic groups defined over number fields obtained by restriction of scalars. (Recall that if $\mathbf{K}$ is a number field of degree d over $\mathbf{Q}$ with integers $\mathcal{O}$ and $\mathbf{G} \subset \mathbf{GL}(n, \mathbf{C})$ is a semi-simple algebraic $\mathbf{K}$-group, then the restriction of scalars functor (due to A. Weil) provides a realization of $\mathbf{G}$ as a $\mathbf{Q}$-group, usually denoted $R_{\mathbf{K}/\mathbf{Q}}(\mathbf{G})$, in $\mathbf{GL}(dn, \mathbf{C})$ such that $\mathbf{G}_\mathbf{K} = \mathbf{G} \cap \mathbf{GL}(n, \mathbf{K})$ corresponds to $(R_{\mathbf{K}/\mathbf{Q}}(\mathbf{G}))_\mathbf{Q}$ and $\mathbf{G}_\mathcal{O} = \mathbf{G} \cap \mathbf{GL}(n, \mathcal{O})$ to $(R_{\mathbf{K}/\mathbf{Q}}(\mathbf{G}))_\mathbf{Z}$. For more details see chapter 6 of [Zi3].) We are currently investigating the extent to which our technique can be used to establish local rigidity for these examples.

Ultimately, for any given lattice Γ and compact manifold M, we would like to obtain a complete description of the "representation variety"

$$\tilde{R}(\Gamma, \mathrm{Diff}(M)) = R(\Gamma, \mathrm{Diff}(M))/\mathrm{Diff}(M),$$

the space of actions of Γ on M up to equivalence. In case Γ is a subgroup of finite index in $\mathbf{SL}(n, \mathbf{Z})$, $n \geq 3$, or $\mathbf{Sp}(n, \mathbf{Z})$, $n \geq 2$, and M is $\mathbf{T}^n$ or $\mathbf{T}^{2n}$, respectively, then (modulo finite quotients) we know of only three inequivalent actions of Γ on M: the standard action, the trivial action, and the contragredient action (given by composing the standard action with the automorphism of Γ obtained by taking inverse conjugate transpose). Obviously the argument we have given for the standard action applies equally to the contragredient action, and version (2.5) of Stowe's theorem implies directly that the trivial action of any Kazhdan group on any compact manifold is locally rigid. Moreover, it follows from super-rigidity and classical finite-dimensional representation theory that for any action of Γ on M, the induced action on $\pi_1(M)$ must coincide (up to conjugacy and finite quotients) with one of these three.

Thus we are led to consider the following problem: Suppose $\rho\colon \Gamma \to \mathrm{Diff}(M)$ is any smooth action whatsoever. Must ρ be conjugate in Diff(M) to the linear action corresponding to the induced action on π_1? In this connection it is probably worth remarking that for $n \geq 5$, a variant of the preceding argument can be used to obtain a smooth conjugacy without assuming that ρ is a small perturbation, provided that we impose the following two hypotheses directly:

(i) There exists a finite orbit for Γ under ρ.

(ii) For each of the (finitely many) hyperbolic generators γ for Γ, for the various "split subgroups" (cf. 3.4), and for the "Cartan subgroup" $\mathcal{A}$ (cf. 4.3), $\rho(\gamma)$ is Anosov.

Details will be provided in [K-L].*

Finally, we shall comment briefly on the relationship of our results to the general program. As discussed by Zimmer in [Zi1], it is a striking fact that the only known examples of smooth, volume-preserving actions of lattices in semi-simple Lie groups of higher rank on compact manifolds are essentially of three types:

(i) Isometric actions (i.e., $\rho(\Gamma)$ has compact closure in Diff(M)).

(ii) Γ acts on $M = H/\Lambda$ via ρ, where $\Gamma \subset G$ and $\Lambda \subset H$ are lattices, with Λ co-compact, and $\rho\colon G \to H$ is a homomorphism.

(iii) Γ acts on $M = N/\Lambda$, where Λ is a (necessarily co-compact) lattice in a nilpotent Lie group N, and Γ is a lattice in G, where G is a semi-simple group of automorphisms of N, such that Γ preserves Λ.

