Local Similarity Manifolds (*) (**).

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Summary. – *A local similarity manifold is defined as a locally affine manifold for which the transition functions of an affine atlas are similarity transformations in $\boldsymbol{R}^n$. The main result of this paper is that, for $n \geqq 3$, the compact local similarity manifolds (which are not locally Euclidean) are given by the formula $M = (\boldsymbol{R}^n \setminus \{0\})/G$, where G is a group of covering transformations such that*

$$G = \{ht_0^k | h \in H,\ k \in \boldsymbol{Z}\}\ ,$$

H being a finite orthogonal group without fixed points in $\boldsymbol{R}^n \setminus \{0\}$, and t_0 being some conformal linear transformation of $\boldsymbol{R}^n$ which commutes with H.

1. – Introduction.

Locally affine and locally Euclidean (i.e., flat Riemannian) manifolds have been studied by many authors. (See, for instance, J. A. WOLF'S book [13]). Therefore, it is sensible to discuss also manifolds which have an atlas with transition functions in other affine subgroups. A natural such subgroup is that of the *similarity transformations* in $\boldsymbol{R}^n$

$$\tilde{x}^i = \varrho \sum_{i=1}^{n} a_j^i x^j + b^i\ , \tag{1.1}$$

where (a_j^i) is a real orthogonal matrix which yields the *orthogonal part* of (1.1), and $\varrho > 0$ is called the *module* of (1.1).

For the sake of simplicity, we agree to consider C^∞ connected manifolds only in this paper.

A differentiable manifold M^n will be called a *local similarity manifold*, shortly an l.s.m., if it is endowed with an l.s. structure i.e. an atlas $\{(U_\alpha, x_\alpha^i)/\alpha \in A,\ (i = 1, ..., ..., n)\}$, whose transition functions are locally of the form (1.1).

Obviously, an l.s. structure is locally affine, but it is not necessarilly locally Euclidean. Moreover, we shall agree to discuss only those l.s.m., which are not locally Euclidean i.e., unless the contrary is explicitely stated, the maximal l.s. atlas of M cannot be reduced to one with transition functions (1.1) of module 1 only.

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(But the same M may have, perhaps, some locally Euclidean structure non-related to the given l.s. structure). It is also clear that an l.s.m. is a locally conformally flat manifold, but the converse may not be true [8, 9].

In the context of the complex manifolds, l.s.m. have been studied as locally conformally Kähler-flat manifolds [11, 12], while an essential use of a procedure of KODAIRA [7] and KATO [5] has been made. In this paper, we shall transfer the same method to real l.s.m., and our main result yields the universal covering of a compact l.s.m. (it must be $\boldsymbol{R}^n\setminus\{0\}$), and a description of the corresponding group of covering transformations. Since we feel that the acces to the l.s.m. should be direct, and not via complex manifolds, we have written this paper as self-contained in spite of the fact that this made us repeat some of the proofs of [11, 12].

The basic example of a compact l.s.m. can be obtained as follows. Consider the transformation $\varphi_\lambda\colon \boldsymbol{R}^n\setminus\{0\} \to \boldsymbol{R}^n\setminus\{0\}$ defined by

$$\tilde{x}^i = \lambda x^i\,, \qquad \lambda \in \boldsymbol{R},\ 0<\lambda<1\,, \tag{1.2}$$

and denote by Φ_λ the infinite cyclic group generated by φ_λ. Then set

$$\boldsymbol{R}H^n = (\boldsymbol{R}^n\setminus\{0\})/\Phi_\lambda\,. \tag{1.3}$$

This will be called the *real Hopf manifold* (see [6, 11] for the complex Hopf manifolds). Using the diffeomorphism

$$\boldsymbol{R}^n\setminus\{0\} \approx S^{n-1}\times \boldsymbol{R}\,, \tag{1.4}$$

given by $(x^i)\mapsto(x^i/|x|, \ln|x|/\ln\lambda)$, we get $\boldsymbol{R}H^n \approx S^{n-1}\times S^1$ (S^h denotes always the h-dimensional unit sphere), which proves that $\boldsymbol{R}H^n$ is a compact, connected for $n>1$, differentiable manifold, and (1.2) shows that $\boldsymbol{R}H^n$ is an l.s.m.

2. – The l.s.m. as Riemannian manifolds.

