Local Risk-Minimization for Defaultable Markets

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Stochastic processes: theory and applications
A conference in honor of Wolfgang J. Runggaldier
July 16 - 20, 2007

Credit Risk

A **default risk** is a possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract.
If this happens, we say that the party defaults, or that a default event occurs.

More generally, by **credit risk** we mean the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads and default events (bankruptcy, insolvency, missed payments).

Defaultable Claims: contingent agreements that are traded over-the-counter between default-prone parties. Each side of contract is exposed to the *counterparty risk* of the other party. *The underlying assets are assumed to be insensitive to credit risk.*

As for all the contingent claims, the most important problems to solve for the defaultable claims are the following ones:

- **pricing**

  What price should the seller of a contingent claim H charge the buyer at time 0? (Contract's Valuation)

- **hedging**

  How can the seller cover himself against the potential losses at time T (maturity) arising from a sale of H?

General Setting

- **Financial Market**

  - *primary assets* on $(\Omega, \mathcal{G}, \mathbb{P})$:

    1. risky asset S_t
    2. money market account $B_t = \exp(\int_0^t r_s \mathrm{d}s)$

  - *default time*: τ

- $H_t = \mathbb{I}_{\{\tau \leq t\}}$ **default process**

- W_t Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$

- $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with the enlarged filtration

$$\boxed{\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t}$$

  where $\mathcal{F}_t = \sigma(W_u : u \leq t)$ and $\mathcal{H}_t = \sigma(H_u : u \leq t)$

- Hypothesis (H): W_t *remains a (continuous) martingale (and then a Brownian motion) with respect to the enlarged filtration* $(\mathcal{G}_t)_{0 \leq t \leq T}$

- τ totally inaccessible $\mathcal{G}_t$-stopping time

- The **hazard process** under $\mathbb{P}$

$$\Gamma_t = -\ln(1 - F_t), \quad \forall t \in [0, T]$$

where $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$ is the cumulative distribution function of τ. We assume that there exists a non-negative integrable process λ_t (**hazard rate** or **intensity**) such that

$$\Gamma_t = \int_0^t \lambda_s \mathrm{d}s, \quad \forall t \in [0, T]$$

- The **compensated process**

$$\hat{M}_t = H_t - \int_0^{t \wedge \tau} \lambda_u \mathrm{d}u = H_t - \int_0^t \tilde{\lambda}_u \mathrm{d}u,$$

with $\tilde{\lambda}_t := \mathbb{I}_{\{\tau \geq t\}} \lambda_t$, is a $\mathcal{G}_t$-martingale under $\mathbb{P}$.

- The risky asset dynamics is given by:

$$\begin{cases} \mathrm{d}S_t & = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \\ S_0 & = s_0, \quad s_0 \in R_+ \end{cases}$$

where $\sigma_t > 0, \ \forall t \in [0, T]$ and $\mu_t, \ \sigma_t, \ r_t$ are $\mathcal{G}_t$-adapted processes s.t. $X_t = \dfrac{S_t}{B_t} \in L^2(\mathbb{P}), \ \forall t \in [0, T]$.

- We denote by

$$\boxed{\theta_t = \frac{\mu_t - r_t}{\sigma_t}}$$

the **market price of risk**. We assume that $\mu, \ \sigma$ and r are such that $\mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T$ is square-integrable (N.A.).

- **Defaultable Claim** $(\bar{X}, A, Z, \tilde{X}, \tau)$, where
  - $\bar{X}$ is the promised contingent claim,
  - A represents the promised dividends,
  - Z is the recovery process,
  - $\tilde{X}$ is the recovery claim.
- In particular we assume $A \equiv 0$.
- Discounted payoff:

$$H = \frac{\bar{X}}{B_T} \mathbb{I}_{\{\tau > T\}} + \left(\frac{Z_\tau}{B_\tau} + \frac{\tilde{X}}{B_T} \right) \mathbb{I}_{\{\tau \leq T\}}$$

- The market extended with a defaultable claim is not complete!

- The process $\hat{M}_t$ is NOT a tradeable asset.

- It makes sense to apply techniques used for pricing and hedging in incomplete markets.

- We choose **Quadratic Hedging Methods**.

