LOCAL TOPOLOGICAL INVARIANTS, II

BY

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INTRODUCTION

The present paper arose out of an earlier draft, submitted in 1954 under the title *Homotopy and singular homology in "local" topology*, whose purpose was to consider relationships between the local groups occurring in the Vietoris, singular and homotopy theories. The referee suggested that the local "C" and "D" groups occurring in the theory and defined in [5] and [6] (hereafter referred to as LTI and CTM respectively), were not "functorial" in the sense that the isomorphisms connected with them were merely "abstract," not induced by maps of one space into another and so not natural. He outlined a new approach using inverse and direct systems of groups, and in many cases the limits of these were isomorphic to the corresponding "C" and "D" groups; but in some cases the limits gave the "wrong" results. To overcome this, he suggested the idea of a *stable* system, where to postulate stability is to postulate something rather stronger than, but often equivalent to, *existence* of the "C" and "D" groups. (In locally Euclidean spaces and the generalized manifolds of Wilder [15], stability occurs at each point in each dimension.) We have therefore re-cast the whole of our previous theory in terms of these new concepts, thereby obtaining a more harmonious theory than before; and many of the results of the earlier draft together with analogues of results in LTI and CTM are here obtained. The plan of the paper is as follows. There are four sections: in §I we prove all the basic results we later need on inverse and direct systems of groups, concerning their "stability" under mappings of various sorts. §II is devoted to a discussion of certain relationships berween Singular and Vietoris homology. In §III, we derive certain results concerning homotopy, which are applied in §IV with the earlier ones to prove theorems concerning the local groups there. Corollaries of theorems in II and III give useful global results of the form:—if $X \subseteq Y$, then under certain conditions and with different values of the functor G, the image of the injection $G(X) \to G(Y)$ is finitely generated (see 2.33, 3.14, 3.15). §IV is concerned essentially with three matters: first the proof that the Wilder manifolds, as mentioned above, have the stability property; second, implications between the various types of local connectivity, with some pathology; and third, proofs that for Singular and Vietoris homology, all the local groups we define (using stability) give the same end-product, i.e. the same class of manifolds,—with a similar but more restricted result for homotopy. Moreover, a "local" theorem of Hurewicz type is proved in 4.35.

Received by the editors March 4, 1957.

I. Abstract theorems on inverse and direct limits

In this section we shall prove several theorems on Inverse and Direct limits of groups. When given topological interpretations, these theorems will become the topological theorems of the later sections. We shall normally use the notation and terminology of [4, Chapter VIII] (in future we denote this reference by E-S).

1.1. Let M be a set directed by $\leqq$ and let $(P, p)_M$ or simply (P, p), be an inverse system of groups P_α and homomorphisms $p_\alpha^\beta: P_\beta \to P_\alpha$, for each α, β in M such that $\alpha \leqq \beta$. By definition

(i) $$p_\alpha^\alpha = \text{identity on } P_\alpha, \qquad \text{all } \alpha \in M,$$

(ii) $$p_\alpha^\beta \circ p_\beta^\gamma = p_\alpha^\gamma, \qquad \text{if } \alpha \leqq \beta \leqq \gamma \text{ in } M.$$

Hence, if in M, α, β, γ, δ satisfy

(iii) $$\alpha \leqq \beta \leqq \gamma, \qquad \alpha \leqq \delta \leqq \gamma$$

we have a diagram

$$\begin{array}{ccc} P_\gamma & \rightarrow & P_\delta \\ \downarrow & \searrow & \downarrow \\ P_\beta & \rightarrow & P_\alpha \end{array}$$

and using (ii) twice we have

$$p_\alpha^\gamma = p_\alpha^\delta p_\delta^\gamma = p_\alpha^\beta p_\beta^\gamma,$$

i.e. the diagram is commutative.

In the interpretations, the p's will usually be injections of homology or homotopy groups, and for our purposes it is the images of these which are important. We therefore now consider the groups

(iv) $$P_{\beta\gamma} = p_\beta^\gamma P_\gamma \subseteq P_\beta.$$

From the above diagram we obtain

$$\begin{aligned} P_{\alpha\gamma} = p_\alpha^\gamma P_\gamma &= p_\alpha^\beta(p_\beta^\gamma P_\gamma) \qquad \text{(by (ii))} \\ &\subseteq p_\alpha^\beta P_\beta, \end{aligned}$$

and so

(v) $$P_{\alpha\gamma} \subseteq P_{\alpha\beta}.$$

Moreover, write temporarily

$$q_\alpha^{\beta\gamma} = p_\alpha^\beta \mid P_{\beta\gamma},$$

so that

$$\begin{aligned} \overset{\beta\gamma}{q_\alpha} P_{\beta\gamma} &= \overset{\beta}{p_\alpha}\overset{\gamma}{p_\beta} P_\gamma && \text{(by (ii))} \\ &= \overset{\gamma}{p_\alpha} P_\gamma \\ &= P_{\alpha\gamma}. \end{aligned}$$

Thus

(vi) *the homomorphism* $\overset{\beta\gamma}{q_\alpha}: P_{\beta\gamma} \to P_{\alpha\gamma}$ *is onto.*

Further, by (v) and (vi)

$$\overset{\beta\gamma}{q_\alpha} P_{\beta\gamma} = P_{\alpha\gamma} \subseteq P_{\alpha\delta},$$

and so we can write

(vii) $$\overset{\beta\gamma}{q_\alpha}: P_{\beta\gamma} \to P_{\alpha\delta}.$$

Hence, if $\lambda, \mu \in M$ satisfy $\lambda \leqq \delta$, $\mu \leqq \alpha$, $\mu \leqq \lambda$, then

$$\overset{\alpha\delta}{q_\mu}: P_{\alpha\delta} \to P_{\mu\lambda}$$

and

$$\overset{\beta\gamma}{q_\mu}: P_{\beta\gamma} \to P_{\mu\lambda},$$

and one can verify that

(viii) $$\overset{\beta\gamma}{q_\mu} = \overset{\alpha\delta}{q_\mu} \circ \overset{\beta\gamma}{q_\alpha}.$$

For typographical reasons, we shall denote the Inverse limit of the system (P, p) by

$$\operatorname{Ilim}(P, p).$$

Now, this limit depends not so much on the actual groups P_α as on the images of the form $P_{\alpha\beta}$. This causes us to consider the set $\overline{M}$ of pairs (β, γ), with $\beta \leqq \gamma$ in M, and we make $\overline{M}$ into a quasi-ordered set by writing $(\alpha, \delta) \leqq (\beta, \gamma)$ whenever (iii) holds, so that then we can form the above diagram. Since M is directed, it can be verified that $\overline{M}$ is directed also. Next, for each $\mu \in \overline{M}$, of the form $\mu = (\beta, \gamma)$, define

(ix) $$\overline{P}_\mu = P_{\beta\gamma}$$

and for each pair $\lambda \leqq \mu$ in $\overline{M}$, with $\lambda = (\alpha, \delta)$, take

(x) $$\overset{\mu}{\bar{p}_\lambda}: \overline{P}_\mu \to \overline{P}_\lambda$$

to be the homomorphism in (vii), i.e.

$$\overset{\mu}{\bar{p}_\lambda} = \overset{\beta\gamma}{q_\alpha}: P_{\beta\gamma} \to P_{\alpha\delta}.$$

Condition (i) holds for $\bar{p}$ since it holds for p, of which $\bar{p}$ is a restriction. Condition (ii) holds for $\bar{p}$ by (viii) above; and $\bar{p}_\lambda^\mu$ is always defined if $\lambda \leqq \mu$ in $\overline{M}$. Hence $(\overline{P}, \bar{p})$ is an inverse system over $\overline{M}$. Moreover, we can identify M with the diagonal of $\overline{M}$ by means of the correspondence $\alpha \to (\alpha, \alpha)$. Since M is directed, it follows that M is cofinal in $\overline{M}$. But, by (i), $P_\alpha = P_{\alpha\alpha}$, and therefore there is a natural isomorphism

$$\text{(xi)} \qquad \operatorname{Ilim}(\overline{P}, \bar{p})_{\overline{M}} \approx \operatorname{Ilim}(P, p)_M.$$

The previous discussion shows that the homomorphisms $\bar{p}$ in (ix) are either inclusions or onto; hence the system $(\overline{P}, \bar{p})$ is "tidier" than (P, p).

$$(\overline{\overline{P}}, \bar{\bar{p}}) = (\overline{P}, \bar{p}).$$

Thus repetition of the construction of $(\overline{P}, \bar{p})$ from (P, p) yields nothing new.

1.2. **Stability.** Recall that a subset A of a directed set $(B, \leqq)$ is *cofinal* in B, written A cof B, whenever given $\beta \in B$ there exists $\alpha \in A$ with $\beta \leqq \alpha$. Define the *saturation* A^s of A to be the set of all $\beta \in B$, such that there exists $\alpha \in A$ and $\alpha \leqq \beta$. Clearly $A \subseteq A^s$ and if A cof B then A^s cof B also.

With $(\overline{P}, \bar{p})$ on $\overline{M}$ as in 1.1, we shall say that "$(\overline{P}, \bar{p})$ is stable rel Λ" if and only if Λ is a cofinal subset of $\overline{M}$, and for all $\lambda, \mu \in \Lambda$ with $\lambda \leqq \mu$ then

$$\text{(i)} \qquad \bar{p}_\lambda^\mu \colon \overline{P}_\mu \approx \overline{P}_\lambda.$$

In this event, of course,

$$\text{(ii)} \qquad \operatorname{Ilim}(\overline{P}, \bar{p})_{\overline{M}} \approx \operatorname{Ilim}(\overline{P}, \bar{p})_\Lambda \approx \overline{P}_\lambda, \qquad \lambda \in \Lambda.$$

Clearly,

(iii) *if* $\Delta \subseteq \Lambda$ *and* Δ *cof* M, *then* $(\overline{P}, \bar{p})$ *is also stable rel* Δ.

