

## Local Stability Conditions for the Babuška Method of Lagrange Multipliers\*

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**Abstract.** We consider the so-called Babuška method of finite elements with Lagrange multipliers for numerically solving the problem  $\Delta u = f$  in  $\Omega$ ,  $u = g$  on  $\partial\Omega$ ,  $\Omega \subset R^n$ ,  $n \geq 2$ . We state a number of local conditions from which we prove the uniform stability of the Lagrange multiplier method in terms of a weighted, mesh-dependent norm. The stability conditions given weaken the conditions known so far and allow mesh refinements on the boundary. As an application, we introduce a class of finite element schemes, for which the stability conditions are satisfied, and we show that the convergence rate of these schemes is of optimal order.

**1. Introduction.** Let  $\Omega$  be a bounded simply connected domain in  $R^n$ ,  $n \geq 2$ , with smooth boundary  $\partial\Omega$ . We consider the second-order Dirichlet problem

$$(1.1) \quad \Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega.$$

If  $f \in L_2(\Omega)$  and  $g \in H^r(\partial\Omega)$ ,  $r > 1$ , the problem (1.1) admits the following variational formulation: find a pair  $(u, \psi)$ ,  $u \in H^1(\Omega)$ ,  $\psi \in L_2(\partial\Omega)$ , such that

$$(1.2) \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \psi \, ds + \int_{\partial\Omega} u \varphi \, ds = - \int_{\Omega} f v \, dx + \int_{\partial\Omega} g \varphi \, ds,$$

for all  $(v, \varphi) \in H^1(\Omega) \times L_2(\partial\Omega)$ .

If  $(u, \psi)$  is the solution of problem (1.2), then  $u$  is the weak solution of problem (1.1) and  $\psi$  equals  $\partial u / \partial n$ , the normal derivative of  $u$  on  $\partial\Omega$  (cf. [1]).

If  $M_1^h \subset H^1(\Omega)$  and  $M_2^h \subset L_2(\partial\Omega)$  are finite-dimensional, the Lagrange multiplier method [1], [2] for the approximate solution of problem (1.1) consists of seeking for a pair  $(u_h, \psi_h) \in M_1^h \times M_2^h$  such that

$$(1.3) \quad \int_{\Omega} \nabla u_h \cdot \nabla v \, dx - \int_{\partial\Omega} v \psi_h \, ds + \int_{\partial\Omega} u_h \varphi \, ds = - \int_{\Omega} f v \, dx + \int_{\partial\Omega} g \varphi \, ds,$$

for all  $(v, \varphi) \in M_1^h \times M_2^h$ .

It is well known, cf. [1], [2], [4], that the uniform stability of the Lagrange multiplier method can be achieved only by rather careful choices of the subspaces  $M_1^h$  and  $M_2^h$ . In particular, if  $M_1^h$  is fixed, then the choice of  $M_2^h$  is critical [2]. In [1] it is shown that for finite element subspaces the stability can be achieved provided, essentially, that the diameter of the smallest element on  $\partial\Omega$  and that of the largest element in the

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interior of  $\Omega$  are related as

$$(1.4) \quad d_{\min}(\partial\Omega) \geq C d_{\max}(\Omega),$$

where  $C$  depends on  $\Omega$ .

The problem of weakening the condition (1.4) was studied in [8], where it was shown that for quasiuniform finite element meshes in  $R^2$  the constant  $C$  in (1.4) can be allowed a value close to unity, with certain natural choices of the boundary subspaces. In the present paper we consider the more general situation where mesh refinements are allowed. Neither (1.4) nor the conditions given in [8] allow mesh refinements on  $\partial\Omega$ .

The technique we use is somewhat different from that of [1] or [8]. Instead of working with the Sobolev space  $H^{-1/2}(\partial\Omega)$ , we carry out the analysis using  $L_2(\partial\Omega)$ , supplied with certain weighted, mesh-dependent norms.

In Section 2 a number of technical assumptions are made, which are sufficient for the uniform stability of the Lagrange multiplier method. The assumptions are given in a form allowing them to be checked locally for subspaces of finite element type.

In Section 3 the uniform stability of the Lagrange multiplier method is proved from the assumptions of Section 2. Here the key result, which allows local conditions in the case of mesh refinements, is Lemma 3.2. The uniform stability yields immediately an abstract quasioptimal error bound for the approximate solution. As a by-product we obtain here a weighted convergence result for  $u - u_h$  on the boundary.

The verification of the stability conditions in practice is finally considered in Section 4, where compatible finite element-type subspaces are constructed assuming a simplicial mesh in the interior of  $\Omega$ . Quasioptimal convergence rates are proved for the resulting finite element schemes.

Among other cases, where the conditions of Section 2 can be verified, we mention the boundary subspaces constructed in [8] and [9] for plane domains.

**2. The Basic Assumptions.** For a domain  $\Omega \subset R^n$  we let  $H^k(\Omega)$ ,  $k \geq 0$ ,  $k$  integer, denote the usual Sobolev space with the norm

$$\|u\|_{H^k(\Omega)}^2 = \sum_{j=0}^k |u|_{j,\Omega}^2,$$

where

$$|u|_{j,\Omega}^2 = \sum_{|\alpha|=j} \int_{\Omega} (D^{\alpha}u)^2 dx, \quad \alpha = (\alpha_1, \dots, \alpha_n), |\alpha| = \alpha_1 + \dots + \alpha_n.$$

For nonintegral  $k$ , the norm  $\|\cdot\|_{H^k(\Omega)}$  is defined by interpolation [7].

For a definition of the spaces  $H^k(\partial\Omega)$ , cf. [1]. To define a convenient norm for  $H^k(\partial\Omega)$ , when  $k$  is an integer, let  $\{S_{\nu}\}$  be a finite family of open, disjoint subsets of  $\partial\Omega$  such that  $\bigcup_{\nu} \bar{S}_{\nu} = \partial\Omega$ . For each  $S_{\nu}$ , let  $\{x_1^{(\nu)}, \dots, x_n^{(\nu)}\}$  be a system of local coordinates, chosen so that  $S_{\nu}$  has the representation

$$(2.1) \quad S_{\nu} = \{(x_1^{(\nu)}, \dots, x_n^{(\nu)}) \in R^n; x_n^{(\nu)} = F_{\nu}(x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}), \\ (x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}) \in U_{\nu}\}.$$

If the diameters of the subsets are sufficiently small, the local coordinates can be chosen so that each  $F_v$  satisfies

$$(2.2) \quad |D^\alpha F_v(x)| \leq C, \quad x \in U_v,$$

where  $C$  depends only on  $|\alpha|$ , not on  $v$ . Assuming this is the case, we can define

$$(2.3) \quad \|\psi\|_{H^k(\partial\Omega)}^2 = \sum_v \|\psi_v\|_{H^k(U_v)}^2,$$

where

$$(2.4) \quad \psi_v(x_1^{(v)}, \dots, x_{n-1}^{(v)}) = \psi(x_1^{(v)}, \dots, x_{n-1}^{(v)}, F_v(x_1^{(v)}, \dots, x_{n-1}^{(v)})).$$

Henceforth we consider a family  $\{\tau^h(\partial\Omega)\}_{0 < h < 1}$  of partitionings of  $\partial\Omega$  satisfying some further conditions as given below. We let  $K$  and  $\lambda$  be real and  $L$  an integral parameter,  $K \geq 1$ ,  $L \geq 1$ ,  $\lambda \geq 0$ . The subsequent conditions are assumed to hold for any given  $h$ ,  $0 < h < 1$ , and for some fixed, finite values of  $K$ ,  $\lambda$ , and  $L$ .

Consider a partitioning  $\tau^h(\partial\Omega)$ ,  $h$  fixed. For a set  $S \subset R^n$ , we let  $d(S)$  denote the diameter of  $S$ . Our first assumptions are the following:

A1. (i) For all  $S \in \tau^h(\partial\Omega)$ ,  $d(S) \leq h$ .

(ii) To the partition  $\tau^h(\partial\Omega) = \{S_v\}$  there corresponds a collection  $\{C_v\}$  of spheres,  $C_v \subset R^n$ , such that for each  $v$ , the center of  $C_v$  is in  $S_v$ ,  $d(C_v) = 8Kd(S_v)$ , and for each  $C \in \{C_v\}$ ,  $C \cap C_v \neq \emptyset$  for at most  $L$  spheres  $C_v \in \{C_v\}$ .