- We apply the Local Risk-Minimization approach to defaultable markets.

Local Risk-Minimization

Local Risk-Minimization

Problem: we look for a hedging strategy φ with minimal cost which replicates the contingent claim H, i.e. $\bar{V}_T(\varphi) = H$.

- $X \in \mathcal{S}^2(\mathbb{P})$

$$X_t = X_0 + \int_0^t (\mu_s - r_s) X_s \mathrm{d}s + \int_0^t \sigma_s X_s \mathrm{d}W_s, \quad t \in [0, T]$$

- **(SC)**: the *mean-variance tradeoff* $\widehat{K}_t := \int_0^t \theta_s^2 \mathrm{d}s$ is almost surely finite.

- We assume that $\widehat{K}$ **is uniformly bounded** $in\ t, \omega$.

We denote by Θ_s the space of $\mathcal{G}$-predictable processes ξ on Ω such that

$$E\left[\int_0^T (\xi_s)^2 \sigma_s^2 X_s^2 \mathrm{d}s\right] + E\left[\left(\int_0^T |\xi_s \mu_s X_s| \mathrm{d}s\right)^2\right] < \infty; \tag{1}$$

Definition

- *An L^2-strategy is a pair $\varphi = (\xi, \eta)$ such that*

  1. $\xi \in \Theta_s$;
  2. η *is a real-valued $\mathcal{G}$-adapted process such that the discounted portfolio value*

$$\bar{V}_t(\varphi) = \xi_t \cdot X_t + \eta_t, \quad t \in [0, T]$$

is right-continuous and square-integrable.

- The **cost process** is defined by:

$$C_t = \bar{V}_t - \int_0^t \xi_s \mathrm{d}X_s, \quad 0 \leq t \leq T.$$

- An L^2-strategy φ is called **mean-self-financing** if its cost process $C_t(\phi)$ is a $\mathbb{P}$-martingale.

Definition

- *Let $H \in L^2(\mathcal{G}_T, \mathbb{P})$. An L^2-strategy φ with $\bar{V}_T(\varphi) = H$ $\mathbb{P}$-a.e. is* **pseudo-locally risk minimizing (plrm)** *for H if φ is mean-self-financing and the martingale $C(\varphi)$ is strongly orthogonal to the martingale part of X.*

Proposition

- *A contingent claim $H \in L^2(\mathbb{P})$ admits a plrm-strategy $\varphi = (\xi, \eta)$ with $\bar{V}_T(\varphi) = H$ $\mathbb{P}$-a.s. if and only if H can be written as*

$$H = H_0 + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H \quad \mathbb{P} - \text{a.s.} \tag{2}$$

Föllmer-Schweizer (FS) decomposition.

- **Plrm-strategy** (with respect to the discounted risky asset): $\boxed{\xi_t = \xi_t^H}$,

- **Minimal cost**: $\boxed{C_t(\varphi) = H_0 + L_t^H}$.

- **Optimal discounted portfolio value**: $\bar{V}_t(\varphi) = H_0 + \int_0^t \xi_s^H \mathrm{d}X_s + L_t^H$ and $\eta_t = \bar{V}_t(\varphi) - \xi_t^H X_t$.

The Minimal Martingale Measure

We see now how one can often obtain the FS decomposition by choosing a good martingale measure for X.

- A martingale measure $\widehat{\mathbb{P}}$ equivalent to $\mathbb{P}$ with square-integrable density is is called **minimal** if $\widehat{\mathbb{P}} \equiv \mathbb{P}$ on $\mathcal{G}_0$ and if any square-integrable $\mathbb{P}$-local martingale which is strongly orthogonal to the martingale part of X under $\mathbb{P}$ remains a local martingale under $\widehat{\mathbb{P}}$.

It can be shown ($\rightarrow$ Föllmer-Schweizer) that the following probability measure $\widehat{\mathbb{P}} \approx \mathbb{P}$

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} := \widehat{Z}_T = \mathcal{E}\left(-\int \theta dW\right)_T \in L^2(\mathbb{P})$$

with $\widehat{Z} \in \mathcal{M}^2(\mathbb{P})$ and strictly positive on $[0,T]$, is the MMM for X.