1.21 Theorem. *If* $(\overline{P}, \bar{p})$ *is stable rel* Λ, *then it is stable rel* Λ^s.

To prove this result we need the

1.22. Lemma. *If* $\lambda \leqq \mu \leqq \nu$ *in* $\overline{M}$, *and* $\lambda, \nu \in \Lambda$, *then* $\bar{p}_\lambda^\mu$ *and* $\bar{p}_\mu^\nu$ *are isomorphisms.*

Proof. Let $\lambda = (\alpha, \beta)$, $\nu = (\gamma, \delta)$, $\mu = (\sigma, \tau)$. Then we have a commutative diagram

$$\begin{tikzcd}
(\alpha, \delta) \arrow[d, "c"'] & (\sigma, \delta) \arrow[l, "a"'] \arrow[d, "f"] & (\gamma, \delta) \arrow[l, "b"'] \arrow[dl, "g"] \arrow[ddll, "i"] \\
(\alpha, \tau) \arrow[d, "d"'] & (\sigma, \tau) \arrow[l, "e"] \arrow[dl, "h"] & \\
(\alpha, \beta) & &
\end{tikzcd}$$

in which $\bar{p}_\lambda^\mu = h$, $\bar{p}_\mu^\nu = g$, etc. Then $(dc)(ab) = i$, by 1.1(ii); but i is an isomorphism, by stability, (dc) and (ab) are a monomorphism and epimorphism by 1.1(v) and (vi), and so each is an isomorphism. Hence a, b, c, d are all(1) iso, since a, b are epi, and c, d are mono. Also by 1.1(ii) $hg = i$, whence g is mono, and h epi. Next, $eg = cab$, and so eg is epi whence the monomorphisms e, g are each iso. Finally, since $h = dc$, and d, c are iso, so is h. This completes the proof of the lemma.

The proof of Theorem 1.21 now proceeds as follows. We have to show that given $\lambda, \mu \in \Lambda^s$ with $\lambda \leqq \mu$, then $\bar{p}_\lambda^\mu$ is an isomorphism. By definition of Λ^s, there exists $\alpha \leqq \lambda$ in Λ, and since Λ cof $\overline{M}$, there exists γ with $\mu \leqq \gamma \in \Lambda$. Then

$$\bar{p}_\alpha^\lambda \bar{p}_\lambda^\mu \bar{p}_\mu^\gamma = \bar{p}_\alpha^\gamma$$

and $\bar{p}_\alpha$ is iso by stability on Λ, while $\bar{p}_\alpha^\lambda$ and $\bar{p}_\mu^\gamma$ are iso by Lemma 1.22. Hence $\bar{p}_\lambda^\mu$ is an isomorphism as required. Thus, the theorem is established.

If $\mu \in \overline{M}$, let Λ_μ denote the set of all $\lambda \in \Lambda$ with $\mu \leqq \lambda$. Then a Corollary of 1.22 is immediately

1.23. Lemma. *If $(\overline{P}, \bar{p})$ is stable rel Λ, and Δ cof $\overline{M}$, then for each $\lambda \in \Lambda$, $(\overline{P}, \bar{p})$ is stable rel Δ_λ.*

Thus, given any two cofinal subsets of $\overline{M}$, then if $(\overline{P}, \bar{p})$ is stable on one, it is stable on "almost the whole" of the other. Hence the stability is essentially independent of the cofinal subsets of $\overline{M}$, and from now on we can say merely that $(\overline{P}, \bar{p})$ is stable.

1.3. **Direct limits.** Let $\{P, p\}$ denote a direct system of groups P^α and homomorphisms $p_\alpha^\beta: P^\alpha \to P^\beta$ on the directed set $(M, \leqq)$. Thus, by definition, p_α^β is defined whenever $\alpha \leqq \beta$; and 1.1(i) and (ii) are replaced by

(i) $$p_\alpha^\alpha = \text{identity on } P^\alpha, \qquad \text{all } \alpha \in M;$$

(ii) $$p_\beta^\gamma p_\alpha^\beta = p_\alpha^\gamma, \qquad \text{if } \alpha \leqq \beta \leqq \gamma \text{ in } M.$$

By analogy with the treatment in 1.1, we define

$$P^{\alpha\beta} = p_\alpha^\beta P^\alpha \subseteq P^\beta, \qquad (\alpha \leqq \beta)$$
$$q_{\alpha\beta}^\gamma = p_\beta^\gamma \mid P^{\alpha\beta}, \qquad (\alpha \leqq \beta \leqq \gamma);$$

and then using (ii) it can be verified that we get, when $\alpha \leqq \beta \leqq \gamma$,

(iii) $q_{\alpha\beta}^\gamma: P^{\alpha\beta} \to P^{\alpha\gamma}$ is an epimorphism,

(iv) $P^{\alpha\gamma} \subseteq P^{\beta\gamma}$.

(1) For brevity, we define "θ is epi, mono, or iso" to mean that θ is respectively an epimorphism, a monomorphism, or an isomorphism.

Note that these are the "duals" of 1.1(v) and (vi), respectively, in the sense of MacLane [12].

With $\overline{M}$ as in 1.1, we now form a new direct system, $\{\overline{P}, \bar{p}\}$ on $\overline{M}$ by taking

$$\overline{P}^{\mu} = P^{\beta\gamma}, \qquad \mu = (\beta, \gamma),$$
$$\bar{p}_{\lambda}^{\mu} = q_{\alpha\beta}^{\gamma}\colon \overline{P}^{\lambda} \to \overline{P}^{\mu}$$

whenever $\lambda \leqq \mu$ in M, and $\lambda = (\alpha, \delta)$; and recalling that $(\alpha, \delta) \leqq (\beta, \gamma)$ means that (iii) of 1.1 holds. As before, we have, from (iii) and (iv) that repetition of the construction of $\{P, p\}$ from $\{P, p\}$ yields nothing new;

$$\{P, p\} = \{\overline{P}, \bar{p}\},$$

and we identify M with the diagonal of $\overline{M}$, and use (i) to write $P^{\alpha} = P^{\alpha\alpha}$. Hence there is a natural isomorphism of the direct limits

(v) $$\operatorname{Dlim}\{P, p\}_M \approx \operatorname{Dlim}\{\overline{P}, \bar{p}\}_{\overline{M}}.$$

By analogy with 1.2 we say that $\{\overline{P}, \bar{p}\}$ is "stable rel Λ" if and only if Λ is a cofinal subset of $\overline{M}$ such that whenever $\lambda, \mu \in \Lambda$ and $\lambda \leqq \mu$, then

(vi) $$\bar{p}_{\lambda}^{\mu}\colon \overline{P}_{\lambda} \approx \overline{P}_{\mu}.$$

We then have

1.31. THEOREM. *The statements of* 1.21, 1.22 *and* 1.23 *hold whenever* $(\overline{P}, \bar{p})$ *is replaced throughout by* $\{\overline{P}, \bar{p}\}$.

Proof. Using the same inequalities in M as in the original proofs, we obtain the same diagrams as before, except that the directions of all arrows are reversed while inclusions and epimorphisms are interchanged (by (iii) and (iv)). Hence, by the "duality" described in MacLane [12] the theorem follows.

1.4. **Mappings of systems.** In locally compact spaces, one often obtains commuting diagrams of groups and homomorphisms of the form

$$\begin{tikzcd} G_x \arrow[r] \arrow[d, "k_x"'] & G_y \arrow[d, "k_y"] \\ H_x \arrow[r] \arrow[ur] & H_y \end{tikzcd}$$

for each pair x, y with $x < y$ in some ordered set. The G's and H's may form either an inverse or a direct system over the set, and one wants to conclude that the k-homomorphism induces an isomorphism of $\lim G_x$ on $\lim H_x$. In this paper we shall need two particular theorems of this sort, and both are for inverse systems; but the omitted proof of 2.32 below requires both 1.42 and its analogue for direct systems.

1.41. Let then (P, p), (Q, q) be inverse systems over M with the property that, for each $\alpha \in M$, there is a homomorphism $\phi_\alpha \colon P_\alpha \to Q_\alpha$ satisfying

$$\text{(i)} \qquad \overset{\beta}{q_\alpha}\phi_\beta = \phi_\alpha \overset{\beta}{p_\alpha},$$

whenever $\alpha \leqq \beta$. We then say that there is a homomorphism $\phi\colon (P, p) \to (Q, q)$; and [4, p. 223] there results an induced homomorphism

$$\text{(ii)} \qquad \phi_\infty \colon \text{Ilim}\,(P, p) \to \text{Ilim}(Q, q)$$

defined for each $\{x_\alpha\} \in \text{Ilim}\,(P, p)$, by

$$\phi_\infty\{x_\alpha\} = \{\phi_\alpha x_\alpha\}.$$

Further, if $(\alpha, \beta) \in \overline{M}$, then $P_{\alpha\beta} \subseteq P_\alpha$ by definition, and

$$\begin{aligned} \phi_\alpha(P_{\alpha\beta}) &= \phi_\alpha \overset{\beta}{p_\alpha} P_\beta = \overset{\beta}{q_\alpha}\phi_\beta P_\beta && \text{(by 1.41)} \\ &\subseteq \overset{\beta}{q_\alpha} Q_\beta = Q_{\alpha\beta}. \end{aligned}$$

Thus, for each $\lambda \leqq \mu$ in $\overline{M}$, ϕ induces homomorphisms

$$\text{(iii)} \qquad \bar{\phi}_\lambda \colon \overline{P}_\lambda \to \overline{Q}_\lambda, \qquad \bar{\phi}_\mu \colon \overline{P}_\mu \to \overline{Q}_\mu$$

such that, using 1.41,

$$\text{(iv)} \qquad \overset{\mu}{\bar{q}_\lambda}\bar{\phi}_\mu = \bar{\phi}_\lambda \overset{\mu}{\bar{p}_\lambda}.$$

Let J denote the set of integers >0, directed by the natural ordering $\leqq$.