Let M^n be an l.s.m. with the maximal structural atlas $\{(U_\alpha, x^i_\alpha)\}$, and let us define over each U_α the local flat Riemannian metric g_α given by

$$ds^2_\alpha = \sum_{i=1}^n (dx^i_\alpha)^2\,. \tag{2.1}$$

Over every intersection $U_\alpha\cap U_\beta$ we have then the transition relation

$$g_\beta = c_{\alpha\beta}\, g_\alpha\,, \tag{2.2}$$

where $c_{\alpha\beta} = \varrho^2_{\alpha\beta} > 0$, and $\varrho_{\alpha\beta}$ is the module of the corresponding transition (1.1), and is locally constant over $U_\alpha \cap U_\beta$. Clearly, $\{c_{\alpha\beta}\}$ satisfies the *cocycle condition*

$$c_{\alpha\beta} c_{\beta\gamma} = c_{\alpha\gamma} , \tag{2.3}$$

and we shall say that it defines a *twisted system of coefficients* on M.

Generally, any maximal system of local Riemannian metrics g_α defined on a differentiable manifold M, and satisfying (2.2), (2.3) will be called a *twisted Riemannian (t.R.) metric* on M. Hence, an l.s.m. has a canonically associated t.R. metric $g = \{g_\alpha\}$.

Furthermore, it is important that the Levi-Civita connections ∇_α of g_α satisfy $\nabla_\beta = \nabla_\alpha$ over $U_\alpha \cap U_\beta$, because of the fact that $c_{\alpha\beta}$ are locally constant. Hence, these connections can be glued up into a global connection ∇ ($= \nabla_\alpha$ on U_α), which will be called a *t.R. connection.* The geodesic lines of ∇ are well defined, and will be called *twisted geodesics.* Locally, they are the usual geodesics of g_α. (We recall that systems of local metrics with a global connection were studied in [10]).

Now, let (M, g) be a t.R. manifold. Then (2.3) shows that $\{\ln c_{\alpha\beta}\}$ is a 1-cocycle with values in the sheaf of germs of differentiable functions of M. Therefore,

$$\ln c_{\alpha\beta} = \sigma_\alpha - \sigma_\beta , \tag{2.4}$$

where $\sigma_\alpha \colon U_\alpha \to \boldsymbol{R}$ are differentiable functions defined up to the addition of a term of the form $\psi|_{U_\alpha}$, $\psi \colon M \to \boldsymbol{R}$. Since by (2.4) we have $d\sigma_\alpha = d\sigma_\beta$ on $U_\alpha \cap U_\beta$, we see that the local system $\{d\sigma_\alpha\}$ defines a closed 1-form ω on M ($\omega = d\sigma_\alpha$ on U_α). ω is determined up to cohomology, and we call it the *characteristic* 1-*form* of the twisted metric. (In [11, 12] this was the *Lee form*).

From (2.4) and (2.2) it follows that

$$\gamma|_{U_\alpha} = e^{\sigma_\alpha} g_\alpha \tag{2.5}$$

provides us with a global Riemannian metric on M which is defined up to a global conformal change $\gamma \mapsto e^\psi \gamma$. These metrics γ will be called *untwisting metrics* of g, and they are locally conformal to the metrics g_α. If we fix one such metric γ, σ_α and, hence, ω is also fixed, and we shall refer to it as to the *characteristic* 1-*form* of γ.

It is easy to understand that the system g contains one of its untwisting metrics γ iff ω is an exact form. In this case, we just have to consider a Riemannian manifold (M, γ), which is nothing new. Accordingly, we shall assume hereafter that ω is not exact, unless the contrary is explicitely stated. In particular, the characteristic 1-form of an l.s.m. is non-exact (unless the contrary is specified) since, otherwise, some untwisting metric γ belongs to g, and since this γ is flat, the structure of the manifold is, in fact, locally Euclidean, which contradicts the convention of Section 1.

Conversely, if γ is a Riemannian metric on a manifold M, a system $g = \{g_\alpha\}$ of local Riemannian metrics will be called *conformally compatible* with γ if $\gamma|_{U_\alpha} = e^{\sigma_\alpha} g_\alpha$.

Then a relation like (2.2) holds, but the $c_{\alpha\beta}$ there may not be locally constant functions, and we may have $d\sigma_\alpha \neq d\sigma_\beta$ on $U_\alpha \cap U_\beta$. However, if it happens that $d\sigma_\alpha = d\sigma_\beta$, one has $c_{\alpha\beta} =$ const, and g is a t.R. metric. Using classical formulas (e.g., [2]), it is easy to see that this happens iff, for every α, β, g_α and g_β have the same parametrized geodesics.