Theorem

Suppose X is continuous (and hence in our model satisfies (SC)). Consider the strictly positive local $\mathbb{P}$-martingale $\widehat{Z} := \mathcal{E}(-\int \theta \mathrm{d}W)$ and suppose that $\widehat{Z} \in \mathcal{M}^2(\mathbb{P})$.

Define the the process $\widehat{V}^H$ as follows

$$\widehat{V}_t^H := \widehat{E}[H|\mathcal{G}_t], \quad 0 \leq t \leq T.$$

Consider the GKW decomposition of $\widehat{V}^H$ with respect to X under $\widehat{\mathbb{P}}$

$$\widehat{V}_t^H = \widehat{E}[H|\mathcal{G}_t] = \widehat{V}_0^H + \int_0^t \widehat{\xi}_s^H \mathrm{d}X_s + \widehat{L}_t^H. \tag{3}$$

If $\widehat{\xi}^H \in \Theta_S, \widehat{L}^H \in \mathcal{M}^2(\mathbb{P})$, then (3) for $t = T$ gives the FS decomposition of H and $\widehat{\xi}^H$ gives a plrm-strategy for H. A sufficient condition to guarantee that $\widehat{Z} \in \mathcal{M}^2(\mathbb{P})$ and the existence of a FS decomposition for H is that $\widehat{K}$ is uniformly bounded.

Local Risk-Minimization for Defaultable Markets

Recovery Scheme at Maturity

The dynamics of the risky asset S_t may be influenced by the occurring of a default event and also the default time τ itself may depend on the risky asset price behavior.

- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \quad \forall t \in [0, T]$.

- $H = \frac{\bar{X}}{B_T} I_{\{\tau > T\}} + h(\tau \wedge T) \frac{\bar{X}}{B_T} I_{\{\tau \leq T\}} = \frac{\bar{X}}{B_T}(1 + (h(\tau \wedge T) - 1)H_T)$,
  where $\bar{X}$ is $\mathcal{G}_T$-measurable.

- **Proposition**. For any $\mathcal{G}_t$-martingale N_t under $\mathbb{P}$ we have

$$N_t = N_0 + \int_0^t \xi_u^N \mathrm{d}W_u + \int_0^t \zeta_u^N \mathrm{d}\hat{M}_u$$

  where W and $\hat{M}$ are strongly orthogonal.

- **Proposition**. The MMM $\widehat{\mathbb{P}}$ for X_t wrt $(\mathcal{G}_t)_{0\le t\le T}$ exists and its density is equal to $\mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T$.

- $\hat{M}_t$ is also a $\widehat{\mathbb{P}}$-martingale.

- Hence for any $\mathcal{G}_t$-martingale N_t under $\widehat{\mathbb{P}}$ we have

$$\hat{N}_t = \hat{N}_0 + \int_0^t \xi_u^{\hat{N}} \mathrm{d}\hat{W}_u + \int_0^t \zeta_u^{\hat{N}} \mathrm{d}\hat{M}_u. \tag{4}$$

- Problem: find the FS decomposition for H by computing (4) for

$$\widehat{V}_t^H = \widehat{E}[H|\mathcal{G}_t] = \widehat{E}\left[\frac{\bar{X}}{B_T}(1 + (h(\tau \wedge T) - 1)H_T)\middle|\mathcal{G}_t\right].$$

Solution

- $\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{G}_t\right] = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^t \bar{\xi}_s \mathrm{d}\hat{W}_s + \int_0^t \bar{\eta}_s \mathrm{d}\hat{M}_s$

- $\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau \wedge T) - 1)H_T\middle|\mathcal{G}_t\right] = H_t(h(\tau \wedge T) - 1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t \vee \mathcal{H}_T\right] + (1 - H_t)\underbrace{e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[\int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\mathcal{F}_t\right]}_{D_t}$,

  where $\bar{Z}_t$ is an $\mathcal{F}_t$-predictable process such that

$$\bar{Z}_\tau = \widehat{E}\left[(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T}\middle|\mathcal{F}_{\tau-}\right].$$

