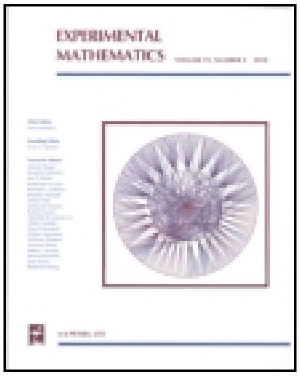

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# Local Rigidity for -Representations of 3-Manifold Groups

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# Local Rigidity for PGL(3, $\mathbb{C}$)-Representations of 3-Manifold Groups

Nicolas Bergeron, Elisha Falbel, Antonin Guilloux, Pierre-Vincent Koseleff, and Fabrice Rouillier



**CONTENTS**



Let $M$ be a noncompact hyperbolic 3-manifold that has a triangulation by positively oriented ideal tetrahedra. We explain how to produce local coordinates for the variety defined by the gluing equations for PGL(3, $\mathbb{C}$)-representations. In particular, we prove local rigidity of the "geometric" representation in PGL(3, $\mathbb{C}$), recovering a recent result of Menal-Ferrer and Porti. More generally, we give a criterion for local rigidity of PGL(3, $\mathbb{C}$)-representations and provide detailed analysis of the figure-eight-knot sister manifold exhibiting the different possibilities that can occur.

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## 1. INTRODUCTION

Let $M$ be a compact orientable 3-manifold with boundary a union of $\ell$ tori. Assume that the interior of $M$ carries a hyperbolic metric of finite volume and let $\rho : \pi_1(M) \to \mathrm{PGL}(3, \mathbb{C})$ be the corresponding holonomy composed with the 3-dimensional irreducible representation of $\mathrm{PGL}(2, \mathbb{C})$ (this representation is usually called the geometric or adjoint representation).

Building on [Bergeron et al. 12], we give a combinatorial proof of the following theorem, first proved in [Menal-Ferrer and Porti 11].

**Theorem 1.1.** *The class $[\rho]$ of $\rho$ in the algebraic quotient of $\mathrm{Hom}(\pi_1(M), \mathrm{PGL}(3, \mathbb{C}))$ by the action of $\mathrm{PGL}(3, \mathbb{C})$ by conjugation is a smooth point with local dimension $2\ell$.*

Our main theorem, Theorem 6.2, is in fact more general. We do not consider solely the geometric representation, and in fact, our proof applies to an explicit open subset (called $\mathcal{R}(M, \mathcal{T}^+)$; see the beginning of Section 6) of the (decorated) representation variety into $\mathrm{PGL}(3, \mathbb{C})$. It also provides explicit coordinates and a description of the possible deformations. We analyze in the last section the figure-eight-knot sister manifold: we describe all the (decorated) representations whose restrictions to the boundary torus are unipotent. It turns out that there exist rigid points (i.e., isolated points in the (decorated) unipotent representation variety) together with nonrigid components.

There is a natural holonomy map (see Section 4) from the (decorated) representation variety of $M$ to the representation variety of its boundary. It is known that its image is a Lagrangian subvariety and that the map is a local isomorphism on a Zariski open set. Remark 6.4 proves these facts in a combinatorial way. When $M$ is a knot complement and one considers the group $\mathrm{PGL}(2, \mathbb{C})$ instead of $\mathrm{PGL}(3, \mathbb{C})$, this image is the algebraic variety defined by the $A$-polynomial of the knot. In this paper, we explore more precisely the map hol and exhibit a fiber that is not discrete.

## 2. IDEAL TRIANGULATION

An *ordered simplex* is a simplex with a fixed vertex ordering. Recall that an orientation of a set of vertices is a numbering of the elements of this set up to even permutation. The face of an ordered simplex inherits an orientation. We define an *abstract triangulation* to be a pair $\mathcal{T} = ((T_\mu)_{\mu=1,\dots,\nu}, \Phi)$, where $(T_\mu)_{\mu=1,\dots,\nu}$ is a finite family of abstract ordered simplicial tetrahedra and $\Phi$ is a matching of the faces of the $T_\mu$'s reversing the orientation. For a simplicial tetrahedron $T$, we define $\mathrm{Trunc}(T)$ to be the tetrahedron truncated at each vertex. The space obtained from $\mathrm{Trunc}(T_\mu)$ after matching the faces will be denoted by $K$.

We call an abstract triangulation $\mathcal{T}$ together with an oriented homeomorphism

$$K = \bigsqcup_{\mu=1}^{\nu} \mathrm{Trunc}(T_\mu)/\Phi \to M,$$

where $M$ is a compact 3-manifold with boundary, a *triangulation*—or rather an *ideal triangulation*.

In the following, we will always assume that the boundary of $M$ is a disjoint union of a finite collection of 2-dimensional tori. Recall that by a simple Euler characteristic count, the number of edges of $K$ is equal to the number $\nu$ of tetrahedra, the most important family of examples being the compact 3-manifolds whose interior carries a complete hyperbolic structure of finite volume. The existence of an ideal triangulation for $M$ still appears to be an open question.[1] Nevertheless, it was proved in

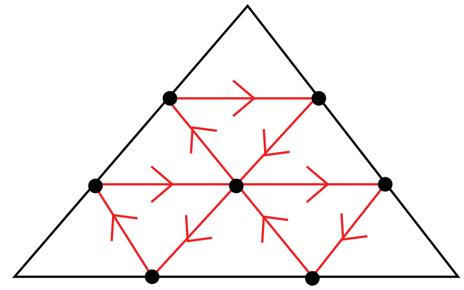

**FIGURE 1.** Combinatorics of $W$ (color figure available online).

[Luo et al. 08] that by passing to a finite regular cover, we may assume that $M$ admits an ideal triangulation. In the following paragraphs, we assume that $M$ itself admits an ideal triangulation $\mathcal{T}$ and postpone to the proof of Theorem 1.1 the task of reducing to this case (see Lemma 6.6).

### 2.1. Parabolic Decorations

We recall from [Bergeron et al. 12] the notion of a *parabolic decoration* of the pair $(M, \mathcal{T})$: to each tetrahedron $T_\mu$ of $\mathcal{T}$ we associate nonzero complex coordinates $z_\alpha(T_\mu)$ $(\alpha \in I)$, where $I$ is equal to the set of vertices of the arrows in the triangulation given by Figure 1.

Let $J^2_{T_\mu} = \mathbb{Z}^I$ denote the 16-dimensional abstract free $\mathbb{Z}$-module and denote the canonical basis of $J^2_{T_\mu}$ by $\{e_\alpha\}_{\alpha\in I}$. It contains *oriented edges* $e_{ij}$ (edges are oriented from $j$ to $i$) and *faces* $e_{ijk}$. Using this notation, the 16-tuple of complex parameters $(z_\alpha(T_\mu))_{\alpha\in I}$ is better viewed as an element

$$z(T_\mu) \in \mathrm{Hom}(J^2_{T_\mu}, \mathbb{C}^\times) \cong \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^*.$$

We refer to [Bergeron et al. 12] for details. Such an element uniquely determines a tetrahedron of flags if and only if the following relations are satisfied:

$$z_{ijk} = -z_{il} z_{jl} z_{kl} \tag{2–1}$$

and

$$z_{ik} = \frac{1}{1 - z_{ij}}. \tag{2–2}$$

Note that the second relation implies the following one:

$$z_{ij} z_{ik} z_{il} = -1, \tag{2–3}$$

Let $J^2$ denote the direct sum of the $J^2_{T_\mu}$'s, and consider an element $z \in \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2)^*$ to be a set of parameters of the triangulation $\mathcal{T}$. As usual, these coordinates are

[1] Note, however, that starting from the Epstein–Penner decomposition of $M$ into ideal polyhedra, [Petronio and Porti 00] produces a degenerate triangulation of $M$.