More simple, let us start with the metric γ, and with an arbitrary closed but not exact 1-form ω on M. Then, we can take the local convex neighbourhoods U_α, and functions $\sigma_\alpha \colon U_\alpha \to \boldsymbol{R}$ such that $\omega|_{U_\alpha} = d\sigma_\alpha$, and it is clear that $g_\alpha = e^{-\sigma_\alpha}\gamma|_{U_\alpha}$ will provide us with a t.R. metric g on M, for which γ is an untwisting metric. If we replace γ by $e^\psi\gamma$ and, simultaneously, ω by $\omega + d\psi$, we shall arrive at the same t.R. metric g. Hence, there is a bijection between the set of the twisted metrics g and the set of equivalence classes $[(\gamma, \omega)]$, where (γ_i, ω_i) $(i = 1, 2)$ are equivalent if $\gamma_2 = e^\psi\gamma_1$, $\omega_2 = \omega_1 + d\psi$, $\psi \in \boldsymbol{C}^\infty(M)$. This means that, up to inessential changes, a twisted metric can be seen as a pair consisting of an usual Riemannian metric and a closed (non-exact) 1-form.

In particular, this viewpoint can be used in the study of the l.s.m. by replacing there the associated twisted metric g with a corresponding pair (γ, ω). Indeed, if we start with the pair (γ, ω), we can refind the metrics g_α, and, if these are flat, they can be put under the form (2.1). Then, (2.2) ensures the transition relations (1.1), and we are done. The only remaining thing is to write down the fact that g_α are flat metrics. Since $\gamma = e^{\sigma_\alpha}g_\alpha$, $\omega = d\sigma_\alpha$, there is a well known relation between the curvature tensors R_γ and R_{g_α} [2], and it follows from it that $R_{g_\alpha} = 0$ means

$$(2.6) \quad R_\gamma(X, Y, Z, W) = \tfrac{1}{2}\{L(X, Z)\gamma(Y, W) - L(Y, Z)\gamma(X, W) + L(Y, W)\gamma(X, Z) - L(X, W)\gamma(Y, Z)\} + (|\omega|^2/4)\{\gamma(Y, Z)\gamma(X, W) - \gamma(X, Z)\gamma(Y, W)\},$$

where the sign of the curvature tensor is like in [2], and

$$(2.7) \qquad L(X, Y) = (D_X\omega)(Y) + \tfrac{1}{2}\omega(X)\omega(Y),$$

D being the Levi-Civita connection of γ. In componentwise form, these two formulas are

$$(2.6') \quad R_{(\gamma)ijkl} = \tfrac{1}{2}\{L_{ik}\gamma_{jl} - L_{jk}\gamma_{il} + L_{jl}\gamma_{ik} - L_{il}\gamma_{jk}\} + (|\omega|^2/4)(\gamma_{jk}\gamma_{il} - \gamma_{ik}\gamma_{jl}),$$

$$(2.7') \qquad L_{ij} = D_i\omega_j + \tfrac{1}{2}\omega_i\omega_j.$$

We shall summarize this discussion in

PROPOSITION 2.1. – *Up to inessential changes, an l.s.m. M is a Riemannian manifold (M, γ), on which there is a closed and non-exact 1-form ω such that the relation* (2.6) *holds good.*

REMARKS. – 1) For the dimension $n = 1$, (2.6) holds for every γ and ω, which makes 1-dimensional l.s. structures uninteresting. Therefore, we shall assume hereafter $n \geqq 2$.

2) Of course, the conformal curvature tensor of a t.R. metric is well defined, and it vanishes in the case of an l.s.m. (Hence, the Pontrjagin classes of an l.s.m. are zero). But we saw no way of using this property such as to obtain a characterization of the l.s.m. which would not contain ω explicitely.

3. – The co-closedness lemma.

In some problems, a more precise determination of the untwisting metric γ is necessary. In the compact orientable case, this can be done by using the following

LEMMA 3.1. (The *Co*-closedness Lemma). – *Let (M^n, γ) $(n \geqq 2)$ be a compact orientable Riemannian manifold, and ω an arbitrary* 1*-form on M. Then, there is a function $\psi\colon M \to \boldsymbol{R}$ such that $\omega + d\psi$ is co-closed with respect to the metric $e^{\psi}\gamma$, and ψ is defined up to the addition of a constant term.*

This is a generalization of Gauduchon's vanishing eccentricity theorem [1], and Gauduchon's proof can be applied to it in a form, which is independent of his Hermitian framework. For the reader's convenience, we shall repeat the basic details.