- Then
  * $H_t(h(\tau \wedge T) - 1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle| \mathcal{F}_t \vee \mathcal{H}_T\right] = H_t Z_\tau,$
  * $(1 - H_t)D_t = m_0 + \int_0^t \psi_s \mathrm{d}\hat{W}_s - \int_0^t D_s \mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau} Z_s \lambda_s \mathrm{d}s.$

- The FS decomposition for H is given by:

$$H = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_0^T \frac{1}{\sigma_s X_s}(\bar{\xi}_s + \mathbb{I}_{\{\tau \geq s\}} \xi_s^m e^{\int_0^s \lambda_u \mathrm{d}u})\mathrm{d}X_s$$
$$+ \int_0^T (\bar{Z}_s - D_s + \bar{\eta}_s)\mathrm{d}\hat{M}_s,$$

where

- $\bar{Z}_t$ is an $\mathcal{F}_t$-predictable process such that $\bar{Z}_\tau = \widehat{E}\left[(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T} \middle| \mathcal{F}_{\tau-} \right]$,

- $m_t := \widehat{E}\left[\int_0^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right] = m_0 + \int_0^t \xi_s^m \mathrm{d}\hat{W}_s$,

- $D_t := e^{\int_0^t \lambda_s \mathrm{d}s} \widehat{E}\left[\int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right]$,

- $\widehat{E}\left[\frac{\bar{X}}{B_T} \middle| \mathcal{G}_t \right] = \widehat{E}\left[\frac{\bar{X}}{B_T} \right] + \int_0^t \bar{\xi}_s \mathrm{d}\hat{W}_s + \int_0^t \bar{\eta}_s \mathrm{d}\hat{M}_s$.

- The plrm-strategy is given by:

$$\boxed{\xi_t^H = \frac{1}{\sigma_t X_t}\left(\bar{\xi}_t + \mathbb{I}_{\{\tau \geq t\}} \xi_t^m e^{\int_0^t \lambda_s \mathrm{d}s}\right)}$$

and the minimal cost is

$$\boxed{C_t^H = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_0^t (Z_s - D_s + \bar{\eta}_s)\mathrm{d}\hat{M}_s.}$$

Example 1: τ dependent of X

- Corporate bond: $\boxed{H = 1 + (h(\tau \wedge T) - 1)H_T}$ $\quad (\bar{X} = 1,\ B_t \equiv 1)$
- $\mathrm{d}\lambda_t = (b + \beta\lambda_t)\mathrm{d}t + \alpha\sqrt{\lambda_t}\mathrm{d}\hat{W}_t,$ with $\lambda_0 = 0$ (λ is affine).
- $h(x) = \alpha_0 \mathbb{I}_{\{x \le T_0\}} + \alpha_1 \mathbb{I}_{\{x > T_0\}}$.
- The FS decomposition for H is given by:

$$H = \alpha_0 + (\alpha_1 - \alpha_0)e^{-A(0,T_0)} + (1 - \alpha_1)e^{-A(0,T)}$$
$$- \int_0^T \frac{1}{\sigma_s X_s}\Big((\alpha_1 - \alpha_0)\mathbb{I}_{\{s \le T_0\}} e^{-A(s,T_0) - B(s,T_0)\lambda_s} B(s, T_0)$$
$$+(1 - \alpha_1)e^{-A(s,T) - B(s,T)\lambda_s} B(s, T)\Big)\sqrt{\lambda_s}\mathrm{d}X_s + \int_0^T (h(s) - D_s - 1)\mathrm{d}\hat{M}_s,$$

where the functions $A(t,T)$, $B(t,T)$ satisfy the following equations:

$$\partial_t B(t,T) = \frac{\alpha^2}{2} B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0 \tag{5}$$

$$\partial_t A(t,T) = -bB(t,T), \quad A(T,T) = 0, \tag{6}$$

that admit explicit solutions.

Example 2: X dependent on τ

- $\mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{H}_t, \ \forall t \in [0, T]$ and $\mathrm{d}S_t = S_t[\mu_t(\tau)\mathrm{d}t + \sigma_t(\tau)\mathrm{d}W_t]$, i.e. drift and volatility depend only on τ, seen as an exterior source of randomness.