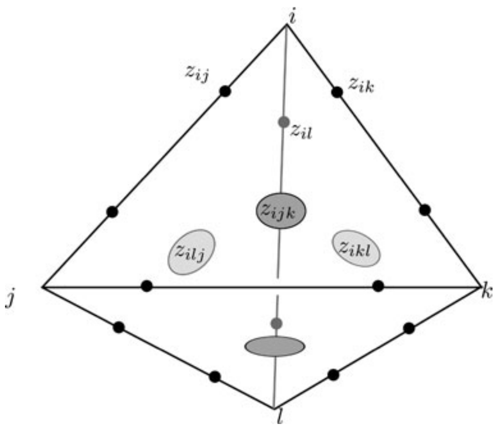


**FIGURE 2.** The $z$-coordinates for a tetrahedron.

subject to consistency relations after gluing by $\Phi$: given two adjacent tetrahedra $T_\mu, T_{\mu'}$ of $T$ with a common face $(ijk)$, then

$$z_{ijk}(T_\mu)z_{ikj}(T_{\mu'}) = 1. \tag{2–4}$$

And given a sequence $T_1, \dots, T_\mu$ of tetrahedra sharing a common edge $ij$ and such that $ij$ is an inner edge of the subcomplex comprising $T_1 \cup \dots \cup T_\mu$, then

$$z_{ij}(T_1)\cdots z_{ij}(T_\mu) = z_{ji}(T_1)\cdots z_{ji}(T_\mu) = 1. \tag{2–5}$$

**Remark 2.1.** Consider a fundamental domain of the triangulation of the universal cover $\tilde{M}$ lifted from that of $M$. A decoration of the complex is then equivalent to an assignment of a flag to each of its vertices together with an additional transversality condition on the flags to ensure that the $z_\alpha$'s do not vanish.

## 3. THE REPRESENTATION VARIETY

Given $M$ and a triangulation $\mathcal{T}$, we consider the space of parabolic decorations and denote it by $\mathcal{R}(M, \mathcal{T})$ (we call it the *representation variety* associated to the parabolic decorations of the triangulation). It is observed in the next subsection that it can be identified with an open subset of $\mathrm{Hom}(\pi_1(M), \mathrm{PGL}(3, \mathbb{C}))/\mathrm{PGL}(3, \mathbb{C})$.

More explicitly, we define $\mathcal{R}(M, \mathcal{T})$ as

$$\mathcal{R}(M, \mathcal{T}) = g^{-1}(1, \dots, 1),$$

where

$$g = (h, a, f) : \mathbb{C}^\times \otimes (J^2)^* \to (\mathbb{C}^\times)^{8\nu} \times (\mathbb{C}^\times)^{4\nu} \times (\mathbb{C}^\times)^{4\nu} \cong (\mathbb{C}^\times)^{16\nu}$$

is the product of the three maps $h$, $a$, $f$, defined below.

First of all, $h = (h_1, \dots, h_\nu)$ is the product of the maps $h_\mu : \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^* \to (\mathbb{C}^\times)^8$ $(\mu = 1, \dots, \nu)$ associated to the $T_\mu$'s and defined by

$$h_\mu(z) = \Big( -\frac{z_{ijk}}{z_{il}z_{jl}z_{kl}}, -\frac{z_{ikl}}{z_{ij}z_{kj}z_{lj}}, -\frac{z_{ilj}}{z_{ik}z_{lk}z_{jk}}, -\frac{z_{kjl}}{z_{ki}z_{ji}z_{lj}}, - z_{ij}z_{ik}z_{il}, -z_{ji}z_{jk}z_{jl}, -z_{ki}z_{kj}z_{kl}, -z_{li}z_{lj}z_{lk} \Big).$$

Here $z = z(T_\mu) \in \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^*$; cf. (2–1) and (2–3).

Next, we define the map $a$; cf. (2–2). Let $a_\mu : \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2_{T_\mu})^* \to \mathbb{C}^4$ $(\mu = 1, \dots, \nu)$ associated to $T_\mu$ be the map defined by

$$a_\mu(z) = \big( z_{ik}(1 - z_{ij}), z_{jl}(1 - z_{ji}), z_{ki}(1 - z_{kl}), \times z_{lj}(1 - z_{lk}) \big).$$

We define $a = (a_1, \dots, a_\nu)$.

Finally, we let $C_1^{\mathrm{or}}$ denote the free $\mathbb{Z}$-module generated by the oriented 1-simplices of $K$, and $C_2$ the free $\mathbb{Z}$-module generated by the 2-faces of $K$. As observed before, if $K$ has only tori as ideal boundaries, then the number of edges in $K$ is $\nu$ and the number of faces is $2\nu$. Therefore, the $\mathbb{Z}$-module $C_1^{\mathrm{or}} + C_2$ has rank $4\nu$, and therefore $\mathrm{Hom}(C_1^{\mathrm{or}} + C_2, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{4\nu}$.

As in [Bergeron et al. 12], we define, for $\bar{e}_{ij}$ an oriented edge of $K$, a map

$$F : C_1^{\mathrm{or}} + C_2 \to J^2$$

by

$$F(\bar{e}_{ij}) = e^1_{ij} + \dots + e^\mu_{ij},$$

where $T_1, \dots, T_\mu$ is a sequence of tetrahedra sharing the edge $\bar{e}_{ij}$ such that $\bar{e}_{ij}$ is an inner edge of the subcomplex $T_1 \cup \dots \cup T_\mu$ and each $e^\mu_{ij}$ gets identified with the *oriented* edge $\bar{e}_{ij}$ in $\mathcal{T}$. And for a 2-face $\bar{e}_{ijk}$,

$$F(\bar{e}_{ijk}) = e^\mu_{ijk} + e^{\mu'}_{ikj},$$

where $\mu$ and $\mu'$ index the two 3-simplices having the common face $\bar{e}_{ijk}$. We then define the map

$$f : \mathrm{Hom}(J^2, \mathbb{C}^\times) \to \mathrm{Hom}(C_1^{\mathrm{or}} + C_2, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{4\nu}$$

by $f(z) = z \circ F$; compare (2–4) and (2–5). A decoration $z \in \mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2)^*$ satisfies the edge and face equations (2–5) and (2–4) if and only if $f(z) = 1$ (compare with the map $F^*$ defined in the next section, so we can write equivalently $z \in \mathbb{C}^\times \otimes_{\mathbb{Z}} \mathrm{Ker}(F^*)$).

From an element in $\mathcal{R}(M, \mathcal{T})$, one may reconstruct a representation (up to conjugacy) by computing the holonomy of the complex of flags (see [Bergeron et al. 12, Section 5]). Restating Remark 2.1, a decoration is equivalent to a map, equivariant under $\pi_1(M)$, from the space of cusps of $\tilde{M}$ to the space of flags with a transversality condition. Note that each flag is then invariant by the holonomy of the cusp.