Of course, the examples we have been considering are of type (iii), with $N = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$. We expect our argument to yield at least partial results for some more general actions of type (iii), where hyperbolicity is still available, although it appears that there may be substantial new obstacles to obtaining smoothness for the conjugacy. A variant of our technique may also be useful for examples of type (ii), although in this case no element of Γ acts hyperbolically, and we will

*In fact, we have recently succeeded in extending our technique to obtain global results under more general conditions. These will appear in [K-L]. An alternative argument, making use of Zimmer's super-rigidity theorem for cocycles [Zi3], will appear in [H-K-L-Z]. This approach provides a more general proof of local rigidity, which in particular covers the case of finite-index subgroups in $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, and $\mathrm{Sp}(n,\mathbb{Z})$, $n \geq 2$, and yields global results without postulating the existence of a periodic orbit, but assuming instead that the action preserves a finite volume.

need to introduce some entirely new ingredients in order to exploit the normal hyperbolicity which is present instead.

Appendix. Non-stationary Sternberg Linearization

In this appendix, we establish the following extension of the standard linearization lemma for contracting diffeomorphisms. Although the statement will admit considerable generalization, we restrict ourselves to the special case which we need in Section 4. The argument is patterned on that given by Sternberg in [Ste].

PROPOSITION A.1: *Let M be a compact manifold, $\mathcal{L} = M \times \mathbb{R}$ the trivial real line bundle over M, and let*

$$F\colon \mathcal{L} \to \mathcal{L}, \quad (x,t) \mapsto \big(f(x), F_x(t)\big),$$

with F_x a C^∞ diffeomorphism of $\mathbb{R}$ for each $x \in M$, satisfy
(i) $F_x(0) = 0$ for every $x \in M$ (F preserves the zero section),
(ii) $0 < F'_x(t) < 1$ for every $x \in M$, $t \in \mathbb{R}$, and
(iii) $x \mapsto F_x$ is a continuous map $M \to C^\infty(\mathbb{R})$. Then there exists a unique reparameterization

$$G\colon \mathcal{L} \to \mathcal{L}, \quad (x,t) \mapsto \big(x, G_x(t)\big)$$

such that
(iv) each G_x is a C^∞ diffeomorphism of $\mathbb{R}$,
(v) $G_x(0) = 0$, $G'_x(0) = 1$ for every $x \in M$,
(vi) $x \mapsto G_x$ is a continuous map $M \to C^\infty(\mathbb{R})$, and
(vii) $GFG^{-1}(x,t) = \big(f(x), F'_x(0)t\big)$ for every $x \in M$, $t \in \mathbb{R}$.

We begin by establishing uniqueness. So suppose G_1 and G_2 both satisfy (iv)–(vii). Then $G_1FG_1^{-1}$ is linear, $G_2G_1^{-1}$ satisfies (iv)–(vi), and $(G_2G_1^{-1})G_1FG_1^{-1} = G_1FG_1^{-1}(G_2G_1^{-1})$. So it will suffice to prove the following:

LEMMA A.2: *Suppose $F\colon \mathcal{L} \to \mathcal{L}$ is linear, i.e., $F(x,t) = \big(f(x), \alpha_x \cdot t\big)$, where $x \mapsto \alpha_x$ is a continuous map $M \to (0,1)$, and suppose G satisfies (iv)–(vii) for this F, so that $GF = FG$. The G is the identity map on $\mathcal{L}$.*

Proof: The condition $GF = FG$ is equivalent to

$$G_{f(x)}(\alpha_x \cdot t) = \alpha_x \cdot G_x(t),$$

hence

$$G_x(t) = \alpha_x^{-1} \cdot G_{f(x)}(\alpha_x \cdot t) = \cdots$$
$$= \alpha_x^{-n} \cdot G_{f^n(x)}(\alpha_x^n \cdot t) \quad \text{for every} \quad n \geq 1.$$

Since $x \mapsto \alpha_x$ is continuous and M is compact, there exists $\epsilon > 0$ with $\alpha_x \leq 1 - \epsilon$ for every $x \in M$. Also, since G_x varies continuously with x in the C^∞ (C^2 is enough) topology, the difference quotients $G_x(\delta)/\delta$ converge uniformly in x to $G_x'(0) = 1$. Consequently

$$G_x(t) = \lim_{n\to\infty} \frac{G_{f^n(x)}(\alpha_x^n \cdot t)}{\alpha_x^n} = t.$$

□

We now establish existence. We begin by solving the corresponding problem for formal power series.