Here A1(ii) may be regarded as a local regularity assumption. We will need an additional global assumption which restricts the possible global mesh refinements. To this end, let us define, for a given  $S \in \tau^h(\partial\Omega)$ , the sets  $\Sigma_k(S)$  and  $R_k(S)$  as follows:

$$(2.5) \quad \begin{aligned} \Sigma_0(S) &= S, \\ \Sigma_k(S) &= \Sigma_{k-1}(S) \cup \{S_v \in \tau^h(\partial\Omega); \bar{S}_v \cap \overline{\Sigma_{k-1}(S)} \neq \emptyset\}, \quad k = 1, 2, \dots, \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} R_0(S) &= S, \\ R_k(S) &= \Sigma_k(S) - \Sigma_{k-1}(S), \quad k = 1, 2, \dots \end{aligned}$$

Each  $S_\mu \in \tau^h(\partial\Omega)$  is obviously a subset of  $R_k(S_v)$ ,  $S_v \in \tau^h(\partial\Omega)$ , for exactly one value of  $k$ . We assume:

A2. If  $S_v, S_\mu \in \tau^h(\partial\Omega)$  and  $S_\mu \subset R_k(S_v)$ , then  $d(S_\mu)/d(S_v) \leq K \cdot k^\lambda$ .

In practice, A2 means that the refinements should be globally subexponential.

For situations where refinements of this type are used, cf. [3].

We associate next a family of finite-dimensional subspaces  $M_2^h$  to the partitionings  $\tau^h(\partial\Omega)$ . For each  $h$ ,  $M_2^h$  is assumed to have a basis  $\{\psi_1, \dots, \psi_{m_h}\}$  which satisfies:

A3. (i) For all  $j$ ,  $1 \leq j \leq m_h$ ,  $\text{supp}\{\psi_j\} \subset \Sigma_k(S_v)$  for some  $S_v \in \tau^h(\partial\Omega)$  and  $k \leq L$ .

(ii) If  $\psi = \sum_1^{m_h} \beta_j \psi_j$ ,  $\beta_j \in R^1$ , and  $\Lambda_v = \{j; \psi_j \neq 0 \text{ on } S_v\}$ ,  $S_v \in \tau^h(\partial\Omega)$ , then

$$K \sum_{j \in \Lambda_v} \beta_j^2 \|\psi_j\|_{L_2(S_v)}^2 \geq \|\psi\|_{L_2(S_v)}^2 \geq K^{-1} \sum_{j \in \Lambda_v} \beta_j^2 \|\psi_j\|_{L_2(S_v)}^2.$$

In addition, we assume the following:

A4. For each  $S_v \in \tau^h(\partial\Omega)$ , there exists  $\theta_v \in M_2^h$  such that  $S_v \cap \text{supp}\{\theta_v\} \neq \emptyset$ ,

$d(\text{supp}\{\theta_v\}) \leq Kd(S_v)$ , and

$$\left| \int_{\partial\Omega} \theta_v ds \right|^2 \geq K^{-1} d(S_v)^{n-1} \int_{\partial\Omega} \theta_v^2 ds.$$

Finally, let  $\{M_1^h\}_{0 < h < 1}$  be a family of subspaces of  $H^1(\Omega)$ . We assume that for each  $h$  the following compatibility conditions hold between  $M_1^h$  and  $M_2^h$ .

A5. To the basis  $\{\psi_1, \dots, \psi_{m_h}\}$ , of  $M_2^h$ , there corresponds the set  $\{v_1, \dots, v_{m_h}\} \subset M_1^h$  such that

(i) For any  $v_i$ ,  $1 \leq i \leq m_h$ ,  $v_i v_j \neq 0$  for at most  $L$  functions  $v_j$ ,  $1 \leq j \leq m_h$ .

(ii) If  $S_v \in \tau^h(\partial\Omega)$  then

$$\{j; v_j \neq 0 \text{ on } S_v\} = \{j; \psi_j \neq 0 \text{ on } S_v\} = \Lambda_v.$$

(iii) If  $j \in \Lambda_v$ , then

$$K^{-1} d(S_v) |v_j|_{1,\Omega}^2 \leq \|v_j\|_{L_2(S_v)}^2 \leq K \|\psi_j\|_{L_2(S_v)}^2.$$

(iv) If  $\psi = \sum_1^{m_h} \beta_j \psi_j$ ,  $v = \sum_1^{m_h} \beta_j v_j$ , and  $S_v \in \tau^h(\partial\Omega)$ , then

$$K \sum_{j \in \Lambda_v} \beta_j^2 \|v_j\|_{L_2(S_v)}^2 \geq \|v\|_{L_2(S_v)}^2 \geq K^{-1} \sum_{j \in \Lambda_v} \beta_j^2 \|\psi_j\|_{L_2(S_v)}^2,$$

and

$$\int_{S_v} \psi v ds \geq K^{-1} \|\psi\|_{L_2(S_v)}^2.$$

Qualitatively, A5 imposes an upper bound for the dimension of  $M_2^h$ , as compared with that of the trace space of  $M_1^h$  on  $\partial\Omega$ . In this sense A5 is a condition of the same spirit as (1.4). The difference is that A5 is only a local constraint.

**3. A Stability Result.** In this section,  $C$  or  $C_j$  denotes a positive constant which may take different values on different usages and may depend on  $n$  and on the parameters  $K$ ,  $\lambda$ , and  $L$  of the preceding section without explicit indication. We write  $C(\Omega)$  if the constant depends also on  $\Omega$ .

If  $q \in R^1$ , denote by  $\|\cdot\|_{h,q,\partial\Omega}$  the weighted  $L_2$ -norm

$$\|\varphi\|_{h,q,\partial\Omega}^2 = \sum_v d(S_v)^{2q} \int_{S_v} \varphi^2 ds,$$

where the sum is over all  $S_v \in \tau^h(\partial\Omega)$ . We will denote by  $X_{\tau^h}$  the space  $H^1(\Omega) \times L_2(\partial\Omega)$  supplied with the norm

$$\|(u, \varphi)\|_{X_{\tau^h}}^2 = |u|_{1,\Omega}^2 + \|u\|_{h,-\frac{1}{2},\partial\Omega}^2 + \|\varphi\|_{h,-\frac{1}{2},\partial\Omega}^2.$$

In this notation, our basic stability and convergence result is as follows:

**THEOREM 3.1.** *Let the partitionings  $\tau^h(\Omega)$  and  $\tau^h(\partial\Omega)$  and the spaces  $M_1^h \subset H^1(\Omega)$  and  $M_2^h \subset L_2(\partial\Omega)$  be defined so that the conditions A1 through A5 are satisfied for some  $K$ ,  $\lambda$ , and  $L$ . Further, let  $f \in L_2(\Omega)$ ,  $g \in H^r(\partial\Omega)$ ,  $r > 1$ , and let  $(u, \psi)$  be the solution of problem (1.2). Then if  $h < h_0$ ,  $h_0 = h_0(\Omega) \in (0, 1)$ , problem (1.3) has a unique solution  $(u_h, \psi_h) \in M_1^h \times M_2^h$  which satisfies*

$$\|(u_h, \psi_h)\|_{X_{\tau^h}} \leq C(\Omega) (\|f\|_{0,\Omega} + \|g\|_{h,-\frac{1}{2},\partial\Omega}),$$

and

$$\|(u, \psi) - (u_h, \psi_h)\|_{X_{\tau h}} \leq C \min_{(v, \varphi) \in M_1^h \times M_2^h} \|(u, \psi) - (v, \varphi)\|_{X_{\tau h}}.$$

The proof of Theorem 3.1 will be based on the following two results.