Moreover, the map from $\mathcal{R}(M,\mathcal{T})$ to

$$\mathrm{Hom}(\pi_1(M), \mathrm{PGL}(3,\mathbb{C}))/\,\mathrm{PGL}(3,\mathbb{C})$$

is open: given a representation $\rho$, its decoration equips each cusp $p$ of $M$ with a flag $F_p$ invariant under the holonomy of the isotropy $\Gamma_p$ of $p$. Now, deforming the representation $\rho$ to $\rho'$, for each cusp $p$, one can deform $F_p$ into a flag $F'_p$ invariant under $\rho'(\Gamma_p)$. The transversality condition being open, this gives a decoration for every decoration $\rho'$ near $\rho$.

Generalizations of this formalism to the case of representations of 3-dimensional fundamental groups to $\mathrm{PGL}(n,\mathbb{C})$ for $n \geq 3$ can be seen in [Garoufalidis et al. 12, Dimofte et al. 13].

## 4. THE SYMPLECTIC ISOMORPHISM

In this section, we recall results of [Bergeron et al. 12] that will be used in the proof of the main theorem. As in that work, each $J^2_{T_\mu}$ is equipped with a bilinear skew-symmetric form given by

$$\Omega^2(e_\alpha, e_\beta) = \varepsilon_{\alpha\beta}.$$

Here given $\alpha$ and $\beta$ in $I$, we set (recall Figure 1)

$$\begin{aligned}\varepsilon_{\alpha\beta} = &\#\{\text{oriented arrows from } \alpha \text{ to } \beta\}\\ &- \#\{\text{oriented arrows from } \beta \text{ to } \alpha\}.\end{aligned}$$

We let $(J^2, \Omega^2)$ denote the orthogonal sum of the spaces $(J^2_{T_\mu}, \Omega^2)$. We denote by $e^\mu_\alpha$ the $e_\alpha$-element in $J^2_{T_\mu}$. Let

$$p : J^2 \to (J^2)^*$$

denote the homomorphism $v \mapsto \Omega^2(v,\cdot)$. In terms of the basis $(e_\alpha)$ and its dual $(e^*_\alpha)$, we can write

$$p(e_\alpha) = \sum_\beta \varepsilon_{\alpha\beta} e^*_\beta.$$

Let $J$ be the quotient of $J^2$ by the kernel of $\Omega^2$. The latter is the subspace generated on each tetrahedron by elements of the form

$$\sum_{\alpha\in I} b_\alpha e_\alpha$$

for all $\{b_\alpha\} \in \mathbb{Z}^I$ such that $\sum_{\alpha\in I} b_\alpha \varepsilon_{\alpha\beta} = 0$ for every $\beta \in I$. Equivalently, it is the subspace generated by $e_{ij} + e_{ik} + e_{il}$ and $e_{ijk} - (e_{il} + e_{jl} + e_{kl})$.

We let $J^* \subset (J^2)^*$ be the dual subspace that consists of the linear maps that vanish on the kernel of $\Omega^2$. Note that we have $J^* = \mathrm{Im}(p)$ and that it is 8-dimensional. The form $\Omega^2$ induces a nondegenerate skew-symmetric (we will call it symplectic) form $\Omega$ on $J$. This yields a canonical identification between $J$ and $J^*$; we denote by $\Omega^*$ the corresponding symplectic form on $J^*$.

Consider the sequence introduced in [Bergeron et al. 12]:

$$C_1^{\mathrm{or}} + C_2 \xrightarrow{F} J^2 \xrightarrow{p} (J^2)^* \xrightarrow{F^*} C_1^{\mathrm{or}} + C_2.$$

The skew-symmetric form $\Omega^*$ on $J^*$ is nondegenerate, but its restriction to $\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)$ has a nontrivial kernel. In [Bergeron et al. 12], we relate this form to “Goldman–Weil–Petersson” forms on the peripheral tori: there is a form $\mathrm{wp}_s$ on each $H^1(T_s, \mathbb{Z}^2)$, $s = 1, \ldots, \ell$, defined as the coupling of the cup product on $H^1$ with the scalar product[2] $\langle\, , \rangle$ on $\mathbb{Z}^2$ defined by

$$\left\langle \begin{pmatrix} n \\ m \end{pmatrix}, \begin{pmatrix} n' \\ m' \end{pmatrix} \right\rangle = \frac{1}{3}(2nn' + 2mm' + nm' + n'm);$$

see [Bergeron et al. 12, Section 7]. For our purpose, we rephrase the content of [Bergeron et al. 12, Corollary 7.3.2] in the following proposition.

**Proposition 4.1.** *We have*

$$\mathrm{Ker}(\Omega^*_{|\mathrm{Im}(p)\cap\mathrm{Ker}(F^*)}) = \mathrm{Im}(p \circ F).$$

*The skew-symmetric form $\Omega^*$ therefore induces a symplectic form on the quotient*

$$(J^* \cap \mathrm{Ker}(F^*))/\,\mathrm{Im}(p \circ F).$$

*Moreover, there is a symplectic isomorphism, defined over $\mathbb{Q}$, between this quotient and the space $\oplus_{s=1}^{\ell} H^1(T_s, \mathbb{Z}^2)$ equipped with the direct sum $\oplus_s \mathrm{wp}_s$, still denoted by* wp.

Let us briefly explain how to understand Proposition 4.1 as a corollary in [Bergeron et al. 12]. First recall from [Bergeron et al. 12, Section 7.2] that given an element $z \in \mathcal{R}(M,\mathcal{T})$, we may compute the holonomy of a loop $c \in H_1(T_s, \mathbb{Z})$ and get an upper triangular matrix; let $(\frac{1}{C^*}, 1, C)$ be its diagonal part. The mapping that takes $c \otimes \binom{n}{m}$ to $C^m (C^*)^n$ yields the *holonomy map*

$$\mathrm{hol} : \mathcal{R}(M,\mathcal{T}) \to \oplus_{s=1}^{\ell} \mathrm{Hom}\left(H_1\left(T_s, \mathbb{Z}^2\right), \mathbb{C}^\times\right).$$

The symplectic map of the proposition is the linearization of this holonomy map.

Here is how it is done: Our variety $\mathcal{R}(M,\mathcal{T})$ is a subvariety of $\mathbb{C}^\times \otimes (J^2)^*$. This latter space may be viewed

[2] This product should be interpreted as the Killing form on the space of roots of $\mathfrak{sl}(3,\mathbb{C})$ through a suitable choice of basis.

as the exponential of the $\mathbb{C}$-vector space $\mathbb{C} \otimes (J^2)^*$. Lemma 7.2.1 of [Bergeron et al. 12] expresses the square of hol (there, the holonomy map is denoted by $R$) as the exponential of a linear map:

$$\begin{aligned}\mathbb{C} \otimes \left(J^2\right)^* &\to \oplus_{s=1}^{\ell} H^1\left(T_s, \mathbb{C}^2\right)\\ &\simeq \oplus_{s=1}^{\ell} \operatorname{Hom}\left(H_1\left(T_s, \mathbb{Z}^2\right), \mathbb{C}\right).\end{aligned}$$

Moreover, this map is defined over $\mathbb{Q}$ and at the level of the $\mathbb{Z}$-modules. At that level, it is indeed obtained as the composition of the map $h^*$, dual to the map $h$ defined in [Bergeron et al. 12, Section 7.2.2], with the projection to $\oplus_s H^1(T_s, \mathbb{Z}^2) \cong \mathbb{Z}^{4\ell}$ (using a symplectic basis of $H_1(T_s, \mathbb{Z})$). The symplectic isomorphism of Proposition 4.1 is given by this map [Bergeron et al. 12, Theorem 7.3.1 and Corollary 7.3.2], after restriction to $J^* \cap \operatorname{Ker}(F^*)$ and quotienting by $\operatorname{Im}(p \circ F)$ (see [Bergeron et al. 12, Section 7.4]).