A simple computation shows that the co-closedness condition asked by the Lemma means

$$\Delta\psi - \frac{n-2}{2}\gamma(d\psi, \omega + d\psi) + \delta\omega = 0 \,. \tag{3.1}$$

For $n = 2$, this equation reduces to $\Delta\psi = -\delta\omega$, and ψ exists since, by the Hodge decomposition theorem, one must have $-\delta\omega = k + \Delta\psi$, $k = \text{const}$, and, by integrating this over M, we get $k = 0$.

For $n \neq 2$, the equation (3.1) can be linearized by means of the substitution $\psi = [2/(n-2)]\ln\varphi$, $\varphi > 0$, which replaces (3.1) by

$$L\varphi \overset{\text{def}}{=} \Delta\varphi - \frac{n-2}{2}\gamma(d\varphi, \omega) + \frac{n-2}{2}\varphi\delta\omega = 0 \,. \tag{3.2}$$

Here, of course, the difficulty lies in the condition $\varphi > 0$.

Let us note that, if γ itself satisfies the condition of the Lemma i.e., $\delta\omega = 0$, the equation (3.2) defines all the other solutions, and since it is then an elliptic equation of the HOPF type [4], these solutions are $\varphi = \text{const}$. This proves the last assertion of Lemma 3.1.

The adjoint of the operator L is

$$L^* = \Delta + \frac{n-2}{2} i(\omega) d \,, \tag{3.3}$$

where $i(\omega)$ denotes the interior product by ω, and L^* is an elliptic operator of the HOPF type [4], whence $\ker L^* = \boldsymbol{R}$.

Furthermore, we have index $L =$ index $L^* =$ index $\Delta = 0$, and, therefore

$$\dim_R \ker L = 1 \,.$$

Hence (3.2) has a solution $\varphi_0 \not\equiv 0$, such that all its solutions are given by $k\varphi_0$, $k \in \boldsymbol{R}$. Moreover,

$$\int_M \varphi_0 \, dv = \langle 1, \varphi_0 \rangle \neq 0 \,, \tag{3.4}$$

(dv is the volume element and $\langle \, , \rangle$ is the global scalar product), since, otherwise, $1 \perp \ker L$, hence $1 \in \operatorname{im} L^*$, which is impossible by Hopf's theorem of [4]. Therefore, we can assume without loss of generality that

$$\int_M \varphi_0 \, dv = \operatorname{vol} M > 0 \,. \tag{3.5}$$

Furthermore, let us assume that $\varphi_0(p) < 0$ at some $p \in M$, and let us chose open neighbourhoods U, V, W, such that $p \in V \subset \overline{V} \subset W \subset U$, and $\varphi_0|_U$ is < 0. Then, let χ be a differentiable function on M with values in $[0, 1]$, and such that χ equals 1 on V and 0 outside W. This yields

$$\int_M \chi\varphi_0 \, dv = -a^2 \qquad (a \in \boldsymbol{R}) \,, \tag{3.6}$$

whence (3.5) becomes

$$\int_M \chi\varphi_0 \, dv + \int_M (1-\chi)\varphi_0 \, dv = \operatorname{vol} M \,,$$

and we get

$$\int_M (1-\chi)\varphi_0 \, dv = \operatorname{vol} M + a^2 = b^2 \qquad (b \in \boldsymbol{R}) \,. \tag{3.7}$$

Finally, let us consider

$$\theta = \chi/a^2 + (1-\chi)/b^2 \geqq 0 \,. \tag{3.8}$$

Then, (3.6) and (3.7) give

$$\int_M \theta\varphi_0\, dv = \langle \theta, \varphi_0 \rangle = 0 .$$

Therefore, $\theta \perp \ker L$, hence $\theta \in \operatorname{im} L^*$, and, since $\theta \geqq 0$, this is impossible by Hopf's theorem [4].

The conclusion is that we must have $\varphi_0 \geqq 0$ everywhere on M, and finally, Lemma 2 of GAUDUCHON [1] (which is outside the Hermitian framework) provides us with the further conclusion $\varphi > 0$. Q.e.d.