1.42. Theorem. *Let (P, p), (Q, q) be inverse systems over J, and let $\phi\colon (P, p) \to (Q, q)$ be a homomorphism. Suppose that for each $j \in J$, there is a homomorphism $\psi_j \colon Q_{j+1} \to P_j$ such that the diagram*

$$\begin{array}{ccc} P_j & \xleftarrow{p} & P_{j+1} \\ \phi_j \downarrow & \nwarrow & \downarrow \phi_{j+1} \\ Q_j & \xleftarrow[q]{} & Q_{j+1} \end{array}$$

commutes (where $p = p_j^{j+1}$, $q = q_j^{j+1}$). Then

$$\phi_\infty \colon \text{Ilim}\,(P, p) \approx \text{Ilim}\,(Q, q).$$

Proof. To prove ϕ_∞ has kernel zero, suppose $\phi_\infty\{x_j\} = 1$, for some $\{x_j\} \in \text{Ilim}\,(P, p)$. Then since $\phi_\infty\{x_j\} = \{\phi_j x_j\}$, we have $\phi_j x_j = 1_j$ (the unit of Q_j) for all j. Hence

$$\begin{aligned} \psi_j \phi_{j+1} x_{j+1} &= 1_j' \text{ (unit of } P_j) \\ &= \overset{j+1}{p_j}\, x_{j+1} \end{aligned}$$

by the commutativity of the diagram above. By definition of $\{x_j\}$, $p_j^{j+1} x_{j+1}$

$=x_j$. Hence for all j, $x_j=1_j'$, and therefore $\{x_j\}$ is the unit of Ilim (P, p), i.e. ϕ_∞ is mono as required.

To prove that ϕ_∞ is epi, let $y_j \in$ Ilim (Q, q). We define $x_1, x_2, \cdots, x_j, \cdots$, inductively as follows. Put $x_1=\psi_1 y_2$, so that

$$\begin{aligned} \phi_1 x_1 &= \phi_1\psi_1 y_2 \\ &= \overset{2}{q_1} y_2 \text{ by commutativity} \\ &= y_1 \text{ by definition of } y_j. \end{aligned}$$

Now suppose that $x_1, \cdots, x_{j-1}$ have been defined to satisfy

$$x_i \in P_i, \qquad \phi_i x_i = y_i, \qquad\qquad 1 \leqq i \leqq j-1$$

and

$$\overset{i+1}{p_i} x_{i+1} = x_i, \qquad\qquad 1 \leqq i < j-1.$$

Define x_j to be $\psi_j y_{j+1}$, so that $x_j \in P_j$ and $\phi_j x_j = \phi_j \psi_j y_{j+1} = q_j^{j+1} y_{j+1}$ by commutativity, $=y_j$ by definition of $\{y_j\}$. Hence, the inductive definition of x_i is justified, and

$$x_i \in \text{Ilim}\,(P, p) \quad \text{and} \quad \phi_\infty\{x_i\} = \{\phi_i x_i\} = \{y_i\}.$$

Thus ϕ_∞ is epi, and the proof is complete.

A STABILITY THEOREM. A "stable" form of 1.42 is the following result.

1.43. THEOREM. *Let $(M, \leqq)$ be directed by $<$, let (P, p), (Q, q) be inverse systems on M and let $\phi\colon (P, p)\rightarrow(Q, q)$ be a homomorphism. Suppose that for each $\alpha, \beta \in M$ with $\alpha<\beta$, there is a homomorphism*

$$\overset{\beta}{\psi_\alpha}\colon Q_\beta \rightarrow P_\alpha$$

so that the diagram

$$\begin{tikzcd} P_\alpha & P_\beta \arrow[l, "p"'] \arrow[d, "\phi_\beta"] \\ Q_\alpha \arrow[from=u, "\phi_\alpha"'] & Q_\beta \arrow[l, "q"] \arrow[ul, "\psi"'] \end{tikzcd}$$

commutes, i.e.

(i) $$\overset{\beta}{\psi_\alpha}\phi_\beta = \overset{\beta}{p_\alpha},$$

and

(ii) $$\phi_\alpha \psi_\alpha^\beta = q_\alpha^\beta.$$

Then, if A, B cof M, and if $(\overline{P}, \bar{p})$ is stable rel Δ, there is a subset Λ cofinal in the saturation Δ^s of Δ, such that (Q, q) is stable rel Λ; and conversely; and the inverse limits of the two systems are isomorphic.

Proof. Consider the diagram

$$\begin{tikzcd}
P_\alpha \arrow[d, "\tau"'] & P_\beta \arrow[l, "p"'] \arrow[d, "\phi"] & P_\gamma \arrow[l, "s"'] \arrow[d, "\sigma"] & P_\delta \arrow[l, "t"'] \arrow[d, "\rho"] \\
Q_\alpha & Q_\beta \arrow[l] \arrow[ul, "\psi"] & Q_\gamma \arrow[l, "q"] \arrow[ul, "\theta"] & Q_\delta, \arrow[l, "r"] \arrow[ul, "\chi"]
\end{tikzcd}$$

where $\alpha<\beta<\gamma<\delta$ in M; these latter exist unless M is empty (when the theorem has no content) since M is directed by $<$. Then

$$\begin{aligned}
\psi q Q_\gamma &= \psi\phi\theta Q_\gamma && \text{by (ii)}\\
&= p\theta Q_\gamma && \text{by (i)}\\
&\subseteq pP_\beta = P_{\alpha\beta}.
\end{aligned}$$

Thus by restriction, ψ induces a homomorphism $\psi'\colon Q_{\beta\gamma}\to P_{\alpha\beta}$.

Now suppose that $(\overline{P}, \bar{p})$ is stable rel Δ. We construct a $\Lambda\subseteq\overline{M}$ such that $(\overline{Q}, \bar{q})$ is stable rel Λ as follows. Choose $(\alpha, \alpha')\in\Delta$ and define Λ to be the set of $(\beta, \delta)\in\overline{M}$ for which there exists γ with

$$\alpha \leqq \alpha' < \beta < \gamma < \delta;$$

it is easily verified that Λ cof $\overline{M}$ and (since $(\alpha, \alpha')\leqq(\beta, \delta)$) that $\Lambda\subseteq\Delta^s$. Hence, by 1.21, $(\overline{P}, \bar{p})$ is stable rel Λ. Moreover, $(\alpha, \alpha')\leqq(\alpha, \beta)\leqq(\beta, \gamma)$ and so (α, β), $(\beta, \gamma)\in\Delta^s$ since $(\alpha, \alpha')\in\Delta$. Hence there is an isomorphism $\bar{p}\colon P_{\alpha\beta}\to P_{\beta\gamma}$, so that by (i) $\psi'\phi'=\bar{p}$, where ψ' is defined above and—using 1.41(iii)—$\phi|P_{\beta\gamma}=\phi'\colon P_{\beta\gamma}\to Q_{\beta\gamma}$. Therefore ϕ' is mono and ψ' is epi. Similarly, since M is directed by $<$, we have on putting $\phi_0=\phi|P_{\beta\delta}$, that

(a) $$\phi_0\colon P_{\beta\delta}\to Q_{\beta\delta}$$

is mono. We assert that ϕ_0 is also epi. For, from the above diagram,

$$Q_{\beta\xi} = qrQ_\delta = q\sigma\chi Q_\delta = \phi s\chi Q_\delta \subseteq \phi s P_\gamma = \phi P_{\beta\gamma},$$

while by stability and 1.21, $P_{\beta\gamma}=P_{\beta\delta}$ because $(\alpha, \alpha')\leqq(\beta, \gamma)\leqq(\beta, \delta)$ in Δ^s. Therefore $Q_{\beta\delta}=\phi P_{\beta\delta}=\phi_0 P_{\beta\delta}$, which proves ϕ_0 to be epi. Thus we have shown that for each $\lambda\in\Lambda$, the maps $\bar{\phi}_\lambda$ of 1.41(iii) are isomorphisms; and hence by the commutativity relation 1.41(iv), $\bar{q}_\lambda^\mu$ is an isomorphism if $\lambda\leqq\mu$ in Λ, since $\bar{\phi}_\lambda$, $\bar{\phi}_\mu$ and $\bar{p}_\lambda^\mu$ are. Hence $(\overline{Q}, \bar{q})$ is stable rel Λ and by 1.2(ii) ϕ induces an isomorphism Ilim $(\overline{P}, \bar{p})\approx$ Ilim $(\overline{Q}, \bar{q})$ as required.

To prove the converse, assume that $(\overline{Q}, \bar{q})$ is stable rel Δ. Since $(\alpha, \alpha') \leqq (\alpha, \beta) \leqq (\beta, \gamma)$ by the above inequalities, then (α, β), (β, γ) are in Δ^s. Hence there is, by 1.21, an isomorphism $\bar{q}: Q_{\beta\gamma} \to Q_{\alpha\beta}$, so that with $\psi': Q_{\beta\gamma} \to P_{\alpha\beta}$ as above and $\phi_1 = \tau | P_{\alpha\beta}$ (see diagram), we have $\phi_1\psi' = \bar{q}$, by (ii). Similarly there is an epimorphism $\phi_0: P_{\beta\delta} \to Q_{\beta\delta}$, which we shall now prove to be mono. For, suppose $\phi_0 x = 0$, $x \in P_{\beta\delta}$. Then $x = st(y)$ for some $y \in P_\delta$, and so from the diagram we get

$$0 = \phi_0 st(y) = qr\rho(y).$$

But $(\alpha, \alpha') \lneqq (\beta, \delta) \leqq (\gamma, \delta)$, so that (β, δ), $(\gamma, \delta) \in \Delta^s$; hence by 1.21 and the stability of $(\overline{Q}, \bar{q})$ rel Δ^s, $r\rho(y) = 0$. Thus

$$0 = r\rho(y) = \sigma t(y) = \theta\sigma t(y) = st(y) \qquad \text{by (ii)},$$

whereat ϕ_0 is mono, as required. We have proved, then, that ϕ induces isomorphisms $\overline{P}_\lambda \approx \overline{Q}_\lambda$ for each $\lambda \in \Lambda$, and hence by the commutativity relation 1.41(iv), $(\overline{Q}, \bar{q})$ is stable rel Λ; and then ϕ induces an isomorphism

$$\text{Ilim } (\overline{P}, \bar{p}) \approx \text{Ilim } (\overline{Q}, \bar{q}).$$

This completes the proof of the theorem.

1.5. **Non-Abelian groups.** For application to the local Fundamental groups on a space, we shall need the following result. Let $\phi: (P, p) \to (Q, q)$ be as in 1.41, with the additional properties that for each $\alpha \in M$,

(i) $$\operatorname{Ker} \phi_\alpha = [P_\alpha, P_\alpha] = \text{commutator subgroup of } P_\alpha;$$

(ii) $$\operatorname{Im} \phi_\alpha = Q_\alpha,$$

(thus, Q_α is P_α made Abelian). Then

1.51. THEOREM. *If $(\overline{P}, \bar{p})$ is stable rel Δ there exists a subset Λ cof $\overline{M}$, such that $(\overline{Q}, \bar{q})$ is stable rel Λ, and*

$$\text{Ilim } (Q, q) \textit{ is } \text{Ilim } (P, p) \textit{ made Abelian.}$$

(The converse is false: see 4.37 below.)