Lemma A.3: *Suppose*

$$F\colon (x,t) \mapsto \big(f(x), F_x(t)\big),\ F_x(t) = \sum_{i=1}^{\infty} a_i(x)t^i$$

is a formal power series based at the zero section in $\mathcal{L}$, with $a_i\colon M \to \mathbb{R}$ continuous for each i and $0 < a_1(x) < 1$ for every $x \in M$. Then there exists a formal power series

$$G\colon (x,t) \mapsto \big(x, G_x(t)\big),\ G_x(t) = t + \sum_{i=2}^{\infty} b_i(x)t^i,$$

with $b_i\colon M \to \mathbb{R}$ continuous for each i, such that

$$GFG^{-1}(x,t) = \big(f(x), a_1(x)t\big),$$

i.e.,

$$(*) \qquad G_{f(x)}\big(F_x(t)\big) = a_1(x) \cdot G_x(t) \quad \textit{for every } x \in M,\ t \in \mathbb{R}.$$

Proof: We solve for the functions b_i inductively. To begin, we solve for b_2 by equating quadratic terms in $(*)$:

$$b_2\big(f(x)\big)a_1(x)^2 + a_2(x) = a_1(x)b_2(x),$$
$$b_2(x) = \frac{a_2(x)}{a_1(x)} + a_1(x)b_2\big(f(x)\big).$$

Since M is compact, there exists $\epsilon > 0$ such that $\epsilon < a_1(x) < 1 - \epsilon$ for every $x \in M$, and

$$b_2(x) = \frac{a_2(x)}{a_1(x)} + \sum_{i=1}^{\infty} \left\{ a_1(x) a_1\big(f(x)\big) \cdots a_1\big(f^{i-1}(x)\big) \frac{a_2\big(f^i(x)\big)}{a_1\big(f^i(x)\big)} \right\}$$

converges uniformly to give a continuous solution for b_2.

Now suppose that we have obtained continuous solutions for b_2 through b_{n-1} such that the first $n-1$ coefficients in $(*)$ agree. We solve for b_n by equating the nth term in $(*)$:

$$b_n\big(f(x)\big) a_1(x)^n + r(x) = a_1(x) b_n(x),$$
$$b_n(x) = \frac{r(x)}{a_1(x)} + a_1(x)^{n-1} b_n\big(f(x)\big),$$

where $r\colon M \to \mathbb{R}$ is continuous; $r(x)$ is a polynomial in $b_i\big(f(x)\big)$, $2 \le i \le n-1$, and $a_i(x)$, $1 \le i \le n$. Since r is continuous and M is compact, r is uniformly bounded, hence

$$b_n(x) = \frac{r(x)}{a_1(x)} + \sum_{i=1}^{\infty} \left\{ [a_1(x) a_1\big(f(x)\big) \ldots a_1\big(f^{i-1}(x)\big)]^{n-1} \frac{r\big(f^i(x)\big)}{a_1\big(f^i(x)\big)} \right\}$$

converges uniformly to a continuous solution for b_n. □

COROLLARY A.4: *Given F satisfying (i)–(iii) above, there exists*

$$H\colon \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto (x, H_x(t))$$

such that
(viii) *each H_x is a C^∞ diffeomorphism of $\mathbb{R}$,*
(ix) *$H_x(0) = 0$, $H_x'(0) = 1$ for every $x \in M$,*
(x) *$x \mapsto H_x$ is continuous $M \to C^\infty(\mathbb{R})$, and*
(xi) *$H_{f(x)} F_x H_x\colon \mathbb{R} \to R$ has a tangency of infinite order with the identity map on $\mathbb{R}$ at 0 for every $x \in M$.*

Proof: Fix $\alpha \in C^\infty(\mathbb{R}, [0,1])$ such that $\alpha(x) \equiv 1$ for $|x| < 1/4$, $\alpha(x) \equiv 0$ for $|x| > 3/4$. Let $\Sigma_{i=1}^{\infty} a_i(x) t^i$ be the Taylor series expansion for F_x at 0, so that $a_i\colon M \to \mathbb{R}$ is continuous for each i and $0 < a_1(x) < 1$ for every $x \in M$. Then define $H_x \in C^\infty(\mathbb{R})$ via

$$H_x(t) = t + \sum_{i=2}^{\infty} b_i(x) t^i \alpha(i! b_i(x) t),$$

with b_i as in (A.3). □

The next step is to obtain a C^k solution for G in a suitable neighborhood of the zero section. Fix a compact neighborhood K of 0 in $\mathbb{R}$, and set

$$s = \min\{F_x'(t) | x \in M,\ t \in K\},$$

$$S = \max\{F_x'(t) | x \in M,\ t \in K\},$$

so that $0 < s \leq S < 1$.