From Lemma 3.1 and Section 2 it follows

PROPOSITION 3.2. – *Let M be a compact orientable manifold, and let $g = \{g_\alpha\}$ be a twisted metric on M. Then, it is always possible to chose an untwisting metric γ such that its characteristic* 1-*form ω is γ-harmonic. This metric γ is defined up to a global homothety.*

Furthermore, under particular circumstances ω satisfies an even stronger restriction. Namely, we have

PROPOSITION 3.3. – *Let (M^n, g) be a compact orientable t.R. manifold, as defined in Section* 2. *If all the local metrics g_α have a non-negative Ricci curvature, and if (γ, ω) are as in Proposition* 3.2, *then the* 1-*form ω is parallel with respect to γ.*

Indeed, from $\gamma = e^{\sigma_\alpha} g_\alpha$, $d\sigma_\alpha = \omega$, $\delta\omega = 0$ (see Section 2 for notation) the following relation between the corresponding Ricci tensors holds good [2, 12]

$$R_{(g_\alpha)jk} = R_{(\gamma)jk} - \frac{n-2}{4}\left(|\omega|^2\gamma_{jk} - \omega_j\omega_k\right) + \frac{n-2}{2} D_j\omega_k , \tag{3.9}$$

where D denotes the Levi-Civita connection of γ.

Now, let us remark that

$$\omega^k D_j\omega_k = \tfrac{1}{2} D_j(|\omega|^2) ,$$

whence

$$\begin{aligned}\int_M (D_j\omega_k)\omega^j\omega^k\, dv &= \tfrac{1}{2}\langle \omega, d(|\omega|^2)\rangle = \\ &= \tfrac{1}{2}\langle \delta\omega, |\omega|^2\rangle = \\ &= 0 .\end{aligned}$$

Consequently, (3.9) yields

$$\int_M R_{(\gamma)jk}\omega^j\omega^k\, dv = \int_M R_{(g_\alpha)jk}\omega^j\omega^k\, dv \geqq 0 , \tag{3.10}$$

and a well known result of BOCHNER [2] tells us that $D_j\omega_k = 0$. Q.e.d.

COROLLARY 3.4. – *A compact orientable l.s.m. M has an untwisting metric γ whose characteristic 1-form ω is γ-parallel. More precisely, an l.s. structure of a compact orientable manifold can be defined as a pair (γ, ω) consisting of a Riemannian metric γ and a nonzero parallel 1-form ω such that*

$$R_{(\gamma)ijkl} = \tfrac{1}{4}\{\gamma_{jl}\omega_i\omega_k - \gamma_{il}\omega_j\omega_k + \gamma_{ik}\omega_j\omega_l - \gamma_{jk}\omega_i\omega_l + |\omega|^2(\gamma_{jk}\gamma_{il} - \gamma_{ik}\gamma_{jl})\}\,. \tag{3.11}$$

Moreover, a normation condition $|\omega|^2 = a \in \boldsymbol{R}$ can be also assumed.

REMARK. – If $\dim M = n = 2$, (3.11) yields $R_{(\gamma)ijkl} = 0$, i.e., γ is a flat metric on M, but (if $\omega \neq 0$) k does not belong to the system $\{g_\alpha\}$. Anyway, we see that M is either a torus or a Klein bottle, with an l.s. structure.

In view of this Remark, we shall always assume hereafter that $\dim M = n \geqq 3$.

Furthermore, (3.11) can be used for the determination of the Betti numbers, and one gets

PROPOSITION 3.5. – *Let M^n $(n \geqq 3)$ be a compact l.s.m. Then, the Betti numbers of M^n are:*

1) $b_0 = b_1 = b_{n-1} = b_n = 1$, $b_i = 0$ *for* $2 \leqq i \leqq n-2$, *if M is orientable;*

2) $b_0 = b_1 = 1$, $b_i = 0$ *for the other dimensions i, if M is nonorientable.*

For an orientable M the results follow from (3.11) by the so-called Bochner technique. The concrete computations needed are those contained in the proof of Theorem 3.9 of [11], and we do not repeat them here again. Of course, the vanishing of the Euler-Poincaré characteristic $\chi(M)$ (which follows from $|\omega| = \text{const} \neq 0$) will also be used where needed.

In the nonorientable case, there is a connected oriented double covering M'^n, and it has the Betti numbers given above: $b'_0 = b'_1 = b'_{n-1} = b'_n = 1$, $b'_i = 0$ $(2 \leqq i \leqq n-2)$. By well-known properties of finite coverings of a compact manifold (e.g., [3]), one has:

i) $\chi' = 2\chi$, hence $\chi(M) = 0$;

ii) $b_k \leqq b'_k$, $(k = 0, \ldots, n)$, hence $b_i = 0$ for $i = 2, \ldots, n-2$;

iii) $1 \leqq b_1 \leqq b'_1 = 1$, hence $b_1 = 1$. Then, $b_n = 0$ since M is nonorientable, and, finally $b_{n-1} = 0$ since $\chi(M) = 0$. Q.e.d.