Proof. Consider the diagram

$$\begin{array}{ccccc} C_\beta & \subseteq & P_\beta & \xrightarrow{\theta} & Q_\beta \\ & & p\downarrow & & \downarrow q \\ C_\gamma & \subseteq & P_\gamma & \xrightarrow{\phi} & Q_\gamma \\ & & p'\downarrow & & \downarrow \\ C_\delta & \subseteq & P_\delta & \to & Q_\delta \end{array}$$

where $\delta \leqq \gamma \leqq \beta$ in M and

$$C_\beta = [P_\beta, P_\beta], \qquad \theta = \phi_\beta \text{ etc.}$$

As in 1.41(iii) we have a homomorphism

$$\phi_{\gamma\beta}: P_{\gamma\beta} \to Q_{\gamma\beta}.$$

Then $\phi_{\gamma\beta}$ is onto; for, any element u in $Q_{\gamma\beta}$ is of the form qx, $x \in Q_\beta$, while $x = \theta y$ for some $y \in P_\beta$ (by (ii)), so that

$$\begin{aligned} u = q\theta y &= \phi p y \qquad &\text{(by 1.41(i))} \\ &= \phi_{\gamma\beta}(py), \end{aligned}$$

which establishes the assertion.

We shall now prove

(iii) if there exists $\mu \in M$ such that $\gamma \leqq \mu \leqq \beta$ then

$$\operatorname{Ker} \phi_{\gamma\beta} = [P_{\gamma\beta}, P_{\gamma\beta}].$$

For let $v \in P_{\gamma\beta}$ be such that $\phi_{\gamma\beta} v = 0$. Then v is of the form pw, $w \in P_\alpha$, so that

$$0 = \phi_{\gamma\beta} p w = \phi p w,$$

whence

$$pw \in C_\beta \qquad \text{(by (i))}.$$

Therefore pw is of the form

$$pw = [x_1, y_1][x_2, y_2] \cdots [x_n, y_n] = \prod [x_i, y_i],$$

where $x_i, y_i \in P_\gamma$, $1 \leqq i \leqq n$. Hence

(iv) $$p'pw = \prod [p'x_i, p'y_i].$$

Fix $(\alpha, \alpha') \in \Delta$, and suppose $\alpha' \leqq \delta$. Then $(\alpha, \alpha') \leqq (\delta, \gamma) \leqq (\delta, \beta)$, and so (δ, β), $(\delta, \gamma) \in \Delta^s$; then by 1.21 and the stability of $(\overline{P}, \bar{p})$, the inclusion $P_{\delta\beta} \subseteq P_{\delta\gamma}$ (see 1.1(iv)) is an equality. Hence there exist $a_i, b_i \in P_\beta$ such that

$$p'x_i = p'pa_i, \qquad p'y_i = p'pb_i \qquad (1 \leqq i \leqq n).$$

Therefore (iv) becomes

(v) $$p'pw = p'(\prod [pa_i, pb_i)].$$

Now, by definition of μ in (iii) we have $(\alpha, \alpha') \leqq (\delta, \beta) \leqq (\gamma, \beta)$; hence by 1.21 the epimorphism

$$\pi: P_{\gamma\beta} \to P_{\delta\beta}$$

defined by $\pi = p' \,|\, P_{\gamma\beta}$, is an isomorphism. Then in (v), $p'(pw) = \pi(pw)$, so that, since π is mono,

$$\begin{aligned} pw = \prod [pa_i, pb_i] &= p\prod [a_i, b_i] \\ &\in pC_\beta. \end{aligned}$$

But

$$pC_\beta = p[P_\beta, P_\beta] = [pP_\beta, pP_\beta]$$
$$= [P_{\gamma\beta}, P_{\gamma\beta}],$$

so that $v = pw \in [P_{\gamma\beta}, P_{\gamma\beta}]$, i.e.

$$\text{Ker } \phi_{\gamma\beta} = [P_{\gamma\beta}, P_{\gamma\beta}],$$

as asserted in (iii).

Now let Λ be the set of all pairs $(\gamma, \beta) \in \overline{M}$ for which there exists μ with $\gamma \leqq \mu \leqq \beta$ and $(\alpha, \alpha') \leqq (\gamma, \beta)$. One verifies that Λ cof $\overline{M}$ and $\Lambda \subseteq \Delta^\bullet$, and so $(\overline{P}, \bar{p})$ is stable rel Λ. We have shown that if

$$C_{\gamma\beta} = [P_{\gamma\beta}, P_{\gamma\beta}],$$

then for all $(\gamma, \beta) \leqq (\tau, \sigma)$ in this subset Λ we have a diagram

$$\begin{array}{ccccc} C_{\gamma\beta} & \subseteq & P_{\gamma\beta} & \xrightarrow{\psi} & Q_{\gamma\beta} \\ & & p \uparrow & & \uparrow q \\ C_{\tau\sigma} & \subseteq & P_{\tau\sigma} & \xrightarrow[\psi']{} & Q_{\tau\sigma} \end{array}$$

where $\psi = \phi_{\gamma\beta}$, $\psi' = \phi_{\tau\sigma}$, p and q are homomorphisms belonging to the systems $(\overline{P}, \bar{p})$, $(\overline{Q}, \bar{q})$ respectively, and p is an isomorphism (since $(\overline{P}, \bar{p})$ is stable rel Λ). It now follows easily, using (iii) and the fact that $p^{-1}C_{\gamma\beta} = C_{\tau\sigma}$, that q is also an isomorphism. Thus $(\overline{Q}, \bar{q})$ is stable rel Λ, and the proof is complete.

1.6. **Abstract relative theory.** For use with relative homology and homotopy, consider the following situation. Let $(M, \leqq)$ be the basic directed set as usual, and if $\xi \in \overline{M}$ is of the form $\xi = (\alpha, \beta)$, define

$$\xi' = \beta.$$

Suppose that (R, r) is an inverse system over $\overline{M}$, (A, a) an inverse system over M, and that for each $\xi \in \overline{M}$ there is a homomorphism

$$d_\xi \colon R_\xi \to A_{\xi'}$$

such that the diagram

$$\begin{array}{ccc} R_\xi & \xrightarrow{d} & A_{\xi'} \\ r \uparrow & & \uparrow a \\ R_\eta & \xrightarrow[d_1]{} & A_{\eta'}, \end{array} \qquad \xi < \eta \text{ in } \overline{M},$$

is commutative, i.e.

(i) $$a_{\xi'}^{\eta'} d_\eta = d_\xi r_\xi^\eta.$$

Suppose further that there is a monotone(²) map $p: M \to M$ such that for each $\gamma \in M$ and all $\delta \in M$ with $p(\gamma) \leqq \delta$, the homomorphism

$$\text{(ii)} \qquad d_\xi: R_\xi \to A_\delta, \qquad \xi = (\gamma, \delta),$$

is onto.

Finally suppose that there is a second monotone map $q: M \to M$, such that for each $\mu \in M$ and all $\alpha \in M$ with $q(\mu) \leqq \alpha$, the above diagram satisfies

$$\text{(iii)} \qquad \operatorname{Ker}(dr) \subseteq \operatorname{Ker}(\overset{\zeta}{r}_\eta), \qquad \zeta \leqq \zeta \leqq \eta$$

whenever ξ, ζ are of the forms (α, β), (μ, γ), respectively.

For each $\sigma, \tau \in \overline{M}$ of the forms $\sigma = (\mu, \mu)$, $\tau = (\alpha, \alpha)$ with $\sigma \leqq \tau$ define

$$S_\mu = R_\sigma, \qquad \overset{\alpha}{s}_\mu = \overset{\tau}{r}_\sigma,$$

so that (S, s) is an inverse system over M. As in 1.2 we identify M with the diagonal of $\overline{M}$, so that M cof $\overline{M}$, and therefore using 1.1(xi)

$$\text{(iv)} \qquad \operatorname{Ilim}(\overline{S}, \bar{s})_{\overline{M}} \approx \operatorname{Ilim}(S, s)_M \approx \operatorname{Ilim}(R, r)_{\overline{M}}.$$

We shall prove the following result.

1.61. THEOREM. *If $(\overline{S}, \bar{s})$ is stable rel Δ, then there is a subset Λ cofinal in Δ^s such that $(\overline{A}, \bar{a})$ is stable rel Λ; and conversely. In both cases*

$$\text{(v)} \qquad \operatorname{Ilim}(\overline{S}, \bar{s}) \approx \operatorname{Ilim}(\overline{A}, \bar{a}).$$

Proof. From the diagram preceding (i) we obtain

$$\begin{array}{ccc} S_\alpha & \xrightarrow{d} & A_\alpha \\ s\downarrow & & \downarrow a \\ S_\mu & \xrightarrow[d_1]{} & A_\mu \end{array} \qquad \mu \leqq \alpha \text{ in } M,$$

and so we obtain an induced homomorphism

$$\partial_{\mu\alpha}: S_{\mu\alpha} \to A_{\mu\alpha}$$

given by

$$\partial_{\mu\alpha} = d_1 \,|\, S_{\mu\alpha}.$$

Pick $(\sigma, \lambda) \in \Delta$ and define Λ to be the set of all pairs $(\mu, \alpha) \in \Delta^s$ for which there exists $\beta \in X$ such that $\lambda \leqq \mu \leqq q(\mu) \leqq \beta \leqq \alpha$ (with q as for (iii)). Define Λ_0 to be the set of all pairs $(\mu, \alpha) \in \Delta^s$ for which there exists $\beta \in M$ such that $\lambda \leqq q(\lambda) \leqq \mu \leqq \beta \leqq p(\beta) \leqq \alpha$ (p as in (ii)). It is easily verified that Λ, Λ_0 cof $\overline{M}$. From the commutativity of the last diagram (following from (i)), we obtain

$$\text{(vi)} \qquad \partial_\sigma \overset{\tau}{\bar{s}}_\sigma = \overset{\tau}{\bar{a}}_\sigma \partial_\tau, \qquad \sigma \leqq \tau \text{ in } \overline{M},$$

(²) I.e. for all α, $\alpha \leqq p(\alpha)$.

and therefore the stability of $(\overline{S}, \bar{s})$ on Λ will follow if we can prove $\partial_{\mu\alpha}$ an isomorphism for each $(\mu, \alpha)\in\Lambda$; and similarly for $(\overline{A}, \bar{a})$ on Λ_0.