LEMMA A.5: *Suppose $k > \log s / \log S$. Then there exists a neighborhood $U \subset K$ of 0 in $\mathbb{R}$ and a unique $G\colon M \times U \to \mathcal{L}$, $(x,t) \mapsto (x, G_x(t))$ such that*
(iv') *each G_x is a C^k diffeomorphism of U onto its image,*
(v) *$G_x(0) = 0$, $G_x'(0) = 1$ for every $x \in M$,*
(vi') *$x \mapsto G_x$ is continuous $M \to C^k(U)$, and*
(vii) *$GFG^{-1}(x,t) = \big(f(x), F_x'(0) \cdot t\big)$ for every $x \in M$, $t \in G_x(U)$.*

Proof: Given a neighborhood V of 0 in $\mathbb{R}$, let A_V^k denote the space of functions $R\colon M \times V \to \mathcal{L}$, $(x,t) \mapsto \big(x, R_x(t)\big)$ such that $x \mapsto R_x$ is continuous $M \to C^k(V)$ and R_x vanishes to order k at 0 for every $x \in M$

For $R \in A_V^k$ and $0 \leq i \leq k$, set

$$||R||_V^i = \sup\big\{|D^i R_x(t)| : x \in M,\ t \in V\big\}.$$

A standard application of the mean value theorem yields the following

LEMMA A.6: *Given V and $\epsilon > 0$, there exists a neighborhood $V^\epsilon \subset V$ of 0 such that*

$$||R||_{V^\epsilon}^0 + ||R||_{V^\epsilon}^1 + \cdots + ||R||_{V^\epsilon}^{k-1} < \epsilon ||R||_{V^\epsilon}^k$$

for every $R \in A_V^k$.

Now define an operator $\Phi\colon A_V^k \to A_V^k$ via

$$(\Phi R)(x,t) = \big(x, F_{f(x)}'(0)^{-1} \cdot R_{f(x)}(F_x(t))\big).$$

LEMMA A.7: *Given $V \subset K$, there exists a neighborhood $W \subset V$ of 0 such that*

$$||\Phi R||_W^k \leq \lambda ||R||_W^k \quad \text{for every} \quad R \in A_V^k$$

for some $\lambda < 1$.

Proof:

$$D^k(R_{f(x)} \circ F_x)(t) = (D^k R_{f(x)})\big(F_x(t)\big) \cdot \big(F_x'(t)\big)^k + P,$$

where $P = P(x,t)$ is a polynomial in derivatives of $R_{f(x)}$ of order $< k$ and in derivatives of F_x of order $\leq k$. Thus by (A.5), given $\epsilon > 0$ there exists $W \subset V$ (depending on F but not on R) such that

$$|P(x,t)| < \epsilon \cdot ||R||_W^k \quad forevery \quad x \in M,\ t \in W.$$

Thus

$$||\Phi R||_W^k \leq s^{-1}(S^k + \epsilon)||R||_W^k \quad \text{for every} \quad R \in A_V^k.$$

In view of the hypothesis on k, we can choose ϵ small enough so that

$$\lambda = s^{-1}(S^k + \epsilon) < 1.$$

□

To complete the proof of (A.5), observe that for H as in (A.4), $H - \Phi H$ (restricted to V) is in A_V^k. Then by (A.6) and (A.7), the sequence of mappings

$$G_n = \Phi^n H = \sum_{i=0}^{n-1} \Phi^i(\Phi H - H) + H$$

converges uniformly on some neighborhood $M \times U$, $U \subset K$ of the zero section, and the limit $G\colon M \times U \to \mathcal{L}$ clearly satisfies (iv′)–(vii). Uniqueness follows from (A.2).

To complete the proof of (A.1), we first observe that the C^k reparameterization G in (A.5) can be extended from the neighborhood $M \times U$ to all of $\mathcal{L}$. In fact, if K is any compact neighborhood of the zero section in $\mathcal{L}$, then $F^r(K) \subset M \times U$ for large enough r, and we can extend G to $\overline{G}$ on K by setting $\overline{G} = \Phi^r G$.

Finally, since the C^k solution is unique, it coincides with the solution in C^{k+1} for each $k > \log s/\log S$, hence G_x is C^∞ for each $x \in M$. This completes the proof of (A.1).

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