PROPOSITION 3.1. *If  $u \in H^1(\Omega)$  is such that*

$$(3.1) \quad \int_{\partial\Omega} u \varphi \, ds = 0 \quad \text{for all } \varphi \in M_2^h,$$

*then if  $h$  is small enough,  $u$  satisfies*

$$\|u\|_{h, -1/2, \partial\Omega} \leq C|u|_{1, \Omega}.$$

PROPOSITION 3.2. *For any  $\varphi \in M_2^h$  there exists  $v \in M_1^h$  such that*

$$|v|_{1, \Omega} + \|v\|_{h, -1/2, \partial\Omega} \leq 1 \quad \text{and} \quad \int_{\partial\Omega} v \varphi \, ds \geq C\|\varphi\|_{h, 1/2, \partial\Omega}.$$

Before proving the propositions, let us show that Theorem 3.1 follows from them. For this, note first that we have

$$(3.2) \quad \|u\|_{L_2(\Omega)} \leq C(\Omega) (|u|_{1, \Omega}^2 + \|u\|_{h, -1/2, \partial\Omega}^2)^{1/2}, \quad u \in H^1(\Omega).$$

This follows, since  $\|u\|_{h, -1/2, \partial\Omega} \geq d(\Omega)^{-1/2} \|u\|_{L_2(\partial\Omega)}$  and since

$$\|u\|_{H^1(\Omega)} \leq C(\Omega) (|u|_{1, \Omega}^2 + \|u\|_{L_2(\partial\Omega)}^2)^{1/2}, \quad u \in H^1(\Omega).$$

Now define on  $X_{\tau h} \times X_{\tau h}$  the bilinear form

$$B(u, \psi; v, \varphi) = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \psi \, ds + \int_{\partial\Omega} u \varphi \, ds,$$

so that Eq. (1.3) can be rewritten as

$$(3.3) \quad B(u_h, \psi_h, v, \varphi) = - \int_{\Omega} f v \, dx + \int_{\partial\Omega} g \varphi \, ds, \quad (v, \varphi) \in M_1^h \times M_2^h.$$

The bilinear form  $B$  is continuous on  $X_{\tau h} \times X_{\tau h}$ :

$$(3.4) \quad |B(u, \psi; v, \varphi)| \leq \|(u, \psi)\|_{X_{\tau h}} \|(v, \varphi)\|_{X_{\tau h}}.$$

Also, using (3.2) we have

$$(3.5) \quad \left| - \int_{\Omega} f v \, dx + \int_{\partial\Omega} g \varphi \, ds \right| \leq C(\Omega) (\|f\|_{L_2(\Omega)} + \|g\|_{h, -1/2, \partial\Omega}) \|(v, \varphi)\|_{X_{\tau h}}.$$

In view of (3.3) through (3.5), the proof of Theorem 3.1 is complete if we can show that [2, pp. 186–188]

$$(3.6) \quad \inf_{(u, \psi) \in M_1^h \times M_2^h} \sup_{(v, \varphi) \in M_1^h \times M_2^h} \frac{B(u, \psi; v, \varphi)}{\|(u, \psi)\|_{X_{\tau h}} \|(v, \varphi)\|_{X_{\tau h}}} \geq C > 0.$$

For (3.6) to be valid in the present situation, the following two conditions are sufficient (see [4]):

$$(3.7) \quad \begin{aligned} u \in M_1^h \text{ \& } \int_{\partial\Omega} u \varphi \, ds = 0 \text{ for all } \varphi \in M_2^h \\ \Rightarrow |u|_{1, \Omega}^2 \geq C(|u|_{1, \Omega}^2 + \|u\|_{h, -1/2, \partial\Omega}^2), \end{aligned}$$

and

$$(3.8) \quad \inf_{\psi \in M_2^h} \sup_{v \in M_1^h} \frac{\int_{\partial\Omega} \psi v \, ds}{\|\psi\|_{h, \frac{1}{2}, \partial\Omega} (\|v\|_{1, \Omega}^2 + \|v\|_{h, -\frac{1}{2}, \partial\Omega}^2)^{1/2}} \geq C > 0.$$

But (3.7) follows from Proposition 3.1, and (3.8) is equivalent to Proposition 3.2.

Thus, if the propositions are true, the proof of Theorem 3.1 is complete.  $\square$

*Proof of Proposition 3.1.* Let  $\mathcal{D}_0$  be the unit sphere of  $R^{n-1}$ ,  $\mathcal{D}_0 = \{x \in R^{n-1}; |x| < 1\}$ , and let  $E_0$  be the cylinder

$$E_0 = \{x = (z, x_n) \in R^n; z \in \mathcal{D}_0, 0 < x_n < 1\}.$$

We prove first a preliminary result.

LEMMA 3.1. *Let  $\delta > 0$  and let  $\theta \in L_2(\mathcal{D}_0)$  be such that*

$$1 = \|\theta\|_{L_2(\mathcal{D}_0)} \leq \delta^{-1} \left| \int_{\mathcal{D}_0} \theta(z) \, dz \right|.$$

*Then if  $u \in H^1(E_0)$ , there is a constant  $C$  depending only on  $\delta$  such that*

$$\int_{\mathcal{D}_0} |u(z, 0)|^2 \, dz \leq C \left\{ \int_{E_0} |\nabla u|^2 \, dx + \left| \int_{\mathcal{D}_0} \theta(z) u(z, 0) \, dz \right|^2 \right\}.$$

*Proof.* If the assertion is not true, then there exist the sequences  $\{u_v\}$  and  $\{\theta_v\}$ ,  $u_v \in H^1(E_0)$ ,  $\theta_v \in L_2(\mathcal{D}_0)$ ,  $v = 1, 2, \dots$ , such that

$$(3.9) \quad 1 = \|\theta_v\|_{L_2(\mathcal{D}_0)} \leq \delta^{-1} \left| \int_{\mathcal{D}_0} \theta_v(z) \, dz \right|,$$

$$(3.10) \quad \|u_v\|_{H^1(E_0)} = 1,$$

$$(3.11) \quad \int_{E_0} |\nabla u_v|^2 \, dx + \left| \int_{\mathcal{D}_0} \theta_v(z) u_v(z, 0) \, dz \right|^2 \leq \frac{1}{v} \int_{\mathcal{D}_0} |u_v(z, 0)|^2 \, dz.$$

By (3.10), there exists a subsequence of  $\{u_v\}$ , still called  $\{u_v\}$ , which converges in  $L_2(E_0)$ . Since (3.10) also implies that  $\int_{\mathcal{D}_0} |u_v(z, 0)|^2 \, dz \leq C$ , we have, by (3.11), that

$$\int_{E_0} |\nabla u_v|^2 \, dx \rightarrow 0, \quad v \rightarrow \infty.$$

Hence,  $\{u_v\}$  converges in  $H^1(E_0)$  to a function  $u_0(x) \equiv C_0 = \text{const}$ . By (3.10),  $C_0 \neq 0$ .

However, taking the limit in (3.11) one has

$$\begin{aligned} 0 &= \lim_{v \rightarrow \infty} \left| \int_{\mathcal{D}_0} \theta_v(z) u_v(z, 0) \, dz \right| \\ &\geq \lim_{v \rightarrow \infty} \left\{ |C_0| \left| \int_{\mathcal{D}_0} \theta_v(z) \, dz \right| - C \|\theta_v\|_{L_2(\mathcal{D}_0)} \|u_v - u_0\|_{H^1(E_0)} \right\}, \end{aligned}$$

so that  $C_0 = 0$ , by (3.9), a contradiction.  $\square$

Now let  $\tau^h(\partial\Omega)$  and  $M_2^h$  be given,  $h$  sufficiently small so that the subsequent assumptions are valid. For each  $S_v \in \tau^h(\partial\Omega)$ , let  $\theta_v \in M_2^h$  be a function referred to in A4, and let the sphere  $C_v \subset R^n$  be as in A1(ii). By A4, if  $h$  is small enough, there exist the systems of local coordinates  $\{x_1^{(v)}, \dots, x_n^{(v)}\}$ , the spheres  $\mathcal{D}_v \subset R^{n-1}$  with center at the origin and diameter  $d(\mathcal{D}_v) = 2Kd(S_v)$ , and the functions  $F_v$  defined on

$\mathcal{D}_\nu$  such that  $S_\nu \subset \Gamma_\nu$  and  $\text{supp}\{\theta_\nu\} \subset \Gamma_\nu$ , where

$$\Gamma_\nu = \{(x_1^{(\nu)}, \dots, x_n^{(\nu)}) \in R^n; x_n^{(\nu)} = F_\nu(x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}), (x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}) \in \mathcal{D}_\nu\}.$$

We may further assume the coordinate systems to be chosen so that each  $F_\nu$  satisfies

$$|D^\alpha F_\nu(z)| \leq C, \quad |\alpha| = 1, z \in \mathcal{D}_\nu,$$

and

$$(3.12) \quad |F_\nu(z_1) - F_\nu(z_2)| \leq Kd(S_\nu), \quad z_1, z_2 \in \mathcal{D}_\nu.$$

For each  $\nu$ , define the set  $E_\nu$  as

$$E_\nu = \{(x^{(\nu)}, \dots, x_n^{(\nu)}) \in R^n; (x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}) \in \mathcal{D}_\nu, \\ F_\nu(x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}) < x_n^{(\nu)} < F_\nu(x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}) + Kd(S_\nu)\}.$$

Then if  $h < h_0$ ,  $h_0 = h_0(\Omega)$  sufficiently small, we have  $E_\nu \subset \Omega$  for each  $\nu$ . From (3.12) and from A1(ii) it is also easily verified that

$$(3.13) \quad E_\nu \subset \mathcal{C}_\nu \cap \Omega, \quad \nu = 1, 2, \dots.$$