4. – The quotient structure of compact l.s.m.

Let M be an l.s.m. which has an untwisting metric γ with a parallel characteristic form ω. Then, (3.11) with $D_i\omega_j = 0$ holds.

The contravariant vector field B of local components ω^i is also parallel; we call it the *characteristic vector field*. Furthermore, the planes orthogonal to B define a foliation $\mathcal{F}$ given by $\omega = 0$, and the leaves L of $\mathcal{F}$ are totally geodesic submanifolds of (M, γ). Hence, the curvature tensor of L is given by

$$R_L(X, Y, Z, W) = R_{(\gamma)}(X, Y, Z, W), \tag{4.1}$$

for every vector fields which satisfy $\omega(X) = \omega(Y) = \omega(Z) = \omega(W) = 0$. Now, (4.1) and (3.11) show that every such leaf L has the constant positive sectional curvature $|\omega|^2/4$, or, by asking that $|\omega| = 2$ (i.e., replacing γ with $|\omega|^2\gamma/4$), L has sectional curvature 1. From these remarks, and using Proposition 3.3, and the classical de Rham decomposition theorem [6], we get

PROPOSITION 4.1. – *Let M^n $(n \geqq 3)$ be a compact orientable l.s.m. Then, M^n has such an untwisting metric γ that $S^{n-1}\times \boldsymbol{R}$ is the Riemannian universal covering of M^n.*

(The condition $n \geqq 3$, ensures that $S^{n-1}\times \boldsymbol{R}$ is simply connected).

In fact, we have to be more precise about the metric of $S^{n-1}\times \boldsymbol{R}$. This will be the product metric

$$ds^2 = d\sigma^2 + 4\,dt^2, \tag{4.2}$$

because of the normation condition $|\omega| = 2$. Here, t is the coordinate on $\boldsymbol{R}$, and $B = \partial/\partial t$, $\omega = 4\,dt$.

Furthermore, let us consider the diffeomorphism of $S^{n-1}\times \boldsymbol{R}$ onto $\boldsymbol{R}^n\setminus\{0\}$, given by

$$x^i = e^{-2t}u^i \qquad (i = 1, \ldots, n), \tag{4.3}$$

where u^i are cartesian coordinates in the copy of $\boldsymbol{R}^n$ which contains S^{n-1}, t is an abscissa on $\boldsymbol{R}$, and x^i are cartesian coordinates in $\boldsymbol{R}^n\setminus\{0\}$. Then, an easy computation shows that the metric (4.2) is transformed into

$$ds^2 = \left[1/\sum_{i=1}^{n}(x^i)^2\right]\sum_{j=1}^{n}(dx^j)^2. \tag{4.4}$$

Accordingly, Proposition 4.1, can be reformulated as

PROPOSITION 4.2. – *If M^n $(n \geqq 3)$ is a compact orientable l.s.m., then $M^n = (\boldsymbol{R}^n\setminus\{0\})/G$, where G is a group of covering transformations, consisting of isometries of the metric* (4.4).

Moreover, from (4.3), and $\omega = 4\,dt$, we get

$$\omega = -\,d\ln\left[\sum_{i=1}^{n}(x^i)^2\right], \tag{4.5}$$

and by the relation between g_α, γ, ω as described in Section 2, we see that, if the coordinates x^i of (4.4) are used as local coordinates about $x \in M$, the metric g_α about x is

$$g_\alpha = \sum_{j=1}^{n} (dx^j)^2 . \tag{4.6}$$

Hence, by (2.1) the present coordinates x^j are related by an affine transformation with linear part in $O(n)$ (n-dimensional orthogonal group) to the coordinates x^i_α of (2.1) about x. Next, because of (1.1), the transformations of G must be themselves of the form (1.1). But, since these transformations fix the origin of $\boldsymbol{R}^n$, G must consist of transformations of the form

$$\tilde{x}^i = \varrho \sum_{j=1}^{n} a^i_j x^j , \tag{4.7}$$

where $\varrho > 0$ is the *module* and $(a^i_j) \in O(n)$ is the *orthogonal component*.