We shall therefore prove

$$\partial_{\mu\alpha}: S_{\mu\alpha} \approx A_{\mu\alpha}$$

provided (a) $(\mu, \alpha)\in\Lambda$ and $(\overline{A}, \bar{a})$ is stable rel Δ, or (b) $(\mu, \alpha)\in\Lambda_0$ and $(\overline{S}, \bar{s})$ is stable rel Δ, and then by (vi) it follows immediately that

$$\partial_\infty: \text{Ilim } (\overline{S}, \bar{s}) \approx \text{Ilim } (\overline{A}, \bar{a}),$$

as required by (v).

Proof for (a). $\partial_{\mu\alpha}$ is onto. For, since M is directed, there exists $\beta\in M$ such that

$$\mu \leqq \alpha \leqq p(\alpha) \leqq \beta,$$

giving a diagram of the form

$$\begin{array}{ccccc} R_{(\alpha,\beta)} & \xrightarrow{b} & S_\alpha & \xrightarrow{b'} & S_\mu \\ d\downarrow & & & & \downarrow d' \\ A_\beta & \xrightarrow[a]{} & A_\alpha & \xrightarrow[a']{} & A_\mu \end{array}$$

Let $u\in A_{\mu\alpha}$. By 1.1(iv), $A_{\mu\beta}\subseteq A_{\mu\alpha}$, and since (μ, β), $(\mu, \alpha)\in\Delta^s$, the inclusion is an equality, by the stability of $(\overline{A}, \bar{a})$. Hence there exists $v\in A_\beta$, such that $u=a'av$. Since $\alpha\leqq p(\alpha)\leqq\beta$, we can apply (ii) of 1.6, to say that d is onto. Hence there exists $w\in R_{(\alpha,\beta)}$ such that $v=dw$; and so $u=a'adw=d'b'bw$ $=d'b'(bw)=d'u_0$ say, where $u_0=b'(bw)\in S_{\mu\alpha}$ since $bw\in S_\alpha$. But then $d'u_0$ $=\partial_{\mu\alpha}u_0$, whence $\partial_{\mu\alpha}$ is onto, as required.

Let us now prove that $\partial_{\mu\alpha}$ is mono. Since $(\mu, \alpha)\in\Lambda$, there exists by definition $\beta\in M$ such that $\mu\leqq q(\mu)\leqq\beta\leqq\alpha$, giving a diagram of the form

$$\begin{array}{ccccc} S_\alpha & \xrightarrow{b} & R_{(\mu,\beta)} & \xrightarrow{b'} & S_\mu \\ d_1\downarrow & & \downarrow d & & \downarrow d' \\ A_\alpha & \xrightarrow[a]{} & A_\beta & \xrightarrow[a']{} & A_\mu \end{array}$$

Let $x\in S_{\mu\alpha}$ be such that $\partial_{\mu\alpha}x=0$. Then x is of the form $b'by$, $y\in S_\alpha$, so that

$$0 = \partial_{\mu\alpha}x = d'x = d'b'by = a'ad_1y.$$

Now the homomorphism

$$\theta: A_{\beta\alpha} \to A_{\mu\alpha}$$

defined by $\theta=a'\,|\,A_{\beta\alpha}$, is an isomorphism because (β, α), $(\mu, \alpha)\in\Delta^s$ and $(\overline{A}, \bar{a})$ is stable rel Δ. Thus

$$0 = a'(ad_1y) = \theta(ad_1y)$$

and so $ad_1y=0$ because θ is mono, giving $dby=0$. Since $\mu \leqq q(\mu) \leqq \alpha$ the conditions of 1.6(iii) are satisfied with $\gamma=\mu$. Hence $y \in \operatorname{Ker}(db) \subseteq \operatorname{Ker}(b'b)$ and so $x=b'by=0$. Thus $\partial_{\mu\alpha}$ is mono, as asserted. With the previous result, this proves $\partial_{\mu\alpha}$ to be an isomorphism.

Proof for (b). If $(\mu, \alpha) \in \Lambda_0$, then by definition there exists $\beta \in M$ such that $\mu \leqq \beta \leqq p(\beta) \leqq \alpha$, giving the following diagram:

$$\begin{tikzcd}
S_\alpha \arrow[r, "r"] \arrow[d] & R_{\beta\alpha} \arrow[r, "s"] \arrow[dl, "d"] & S_\beta \arrow[r, "s'"] & S_\mu \arrow[d, "d'"] \\
A_\alpha \arrow[rr, "a"'] & & A_\beta \arrow[r, "a'"'] & A_\mu
\end{tikzcd}$$

To prove that $\partial_{\mu\alpha}$ is onto, let $x \in A_{\mu\alpha}$, so that x is of the form $a'ay$, $y \in A_\alpha$. Since $p(\beta) \leqq \alpha$ we can apply (ii) of 1.6 to assert that d is onto; and so $y=dz$ for some $z \in R_{\beta\alpha}$. Hence

$$x = a'adz = d's'sz = d's'u, \text{ say,}$$

where $u=sz \in S_\beta$. Now, since $(\bar{S}, \bar{s})$ is stable rel Δ, the inclusion $S_{\mu\alpha} \subseteq S_{\mu\beta}$ is an equality. Thus $w=s'u \in S_{\mu\beta}=S_{\mu\alpha}$ and

$$x = d'w = \partial_{\mu\alpha}w \qquad (\partial_{\mu\alpha} = d' \mid S_{\mu\alpha}),$$

whence $\partial_{\mu\alpha}$ is onto.

Lastly, to prove $\partial_{\mu\alpha}$ is mono, let $x \in S_{\mu\alpha}$ be such that $\partial_{\mu\alpha}x=0$. By definition of Λ_0, there exists $\lambda \in M$ such that $\lambda \leqq q(\lambda) \leqq \mu \leqq \alpha$, so that we have a diagram

$$\begin{tikzcd}
S_\alpha \arrow[r, "s"] \arrow[d] & S_\mu \arrow[r, "s'"] \arrow[d, "d"] & S_\lambda \\
A_\alpha \arrow[r, "\alpha"'] & A_\mu &
\end{tikzcd}$$

Because $x \in S_{\mu\alpha}$, there exists $y \in S_\alpha$ such that $x=sy$; and then $\partial_{\mu\alpha}x=dsy$, since $\partial_{\mu\alpha}=d \mid S_{\mu\alpha}$. Hence $y \in \operatorname{Ker}(ds) \subseteq \operatorname{Ker}(s's)$, for (iii) can be applied with $\eta=(\alpha, \alpha)$, $\xi=(\mu, \mu)$, $\zeta=(\lambda, \lambda)$, respectively, since $q(\lambda) \leqq \mu$. Therefore $s'sy=0$. But s' induces a homomorphism

$$\sigma: S_{\mu\alpha} \to S_{\lambda\alpha}$$

defined by $\sigma=s' \mid S_{\mu\alpha}$; and since (μ, α), $(\lambda, \alpha) \in \Delta^s$, and $(\bar{S}, \bar{s})$ is stable rel Δ^s, then σ is an isomorphism. Now $0=s'(sy)=\sigma(sy)$, whence $sy=0$ because σ is mono. Therefore $x=sy=0$, whence $\partial_{\mu\alpha}$ is mono, as required.

By the remarks preceding (a) and (b), the proof of the theorem is now complete.

II. Vietoris and Singular homology

We shall later wish to compare the local invariants in a space, defined in each of the Vietoris and the Singular theories. To bring out better the relationships, we shall here express the singular groups as inverse limits, in the spirit of Lefschetz [11] with his *Vietoris singular complex*, and then treat the Vietoris groups similarly. Throughout, R will denote a fixed commutative ring with unit. We let $(M, \prec)$ denote the set of all non-negative real numbers, where

$$\lambda \prec \mu \text{ in } M \text{ means } \mu \leqq \lambda.$$

2.1. **The singular inverse system.** As in [*E-S*, Chapter VII] let Δ_q denote the unit q-simplex in Euclidean space R^{q+1}, and let X be a fixed metric space. If $Y \subseteq X$ define $C_qS(Y, \lambda)$ to be the free R-module generated by all singular q-simplexes $T: \Delta_q \to Y$, such that

(i) $$\operatorname{diam} T(\Delta_q) < \lambda.$$

If $q<0$, there are no such simplexes, and $C_qS(Y, \lambda)=0$. From the definition of the singular boundary operator in [E-S, p. 186], it follows that

$$\partial_q: C_qS(Y, \lambda) \to C_{q-1}S(Y, \lambda),$$

and so $(C_qS(Y, \lambda), \partial_q)$ is a chain complex whose homology groups we denote by $H_qS(Y, \lambda)$. In the notation of [E-S, p. 197] our $C_qS(Y, \lambda)$ is the group $C_q(Y, F)$ where F is the covering of Y consisting of all open sets of diameter $<\lambda$, and Theorem VII 8.2, *op. cit.* proves that the inclusion

(ii) $$C_qS(Y, \lambda) \subseteq C_qS(Y)$$

induces a homotopy equivalence in each dimension. Moreover, if $\mu \in M$ and $0<\mu \leqq \lambda$, there is an inclusion

(iii) $$C_qS(Y, \mu) \subseteq C_qS(Y, \lambda)$$

and an obvious modification(³) of the proof of [4, VII 8.2] shows that this inclusion also induces a homotopy equivalence in each dimension. We therefore have a commuting diagram

(iv) $$\begin{array}{ccc} H_qS(Y, \lambda) & \xrightarrow{a_\lambda} & H_qS(Y) \\ f\uparrow & & \uparrow g \\ H_qS(Y, \mu) & \xrightarrow[a_\mu]{} & H_qS(Y) \end{array}$$

where $H_qS(Y)$ is the ordinary q-dimensional singular homology group of Y, g is the identity map, a_λ and a_μ are homomorphisms induced by inclusions of the sort (ii), and $f=f_\lambda^\mu$ a homomorphism induced by the inclusion (iii). Owing