The set  $E_\nu$  may be mapped onto the cylinder  $E_0$  of Lemma 3.1 by the mapping  $x \rightarrow \hat{x} = J_\nu(x)$  defined as

$$\hat{x}_i = K^{-1}d(S_\nu)^{-1}x_i^{(\nu)}, \quad i = 1, \dots, n-1, \\ \hat{x}_n = K^{-1}d(S_\nu)^{-1}[x_n^{(\nu)} - F_\nu(x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)})].$$

By the properties of  $F_\nu$  assumed above,  $J_\nu: E_\nu \rightarrow E_0$  is an invertible smooth transformation. Moreover, we obtain from the definition of  $J_\nu$  the formula

$$(3.14) \quad \int_{\Gamma_\nu} \varphi(s) ds = \int_{\mathcal{D}_0} e_\nu(z) \hat{\varphi}_\nu(z) dz, \quad \varphi \in L_2(\Gamma_\nu),$$

where  $\hat{\varphi}_\nu(z) = \varphi(J_\nu^{-1}(z, 0))$  and  $e_\nu$  satisfies

$$(3.15) \quad C_1 K^{n-1} d(S_\nu)^{n-1} \leq e_\nu(z) \leq C_2 K^{n-1} d(S_\nu)^{n-1}, \quad z \in \mathcal{D}_0.$$

Now let  $u \in H^1(\Omega)$  be such that (3.1) holds, let  $u_\nu$  be the function  $u$  in the coordinate system  $\{x_1^{(\nu)}, \dots, x_n^{(\nu)}\}$ , and let  $\hat{u}_\nu(x) = u_\nu(J_\nu^{-1}(x))$ ,  $x \in E_0$ . Then, since  $\Gamma_\nu \supset \text{supp}\{\theta_\nu\}$ , we have, using (3.1) and (3.14),

$$(3.16) \quad 0 = \int_{\Gamma_\nu} u_\nu \theta_\nu ds = \int_{\mathcal{D}_0} e_\nu(z) \hat{\theta}_\nu(z) \hat{u}_\nu(z, 0) dz,$$

where  $\hat{\theta}_\nu(z) = \theta_\nu(J_\nu^{-1}(z, 0))$ . By A4 we have

$$\left| \int_{\Gamma_\nu} \theta_\nu ds \right|^2 \geq K^{-1} d(S_\nu)^{n-1} \int_{\Gamma_\nu} |\theta_\nu|^2 ds,$$

or

$$\left| \int_{\mathcal{D}_0} e_\nu \hat{\theta}_\nu dz \right|^2 \geq K^{-1} d(S_\nu)^{n-1} \int_{\mathcal{D}_0} e_\nu |\hat{\theta}_\nu|^2 dz,$$

or, using (3.15),

$$(3.17) \quad \left| \int_{\mathcal{D}_0} e_\nu \hat{\theta}_\nu dz \right|^2 \geq C \int_{\mathcal{D}_0} |e_\nu \hat{\theta}_\nu|^2 dz.$$

Let  $\theta_\nu$  be normalized so that  $\|e_\nu \hat{\theta}_\nu\|_{L_2(\mathcal{V}_0)} = 1$ . Then, choosing  $\theta = e_\nu \hat{\theta}_\nu$  in Lemma 3.1, we conclude from (3.16) and (3.17) and from the lemma that

$$\int_{\mathcal{V}_0} |\hat{u}_\nu(z, 0)|^2 dz \leq C \int_{\mathcal{E}_0} |D^1 \hat{u}_\nu|^2 dx,$$

or, in the original coordinates,

$$d(S_\nu)^{-1} \int_{\Gamma_\nu} |u_\nu|^2 ds \leq C \int_{\mathcal{E}_\nu} |D^1 u_\nu|^2 dx.$$

We sum the last inequality over  $\nu$  and use (3.13), together with A1(ii), to finally obtain

$$\begin{aligned} \|u\|_{h, -1/2, \partial\Omega}^2 &\leq \sum_\nu d(S_\nu)^{-1} \int_{\Gamma_\nu} |u_\nu|^2 ds \\ &\leq C \sum_\nu \int_{\mathcal{E}_\nu} |D^1 u_\nu|^2 dx \leq CL \int_\Omega |D^1 u|^2 dx. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Proposition 3.2.* Let  $V^h$  denote the subspace of  $L_2(\partial\Omega)$  spanned by the functions  $v_{i|\partial\Omega}$ ,  $i = 1, \dots, m_h$ ,  $v_i$  as in A5. It is easily verified from A5 that  $\dim(V^h) = m_h$ , i.e.,  $V^h$  and  $M_2^h$  are of equal dimension. We also recall from A5(i), (iii), and (iv) that if  $v = \sum_1^{m_h} \beta_j v_j$ , then

$$\begin{aligned} \|v\|_{1,\Omega}^2 &\leq L \sum_1^{m_h} \beta_j^2 \|v_j\|_{1,\Omega}^2 \\ &\leq LK \sum_\nu d(S_\nu)^{-1} \sum_{j \in \Lambda_\nu} \beta_j^2 \|v_j\|_{L_2(S_\nu)}^2 \leq LK^2 \|v\|_{h, -1/2, \partial\Omega}^2. \end{aligned}$$

In view of this, the assertion of Proposition 3.2 follows if we can show that

$$(3.18) \quad \inf_{\psi \in M_2^h} \sup_{v \in V^h} \frac{\int_{\partial\Omega} \psi \varphi ds}{\|\psi\|_{h, 1/2, \partial\Omega} \|v\|_{h, -1/2, \partial\Omega}} \geq C > 0.$$

The remaining problem of proving (3.18) can be further modified as follows. First, define the linear mapping  $P_h: L_2(\partial\Omega) \rightarrow M_2^h$  as

$$(3.19) \quad P_h \psi \in M_2^h: b(P_h \psi, v) = b(\psi, v) \quad \forall v \in V^h,$$

where

$$(3.20) \quad b(\psi, \varphi) = \int_{\partial\Omega} \psi \varphi ds, \quad \psi, \varphi \in L_2(\partial\Omega).$$

Then we have

LEMMA 3.2. (3.18) holds, provided that  $P_h$  satisfies

$$(3.21) \quad \|P_h \varphi\|_{h, 1/2, \partial\Omega} \leq C \|\varphi\|_{h, 1/2, \partial\Omega}, \quad \varphi \in L_2(\partial\Omega).$$

*Proof.* Denote by  $H_1$  and  $H_2$  the Hilbert spaces obtained by supplying  $L_2(\partial\Omega)$  with the inner products

$$(\psi, \varphi)_{H_1} = \sum_\nu d(S_\nu) \int_{S_\nu} \psi \varphi ds, \quad \text{and} \quad (\psi, \varphi)_{H_2} = \sum_\nu d(S_\nu)^{-1} \int_{S_\nu} \psi \varphi ds,$$

respectively, where the sums are over  $S_\nu \in \tau^h(\partial\Omega)$ . Then  $\|\cdot\|_{H_1} = \|\cdot\|_{h, 1/2, \partial\Omega}$ , and



$\|\cdot\|_{H_2} = \|\cdot\|_{h, -\frac{1}{2}, \partial\Omega}$ , and  $b$  is a bounded bilinear form on  $H_1 \times H_2$ :

$$(3.22) \quad |b(\psi, \varphi)| \leq \|\psi\|_{H_1} \|\varphi\|_{H_2}, \quad \psi \in H_1, \varphi \in H_2.$$

We also easily contend that

$$(3.23) \quad \inf_{\psi \in H_1} \sup_{\varphi \in H_2} \frac{b(\psi, \varphi)}{\|\psi\|_{H_1} \|\varphi\|_{H_2}} = \inf_{\varphi \in H_2} \sup_{\psi \in H_1} \frac{b(\psi, \varphi)}{\|\psi\|_{H_1} \|\varphi\|_{H_2}} = 1.$$

Assume that (3.21) holds. Then, using (3.23), (3.19), and (3.21) we find that if  $v \in V^h$ , then

$$(3.24) \quad \begin{aligned} \|v\|_{H_2} &= \sup_{\psi \in H_1} \frac{b(\psi, v)}{\|\psi\|_{H_1}} = \sup_{\psi \in H_1} \frac{b(P_h \psi, v)}{\|\psi\|_{H_1}} \\ &\leq C \sup_{\psi \in H_1} \frac{b(P_h \psi, v)}{\|P_h \psi\|_{H_1}} \leq C \sup_{\psi \in M_2^h} \frac{b(\psi, v)}{\|\psi\|_{H_1}}, \quad v \in V^h. \end{aligned}$$