Furthermore, if M is a non-orientable compact l.s.m., then let M' be its double oriented covering, and let $e^\lambda \gamma$ (λ: $M' \to \boldsymbol{R}$) be a metric with a parallel characteristic form, conformal to the lift of the untwisting metric γ of M to M'. Then, $e^\lambda \gamma$ is given by (4.2), (4.4), and it has the characteristic form (4.5). Hence

$$\gamma = \left[e^{-\lambda} / \sum_{i=1}^{n} (x^i)^2\right] \sum_{j=1}^{n} (dx^j)^2$$

has the characteristic form $-d \ln \left(\sum_{i=1}^{n} (x^i)^2 \right) - d\lambda$, and we find again the expression (4.6) for the local metrics g_α associated to γ. Since the universal covering of M and M' is obviously the same, we obtain now the same result (4.7) for the covering group G of M. Since such a group G preserves (4.4), it follows that even in the nonorientable case M has the metric induced by (4.4) which has a parallel characteristic form.

The converse, i.e., every quotient $(\boldsymbol{R}^n \setminus \{0\})/G$, where G is a group of covering transformations of the form (4.7), and not all modules ϱ are 1, is a compact l.s.m., is also obvious. Hence, we have proven

THEOREM 4.3. – *The class of compact l.s.m.* M^n ($n \geq 3$) *is defined by the formula*

$$M^n = (\boldsymbol{R}^n \setminus \{0\})/G , \tag{4.8}$$

where G *is a group of covering transformations of the form* (4.7) *having not all modules* 1. *The associated global metric* γ *of such a manifold is then defined by* (4.4), *where the* x^i *are given by cartesian coordinates in* $\boldsymbol{R}^n$.

In the sequel, we shall proceed like in [5, 7, 12], and get a more precise description of the group G.

A transformation (4.7) is called a *contraction* if $0 < \varrho < 1$. If t is a contraction, it generates an infinite cyclic group $\{t\}$ since the different powers t^h have different modules ϱ^h. Then, $(\boldsymbol{R}^n \diagdown \{0\})/\{t\}$ is a compact manifold M' covered by $\boldsymbol{R}^n \diagdown \{0\}$; in order to see this, it suffices to look at the diffeomorphism (1.4) with λ replaced by ϱ.

LEMMA 4.4 [5, 7, 12]. – *For every group G of (4.8), G contains at least one contraction. Every contraction $t \in G$ generates an infinite cyclic group $\{t\}$ which is a subgroup of finite index in G. There is a contraction $t_0 \in G$ such that $\varrho(t_0)$ is maximal < 1.*

The first assertion is true since otherwise M^n of (4.8) would not be compact. Then $M' = (\boldsymbol{R}^n \diagdown \{0\})/\{t\}$ is a compact covering of M^n whose fibers have $G/\{t\}$ points; therefore, $G/\{t\}$ is a finite set. Furthermore, let us fix such a contraction t. The modules of the elements of a class $[\tau]_{\{t\}}$ $(\tau \in G)$ are $\varrho(\tau)\varrho^h(t)$, and we can find in this class a contraction whose module is the closest possible to 1. Then, since we have only a finite number of classes, a comparison will provide us with the t_0 desired. Q.e.d.

Now, let us denote

$$H = \{\tau \in G | \varrho(\tau) = 1\}\,, \tag{4.9}$$

which is obviously an orthogonal normal subgroup of G.

PROPOSITION 4.5 [5, 12]. – *H is a finite subgroup of G, which commutes with t_0, and*

$$G = \{ht_0^k | h \in H,\ k \in \boldsymbol{Z}\}\,. \tag{4.10}$$

Indeed, consider a class $[\tau]_{\{t_0\}}$ $(\tau \in G)$. Its elements are of the form τt_0^k $(k \in \boldsymbol{Z})$, and let λ be the element of maximal module < 1. Then, since $\varrho(\lambda t_0^{-1}) > \varrho(\lambda)$, we must have $\varrho(\lambda t_0^{-1}) \geqq 1$, whence $\varrho(\lambda) \geqq \varrho(t_0)$. By the definition of t_0 and λ, this is impossible unless $\varrho(\lambda) = \varrho(t_0)$. Therefore, $\lambda t_0^{-1} \in H$, and we proved that every class $[\tau]_{\{t_0\}}$ has an element $h \in H$. The latter must be unique since all the other elements of the same class are of the form ht_0^k $(k \neq 0)$, and they have a module $\neq 1$. Hence H is finite, (4.10) is justified, and H commutes with t_0 since H is a normal subgroup of G. Q.e.d.

Conversely, for any finite subgroup $H \subset O(n)$, and any contraction t_0 of $\boldsymbol{R}^n$ which satisfies

$$t_0 H = H t_0\,, \tag{4.11}$$

formula (4.10) is a meaningful definition of a group G, and, if G is a covering transformation group, (4.8) yields a corresponding compact l.s.m. M^n.