## 5. INFINITESIMAL DEFORMATIONS

Let $z = (z(T_\mu))_{\mu=1,\dots,\nu} \in \mathcal{R}(M, \mathcal{T})$. The exponential map identifies $T_z(\mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2)^*)$ with $\mathbb{C} \otimes (J^2)^* = \operatorname{Hom}(J^2, \mathbb{C})$. Under this identification, the differential $d_z g$ defines a linear map, which we write as a direct sum $d_z h \oplus d_z a \oplus d_z f$.

In the following three lemmas, we identify the kernel of each of these three linear maps in order to prove Proposition 5.4.

**Lemma 5.1.** *As a subspace of $\mathbb{C} \otimes (J^2)^*$, the kernel of $d_z h$ is equal to $\mathbb{C} \otimes J^*$.*

*Proof.* The lemma follows from the definitions that $\xi \in \mathbb{C} \otimes (J^2)^*$ belongs to the kernel of $d_z h$ if and only if it vanishes on the subspace $\operatorname{Ker}(\Omega^2)$ generated by $e^\nu_{ij} + e^\nu_{ik} + e^\nu_{il}$ and $e^\nu_{ijk} - (e^\nu_{il} + e^\nu_{jl} + e^\nu_{kl})$. This concludes the proof. □

**Lemma 5.2.** *As a subspace of $\mathbb{C} \otimes (J^2)^*$, the kernel of $d_z a$ is equal to the subspace $\mathcal{A}(z)$ defined as*

$$\begin{aligned}\bigl\{\xi \in \operatorname{Hom}(J^2, \mathbb{C}) : \xi(e^\mu_{ij}) + z_{il}(T_\mu)\xi(e^\mu_{ik}) = 0,\\ \xi(e^\mu_{ji}) + z_{jk}(T_\mu)\xi(e^\mu_{jl}) = 0,\ \xi(e^\mu_{ki}) + z_{kl}(T_\mu)\xi(e^\mu_{kj}) = 0,\\ \xi(e^\mu_{lj}) + z_{lk}(T_\mu)\xi(e^\mu_{li}) = 0,\ \forall \mu\bigr\}.\end{aligned}$$

*Proof.* Here again, we have only to check the assertion on each tetrahedron $T_\mu$ of $\mathcal{T}$. All four coordinates of $a_\mu$ can be dealt with in the same way, so we consider here only the first coordinate:

$$z \mapsto z_{ik}(1 - z_{ij}).$$

Taking the differential of the logarithm, we obtain

$$\frac{dz_{ik}}{z_{ik}} - \frac{dz_{ij}}{1 - z_{ij}} = 0.$$

Equivalently,

$$\frac{dz_{ij}}{z_{ij}} = \left(\frac{1 - z_{ij}}{z_{ij}}\right)\frac{dz_{ik}}{z_{ik}}.$$

Since $z \in \mathcal{R}(M, \mathcal{T})$, we have $h_\nu(z) = a_\mu(z) = 1$. In particular,

$$(1 - z_{ij}) = \frac{1}{z_{ik}} \quad \text{and} \quad z_{ij}z_{ik} = -\frac{1}{z_{il}}.$$

We conclude that

$$\frac{dz_{ij}}{z_{ij}} + z_{il}\frac{dz_{ik}}{z_{ik}} = 0.$$

Under the identification of $T_z(\mathbb{C}^\times \otimes_{\mathbb{Z}} (J^2)^*)$ with $\mathbb{C} \otimes (J^2)^* = \operatorname{Hom}(J^2, \mathbb{C})$, this proves the lemma. □

We denote by $F^* : (J^2)^* \to C_1^{\mathrm{or}} + C_2$ the dual map to $F$ (here we identify $C_1^{\mathrm{or}} + C_2$ with its dual using the canonical basis). It is the "projection map"

$$(e^\mu_\alpha)^* \mapsto \bar{e}_\alpha$$

when $(e^\mu_\alpha)^* \in (J^2)^*$. By definition of $f$ we have the following.

**Lemma 5.3.** *As a subspace of $\mathbb{C} \otimes (J^2)^*$, the kernel of $d_z f$ is equal to $\mathbb{C} \otimes \operatorname{Ker}(F^*)$.*

Lemmas 5.1, 5.2, and 5.3 clearly imply the next proposition.

**Proposition 5.4.**

$$\operatorname{Ker} d_z g = (\mathbb{C} \otimes (\operatorname{Im}(p) \cap \operatorname{Ker}(F^*)) \cap \mathcal{A}(z). \quad (5\text{–}1)$$

Note that among these three spaces, two are defined over $\mathbb{Z}$ and do not depend on the point $z$, but the last one, $\mathcal{A}(z)$, actually depends on $z$. We shall give examples in which the dimension of the intersection varies and describe the corresponding deformations in $\mathcal{R}(M, \mathcal{T})$. But first, we consider an open subset of $\mathcal{R}(M, \mathcal{T})$ that we prove to be a manifold.

## 6. THE COMPLEX MANIFOLD $\mathcal{R}(M, \mathcal{T}^+)$

Let

$$\mathcal{R}(M,\mathcal{T}^+) = \big\{z = (z(T_\mu))_{\mu=1,\dots,\nu} \in \mathcal{R}(M,\mathcal{T}) \\ : \operatorname{Im} z_{ij}(T_\mu) > 0,\ \forall \mu, i, j\big\}$$

be the subspace of $\mathcal{R}(M,\mathcal{T})$ whose *edge* coordinates have positive imaginary parts. Note that coordinates corresponding to the geometric representation belong to $\mathcal{R}(M,\mathcal{T}^+)$.

**Remark 6.1.** Observe that in the case of an ideal triangulation of a hyperbolic manifold with shape parameters all having positive imaginary part and satisfying the edge conditions and unipotent holonomy conditions, we obtain as holonomy the geometric representation $\rho_{\text{geom}}$. The shape parameters in the PSL(2, $\mathbb{C}$) case give rise to a parabolic decoration of the ideal triangulation in the sense of this paper, which is clearly contained in $\mathcal{R}(M,\mathcal{T}^+)$. This is explained in detail in [Bergeron et al. 12].

The main theorem of this section is a generalization of a theorem of [Choi 04]; it states that $\mathcal{R}(M,\mathcal{T}^+)$ is a smooth complex manifold and gives local coordinates.