Proceeding as in [2, p. 113], we now define the linear mapping  $E_h: V^h \rightarrow M_2^h$  through the formula

$$b(\psi, v) = (\psi, E_h v)_{H_1}, \quad \psi \in M_2^h, v \in V^h.$$

Then, by (3.22) and (3.24),  $E_h$  is bounded and 1:1 [2, pp. 113–115]. Moreover, as  $M_2^h$  and  $V^h$  are of equal dimension,  $E_h$  is onto  $M_2^h$ . Then if  $E_h^*: M_2^h \rightarrow V^h$  is the adjoint mapping defined as

$$b(\psi, v) = (E_h^* \psi, v)_{H_2}, \quad \psi \in M_2^h, v \in V^h,$$

we have, using (3.24) and a well-known argument (cf. [2, pp. 112–115] or [4]), that

$$\begin{aligned} \inf_{\psi \in M_2^h} \sup_{v \in V^h} \frac{b(\psi, v)}{\|\psi\|_{H_1} \|v\|_{H_2}} &= \|(E_h^*)^{-1}\|^{-1} = \|E_h^{-1}\|^{-1} \\ &= \inf_{v \in V^h} \sup_{\psi \in M_2^h} \frac{b(\psi, v)}{\|\psi\|_{H_1} \|v\|_{H_2}} \geq C > 0. \end{aligned}$$

This proves the lemma.  $\square$

To prove (3.21), let  $\varphi \in L_2(\partial\Omega)$  be given and define, for  $S_v \in \tau^h(\partial\Omega)$ , the function  $\varphi_v$  as

$$\begin{aligned} \varphi_v(x) &= \varphi(x), \quad x \in S_v, \\ &= 0, \quad x \in \partial\Omega \setminus S_v. \end{aligned}$$

With the sets  $R_k(S_v)$ ,  $k = 0, 1, 2, \dots$ , defined as in (2.6), let us first prove that  $P_h \varphi_v(x)$  decays exponentially in the following sense:

$$(3.25) \quad \|P_h \varphi_v\|_{L_2(R_k(S_v))} \leq C e^{-\alpha k} \|\varphi_v\|_{L_2(S_v)}, \quad \alpha > 0,$$

where  $\alpha$  does not depend on  $h$ .

Note that, in the case of a quasiuniform partitioning, a result analogous to (3.25) was proved in [6] in a different context.

We start the proof of (3.25) by first introducing a real parameter  $q$ ,  $q > 1$ , and defining

$$(3.26) \quad \tilde{v}_j(x) = q^{m_j} v_j(x), \quad j = 1, \dots, m_h,$$

where  $m_j = 0$  if  $\text{supp}\{v_j\} \cap S_v \neq \emptyset$ , and otherwise  $m_j$  is chosen so that  $\text{supp}\{v_j\} \cap \Sigma_{m_j-1}(S_v) = \emptyset$  and  $\text{supp}\{v_j\} \cap \Sigma_{m_j}(S_v) \neq \emptyset$ , where the sets  $\Sigma_k(S_v)$  are as in (2.5).

Consider a given subset  $S_\mu \in \tau^h(\partial\Omega)$  such that  $S_\mu \subset R_k(S_v)$ ,  $k \geq 1$ . By the assumptions A3(i) and A5(ii), if  $v_j \neq 0$  on  $S_\mu$ , then  $\text{supp}\{v_j\} \cap \partial\Omega \subset \Sigma_L(S_\mu)$ .

In view of (2.5) and the above definition of  $m_j$ , this implies that  $k - L \leq m_j \leq k$ , and therefore, by (3.26),

$$(3.27) \quad \tilde{v}_j(x) - q^k v_j(x) = q^k \Delta_j v_j(x), \quad |\Delta_j| \leq 1 - q^{-L}.$$

Now let  $P_h \varphi_v$  be expanded as  $P_h \varphi_v = \sum_1^{m_h} \beta_j \psi_j$ , and let  $\tilde{v} = \sum_1^{m_h} \beta_j \tilde{v}_j$ . Then if  $\Lambda_\mu$  is as in A5(ii), we get, using A3(ii), A5(iii), A5(iv) and (3.27),

$$\begin{aligned} (3.28) \quad & q^{-k} \int_{S_\mu} (P_h \varphi_v) \tilde{v} \, ds \\ &= \int_{S_\mu} (P_h \varphi_v) \left( \sum_{j \in \Lambda_\mu} \beta_j v_j \right) ds + \int_{S_\mu} (P_h \varphi_v) \left( \sum_{j \in \Lambda_\mu} \Delta_j \beta_j v_j \right) ds \\ &\geq [K^{-1} - \tfrac{1}{2}(1 - q^{-L})] \|P_h \varphi_v\|_{L_2(S_\mu)}^2 \\ &\quad - \frac{1}{2(1 - q^{-L})} \int_{S_\mu} \left( \sum_{j \in \Lambda_\mu} \Delta_j \beta_j v_j \right)^2 ds \\ &\geq [K^{-1} - \tfrac{1}{2}(1 - q^{-L})] \|P_h \varphi_v\|_{L_2(S_\mu)}^2 \\ &\quad - \tfrac{1}{2} K (1 - q^{-L}) \sum_{j \in \Lambda_\mu} \beta_j^2 \int_{S_\mu} v_j^2 \, ds \\ &\geq [K^{-1} - \tfrac{1}{2}(K^3 + 1)(1 - q^{-L})] \|P_h \varphi_v\|_{L_2(S_\mu)}^2. \end{aligned}$$

Taking  $q$  sufficiently close to (but still larger than) unity, we can have

$$(3.29) \quad K^{-1} - \tfrac{1}{2}(K^3 + 1)(1 - q^{-L}) > \tfrac{1}{2} K^{-1}.$$

Upon summing over  $\mu$ , (3.28) and (3.29) then imply

$$\begin{aligned} (3.30) \quad & \int_{\partial\Omega} (P_h \varphi_v) \tilde{v} \, ds = \sum_k \int_{R_k(S_v)} (P_h \varphi_v) \tilde{v} \, ds \\ & \geq \tfrac{1}{2} K^{-1} \sum_k q^k \int_{R_k(S_v)} (P_h \varphi_v)^2 \, ds. \end{aligned}$$

On the other hand, we find from the definition of  $\tilde{v}$  and from A3(ii), A5(iii), and A5(iv) that

$$\int_{S_v} \varphi_v \tilde{v} \, ds \leq K^{3/2} \|\varphi_v\|_{L_2(S_v)} \|P_h \varphi_v\|_{L_2(S_v)}.$$

Recalling that  $\tilde{v} \in V^h$ , we then have, using (3.19) and (3.20),

$$\begin{aligned}
 \int_{\partial\Omega} (P_h \varphi_v) \tilde{v} \, ds &= \int_{\partial\Omega} \varphi_v \tilde{v} \, ds = \int_{S_v} \varphi_v \tilde{v} \, ds \\
 (3.31) \quad &\leq K^{3/2} \|\varphi_v\|_{L_2(S_v)} \|P_h \varphi_v\|_{L_2(S_v)} \\
 &\leq K^{3/2} \|\varphi_v\|_{L_2(S_v)} \left\{ \sum_k q^k \|P_h \varphi_v\|_{L_2(R_k(S_v))}^2 \right\}^{1/2}.
 \end{aligned}$$

Since  $q > 1$ , the estimate (3.25) follows from (3.30) and (3.31).

Combining (3.25) with the assumption A2, we get for all  $S_v, S_\mu \in \tau^h(\partial\Omega)$  the estimate

$$\begin{aligned}
 d(S_\mu) \int_{S_\mu} (P_h \varphi_v)^2 \, ds &\leq d(S_\mu) \int_{R_k(S_v)} (P_h \varphi_v)^2 \, ds \\
 (3.32) \quad &\leq C k^\lambda e^{-\alpha k} d(S_v) \int_{S_v} \varphi_v^2 \, ds \leq C_1 e^{-\alpha_1 k} d(S_v) \int_{S_v} \varphi_v^2 \, ds,
 \end{aligned}$$

where  $k$  is such that  $S_\mu \subset R_k(S_v)$ , and  $0 < \alpha_1 < \alpha$ .