(³) In fact, replace "T in X" on p. 198, line 11 of *op. cit.* by "T in $X' \in F_\lambda$."

to the homotopy equivalence in (ii), a_λ and a_μ are isomorphisms; hence passing to the limit we get

(v) $$a_\infty\colon \operatorname{Ilim}\,(H_qS(Y,\lambda), f_\lambda^\mu)_M \approx H_qS(Y).$$

2.2. **The Vietoris inverse system.** Now let F be a compact subset of X, and let Δ_q^0 denote the set of vertices of Δ_q. Define a "Vietoris q-simplex," or (V, q)-simplex, of F to be a map $\tau\colon\Delta_q^0\to F$, so that using (V, q)-simplexes in place of singular cells in 2.1 (i), we obtain analogously a chain-complex $\Omega(F,\lambda)=(C_q\Omega(F,\lambda), \partial_q)$ of free R-modules. If $F\subseteq K$, K compact, then we regard every (V, q)-simplex of F as being one of K, so that $\Omega(F,\lambda)$ is a subcomplex of $\Omega(K,\lambda)$. Forming the quotient complex

$$\Omega(K, F, \lambda) = \Omega(K,\lambda)/\Omega(F,\lambda),$$

we obtain an exact sequence of homology groups

(i) $$\cdots \xrightarrow{\partial_q} H_q\Omega(F,\lambda) \to H_q\Omega(K,\lambda) \to H_q\Omega(K,F,\lambda) \xrightarrow{\partial_{q-1}} H_{q-1}\Omega(F,\lambda) \to \cdots.$$

Moreover, if $\mu\in M$ and $0<\mu\leqq\lambda$, there is an inclusion

$$C_q\Omega(F,\mu)\subseteq C_q\Omega(F,\lambda)$$

inducing an injection $\phi_\lambda^\mu\colon H_n\Omega(F,\mu)\to H_q\Omega(F,\lambda)$; and it is easily seen that the usual Vietoris qth homology group of F is identical with

(ii) $$H_q\Omega(F) = \operatorname{Ilim}\,(H_q\Omega(F,\lambda),\phi_\lambda^\mu)_M,$$

and similarly for $H_q\Omega(K, F)$. If A is any subset of X, we define

(iii) $$H_q\Omega(X, A) = \operatorname{Dlim}\{H_q\Omega(K,F), \omega_{KF}^{JE}\}_{\mathfrak{K}}$$

where $\mathfrak{K}$ is the system of all compact pairs (K, F) with $F\subseteq K\subseteq X$, $F\subseteq A$, directed by inclusion, and ω_{KF}^{JE} is induced by the inclusion $(K, F)\subseteq(J, E)$. Since singular theory has compact carriers, it is well known that

(iv) $$H_qS(X, A) = \operatorname{Dlim}\,\{H_qS(K,F), s_{KF}^{JE}\}_{\mathfrak{K}},$$

where s_{KF}^{JE} is induced by inclusion.

Given a singular q-cell $T\colon\Delta_q\to F$ in $C_qS(F,\lambda)$, the restriction $\sigma T=T|\Delta_q^0$ defines an element of $C_q\Omega(F,\lambda)$, and if we extend by linearity we get a homomorphism

(v) $$\sigma\colon C_qS(F,\lambda)\to C_q\Omega(F,\lambda)$$

which commutes properly with boundaries and injections. Hence, from (iii) and (iv), there is an induced homomorphism

(vi) $$\sigma_*\colon H_qS(X)\to H_q\Omega(X)$$

which is natural. We shall next consider restrictions on X which will enable us to assert σ_* to be an isomorphism. These restrictions concern the local connectivity of X.

2.3. **Local connectivity.** Let x be a fixed point in the space X, and let $\mathfrak{U}$ denote the set of all neighborhoods(4) of x, directed by $\prec$, where

$$\text{(i)} \qquad U \prec V \cdot \Leftrightarrow \cdot V \subseteq U.$$

We shall suppose that X is locally compact at x, so that the set $\mathfrak{U}_c$ of all *compact* neighborhoods of x satisfies

$$\mathfrak{U}_c \text{ cof } \mathfrak{U}.$$

With U, V as above and with ω and s as in 2.2 (iii) and (iv), respectively, let $s_V^U = s_{V,0}^{U,0}$ (0 = empty set), and similarly for ω; define

$$\text{(ii)} \qquad \begin{aligned} L_qS(x) &= \operatorname{Ilim}\,(H_qS(U), \overset{V}{s}_U)_{\mathfrak{U}_c}, \\ L_q\Omega(x) &= \operatorname{Ilim}\,(H_q\Omega(U), \overset{V}{\omega}_U)_{\mathfrak{U}_c}. \end{aligned}$$

We write, whenever $U \prec V$

$$\text{(iii)} \qquad H_qS(V \mid U) = \overset{V}{s}_U H_qS(V), \qquad H_q\Omega(V \mid U) = \overset{V}{\omega}_U H_q\Omega(V).$$

Then X is said(5) to be $q-\mathrm{lc}_s[q-\mathrm{lc}_v]$ at x if and only if to each $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$, such that $V \subseteq U$ and, using augmented homology in dimension zero,

$$\text{(iv)} \qquad H_qS(V \mid U) = 0, \qquad [H_q\Omega(V \mid U) = 0].$$

X is lc_s^q at x if and only if it is $r-\mathrm{lc}_s$, $0 \leqq r \leqq q$, and X is lc_s^q if and only if it is lc_s^q at all its points. Similarly for lc_v^q.

For brevity we shall write

$$\text{(v)} \qquad \Sigma_{UV} = H_qS(V \mid U), \qquad \Omega_{UV} = H_q\Omega(V \mid U)$$

and

$$\text{(vi)} \qquad L_qS(x) \equiv 0$$

whenever, in the notation of 1.2, there is a subset Λ of(6) $\mathfrak{U}_c^2$ such that $(\overline{\Sigma}, \bar{s})$ is stable rel Λ; and similarly for $L_q\Omega(x)$. It is then easily verified that

$$\text{(vii)} \qquad \begin{aligned} &X \text{ is } q - \mathrm{lc}_s \text{ at } x \cdot \Leftrightarrow \cdot L_qS(x) \equiv 0; \\ &X \text{ is } q - \mathrm{lc}_v \text{ at } x \cdot \Leftrightarrow \cdot L_q\Omega(x) \equiv 0. \end{aligned}$$

Begle [1, 3.1] has given a very useful definition of local connectivity which in our notation can be written as:

(4) Following Bourbaki: U is a neighborhood of x means $x \in$ Interior (U).

(5) Our lc_s is the "H.L.C." of Cartan [2].

(6) For typographical reasons, we write the $\overline{M}$ of 1.1 (ix) as M^2.

(viii) *X is $(V, q)-lc$ at $x \cdot\Leftrightarrow\cdot$ given $U \in \mathfrak{U}_c$ and $\epsilon > 0$, there exist $V \in \mathfrak{U}_c$ (depending only on U) and $\eta > 0$ such that every cycle in $C_q\Omega(V, \eta)$ is homologous to zero in $C_q\Omega(U, \epsilon)$.*

Begle proves that when X is compact metric—and his proof requires only minor modifications if X is locally compact—then X is lc_v^q if and only if it is $(V, r)-lc$, $0 \leqq r \leqq q$. We therefore formulate the analogue of (viii):

(ix) *X is $(S, q)-lc$ at $x \cdot\Leftrightarrow\cdot$ given $U \in \mathfrak{U}_c$ and $\epsilon > 0$, there exist $V \in \mathfrak{U}_c$ (depending only on U) and $\eta > 0$ such that every cycle in $C_qS(V, \eta)$ is homologous to zero in $C_qS(U, \epsilon)$.*

2.31. Lemma. *X is $q-lc_s$ at $x \cdot\Leftrightarrow\cdot X$ is $(S, q)-lc$ at x.*

Proof. Consider the diagram

$$\begin{array}{ccc} H_qS(U, \epsilon) & \xrightarrow{b} & H_qS(U) \\ t\uparrow & & \uparrow s \\ H_qS(V, \eta) & \xrightarrow[c]{} & H_qS(V) \end{array}$$

where $V \subseteq U$ in $\mathfrak{U}_c$, $0 < \eta \leqq \epsilon$, s and t are injections, and b and c are isomorphisms of the sort a_λ in 2.1 (iv). Thus $sc = bt$. If U, V are as in (v), $s = 0$. Hence $bt = 0$, and so $t = 0$ since b is an isomorphism. Therefore $q-\mathrm{lc}_s$ implies $(S, q)-\mathrm{lc}$. Conversely, if U, V, ϵ, η are as in (ix), $t = 0$. Hence $sc = 0$, and so $s = 0$ because c is an isomorphism. Thus $(S, q)-\mathrm{lc}$ implies $q-\mathrm{lc}_s$, and the lemma is proved.

The pair (V, η) in (viii) is clearly a function of the pair (U, ϵ), and similarly in (ix); let us therefore write, respectively,

$$\text{(x)} \qquad V = \overset{v}{\lambda}_q(U), \qquad \eta = \overset{v}{\lambda}_q(U, \epsilon); \qquad V = \overset{s}{\lambda}_q(U), \qquad \eta = \overset{s}{\lambda}_q(U, \epsilon).$$

We can now assert the following result concerning the homomorphism σ_* of 2.2 (vi).

2.32. Theorem. *If the locally compact metric space X is both lc_v^q and lc_s^q, then*

$$\sigma_*: H_rS(X) \approx H_r\Omega(X), \qquad 0 \leqq r \leqq q.$$

We shall not digress to give a proof; a full treatment will be given elsewhere.

Added in proof, September 1958. In a forthcoming paper by S. Mardešić (See Notices Amer. Math. Soc., April, 1958, p. 210, Abstract 544-14) it is proved that the theorem holds for a paracompact Hausdorff, lc_s^n space X and that in dimension $n+1$, σ_* is onto.

The relationships between the types of local connectivity will be discussed further in 4.2 below. Suffice it to say for the present that the conclusion of the theorem would hold if X were LC^q, thus generalizing Lefschetz [11, 22.1].