- **Proposition**. The plrm strategy ξ_t^H coincides with the predictable projection of the $\tilde{\mathcal{G}}_t$-predictable process $\tilde{\xi}_t^H$ s.t. $\int_0^T (\tilde{\xi}_s^H)^2 \mathrm{d}s < \infty$ a.s. and

$$H = \widehat{E}\left[H \middle| \tilde{\mathcal{G}}_0\right] + \int_0^T \tilde{\xi}_s^H \mathrm{d}\hat{W}_s,$$

  where $\tilde{\mathcal{G}}_t = \mathcal{F}_t \otimes \mathcal{H}_T$. (Case of *incomplete information* $\rightarrow$ Föllmer-Schweizer).

- If $\bar{X} = (S_T - K)^+$, i.e. X_1 is given by the standard payoff of a call option, the plrm-strategy for H is given by:

$$\xi_t^H = E^{\widehat{\mathbb{P}},X}\left[\mathbb{I}_{\{S_T \geq K\}}(1 + (h(\tau \wedge T) - 1)H_T) | \mathcal{G}_{t-}\right].$$

Example 3: Computation of $\bar{Z}_t$

- The introduction of the process $\bar{Z}$ in may appear artificial. However it is necessary to find the FS decomposition.

- How can we compute $\bar{Z}_t$?

- If $\dfrac{\bar{X}}{B_T}$ is $\mathcal{F}_T$-measurable, then

$$\bar{Z}_t = (h(t \wedge T) - 1)(\widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^t \bar{\xi}_s \mathrm{d}\hat{W}_s).$$

- $\dfrac{\bar{X}}{B_T}$ (strictly) $\mathcal{G}_T$-measurable: Suppose that under $\widehat{\mathbb{P}}$, the discounted asset price X_t is of the form

$$X_t = x_0 e^{\int_0^t \sigma(\tau\wedge s)\mathrm{d}\widehat{W}_s - \frac{1}{2}\int_0^t \sigma(\tau\wedge s)^2 \mathrm{d}s}, \quad x_0 > 0,$$

where σ is a sufficiently integrable positive Borel function, and $\dfrac{\bar{X}}{B_T} = X_T^2$.

- We obtain

$$\begin{aligned}\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\,\mathcal{F}_{\tau-}\right] &= \widehat{E}\left[x_0^2 e^{2\int_0^T \sigma(\tau\wedge s)\mathrm{d}\widehat{W}_s - \int_0^T \sigma(\tau\wedge s)^2\mathrm{d}s}\middle|\,\mathcal{G}_{\tau-}\right] \\ &= x_0^2 e^{\int_0^T \sigma(\tau\wedge s)^2 \mathrm{d}s}\widehat{E}\left[e^{2\int_0^T \sigma(\tau\wedge s)\mathrm{d}\widehat{W}_s - 2\int_0^T \sigma(\tau\wedge s)^2\mathrm{d}s}\middle|\,\mathcal{G}_{\tau-}\right] \\ &= x_0^2 e^{\sigma(\tau)^2(T-\tau)} e^{2\int_0^\tau \sigma(s)\mathrm{d}\widehat{W}_s - \int_0^\tau \sigma(s)^2\mathrm{d}s}\end{aligned}$$

and

$$\bar{Z}_t = x_0^2(h(t\wedge T) - 1)e^{\sigma(t)^2(T-t)} e^{2\int_0^t \sigma(s)\mathrm{d}\widehat{W}_s - \int_0^t \sigma(s)^2\mathrm{d}s}$$

Recovery Scheme at Default Time

A random recovery payment is received by the owner of the contract in case of default at time of default.

$$H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau \le T\}}$$

- B_t is an $\mathcal{F}_t$-predictable process;
- Z_t is a $\mathcal{F}_t$-predictable process such that $\frac{Z_t}{B_t}$ is bounded;
- Since in our model we have a single default time, we assume that hedging stops after default.

Local Risk-Minimization with $\mathcal{G}_t$-strategies

The agent information takes into account the possibility of a default event.