Finally, to estimate  $P_h \varphi$ , write  $P_h \varphi = \sum_v P_h \varphi_v$ , and apply the inequality

$$\left( \sum_1^\infty \alpha_k \right)^2 \leq C \sum_1^\infty e^{\beta k} \alpha_k^2, \quad \beta > 0, C = C(\beta), \alpha_k \in R^1,$$

together with (3.32) to first get

$$\begin{aligned}
 d(S_\mu) \int_{S_\mu} (P_h \varphi)^2 \, ds &= d(S_\mu) \int_{S_\mu} \left[ \sum_k \left( \sum_{v \in \Lambda_{k,\mu}} P_h \varphi_v \right) \right]^2 \, ds \\
 (3.33) \quad &\leq C d(S_\mu) \sum_k e^{\beta k} \int_{S_\mu} \left( \sum_{v \in \Lambda_{k,\mu}} P_h \varphi_v \right)^2 \, ds \\
 &\leq C_1 d(S_\mu) \sum_k N_{k,\mu} e^{\beta k} \sum_{v \in \Lambda_{k,\mu}} \int_{S_\mu} (P_h \varphi_v)^2 \, ds \\
 &\leq C_2 \sum_k e^{(\beta - \alpha_1)k} N_{k,\mu} \sum_{v \in \Lambda_{k,\mu}} d(S_v) \int_{S_v} \varphi_v^2 \, ds,
 \end{aligned}$$

where  $\Lambda_{k,\mu} = \{v; S_v \subset R_k(S_\mu)\}$ , and  $N_{k,\mu}$  denotes the number of sets  $S_v \in \tau^h(\partial\Omega)$  which are contained in  $R_k(S_\mu)$ .

$N_{k,\mu}$  can be estimated as follows. First, if  $C_v$  is the sphere associated to  $S_v$  as in A1(ii), then, by A1(ii) and A2, the set  $\bigcup_{v \in \Lambda_{k,\mu}} C_v$  is contained in a sphere of radius

$$(3.34) \quad \rho \leq K d(S_\mu) \sum_{j=0}^k j^\lambda + 4K(K d(S_\mu) k^\lambda) \leq C d(S_\mu) k^{\lambda+1}.$$

Second, by A1(ii) and A2,

$$(3.35) \quad d(C_v) \geq 8K[K^{-1} k^{-\lambda} d(S_\mu)], \quad v \in \Lambda_{k,\mu}.$$

Therefore, using (3.34), (3.35), and A1(ii), we have the estimate

$$L[Cd(S_\mu)k^{\lambda+1}]^n \geq N_{k,\mu}[4k^{-\lambda}d(S_\mu)]^n,$$

so that

$$(3.36) \quad N_{k,\mu} < Ck^{n(2\lambda+1)}.$$

Upon taking  $0 < \beta < \alpha_1$  in (3.33) and using (3.36), we get

$$d(S_\mu) \int_{S_\mu} (P_h \varphi)^2 ds \leq C \sum_k e^{-\alpha_2 k} \sum_{v \in \Lambda_{k,\mu}} d(S_v) \int_{S_v} \varphi^2 ds,$$

with  $\alpha_2 > 0$ . Summing this over  $\mu$  and again using (3.36), we finally have

$$\begin{aligned} \|P_h \varphi\|_{h, \frac{1}{2}, \partial\Omega}^2 &\leq C \sum_v \sum_k \sum_{\mu \in \Lambda_{k,v}} e^{-\alpha_2 k} d(S_v) \int_{S_v} \varphi^2 ds \\ &= C \sum_v \left( \sum_k e^{-\alpha_2 k} N_{k,v} \right) d(S_v) \int_{S_v} \varphi^2 ds \\ &\leq C_1 \sum_v \left( \sum_{k=1}^{\infty} e^{-\alpha_2 k} k^{n(2\lambda+1)} \right) d(S_v) \int_{S_v} \varphi^2 ds \\ &\leq C_2 \|\varphi\|_{h, \frac{1}{2}, \partial\Omega}^2. \end{aligned}$$

This proves (3.21), and the proof of Proposition 3.2 is complete.  $\square$

We point out that the regularity properties of  $\partial\Omega$  did not play any explicit role in the proof of Proposition 3.2. Also, Proposition 3.1 is true under weaker regularity assumptions on  $\Omega$ ; for example, it would suffice to assume that  $\Omega$  is a Lipschitzian domain with piecewise smooth boundary. Theorem 3.1 is then true also for such more general domains.

**4. Application.** Let  $\{\tau^h(\partial\Omega)\}_{0 < h < 1}$  be a family of partitionings of  $\partial\Omega$  satisfying assumptions A1 and A2. Let  $h$  be fixed sufficiently small and assume that for each  $S_v \in \tau^h(\partial\Omega)$  the systems of local coordinates  $\{x_1^{(v)}, \dots, x_n^{(v)}\}$  are defined so that Eqs. (2.1) through (2.4) are satisfied.

Let  $k$  be a fixed integer,  $k \geq 1$ . We define  $M_2^h$  as the maximal subspace of  $L_2(\partial\Omega)$  such that if  $\psi \in M_2^h$ ,  $S_v \in \tau^h(\partial\Omega)$ , and  $\psi_v$  is defined as in (2.4), then  $\psi_v$  is a polynomial of degree  $k-1$  in the variables  $x_1^{(v)}, \dots, x_{n-1}^{(v)}$ .

To define the interior subspaces  $M_1^h$ , consider a family of partitionings  $\pi^h$  of  $R^n$  into  $n$ -simplicial subdomains  $t_v$ . We assume the partitionings to be arranged in the usual way so that any face of any  $t \in \pi^h$  is also a face of another  $t_v \in \pi^h$ . The following additional assumptions will be made:

- B1. (i)  $t \in \pi^h \Rightarrow d(t) \leq h$ .
- (ii) For each  $t \in \pi^h$ , either  $t \cap \Omega = \emptyset$  or  $t \cap \Omega$  contains a sphere of radius  $\rho \geq K^{-1}d(t)$ .
- (iii) If  $t \in \pi^h$ ,  $S \in \tau^h(\partial\Omega)$ , and  $t \cap S \neq \emptyset$ , then  $K^{-1}d(S) \leq d(t) \leq Kd(S)$ .

We note that if  $t \in \Omega$ , then (ii) reduces to the ordinary regularity assumption [5].

We let  $\{\tau^h(\Omega)\}_{0 < h < 1}$  be a family of partitionings of  $\Omega$ , defined as

$$\tau^h(\Omega) = \{T_\nu; T_\nu = t_\mu \cap \Omega, t_\mu \in \pi^h, t_\mu \cap \Omega \neq \emptyset\}.$$

We will further denote by  $\Omega_h$  the polyhedron

$$\Omega_h = \bigcup_{t_\nu \in \pi^h; t_\nu \cap \Omega \neq \emptyset} \overline{t_\nu}.$$

For each  $h$ , let  $M_1^h$  be defined as the maximal subspace of  $H^1(\Omega)$  such that, for each  $T \in \tau^h(\Omega)$ ,  $M_1^h|_T$  equals the space of polynomials of order  $k$  on  $T$ , where  $k$  has the same value as in the definition of  $M_2^h$ .

In order to be able to verify the conditions A3 through A5 for the above subspaces, a further constraint on the partitionings will be required. In particular, A5 is not generally true unless the partitioning  $\tau^h(\partial\Omega)$  is locally by a certain factor coarser than the partitioning induced by  $\tau^h(\Omega)$  on  $\partial\Omega$ . A sufficient condition can be found as follows. Let  $\{y_1, \dots, y_{p_h}\}$  be the set of vertices of the simplices  $t_\nu \in \pi^h$  which are located on  $\partial\Omega_h$ . For each  $i$ ,  $1 \leq i \leq p_h$ , let

$$(4.1) \quad E_i^h = \bigcup_{\nu \in \Lambda_i} \overline{t_\nu},$$

where  $\Lambda_i = \{\nu; t_\nu \in \pi^h, y_i \in \overline{t_\nu}\}$ . Then it suffices to assume

B2. Each  $S_\nu \in \tau^h(\partial\Omega)$  contains at least one of the sets  $\partial\Omega \cap E_i^h$ ,  $1 \leq i \leq p_h$ .