Recall that we assumed that $\partial M$ is the disjoint union of $\ell$ tori. For each boundary torus $T_s$ ($s = 1,\dots,\ell$) of $M$, we fix a symplectic basis $(a_s, b_s)$ of the first homology group $H_1(T_s)$. Given a point $z$ in the representation variety $\mathcal{R}(M,\mathcal{T})$, we may consider the holonomy elements associated to $a_s$ and $b_s$. They preserve a flag associated to the torus by the decoration. In a basis adapted to this flag, those matrices are of the form (for notational simplicity, we write them in PGL(3, $\mathbb{C}$) rather than SL(3, $\mathbb{C}$))

$$\begin{pmatrix} 1/A_s^* & * & * \\ 0 & 1 & * \\ 0 & 0 & A_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/B_s^* & * & * \\ 0 & 1 & * \\ 0 & 0 & B_s \end{pmatrix}.$$

Now the diagonal entries $A_s$ and $A_s^*$ of the first matrix define for each torus a map

$$\mathcal{R}(M,\mathcal{T}) \to (\mathbb{C}^\times)^{2\ell}; \quad z \mapsto (A_s, A_s^*)_{s=1,\dots,\ell}. \tag{6–1}$$

**Theorem 6.2.** *Assume that $\partial M$ is the disjoint union of $\ell$ tori. Then the complex variety $\mathcal{R}(M,\mathcal{T}^+)$ is a smooth complex manifold of dimension $2\ell$. Moreover, the map* (6–1) *restricts to a local biholomorphism from $\mathcal{R}(M,\mathcal{T}^+)$ to $(\mathbb{C}^\times)^{2\ell}$.*

*Proof.* The proof that $\mathcal{R}(M,\mathcal{T}^+)$ is smooth follows immediately if we prove that $g$ is of constant rank at its points. We will show that the complex dimension of $\operatorname{Ker}(dg)$ is $2l$ and relate it to the map (6–1) in order to prove the second part of the theorem.

The key point of the proof of Theorem 6.2 is the following:

**Lemma 6.3.** *Let $z \in \mathcal{R}(M,\mathcal{T}^+)$.*

- *For every $\xi \neq 0$ in $A(z)$, we have $\Omega^*(\xi,\bar{\xi}) \neq 0$.*
- $(\mathbb{C} \otimes \operatorname{Im}(p \circ F)) \cap \mathcal{A}(z) = \{0\}$.

*Proof.* Here $\bar{\xi}$ is the complex conjugate of $\xi$. The second point is a direct consequence of the first. Indeed, let

$$\xi \in (\mathbb{C} \otimes \operatorname{Im}(p \circ F)) \cap \mathcal{A}(z).$$

It follows from the first point in Proposition 4.1 that $\Omega^*(\xi,\bar{\xi}) = 0$. If the first point holds, then it forces $\xi$ to be null.

Now $\Omega^*(\xi,\bar{\xi})$ can be computed locally on each tetrahedron $T_\mu$: Since $\xi$ belongs to the subspace $\mathbb{C} \otimes J^* \subset \mathbb{C} \otimes (J^2)^*$, it is determined by the coordinates $\xi^\mu_{ij} = \xi(e^\mu_{ij})$. Now, with respect to the symplectic form $\Omega$, the basis vector $e^\mu_{ij}$ is orthogonal to all the basis vectors except $e^\mu_{ik}$ and $\Omega(e^\mu_{ij}, e^\mu_{ik}) = 1$. By duality, we therefore have

$$\begin{aligned}\Omega^*(\xi,\bar{\xi}) &= \sum_{\mu=1}^{\nu}\sum_{i=1}^{4}\left(\xi^\mu_{ij}\bar{\xi}^\mu_{ik} - \bar{\xi}^\mu_{ij}\xi^\mu_{ik}\right) \\ &= -\sum_{\mu=1}^{\nu}\sum_{i=1}^{4}\left|\xi^\mu_{ij}\right|^2\left(\frac{1}{\overline{z_{il}(T_\mu)}} - \frac{1}{z_{il}(T_\mu)}\right).\end{aligned}$$

Here the last equality follows from the fact that $\xi \in \mathcal{A}(z)$. Since for each $\mu$ and $i$, we have (up to a nonzero constant)

$$\operatorname{Im}\left(\frac{1}{\overline{z_{il}(T_\mu)}} - \frac{1}{z_{il}(T_\mu)}\right) > 0,$$

the proof is complete. □

Let $\mathcal{L}(z)$ be the image of $\mathcal{A}(z)$ in $\oplus_{s=1}^{\ell} H^1(T_s, \mathbb{C}^2)$. It follows from the previous lemma and the fact that the map is defined over $\mathbb{Q}$ (see Lemma 4.1) that $\mathcal{L}(z)$ is a totally isotropic subspace isomorphic to $\mathcal{A}(z) \cap (\mathbb{C} \otimes (J^* \cap \operatorname{Ker}(F^*)))$ and that it satisfies that for every $\chi \neq 0$ in $\mathcal{L}(z)$, we have $\operatorname{wp}(\chi,\bar{\chi}) \neq 0$.

The space $\oplus_{s=1}^{\ell} H^1(T_s, \mathbb{C}^2)$ decomposes as the sum of two subspaces: $\sum_s [a_s] \otimes \mathbb{C}^2$ and $\sum_s [b_s] \otimes \mathbb{C}^2$ (where $[a_s]$ and $[b_s]$ denote the respective Poincaré duals to $a_s$ and $b_s$). Both are Lagrangian subspaces and are invariant under complex conjugation. To prove Theorem 6.2, it remains to prove that $\mathcal{L}(z)$ projects surjectively onto $\sum_s [a_s] \otimes \mathbb{C}^2$. The dimension $\dim \mathcal{L}(z)$ may be computed. In fact, by duality, we have

$$\begin{aligned}\dim(J^* \cap \operatorname{Ker}(F^*)) &= \dim(\operatorname{Im}(p) \cap \operatorname{Ker}(F^*)) \\ &= \dim(J^2)^* - \dim(\operatorname{Im}(F) + \operatorname{Ker}(p)).\end{aligned}$$

But we obviously have

$$\begin{aligned}&\dim(\operatorname{Im}(F) + \operatorname{Ker}(p)) \\ &\quad = \dim \operatorname{Ker}(p) + \dim \operatorname{Im}(F) - \dim(\operatorname{Ker}(p) \cap \operatorname{Im}(F)).\end{aligned}$$

On the other hand, we have

$$\dim J^2 = 16\nu, \quad \dim \operatorname{Ker}(p) = 8\nu,$$
$$\dim \operatorname{Im}(F) = \dim C_1^{\mathrm{or}} + \dim C_2 = 4\nu$$

(note that the map $F$ is injective). It finally follows from the proof of [Bergeron et al. 12, Lemma 7.4.1] that $\dim(\operatorname{Ker}(p) \cap \operatorname{Im}(F)) = 2\ell$. We conclude that

$$\dim(J^* \cap \operatorname{Ker}(F^*)) = 4\nu + 2\ell.$$

Since $\dim \mathcal{A}(z) = 4\nu$, the intersection $\mathcal{A}(z) \cap J^* \cap \operatorname{Ker}(F^*)$ is of dimension at least $2\ell$, and $\mathcal{L}(z)$ is a totally isotropic subspace of dimension at least $2\ell$ in a symplectic space of dimension $4\ell$: it is a Lagrangian subspace. Theorem 6.2 now immediately follows from the following lemma. □

**Remark 6.4.** The preceding considerations give a combinatorial proof that the image of $\mathcal{R}(M, \mathcal{T})$ is a Lagrangian subvariety of the space of representations of the fundamental group of the boundary of $M$.