Definition

A hedging strategy $\varphi^{\mathcal{G}} = (\xi, \eta)$ *is said a* *$\mathcal{G}$-plrm strategy* *if:*

1. $\xi_t \in \Theta_s$;
2. η_t *is* $\mathcal{G}_t$*-adapted;*
3. *the discounted value process* $V_t(\varphi^{\mathcal{G}}) = \xi_t X_t + \eta_t$ *is such that*

$$V_t(\varphi^{\mathcal{G}}) = \int_0^t \xi_s \mathrm{d}X_s + C_t(\varphi^{\mathcal{G}}), \quad t \in [\![0, \tau \wedge T]\!]$$

where C_t *is a square-integrable martingale strongly orthogonal to the* *martingale part of* X *and* $\boxed{V_{\tau\wedge T}(\varphi^{\mathcal{G}}) = H}$, *i.e.*

$$V_T(\varphi^{\mathcal{G}}) = \frac{\bar{X}}{B_T}, \text{ if } \tau > T, \quad V_\tau(\varphi^{\mathcal{G}}) = \frac{Z_\tau}{B_\tau}, \text{ if } \tau \leq T$$

Proposition

Let $G_t = \widehat{E}[H|\mathcal{G}_t]$. *There exists a pair of* $\mathcal{G}$*-predictable processes* $(\tilde{\xi}, \tilde{\zeta})$ *s.t.*

$$G_t = G_0 + \int_0^t \tilde{\xi}_s \mathrm{d}\hat{W}_s + \int_0^t \tilde{\zeta}_s \mathrm{d}\hat{M}_s.$$

- The FS decomposition for H is given by

$$H = \widehat{E}[F] + m_0 + \int_0^T \mathbb{I}_{\{\tau \geq s\}} \Big(\frac{e^{\int_0^s \lambda_u \mathrm{d}u}(\xi_s + \xi_s^m)}{\sigma_s X_s} \Big) \mathrm{d}X_s$$
$$+ \int_0^T \Big(\mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}[F|\mathcal{F}_s] + \frac{Z_s}{B_s} - D_s \Big) \mathrm{d}\hat{M}_s$$

where

- $F = e^{-\int_0^T \lambda_s \mathrm{d}s} \frac{\bar{X}}{B_T}$,

- $m_t := \widehat{E}\left[\int_0^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right] = m_0 + \int_0^t \xi_s^m \mathrm{d}\hat{W}_s$,

- $\widehat{E}\left[F \middle| \mathcal{F}_t \right] = \widehat{E}[F] + \int_0^t \xi_s \mathrm{d}\hat{W}_s$,

- $D_t := e^{\int_0^t \lambda_s \mathrm{d}s} \widehat{E}\left[\int_t^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right]$.

- The plrm-strategy is given by:

$$\xi_t^H = \mathbb{I}_{\{\tau \geq t\}} \frac{e^{\int_0^t \lambda_s \mathrm{d}s}(\xi_t + \xi_t^m)}{\sigma_t X_t}$$

- The minimal cost is

$$C_t^H = \widehat{E}[F] + m_0 + \int_0^t \left(\mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} E^{\widehat{\mathbb{P}}}[F|\mathcal{F}_s] + \frac{Z_s}{B_s} - D_s \right) \mathrm{d}\hat{M}_s$$

$\forall t \in [\![0, \tau \wedge T]\!]$.

Local Risk-Minimization with $\mathcal{F}_t$-strategies

The agent obtains her information only by observing the non-defaultable assets.

- **Lemma**. For any $\mathcal{G}_t$-predictable process $\tilde{\phi}_t$ there exists a $\mathcal{F}_t$-predictable process ϕ_t such that

$$\mathbb{I}_{\{\tau \geq t\}} \tilde{\phi}_t = \mathbb{I}_{\{\tau \geq t\}} \phi_t, \quad t \in [0, T]$$

- There exists a $\mathcal{F}_t$-predictable process $\tilde{X}_t$ (**pre-default discounted value** of X_t) s.t.

$$\mathbb{I}_{\{\tau \geq t\}} \tilde{X}_t = \mathbb{I}_{\{\tau \geq t\}} X_t, \quad t \in [0, T]$$

- We can consider prices of primary non-defaultable assets stopped at $\tau \wedge T$.

- We can suppose that μ and σ are $\mathcal{F}$-predictable.

- There do NOT exist $\mathcal{F}$-plrm strategies. We cannot hedge against the occurrence of a default by using only the information contained in $\tilde{X}_t$.