To verify the assumptions of Section 2, we need first a basis for  $M_2^h$ . For  $S_\nu \in \tau^h(\partial\Omega)$ , let  $U_\nu$  be as in (2.1), and let  $C_0 > 0$  be a fixed constant. Then, by a scaling argument, there exists for each  $\nu$  a basis  $\{\varphi_1^{(\nu)}, \dots, \varphi_m^{(\nu)}\}$  of the space of polynomials of degree  $\leq k-1$  in  $R^{n-1}$  such that if  $\mathcal{D}_i \subset R^{n-1}$ ,  $i = 1, 2, 3$ , are any spheres satisfying  $\mathcal{D}_i \cap U_\nu \neq \emptyset$  and  $C_0 d(S_\nu) \geq d(\mathcal{D}_i) \geq C_0^{-1} d(S_\nu)$ , then

$$(4.2) \quad \begin{aligned} C^{-1} \sum_{i=1}^m \beta_i^2 \|\varphi_i^{(\nu)}\|_{L_2(\mathcal{D}_1)}^2 &\leq \int_{\mathcal{D}_2} \left( \sum_{i=1}^m \beta_i \varphi_i^{(\nu)} \right)^2 ds \\ &\leq C \sum_{i=1}^m \beta_i^2 \|\varphi_i^{(\nu)}\|_{L_2(\mathcal{D}_3)}^2, \quad \beta_i \in R^1, \end{aligned}$$

where  $C$  depends only on  $C_0$ ,  $k$ , and  $n$ . Assuming such bases to be defined, we take for the basis of  $M_2^h$  the set  $\bigcup_\nu \{\psi_1^{(\nu)}, \dots, \psi_m^{(\nu)}\}$ , where

$$(4.3) \quad \begin{aligned} \psi_i^{(\nu)}(x) &= \varphi_i^{(\nu)}(x_1^{(\nu)}, \dots, x_{n-1}^{(\nu)}), & x \in S_\nu, \\ &= 0, & x \in \partial\Omega \setminus S_\nu. \end{aligned}$$

For the above basis, A3(i) is true with  $L = 0$ . To verify A3(ii), note that, by B2, B1(ii), B1(iii), (2.1), and (2.2), if  $h$  is sufficiently small, then each  $U_\nu$  in (2.1) contains a sphere of radius  $\rho > Cd(S_\nu)$ ,  $C > 0$ ,  $C$  independent of  $\nu$ . Denote this sphere by

$\mathcal{D}_{v,1}$ , and let  $\mathcal{D}_{v,2}$  be a sphere of radius  $\rho = d(S_v)$  such that  $\mathcal{D}_{v,2} \supset U_v$ . Then the validity of A3(ii), for some constant  $K$ , is obvious from the following chain of inequalities:

$$\begin{aligned} \sum_{i=1}^m \beta_i^2 \|\psi_i^{(v)}\|_{L_2(S_v)}^2 &\geq C_1 \sum_{i=1}^m \beta_i^2 \|\varphi_i^{(v)}\|_{L_2(\mathcal{D}_{v,1})}^2 \\ &\geq C_2 \sum_{i=1}^m \beta_i^2 \|\varphi_i^{(v)}\|_{L_2(\mathcal{D}_{v,2})}^2 \geq C_3 \left\| \sum_{i=1}^m \beta_i \varphi_i^{(v)} \right\|_{L_2(\mathcal{D}_{v,2})}^2 \\ &\geq C_4 \left\| \sum_{i=1}^m \beta_i \psi_i^{(v)} \right\|_{L_2(S_v)}^2 \geq C_5 \left\| \sum_{i=1}^m \beta_i \varphi_i^{(v)} \right\|_{L_2(\mathcal{D}_{v,1})}^2 \\ &\geq C_6 \sum_{i=1}^m \beta_i^2 \|\varphi_i^{(v)}\|_{L_2(\mathcal{D}_{v,1})}^2 \geq C_7 \sum_{i=1}^m \beta_i^2 \|\varphi_i^{(v)}\|_{L_2(\mathcal{D}_{v,2})}^2 \\ &\geq C_8 \sum_{i=1}^m \beta_i^2 \|\psi_i^{(v)}\|_{L_2(S_v)}^2, \quad \beta_i \in R^1. \end{aligned}$$

Here the constants  $C_j$  are positive and independent of  $v$ ; the second, third, sixth, and seventh inequalities follow from (4.2) and the remaining ones from (4.3), (2.1), (2.2), and from the inclusions  $\mathcal{D}_{v,1} \subset U_v \subset \mathcal{D}_{v,2}$ .

Assumption A4 is easily verified. Taking

$$\begin{aligned} \theta_v(x) &= 1, \quad x \in S_v, \\ &= 0, \quad x \in \partial\Omega \setminus S_v, \end{aligned}$$

it suffices to verify that, for all  $S_v \in \tau^h(\partial\Omega)$ ,

$$\int_{S_v} ds \geq Cd(S_v)^{n-1},$$

with  $C$  independent of  $v$ . This follows from B2, B1(ii), and B1(iii).

Finally, A5 should be verified. To this end, let  $\{\xi_1, \dots, \xi_{p_h}\}$  be the set of continuous functions in  $R^n$  such that for each  $i$ ,  $1 \leq i \leq p_h$ ,  $\xi_i$  is a polynomial of degree one in each  $t_v \in \pi^h$ ,  $\|\xi_i\|_{L_\infty(R^n)} = 1$ ,  $\xi_i(x) \geq 0$ , and  $\text{supp}\{\xi_i\} = E_i^h$ , where  $E_i^h$  is as in (4.1). As is well known, such functions exist. Then define

$$(4.4) \quad v_j^{(v)}(x) = \xi_{i_v}(x_1^{(v)}, \dots, x_n^{(v)}) \varphi_j^{(v)}(x_1^{(v)}, \dots, x_{n-1}^{(v)}), \quad 1 \leq j \leq m, x \in \Omega,$$

where  $\{x_1^{(v)}, \dots, x_n^{(v)}\}$  and  $\{\varphi_1^{(v)}, \dots, \varphi_m^{(v)}\}$  are as above, and  $i_v$  is chosen so that

$$(4.5) \quad S_v \supset E_{i_v}^h \cap \partial\Omega.$$

Functions  $v_j^{(v)}$  are continuous and piecewise polynomials of degree  $k$ , so  $v_j^{(v)} \in M_1^h$ . Now let us verify that the set  $\bigcup_v \{v_1^{(v)}, \dots, v_m^{(v)}\}$  is what we are looking for. First, A5(i) follows from the assumed regularity property B1(ii). Second, A5(ii) is true since  $\text{supp}\{v_j^{(v)}\} \cap \partial\Omega \subset S_v = \text{supp}\{\psi_j^{(v)}\}$ . Third, A5(iii) follows from a scaling argument, using B1(ii) and (4.2).

Finally, to verify A5(iv), note that if  $h$  is small enough, then (2.1), (2.2), B1(ii), and B1(iii) imply the existence of the spheres  $\mathcal{D}_v \subset R^{n-1}$ , with  $d(\mathcal{D}_v) \geq Cd(S_v)$ ,  $C$  independent of  $v$ , such that  $\mathcal{D}_v \subset U_v$  and so that  $\xi_{i_v}$  satisfies

$$(4.6) \quad \xi_{i_v}(x) \geq C > 0, x \in \Delta_v = \{(x_1^{(v)}, \dots, x_n^{(v)}) \in \partial\Omega; x_1^{(v)}, \dots, x_{n-1}^{(v)} \in \mathcal{D}_v\}.$$

Then, since  $\xi_{i_v}$  is everywhere nonnegative, we get, using (2.1), (2.2), (4.2), and (4.4) through (4.6),

$$\begin{aligned} \int_{S_v} \left( \sum_1^m \beta_j v_j^{(v)} \right) \left( \sum_1^m \beta_j \psi_j^{(v)} \right) ds &\geq \int_{\Delta_v} \xi_{i_v} \left( \sum_1^m \beta_j \psi_j^{(v)} \right)^2 ds \\ &\geq C \int_{\mathcal{D}_v} \left( \sum_1^m \beta_j \varphi_j^{(v)} \right)^2 dx_1^{(v)} \dots dx_{n-1}^{(v)} \\ &\geq C_1 \int_{U_v} \left( \sum_1^m \beta_j \varphi_j^{(v)} \right)^2 dx_1^{(v)} \dots dx_{n-1}^{(v)} \\ &\geq C_2 \int_{S_v} \left( \sum_1^m \beta_j \psi_j^{(v)} \right)^2 ds, \quad \beta_j \in R^1. \end{aligned}$$

The remaining inequalities of A5(iv) follow from a similar reasoning.