**Lemma 6.5.** *We have*

$$\mathcal{L}(z) \cap \sum_s [b_s] \otimes \mathbb{C}^2 = \{0\}.$$

*Proof.* Suppose that $\chi$ belongs to this intersection. Since $\sum_s [b_s] \otimes \mathbb{C}^2$ is a Lagrangian subspace invariant under complex conjugation, the complex conjugate $\bar{\chi}$ also belongs to $\sum_s [b_s] \otimes \mathbb{C}^2$, and we have

$$\mathrm{wp}(\chi, \bar{\chi}) = 0.$$

Since $\chi$ also belongs to $\mathcal{L}(z)$, Lemma 6.3 finally implies that $\chi = 0$. □

### 6.1. Rigid Points

In general, if $z \in \mathcal{R}(M, \mathcal{T})$, the space $\mathcal{L}(z)$ is still a Lagrangian subspace. If we replace Lemma 6.3 by the assumption that

$$\big(\mathbb{C} \otimes \operatorname{Im}(p \circ F)\big) \cap \mathcal{A}(z) = \{0\}, \tag{6–2}$$

the proof of Theorem 6.2 still implies that $\mathcal{R}(M, \mathcal{T})$ is (locally around $z$) a smooth complex manifold of dimension $2\ell$, and the choice of a $2\ell$-dimensional subspace of $\oplus_{s=1}^{\ell} H^1(T_s, \mathbb{C}^2)$ transverse to $\mathcal{L}(z)$ yields a choice of local coordinates.

A point $z$ satisfying (6–2) is called a *rigid point* of $\mathcal{R}(M, \mathcal{T})$: indeed, at such a point, you cannot deform the representation without deforming its trace on the boundary tori. Note that if there exists a point $z \in \mathcal{R}(M, \mathcal{T})$ such that the condition (6–2) is satisfied, then (6–2) is satisfied for almost every point in the same connected component. This transversality condition may be expressed as the nonvanishing of a determinant of a matrix with entries in $\mathbb{C}(z)$. In the next section, we provide explicit examples of all the situations that can occur.

### 6.2. Proof of Theorem 1.1

Theorem 1.1 does not immediately follow from Theorem 6.2, since $M$ may not admit an ideal triangulation. Recall, however, that $M$ has a finite regular cover $M'$ that does admit an ideal triangulation [Luo et al. 08]. We may therefore apply Theorem 1.1 to $M'$, and the proof follows from the following general (certainly well-known) lemma.

**Lemma 6.6.** *Let $M'$ be a finite regular cover of $M$. Let $\rho$ and $\rho'$ be the geometric representations for $M$ and $M'$. Then one cannot deform $\rho$ without deforming $\rho'$.*

*Proof.* Let $\gamma_i$ be a finite set of loxodromic elements generating $\pi_1(M)$. Let $n$ be the index of $\pi_1(M')$ in $\pi_1(M)$. Then $\gamma_i^n$ is a loxodromic element of $\pi_1(M')$.

Hence $\rho'(\gamma_i^n) = (\rho(\gamma_i))^n$ is a loxodromic element in $\mathrm{PGL}(3, \mathbb{C})$. The crucial though elementary point is that its $n$th square roots form a finite set of $\mathrm{PGL}(3, \mathbb{C})$. So once $\rho'$ is fixed, the determination of a representation $\rho$ such that $\rho' = \rho_{|\pi_1(M')}$ requires a finite number of choices: we should choose an $n$th square root for each $\rho'(\gamma_i^n)$ among a finite number of them. □

## 7. EXAMPLES

In this section, we describe exact solutions of the compatibility equations that give all unipotent decorations of the triangulation with two tetrahedra of the figure-eight knot's sister manifold. This manifold has one cusp, so is homotopic to a compact manifold whose boundary consists of one torus. In terms of Theorem 6.2, we are looking at the fiber over $(1, 1)$ of the map $z \mapsto (A, A^*)$. We show that besides rigid decorations (i.e., isolated points in the fiber), we obtain nonrigid ones, namely four 1-parameter families of unipotent decorations.

Among the rigid decorations, one corresponds to the (complete) hyperbolic structure and belongs to $\mathcal{R}(M, \mathcal{T}^+)$. The rigidity then follows from Theorem 6.2. At the other isolated points, the rigidity is simply explained by the transversality between $\mathcal{A}(z)$ and $\mathrm{Im}(p \circ F)$, as explained in Section 6.1.

As for the nonrigid components, their existence shows first that rigidity is not granted at all. Moreover, the geometry of the fiber over a point in $(\mathbb{C}^*)^2$ appears to be complicated, with intersections of components. The map from the (decorated) representation variety $\mathcal{R}(M, \mathcal{T})$ to its image in the representation variety of the torus turns out to be far from trivial from a geometric point of view.

Let us stress that these components also contain points of special interest: there are points corresponding to representations with values in $\mathrm{PSL}(2, \mathbb{C})$ that are rigid inside $\mathrm{PSL}(2, \mathbb{C})$, but no longer inside $\mathrm{PSL}(3, \mathbb{C})$.

The analysis of this simple example seems to indicate that basically anything can happen, at least outside of $\mathcal{R}(M, \mathcal{T}^+)$.

### 7.1. The Figure-Eight Knot's Sister Manifold

The figure-eight knot's sister manifold $M$ and its triangulation $\mathcal{T}$ are described by the gluing of two tetrahedra as in Figure 3. Let $z_{ij}$ and $w_{ij}$ be the coordinates associated to the edge $ij$. We will express all the equations in terms of these edge coordinates (since the face coordinates are monomials in the edge coordinates; see (2–1)).

The variety $\mathcal{R}(M, \mathcal{T})$ is then given by relations (2–3) and (2–2) among the $z_{ij}$ and among the $w_{ij}$ plus the face and edge conditions (2–4) and (2–5).

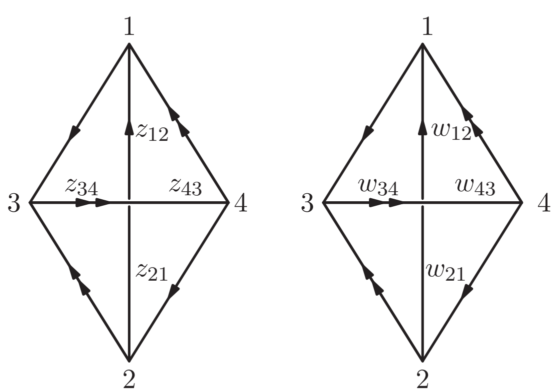


**FIGURE 3.** The figure-eight sister manifold represented by two tetrahedra.

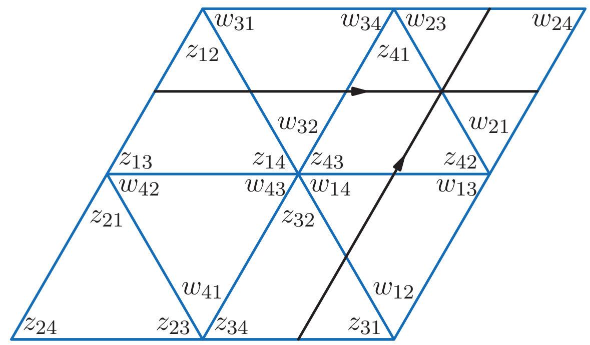


**FIGURE 4.** The boundary holonomy of the figure-eight sister manifold. The horizontal oriented line corresponds to $A, A^*$ and the other oriented line to $B, B^*$ (color figure available online).