Let us also note that t_0 itself has some module ϱ_0 $(0 < \varrho_0 < 1)$, and some orthogonal component $h_0 \in O(n)$, and (4.11) is equivalent to

$$h_0 H = H h_0 . \tag{4.12}$$

The other condition which we had namely, that G is a covering transformation group, is equivalent to the fact that H acts on $\boldsymbol{R}^n \setminus \{0\}$ without fixed points. Indeed, if this happens, G given by (4.10) is discrete and without fixed points in $\boldsymbol{R}^n \setminus \{0\}$, whence it follows that G is a covering transformation group (see, for instance, [13, p. 98]).

Thereby, we have proven

THEOREM 4.6. – *In order to obtain all the compact l.s.m. M^n $(n \geqq 3)$ we have to take:*

a) all the finite subgroups $H \subset O(n)$, which have no fixed points in $\boldsymbol{R}^n \setminus \{0\}$;

b) for every such H, all the elements $h_0 \in O(n)$ which commute with H (i.e. h_0 is in the normalizer of H in $O(n)$);

c) all the numbers $0 < \varrho_0 < 1$.

Then, M^n will be defined by the formulas (4.8) *and* (4.10). *All these manifolds are locally isometric with respect to the metrics defined by* (4.4).

Like in the complex case, we also have some more information about the topology of the manifolds above.

THEOREM 4.7 [5, 12]. – *Every compact l.s.m. M^n $(n \geqq 3)$ is a locally trivial differentiable fibre bundle with base space S^1, fiber S^{n-1}/H, and structure group $\{h_0\}$, where the notation is like in Theorem* 4.6.

Indeed, putting again $\boldsymbol{R}^n \setminus \{0\} \approx S^{n-1} \times \boldsymbol{R}$ by $(x^i) \mapsto [x^i/|x|, (\ln |x|)/\ln \varrho(t_0)]$, we see that a transformation ht_0^k acts by hh_0^k on S^{n-1}, and by $\tilde{r} = r + k$ on $\boldsymbol{R}$. Now, we see that

$$M^n = (\boldsymbol{R}^n \setminus \{0\})/G = [(\boldsymbol{R}^n \setminus \{0\})/H]/(G/H) \approx [(S^{n-1}/H) \times \boldsymbol{R}]/\{t_0\} , \tag{4.13}$$

and these relations yield the following commutative diagram

$$\begin{array}{ccc} (S^{n-1}/H) \times \boldsymbol{R} & \xrightarrow{pr_2} & \boldsymbol{R} \\ \Pi \big\downarrow & & \big\downarrow \Pi' \\ M & \xrightarrow[q]{} & S^1 = \boldsymbol{R}/\boldsymbol{Z} \end{array} \tag{4.14}$$

whose arrows have an obvious significance. Since Π and Π' are covering mappings, it follows that q is the fibration stated by Theorem 4.7. Q.e.d.

We shall end this paper by a few simple remarks about the groups of Theorem 4.6.

a) If $n = 2h + 1$ ($h \geqq 1$), a proper rotation has an axis, therefore, H has no proper rotation except the identity. For the same reason H cannot have two different improper rotations. Therefore, there are only two groups: H_1 which is trivial, and H_2 which consists of the identity and the symmetry with respect to the origin. In both cases, t_0 can be chosen arbitrarily. The corresponding manifolds M are fiber bundles over S^1 whose fiber is either S^{2h} or $\boldsymbol{R}P^{2h}$ (the real projective space).

b) If $n = 2k$ ($k \geqq 2$) then, since H has no fixed points in $\boldsymbol{R}^n \setminus \{0\}$, H is a subgroup of $U(k)$ with the corresponding real action on $\boldsymbol{R}^{2k}$. The determination of the groups H and G in this case is a difficult problem. (See [5] for the case $k = 2$).

Note that we refind in this scheme the real Hopf manifolds $S^{n-1} \times S^1$ (Section 1), as well as the manifolds $\boldsymbol{R}P^{n-1} \times S^1$ (which are non-orientable if n is odd).

Added in Proofs. – Recently, we became aware of the following papers: N. KUIPER, *Compact spaces with a local structure determined by the group of similarity transformations in* E^n, Indagationes Math., **42** (1950), pp. 411-418, and D. FRIED, *Closed similarity manifolds*, Comment. Math. Helvetici, **55** (1980), pp. 576-582, where the l.s.m. are studied by a straightforward geometric method.

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