Having verified that the assumptions of Theorem 3.1 are satisfied, let us evaluate the error of the approximation in terms of the parameter  $h$  of A1(i) and B1(i). We need the following approximation results:

**PROPOSITION 4.1.** *Let  $M_1^h$  and  $M_2^h$  be defined as above, with  $k$  denoting the degree of polynomials in  $M_1^h$ , and let  $u \in H^{k+1}(\Omega)$  and  $\psi \in H^{k+1/2}(\partial\Omega)$ . Then, if  $k > (n-1)/2$ , the following estimates hold*

$$(4.7) \quad \min_{v \in M_1^h} \{ \|u - v\|_{1,\Omega} + \|u - v\|_{h,-1/2,\partial\Omega} \} \leq Ch^k \|u\|_{H^{k+1}(\Omega)},$$

and

$$(4.8) \quad \min_{\varphi \in M_2^h} \{ \|\psi - \varphi\|_{h,1/2,\partial\Omega} \} \leq Ch^k \|\psi\|_{H^{k+1/2}(\partial\Omega)}.$$

*Proof.* Denote by  $N^h$  the finite element space associated to the partitioning of  $\Omega_h$  such that  $M_1^h = N^h|_{\Omega}$ . Further, let  $u^e$  denote the extension of  $u$  to  $\Omega_h$  such that

$$(4.9) \quad \|u^e\|_{H^{k+1}(\Omega_h)} \leq C \|u\|_{H^{k+1}(\Omega)}.$$

Such an extension exists since  $\partial\Omega$  is smooth.

Let  $I_h u^e \in N^h$  denote the  $k$ th-order Lagrange interpolant of  $u^e$  on  $\Omega_h$ . If  $k > (n-1)/2$ ,  $I_h$  is a continuous linear operator on  $H^{k+1}(\Omega_h)$ , and, by the classical results of approximation theory (cf. [5, p. 121]),

$$(4.10) \quad \|u^e - I_h u^e\|_{H^1(\Omega_h)} \leq Ch^k |u^e|_{k+1,\Omega_h}.$$

To estimate  $\|u^e - I_h u^e\|_{h,-1/2,\partial\Omega}$ , let  $t_v \in \pi^h$  be given such that  $t_v \cap \partial\Omega \neq \emptyset$ , and let  $J_v$  denote the scaling mapping

$$J_v: x \longrightarrow \hat{x} = \frac{x}{d(t_v)}.$$

Further, let  $J_v(t_v) = \hat{t}_v$ ,  $J_v(t_v \cap \partial\Omega) = \hat{\Gamma}_v$ ,  $\hat{u}^e(\hat{x}) = u^e(x)$ , and  $\hat{I}_h \hat{u}^e(\hat{x}) = I_h u^e(x)$ ,  $x \in t_v$ .

By the assumption B1(ii), if  $h$  is small enough, the estimate

$$\int_{\hat{\Gamma}_v} (\hat{u}^e - \hat{I}_h \hat{u}^e)^2 d\hat{s} \leq C \|\hat{u}^e - \hat{I}_h \hat{u}^e\|_{H^1(\hat{\Gamma}_v)}^2$$

holds uniformly for all  $\hat{t}_v = J_v(t_v)$ ,  $t_v \in \pi^h$ ,  $t_v \cap \partial\Omega \neq \emptyset$ . By the approximation properties of  $\hat{I}_h$  we then have [5]

$$\int_{\hat{\Gamma}_v} (\hat{u}^e - \hat{I}_h \hat{u}^e)^2 d\hat{s} \leq C |\hat{u}^e|_{k+1, \hat{\Gamma}_v}^2.$$

Upon scaling this back to the original size we get

$$(4.11) \quad d(t_v)^{-1} \int_{t_v \cap \partial\Omega} (u^e - I_h u^e)^2 ds \leq C d(t_v)^{2k} |u^e|_{k+1, t_v}^2.$$

Summing over  $v$  and using B1(i) and B1(iii), (4.11) gives

$$(4.12) \quad \|u^e - I_h u^e\|_{h, -1/2, \partial\Omega} \leq C h^k \|u^e\|_{H^{k+1}(\Omega_h)}.$$

In view of (4.9), (4.10), and (4.12), the estimate (4.7) follows by choosing  $v = I_h u^e|_{\Omega}$ .

To prove (4.8), let  $\psi \in H^k(\partial\Omega)$  and consider a given partitioning  $\tau^h(\partial\Omega)$ . For each  $S_v \in \tau^h(\partial\Omega)$ , let  $\{x_1^{(v)}, \dots, x_n^{(v)}\}$  be as in (4.3), and assume that  $h$  is small enough so that (2.1) through (2.4) hold. For each  $S_v \in \tau^h(\partial\Omega)$ , let  $\mathcal{D}_v \subset \mathbb{R}^{n-1}$  be a sphere such that  $\mathcal{D}_v \supset U_v$ , where  $U_v$  refers to (2.1), and

$$(4.13) \quad d(\mathcal{D}_v) \leq C d(S_v),$$

for some constant  $C$ . Further, let

$$\Gamma_v = \{(x_1^{(v)}, \dots, x_n^{(v)}) \in \partial\Omega; (x_1^{(v)}, \dots, x_{n-1}^{(v)}) \in \mathcal{D}_v\},$$

and let  $\psi_v$  be defined on  $\mathcal{D}_v$  as

$$\psi_v(x_1^{(v)}, \dots, x_{n-1}^{(v)}) = \psi(x_1^{(v)}, \dots, x_n^{(v)}), \quad (x_1^{(v)}, \dots, x_n^{(v)}) \in \Gamma_v.$$

In view of (4.13), there exists for each  $v$  a polynomial  $\varphi_v$  on  $\mathcal{D}_v$  of degree  $k-1$  such that

$$(4.14) \quad \|\psi_v - \varphi_v\|_{L_2(\mathcal{D}_v)} \leq C d(S_v)^k |\psi_v|_{k, \mathcal{D}_v},$$

where  $C$  is independent of  $v$  (cf. [5, p. 115]). On the other hand, by the assumption A1(ii), the constant  $C$  in (4.13) can be chosen so that, for any given  $\Gamma \in \{\Gamma_v\}$ ,  $\Gamma$  intersects  $\Gamma_v$  for at most  $L$  different  $\Gamma_v \in \{\Gamma_v\}$ . Therefore, if  $h$  is small enough, (4.14), A1(i), and (2.3) imply

$$\sum_v d(S_v) \|\psi_v - \varphi_v\|_{L_2(\mathcal{D}_v)}^2 \leq C h^{2k+1} \sum_v |\psi_v|_{k, \mathcal{D}_v}^2 \leq C_1 h^{2k+1} \|\psi\|_{H^k(\partial\Omega)}^2.$$

In view of this we have

$$(4.15) \quad \min_{\varphi \in M_2^h} \|\psi - \varphi\|_{h, 1/2, \partial\Omega} \leq C h^{k+1/2} \|\psi\|_{H^k(\partial\Omega)}.$$



The estimate (4.8) now follows from interpolation between (4.15) and the trivial estimate

$$\min_{\varphi \in M_2^h} \|\psi - \varphi\|_{h, \frac{1}{2}, \partial\Omega} \leq h^{\frac{1}{2}} \|\psi\|_{L_2(\partial\Omega)}.$$

This completes the proof.  $\square$

Using Proposition 4.1 one can now easily estimate the convergence rate of the Lagrange multiplier method.

**THEOREM 4.1.** *Let  $u$  be the solution of problem (1.1) and let  $(u_h, \psi_h)$  be the solution of (1.3), with  $M_1^h$  and  $M_2^h$  defined as above, and so that  $M_1^h$  contains the polynomials of degree  $k$ . Then if  $f \in H^{k-1}(\Omega)$  and  $g \in H^{k+\frac{1}{2}}(\partial\Omega)$  in (1.1), and if  $k > (n-1)/2$ , the following error bounds hold:*

$$\left\| \left( u, \frac{\partial u}{\partial n} \right) - (u_h, \psi_h) \right\|_{X_{\tau h}} \leq Ch^k (\|f\|_{H^{k-1}(\Omega)} + \|g\|_{H^{k+\frac{1}{2}}(\partial\Omega)}),$$

and

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^{k+1} (\|f\|_{H^{k-1}(\Omega)} + \|g\|_{H^{k+\frac{1}{2}}(\partial\Omega)}).$$

*Proof.* The first inequality follows from Theorem 3.1, Proposition (4.1), and the estimate (cf. [7])

$$\|u\|_{H^{k+1}(\Omega)} + \left\| \frac{\partial u}{\partial n} \right\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \leq C(\|f\|_{H^{k-1}(\Omega)} + \|g\|_{H^{k+\frac{1}{2}}(\partial\Omega)}), \quad k \geq 1.$$

The second estimate follows from the ordinary duality argument, see [1].  $\square$

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