- We can think that the agent *invests* in X_t according to the information provided by the asset behavior *before default* and *adjusts* the portfolio value depending on the occurrence or not of the default.

- **Definition**. A strategy $\varphi^{\mathcal{F}} = (\xi, C)$ is said a (pre) $\mathcal{F}$-plrm strategy if:

1. ξ_t is a $\mathcal{F}_t$-predictable process satisfying (1);
2. C_t is a $\mathcal{G}_t$-martingale strongly orthogonal to the martingale part of X;
3. the discounted value process $V_t(\varphi^{\mathcal{F}}) = \xi_t X_t + \eta_t$ is such that

$$V_t(\varphi^{\mathcal{F}}) = \int_0^t \xi_s \mathrm{d}X_s^\tau + C_t(\varphi^{\mathcal{F}}), \quad t \in [\![0, \tau \wedge T]\!]$$

where

$$V_T(\varphi^{\mathcal{F}}) = \frac{\bar{X}}{B_T}, \text{ if } \tau > T, \quad V_\tau(\varphi^{\mathcal{F}}) = \frac{Z_\tau}{B_\tau}, \text{ if } \tau \le T$$

- The plrm-strategy is given by:

$$\boxed{\xi_t^H = \frac{e^{\int_0^t \lambda_s \mathrm{d}s}(\xi_t + \xi_t^m)}{\tilde{\sigma}_t \tilde{X}_t}}$$

- The minimal cost is

$$\boxed{C_t^H = E^{\widehat{\mathbb{P}}}[F] + m_0 + \int_0^t \left(e^{\int_0^s \lambda_u \mathrm{d}u} E^{\widehat{\mathbb{P}}}[F|\mathcal{F}_s] + \frac{Z_s}{B_s} - D_s\right) \mathrm{d}\hat{M}_s}$$

$\forall t \in [\![0, \tau \wedge T]\!]$.

Example

- Corporate bond: $\boxed{H = \frac{1}{B_T}(1 - H_T) + \delta X_\tau H_T}$

  where $X_t = \widehat{E}\left[e^{-\int_0^T r_s \mathrm{d}s} \middle| \mathcal{F}_t\right]$ and $\frac{Z_t}{B_t} = \delta X_t,\ t \in [0, T]$

- $\mathrm{d}r_t = (b + \beta r_t)\mathrm{d}t + \alpha\sqrt{r}\mathrm{d}\hat{W}_t,$ with $r_0 = 0$ (r is affine).

- λ deterministic function

- The FS decomposition for H is given by:

$$H = e^{-A(0,T)}\left[e^{-\int_0^T \lambda(u)\mathrm{d}u} + \delta(1 - e^{-\int_0^T \lambda(u)\mathrm{d}u})\right]$$

$$-\int_0^T \mathbb{I}_{\{\tau \geq s\}} \frac{1}{\sigma_s X_s}\varphi_s \mathrm{d}X_s + (\delta + 1)\int_0^T X_s e^{-\int_s^T \lambda(u)\mathrm{d}u}\mathrm{d}\hat{M}_s,$$

where

$$\varphi_s = e^{-A(s,T)-B(s,T)r_s}\frac{B(s,T)}{B_s}\sqrt{r_s}$$

and the functions $A(t,T)$, $B(t,T)$ satisfy the following equations:

$$\partial_t B(t,T) = \frac{\alpha^2}{2}B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0 \tag{7}$$

$$\partial_t A(t,T) = -bB(t,T), \quad A(T,T) = 0, \tag{8}$$

that admit explicit solutions.

- The plrm-strategy is given by:

$$\boxed{\xi_t^H = -\frac{1}{\sigma_t X_t}\varphi_t}$$

- The minimal cost is

$$C_t^H = e^{-\int_0^T \lambda(u)\mathrm{d}u - A(0,T)} + \delta e^{-A(0,T)}(1 - e^{-\int_0^T \lambda(u)\mathrm{d}u})$$
$$+(\delta+1)\int_0^T X_s e^{-\int_s^T \lambda(u)\mathrm{d}u}\mathrm{d}\hat{M}_s$$

$\forall t \in [\![0, \tau \wedge T]\!]$.

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