In this case, the edge equations are

$$(L_e) \begin{cases} e_1 := z_{23}z_{34}z_{41}w_{23}w_{34}w_{41} - 1 = 0, \\ e_2 := z_{32}z_{43}z_{14}w_{32}w_{43}w_{14} - 1 = 0, \\ e_3 := z_{12}z_{24}z_{31}w_{12}w_{24}w_{31} - 1 = 0, \\ e_4 := z_{21}z_{42}z_{13}w_{21}w_{42}w_{13} - 1 = 0, \end{cases}$$

and the face equations are

$$(L_f) \begin{cases} f_1 := z_{21}z_{31}z_{41}w_{12}w_{32}w_{42} - 1 = 0, \\ f_2 := z_{12}z_{32}z_{42}w_{21}w_{31}w_{41} - 1 = 0, \\ f_3 := z_{13}z_{43}z_{23}w_{14}w_{34}w_{24} - 1 = 0, \\ f_4 := z_{14}z_{24}z_{34}w_{13}w_{23}w_{43} - 1 = 0. \end{cases}$$

Moreover, one may compute the eigenvalues of the holonomy in the boundary torus (see [Bergeron et al. 12, Section 7.2]) by following the two paths representing the generators of the boundary torus homology in Figure 4. The two eigenvalues associated to a path are obtained using the following rule: For the first one, say $A$, we multiply the cross-ratio invariant $z_{ij}$ if the vertex $ij$ of a triangle is seen to the left and by its inverse if it is seen to the right. For the inverse of the second one, say $A^*$, we multiply by $1/z_{ji}$ if the vertex $ij$ of a triangle is seen to the left and by $z_{kl}z_{lk}/z_{ij}$ if it is seen to the right:

$$A = z_{12}\frac{1}{w_{32}}z_{41}\frac{1}{w_{21}}, \quad A^* = \frac{1}{z_{21}}\frac{w_{14}w_{41}}{w_{32}}\frac{1}{z_{14}}\frac{w_{34}w_{43}}{w_{21}},$$

$$B = z_{31}\frac{1}{w_{14}}z_{42}\frac{1}{w_{23}}, \quad B^* = \frac{1}{z_{13}}\frac{w_{23}w_{32}}{w_{14}}\frac{1}{z_{24}}\frac{w_{14}w_{41}}{w_{23}},$$

or equivalently,

$$(L_{h,A,A^*,B,B^*}) \times \begin{cases} h_A := w_{32}w_{21}A - z_{12}z_{41} = 0, \\ h_{A^*} := z_{21}w_{32}z_{14}w_{21}A^* - w_{14}w_{41}w_{34}w_{43} = 0, \\ h_B := w_{14}w_{23}B - z_{31}z_{42} = 0, \\ h_{B^*} := z_{13}w_{14}z_{24}w_{23}B^* - w_{23}w_{32}w_{14}w_{41} = 0. \end{cases}$$

If $A = B = A^* = B^* = 1$, the solutions of the equations correspond to unipotent structures.

### 7.2. Methods

The computational problem to be solved is the description of a constructible set of $\mathbb{C}^{24}$ defined by the union of the edge equations $L_e$, the face equations $L_f$, the equations modeling the unipotent structures $L_{h,1,1,1,1}$ augmented by a set of relations between some of the variables $L_r$, and a set of inequalities (the coordinates are supposed to be different from 0 and 1), with

$$L_r := \begin{cases} w_{13} = \dfrac{1}{1-w_{12}}, & w_{14} = \dfrac{w_{12}-1}{w_{12}}, \\ w_{23} = \dfrac{w_{21}-1}{w_{21}}, & w_{24} = \dfrac{1}{1-w_{21}}, \\ w_{31} = \dfrac{1}{1-w_{34}}, & w_{32} = \dfrac{w_{34}-1}{w_{34}}, \\ w_{41} = \dfrac{w_{43}-1}{w_{43}}, & w_{42} = \dfrac{1}{1-w_{43}}, \\ z_{13} = \dfrac{1}{1-z_{12}}, & z_{14} = \dfrac{z_{12}-1}{z_{12}}, \\ z_{23} = \dfrac{z_{21}-1}{z_{21}}, & z_{24} = \dfrac{1}{1-z_{21}}, \\ z_{31} = \dfrac{1}{1-z_{34}}, & z_{32} = \dfrac{z_{34}-1}{z_{34}}, \\ z_{41} = \dfrac{z_{43}-1}{z_{43}}, & z_{42} = \dfrac{1}{1-z_{43}}. \end{cases} \tag{7–1}$$

After a straightforward substitution of the relations $L_r$ in the equations

$$\{e_1, \ldots, e_4, f_1, \ldots, f_4, h_{A|A=1}, h_{A^*|A^*=1}, h_{B|B=1}, \times\ h_{B^*|B^*=1}\},$$

one shows that the initial problem is then equivalent to describing the constructible set defined by a set of 12 polynomial equations

$$\mathcal{E} := \{x \in \mathbb{C}^8, P_i(x) = 0,\ i = 1, \ldots, 12,\ P_i \in \mathbb{Z}[\mathcal{X}]\}$$

in the eight unknowns

$$\mathcal{X} = \{z_{12}, z_{21}, z_{34}, z_{43}, w_{12}, w_{21}, w_{34}, w_{43}\},$$

and a set of 16 polynomial inequalities

$$\mathcal{F} := \{x \in \mathbb{C}^8,\ u(x) \neq 0,\ u(x) \neq 1,\ u \in \mathcal{X}\}.$$

Classical tools from computer algebra are used to:

- Compute generators of ideals using Gröbner bases. A Gröbner basis of a polynomial ideal $I$ is a set of generators of $I$ such that there is a natural way of reducing canonically a polynomial $P \pmod I$.
- Eliminate variables: Given $\mathcal{Y} \subset \mathcal{X}$ and $I \subset \mathbb{Q}[\mathcal{X}]$, compute $J = I \cap \mathbb{Q}[\mathcal{Y}]$ and note that the set
$$\mathcal{J} = \{x \in \mathbb{C}^{\sharp \mathcal{Y}},\ p(x) = 0,\ p \in J\}$$
is the Zariski closure of the projection of
$$\mathcal{I} = \{x \in \mathbb{C}^{\sharp \mathcal{X}},\ p(x) = 0,\ p \in I\}$$
onto the $\mathcal{Y}$-coordinates.

Combining the items, one can then compute an ideal $I'$ whose zero set is $\overline{\mathcal{E} \setminus \mathcal{F}}$ by computing

$$\left(I + \left\langle T \prod_{f \in \mathcal{F}} f - 1 \right\rangle\right) \cap \mathbb{Q}[\mathcal{X}]$$

(see, for example, [Cox et al. 07, Chapter 4]).

For rather small systems, one then computes straightforwardly (by means of a classical algorithm) a prime or primary decomposition of any ideal defining $\overline{\mathcal{E} \setminus \mathcal{F}}$. This is possible in the present case. In practice, however, for triangulations with more than two tetrahedra, these classical algorithms will not be sufficiently powerful to study these varieties.

We do not go further in the description of the computations, which will be part of a more general contribution by the last three authors [Falbel et al. 13]. Let us just mention that the process gives us an exhaustive description of all the components of the constructible set we study. Moreover, the interested reader may easily check that the given solutions indeed satisfy all the equations.

For the present paper, we just retain that a prime decomposition of an ideal defining $\overline{\mathcal{E} \setminus \mathcal{F}}$ has been computed, and we give the main elements describing the solutions so that the reader can at least check the main properties (essentially dimensions) of the results.