With the notation $H_q(X|Y)$ of 2.3 (iii), a useful consequence of 2.32 is:

2.33. THEOREM. *If X is locally compact metric, lc_v^n and lc_s^n, if G is any neighborhood of a compact set $F \subseteq X$, and if $0 \leqq q \leqq n$, then*

$$H_q S(F \mid G)$$

is finitely generated, $0 \leqq q \leqq n$.

Proof. Since X is locally compact, there is a compact neighborhood W of F such that $W \subseteq G$. Let $U =$ interior W. We have a commutative diagram

$$\begin{array}{ccc} H_q\Omega(U) \xrightarrow{w} H_q\Omega(W) & \xrightarrow{g} & H_q\Omega(G) \\ \sigma \uparrow & & \uparrow \tau \\ H_qS(U) & \xrightarrow[s]{\hspace{6em}} & H_qS(G) \end{array}$$

where the horizontal arrows are injections and σ, τ are isomorphisms of the sort σ_* in 2.32 (they exist since U, G are open in X). Then by Newman [13, Theorem 1], $P = gH_q\Omega(W)$ is finitely generated. Therefore, if $u = gw$, then

$$Q = uH_q\Omega(U) \subseteq P$$

and so Q is finitely generated since all groups are Abelian. But

$$\begin{aligned} uH_q\Omega(U) &= u\sigma H_qS(U) \text{ since } \sigma \text{ is an isomorphism,} \\ &= \tau s H_qS(U) \end{aligned}$$

and so, since τ is an isomorphism, $sH_qS(U) \approx Q$ and is therefore finitely generated. Now, since $F \subseteq U$,

$$H_qS(F \mid G) \subseteq H_qS(U \mid G) = sH_qS(U)$$

and the required result follows.

III. HOMOTOPY

3.1. In order to prove a "local" version of Hurewicz's theorem we shall in this section discuss certain modifications of Eilenberg [3]. Let X, Y be subsets of a topological space, with $X \subseteq Y$, and let $x \in X$ be taken as base-point of homotopy groups until further notice. Thus we write $\pi_n(X)$ for $\pi_n(X, x)$. Following Eilenberg, we denote by $S(X)$ the singular complex of X, and by $S_n(X)$ the subcomplex of $S(X)$ consisting of all singular simplexes $T: \Delta \to X$ such that all the faces of Δ of dimension $< n$ are mapped by T into x. Thus

(i) $$S(X) = S_0(X) \supseteq S_1(X) \supseteq \cdots \supseteq S_n(X) \supseteq \cdots$$

and

$$S_n(X) \subseteq S_n(Y), \qquad n = 0, 1, 2, \cdots.$$

We denote the image of the injection of $\pi_n(X)$ in $\pi_n(Y)$, by

$$\pi_n(X | Y).$$

If we look at Eilenberg's proof, op. cit., of 31.1, p. 440, we see that if "$\pi_n(X) = 0$" is replaced by "$\pi_n(X | Y) = 0$," then in that proof we have to replace p. 441, 1.7 and 31.4 respectively by "$R_T: s\times I \to Y$" and "R_T is in $S_{n+1}(Y)$." We therefore obtain instead of his 1.5, 1.7 on p. 442 the following result.

3.11. LEMMA. *Suppose $\pi_n(X | Y) = 0$. Then there is a diagram*

$$\begin{tikzcd} S_n(X) \arrow[r, "j_n"] \arrow[dr, "p"] & S_n(Y) \\ S_{n+1}(X) \arrow[u, "\eta_X"] \arrow[r, "j_{n+1}"'] & S_{n+1}(Y) \arrow[u, "\eta_Y"'] \end{tikzcd}$$

such that

(i) $$p\eta_X = j_{n+1},$$

(ii) $$\eta_Y p \simeq j_n \text{ rel } x$$

where η_X, η_Y, j_n, j_{n+1} are injections, and p is the analogue of Eilenberg's π.

3.12. If we have a chain of subsets

$$X = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = Y$$

such that

$$\pi_r(A_r | A_{r+1}) = 0, \qquad r = 0, 1, \cdots, n-1,$$

it will be convenient to write

$$X <_{n-1} Y.$$

The last lemma now enables us to prove the following theorem, which becomes the Hurewicz Theorem when $X = Y$.

3.13. THEOREM. *If $X <_{n-1} Y$ and $n > 1$, there is a commutative diagram (for integer coefficients)*

$$\begin{tikzcd} H_nS(X) \arrow[r, "i"] \arrow[dr, "\pi"] & H_nS(Y) \\ \pi_n(X) \arrow[u, "h_X"] \arrow[r, "j"'] & \pi_n(Y) \arrow[u, "h_Y"'] \end{tikzcd}$$

where i, j are injections, the h's are natural Hurewicz homomorphisms, and π is to be constructed.

Proof. By Eilenberg [3, p. 443], there is, since $n>1$, a commutative diagram

$$\begin{tikzcd} H_nS_n(X) \arrow[r, "k"] & H_nS_n(Y) \\ \pi_n(X) \arrow[u, "\nu_1"] \arrow[r, "j"'] & \pi_n(Y) \arrow[u, "\nu_2"'] \end{tikzcd}$$

where k is the injection, and the ν's are isomorphisms. It therefore suffices by commutativity to show the existence of a commutative diagram

(i)
$$\begin{tikzcd} H_nS(X) \arrow[r, "i"] \arrow[dr, "q"] & H_nS(Y) \\ H_nS_n(X) \arrow[u, "\lambda"] \arrow[r, "k"'] & H_nS_n(Y) \arrow[u, "\mu"'] \end{tikzcd}$$

where λ, μ are injections of the sort given by 3.1 (i), and

(ii) $$h_X = \lambda\nu_1, \qquad h_Y = \mu\nu_2, \qquad \pi = \overset{-1}{\nu_2}\, q.$$

Since $X <_{n-1} Y$, there is a chain $X = A_0 \subseteq \cdots \subseteq A_n = Y$, as in 3.12, and so by 3.11 we can form the following diagram, which is commutative in each square and triangle (by (i) and (ii) of 3.11); in it, the α's, β's and p's correspond to the η's, j's and p of 3.11. The diagram is:

$$\begin{tikzcd} S_0(A_0) \arrow[r, "\beta_{01}"] \arrow[dr, "p_1"] & S_0(A_1) \arrow[r, "\beta_{02}"] & \cdots & \cdots \arrow[r, "\beta_{0n}"] & S_0(A_n) \\ S_1(A_0) \arrow[u, "\alpha_{01}"] \arrow[r, "\beta_{11}"] & S_1(A_1) \arrow[u, "\alpha_{11}"'] \arrow[r, "\beta_{12}"] \arrow[dr, "p_2"] & \cdots & \cdots \arrow[r, "\beta_{1n}"] & S_1(A_n) \arrow[u, "\alpha_{n1}"'] \\ \vdots \arrow[u] & \vdots \arrow[u] & \arrow[dr, "p_{n-1}"] & \vdots & \vdots \arrow[u] \\ S_{n-1}(A_0) \arrow[u] \arrow[r] & S_{n-1}(A_1) \arrow[u] \arrow[r] & \cdots \arrow[r] & S_{n-1}(A_{n-1}) \arrow[u] \arrow[r] \arrow[dr, "p_n"] & S_{n-1}(A_n) \arrow[u] \\ S_n(A_0) \arrow[u, "\alpha_{0n}"] \arrow[r, "\beta_{n1}"] & S_n(A_1) \arrow[u, "\alpha_{1n}"'] \arrow[r, "\beta_{n2}"] & \cdots \arrow[r, "\beta_{n,n-1}"] & S_n(A_{n-1}) \arrow[u, "\alpha_{n-1,n}"] \arrow[r, "\beta_{nn}"] & S_n(A_n) \arrow[u, "\alpha_{nn}"'] \end{tikzcd}$$

By induction on m $(0 \leqq m \leqq n)$ we obtain, on using the commutativity of the diagram below and above the diagonal respectively,

(iii) $$\beta_{mm}\beta_{mm-1} \cdots \beta_{m1} = (p_mp_{m-1} \cdots p_1)(\alpha_{01}\alpha_{02} \cdots \alpha_{0m}),$$

(iv) $$\beta_{0m}\beta_{0m-1} \cdots \beta_{01} \simeq (\alpha_{m1}\alpha_{m2} \cdots \alpha_{mm})(p_mp_{m-1} \cdots p_1).$$

Now $H_nS(X)=H_nS_0(A_0)$, so that in diagram (i) above, k is induced by $\beta_{nn}\beta_{nn-1}\cdots\beta_{n1}$; and similarly, i is induced by $\beta_{0n}\beta_{0n-1}\cdots\beta_{01}$, λ by $\alpha_{01}\alpha_{02}\cdots\alpha_{0n}$, and μ by $\alpha_{n1}\alpha_{n2}\cdots\alpha_{nn}$. Hence, at the level of homology groups, (iii) and (iv) with $n=m$ give respectively

$$k = \pi\lambda, \qquad i = \mu\pi$$

where π is induced by $p_np_{n-1}\cdots p_1$. This proves the existence of a commutative diagram of the form (i) and hence the theorem follows.

3.14. COROLLARY. *If in Theorem* 3.13, Y *is locally compact,* lc_v^n *and* lc_s^n, *and if we have also a compact set* U *such that*

$$U \subseteq \text{Interior } (X).$$

then $\pi_n(U| Y)$ *is finitely generated* $(n>1)$.

Proof. By 3.13, we have a commutative diagram

$$\begin{tikzcd}
H_nS(U) \arrow[r, "v"] & H_nS(X) \arrow[r] & H_nS(Y) \\
\pi_n(U) \arrow[u, "h_U"] \arrow[r, "u"'] & \pi_n(X) \arrow[u, "h_X"] \arrow[r, "j"'] & \pi_n(Y) \arrow[u] \arrow[from=2-2, to=1-2] \arrow[from=1-2, to=2-3, "\pi"]
\end{tikzcd}$$

where u, v are injections, and h_U is the Hurewicz homomorphism. Now

$$\begin{aligned}
\pi_n(U \mid Y) &= juG, \qquad G = \pi_n(U) \\
&= \pi h_X uG = \pi v h_U G \\
&\subseteq \pi v H_nS(U).
\end{aligned}$$

Our hypotheses enable us to invoke 2.33, which asserts that $vH_nS(U)$ is finitely generated: hence so is its image $\pi_n(U| Y)$, and the proof is complete.