Each component (0- or 1-dimensional) can be described in the same way: a polynomial $P$ (in one or two variables) over $\mathbb{Q}$ such that each coordinate $z_{ij}$ or $w_{ij}$ is an algebraic (over $\mathbb{Q}$) function of the roots of $P$. In particular, they naturally come in families of Galois conjugates. This is no surprise, since the equations defining $\mathcal{R}(M, \mathcal{T})$ have integer coefficients.

### 7.3. Rigid Unipotent Decorations

We are looking for the isolated points of the set

$$\mathcal{U} = \{z \in \mathcal{R}(M, \mathcal{T}) \mid A = A^* = B = B^* = 1\}.$$

There are four Galois families of such points. They are described by four irreducible polynomials with integer coefficients in one variable. Two of them are of degree 2 and the other two of degree 8.

The first polynomial is the minimal polynomial of the sixth root of unity $\frac{1}{2}\left(1 + i\sqrt{3}\right)$. For a root $\omega^\pm = \frac{1}{2}\left(1 \pm i\sqrt{3}\right)$, the following defines an isolated point in $\mathcal{U}$:

$$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \omega^\pm.$$

The solution associated to $\omega^+$ is easily checked to correspond to the hyperbolic structure on $M$: it is the geometric representation, as we called it. The other one is its complex conjugate.

A point of $\mathcal{R}(M, \mathcal{T})$ corresponding to a representation in $\mathrm{PU}(2, 1)$ (we call such representations CR; see [Falbel 08]) with unipotent boundary holonomy was obtained in [Genzmer 10] and is parameterized by the same polynomial, the $z$- and $w$-coordinates being this time given by

$$\begin{aligned} z_{12} = z_{21} = -\omega, \quad z_{34} = z_{43} = -(\omega^\pm)^2, \\ w_{12} = w_{21} = -\omega^2, \quad w_{34} = w_{43} = -\omega^\pm. \end{aligned}$$

The two other isolated 0-dimensional components have degree 8, and their minimal polynomials are respectively

$$\begin{aligned} P = X^8 - X^7 + 5X^6 - 7X^5 + 7X^4 - 8X^3 + 5X^2 \\ -2X + 1 = 0 \end{aligned}$$

and

$$Q(X) = P(1 - X) = 0.$$

We do not describe all the $z$- and $w$-coordinates in terms of their roots (for the record, let us mention that $z_{43}$ is directly given by the root). None of these 16 representations are in $\mathrm{PSL}(2, \mathbb{C})$ or in $\mathrm{PU}(2, 1)$.

Although the computations above are exact, we could also check that these isolated components are rigid by computing that the tangent space is zero-dimensional. We do not include the computations here.

### 7.4. Nonrigid Components

There exist two 1-dimensional prime components ($S_1$ and $S_2$), each of which can be parameterized by two 1-parameter families.

The four 1-parameter families of solutions are described as follows: Let $\tau^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ be one of the two real roots of $X^2 = X + 1$. Then the roots $X^2 - XY - Y^2$ define two 1-parameter families meeting at $(0, 0)$: $X = \tau^\pm Y$. They parameterize four 1-parameter families of points $(S_1^\pm)$ and $(S_2^\pm)$.

For $S_1$, we obtain

$$(S_1^\pm) \begin{cases} z_{12} = w_{12} = \dfrac{X + Y}{X - 1}, \quad z_{21} = w_{21} = 1 + Y, \\ z_{34} = w_{34} = \dfrac{X^2 + X + Y}{X(X - 1)}, \quad z_{43} = w_{43} = X. \end{cases}$$

By restricting $S_1$ to the condition that the representation be in $\mathrm{PU}(2, 1)$, we obtain (after writing the system as a real system separating the real and imaginary parts) an algebraic set of real dimension 1 entirely characterized by its projection on the coordinates in $\mathbb{R}^2$ of $z_{21} = x + iy$. The projection is a product of two circles:

$$(x - \tau^\pm)^2 + y^2 = 1.$$

Among the solutions (in $S_1$), we obtain only two belonging to $\mathrm{PSL}(2, \mathbb{C})$ (and they even belong to $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PU}(2, 1)$):

$$\begin{aligned} z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} \\ = 1 + \tau^\pm. \end{aligned}$$

These points are then rigid inside $\mathrm{PSL}(2, \mathbb{C})$ but not inside $\mathrm{PSL}(3, \mathbb{C})$ (nor inside $\mathrm{PU}(2, 1)$).

The other two 1-parameter families are parameterized as follows:

$$(S_2^\pm) \begin{cases} z_{12} = w_{21} = 1 + \dfrac{Y}{X} - \dfrac{(X + 1)(Y + 1)}{X^2 + X - 1}, \\ z_{21} = w_{12} = \dfrac{X + Y - 1}{Y - 1}, \\ z_{34} = w_{43} = X + Y, \\ z_{43} = w_{34} = \dfrac{1}{Y}. \end{cases}$$

None of these points gives a representation in $\mathrm{PSL}(2, \mathbb{C})$ or $\mathrm{PU}(2, 1)$.

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## REFERENCES

[Bergeron et al. 12] Nicolas Bergeron, Elisha Falbel, and Antonin Guilloux. "Tetrahedra of Flags, Volume and Homology of SL(3)." Preprint, arXiv:1101.2742, 2012.

[Choi 04] Young-Eun Choi. "Positively Oriented Ideal Triangulations on Hyperbolic Three-Manifolds." *Topology* 43 (2004), 1345–1371.

[Cox et al. 07] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*, Undergraduate Texts in Mathematics. Springer, 2007.

[Dimofte et al. 13] T. Dimofte, M. Gabella, and A. B. Goncharov. "$K$-Decompositions and 3D Gauge Theories." Preprint, 2013.

[Falbel 08] Elisha Falbel. "A Spherical CR Structure on the Complement of the Figure Eight Knot with Discrete Holonomy." *Journal of Differential Geometry* 79 (2008), 69–110.

[Falbel et al. 13] E. Falbel, P.-V. Koseleff, and F. Rouillier. "Representations of Fundamental Groups of 3-Manifolds into PGL(3,$\mathbb{C}$): Exact Computations in Low Complexity." Preprint, arXiv:1307.6697, 2013.

[Garoufalidis et al. 12] S. Garoufalidis, M. Goerner, and Christian K. Zickert. "Gluing Equations for PGL$(n, c)$-Representations of 3-Manifolds." arXiv:1207.6711, 2012.

[Genzmer 10] Juliette Genzmer. "Sur les triangulations des structures CR-Sphériques." PhD thesis, Université Pierre et Marie Curie Paris 6, 2010.

[Luo et al. 08] Feng Luo, Saul Schleimer, and Stephan Tillmann. "Geodesic Ideal Triangulations Exist Virtually." *Proc. Amer. Math. Soc.* 136 (2008), 2625–2630.

[Menal-Ferrer and Porti 11] P. Menal-Ferrer and J. Porti. "Local Coordinates for SL$(n, C)$ Character Varieties of Finite Volume Hyperbolic 3-Manifolds." Preprint, arXiv:1111.4338, 2011.

[Petronio and Porti 00] Carlo Petronio and Joan Porti. "Negatively Oriented Ideal Triangulations and a Proof of Thurston's Hyperbolic Dehn Filling Theorem." *Expo. Math.* 18 (2000), 1–35.

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