To extend 3.14 to the case $n=1$, we have the following result. First, if X is a locally compact metric space, of which F is a compact subset, then there exists $\kappa=\kappa(F)>0$ such that the closure F_κ, of the κ-neighborhood of F, is compact; and then for any $\lambda<\kappa$, F_λ is also compact.

3.15. LEMMA. *In the locally compact metric*(7) LC^1 *space* X, *let* F *be a compact subset,* G *a neighborhood of* F. *Given* $\zeta>0$ *such that* F_ζ *is compact,* $F\subseteq F_\zeta \subseteq G$, *and*

$$\text{(i)} \qquad \pi_1(F \mid G, y_0) = \pi_1(F_\zeta \mid G, y_0)$$

relative to some base-point $y_0 \in F$, *then* $\pi_1(F| G, y_0)$ *is finitely generated.*

Proof. Since X is LC^1 and locally compact metric, there is, by [LTI, 7.1],

(7) The LC^1 property is defined in 4.21 below.

a function $\eta(T, \delta, \epsilon)>0$, defined for any compact subset T of X and all ϵ, $\delta>0$, with this property: every partial realization(8) in T, of mesh $<\eta$, of a finite 2-dimensional complex, can be extended to a full realization, in the δ-neighborhood of T, of mesh $<\epsilon$. Now with ζ as above, let

$$\epsilon = 4^{-1}\eta(F_\zeta, \sigma, \sigma), \qquad \beta = 4^{-1}\eta(F, \zeta, \epsilon)$$

where

$$\sigma = 2^{-1}\operatorname{dist}(F_\zeta, X - G).$$

Let K be the 2-skeleton of the nerve of a finite covering $\{U(x_i, \beta)\}$ of F, where $x_1 = y_0$ and each $x_i \in F$. If p denotes the (1-1) correspondence $k_i \rightarrow x_i$ between the vertices k_i of K and the points x_i of F, then p, as a partial realization in F of K, is of mesh $<2\beta<\eta(F, \zeta, \epsilon)$. By definition of the η-function above, p can now be extended to be a full realization of K in F_ζ, of mesh $<\epsilon$, i.e. $p: K \rightarrow F_\zeta$ is a mapping. We therefore have homomorphisms

$$\pi_1(K, k_1) \xrightarrow{\psi} \pi_1(F_\zeta, x_1) \xrightarrow{j} \pi_1(G, x_1)$$

where ψ is induced by p, and j is the injection, giving

$$j\psi = \theta: \pi_1(K, k_1) \rightarrow \pi_1(F_\zeta \mid G, x_1).$$

We shall shortly prove

(ii) $$\pi_1(F \mid G, x_1) \subseteq \theta\pi_1(K, k_1) \ (\subseteq \pi_1(F_\zeta \mid G, x_1))$$

which, with the above hypothesis (i) gives

$$\pi_1(F \mid G, x_1) = \theta\pi_1(K, k_1).$$

Now K is a finite complex and so has a finitely generated "Kantenweggruppe" (see Seifert-Threlfall [14, p. 158]) isomorphic to $\pi_1(K, k_1)$. Since θ is a homomorphism, $\pi_1(F|G, x_1)$ is therefore finitely generated(9).

To prove (ii), let $f: E^1, \dot{E}^1 \rightarrow F, x$ be a loop in F and let $\{U(x_{i(r)}, \beta)\}$, $r=1, \cdots, s+1$, cover $f(E^1)$, where we assume the numbering to be such that

$$U(x_{i(r)}, \beta) \cap U(x_{i(r+1)}, \beta) \neq 0, \qquad r = 1, \cdots, s;$$

and $i(1)=i(s+1)=1$. For each $r=1, \cdots, s$, choose a point $\xi_r \in E^1$, such that $f(\xi_r) = y_r \in f(E^1) \cap U(x_{i(r)}, \beta)$ with $y_1 = x_1 = y_{s+1}$. Then if the metric in X is ρ,

$$\begin{aligned} \rho(y_r, y_{r+1}) &\leqq \rho(y_r, x_{i(r)}) + \rho(x_{i(r)}, x_{i(r+1)}) + \rho(x_{i(r+1)}, y_{r+1}) \\ &\leqq \beta + 2\beta + \beta \\ &= 4\beta. \end{aligned}$$

(8) The terms employed are defined in Lefschetz [11, Chapter II].

(9) If $\pi_1(F|G, x)$ was Abelian, the hypothesis (i) would not be necessary, by (ii). A subgroup of a finitely generated, non-Abelian group may well not be finitely generated.

Hence, if $\lambda_r(\xi)=f(\xi_r+\xi_{r+1}(\xi-\xi_r)/(\xi_{r+1}-\xi_r))$ denotes the part of the curve $f(E^1)$ between y_r and y_{r+1}, then λ_r is a path in F of diameter $<4\beta$. Also, $\rho(x_{i(r)}, y_r)<\beta<\eta(F, \zeta, \epsilon)$, and so we may join $x_{i(r)}$ to y_r by a path μ_r of diameter $<\epsilon$ in F_ζ. Since $x_{i(1)}=y_1=x_1=x_{i(s+1)}$ we take $\mu_1=\mu_{s+1}$ to be the point x_1; and since $U(x_{i(r)}, \beta)\cap U(x_{i(r+1)}, \beta)\neq 0$, then $x_{i(r)}$ is already joined to $x_{i(r+1)}$ by a path ν_r—the image by p of an edge of K ($r=1, \cdots, s$). The diameter of ν_r is $<\epsilon$, and therefore the loop $\lambda_r-\mu_{r+1}-\nu_{r+1}+\mu_r$ is an image, say by f_r, of E^2 in F_ζ, and of diameter $<3\epsilon+4\beta<4\epsilon=\eta(F_\zeta, \sigma, \sigma)$, (for $\beta=\eta(F, \zeta, \epsilon)/4<\epsilon/4$).

Hence f_r may be extended to a mapping f_r' of the disc E^2, of diameter $<\sigma$ and so in $F_{\zeta+\sigma}\subseteq G$. Using the deformation d_t of E^1 given by $d_t(\xi_0, \xi_1)=(\xi_0, t\xi_1)$ $(0\leqq t\leqq 1)$, λ_r is deformable in $f_r'(E^2)\subseteq G$ to ν_{r+1}, with end-points on $-\mu_{r+1}$ and $-\nu_{r+1}$; hence by combining these deformations in the obvious way,

(iii) $$f\simeq\sum_{r=1}^{s+1}\nu_r \text{ in } G,$$

and this homotopy is rel y_0 since $\nu_1=\nu_{s+1}=x_1$. But by definition of ν_r, $\sum\nu_r$ is the image by p of a closed edge-path

$$\gamma = k_{i(1)}k_{i(2)}\cdots k_{i(s)}k_{i(1)}$$

on K. Denoting homotopy classes in $\pi_1(G, x_1)$ by $[h]$, we thus have from (iii)

$$[f] = [\sum \nu_r] = [p\gamma] = j\psi[\gamma] = \theta[\gamma],$$

and since $[f]$ is the class of f in $\pi_1(F|G, x_1)$, this proves (ii), and completes the proof of the lemma.

If we put $F=X$ in 3.15, we get:

3.16. *The fundamental group of a compact metric LC^1 space is finitely generated.* Neither the "compact" nor the "LC" can be omitted: for counterexamples see Griffiths [9, p. 470].

IV. Local topology

We are now ready to apply the results of the previous sections to the various local groups at a point x of the space X, *which is always taken to be locally compact metric.*

4.1. **Local Betti numbers.** If $\mathfrak{U}$ denotes the system of all neighborhoods of x in X, directed as in 2.3 (i), we shall denote by $\mathfrak{U}_0$, $\mathfrak{U}_c$ respectively the systems of open and of compact members of $\mathfrak{U}$, so that(10)

(i) $$\mathfrak{U}_0,\ \mathfrak{U}_c \text{ cof } \mathfrak{U}.$$

Apart from local connectivity, the earliest algebraic local invariant to be considered in Topology was the Alexandroff-Cech "local Betti number" $p^n(x)$, defined for coefficients in a field $\mathfrak{F}$; and the important case is when

(10) cof was defined at the start of 1.2. For typographical reasons we write the $\overline{M}$ of 1.1 as M^*.

$p^n(x)$ is finite. This occurs, as we see from(11) 6.11 of Wilder [15, p. 192], if and only if there is a subset $\mathfrak{O}$ cof $\mathfrak{U}_0^2$ such that for every pair $(P, Q)\in\mathfrak{O}$, the image of the injection

$$i_{PQ}: H_n\Omega(X, X-P) \to H_n\Omega(X, X-Q)$$

is a vector space over $\mathfrak{F}$ of dimension $p^n(x)$. Putting

$$B^P = H_n(X, X-P), \qquad b_P^Q = i_{PQ},$$

it follows that $\{B, b\}$ is a direct system over $\mathfrak{U}_0$, and in the notation of 1.3,

(ii) $$p^n(x) \text{ finite} \Longleftrightarrow \dim B^{PQ} = p^n(x), \text{ if } (P, Q) \in \mathfrak{O}.$$

Since we are here dealing with vector spaces, (iii) and (iv) of 1.3 imply that if $T\in\mathfrak{U}_0$ and $P\supseteq T\supseteq Q$, then respectively:

(a) if $(T, Q)\in\mathfrak{O}$ then $B^{PQ}=B^{TQ}$;

(b) if $(P, T)\in\mathfrak{O}$, the injection $B^{PT}\to B^{PQ}$ is an isomorphism.

In other words, the system $\{\overline{B}, \bar{b}\}$ is stable rel $\mathfrak{O}$ in the sense of 1.3, and its direct limit—which is Dlim $\{B, b\}$ by 1.3 (v)—is a vector space over $\mathfrak{F}$, of dimension $p^n(x)$.

Now for any coefficient group, whether we have stability or not, Dlim$\{B, b\}$ always exists. We assert

4.11. Lemma. Dlim $\{B, b\}\approx H_n\Omega(X, X-x)$.

Proof. By 2.2 (iv), we have

$$H_n\Omega(X, X-x) = \text{Dlim }\{H_n\Omega(K, F), \pi\}$$

taken over the system $\mathfrak{K}$ of all compact pairs $(K, F)\subseteq(X, X-x)$ directed by inclusion. But every compact K has a compact neighborhood G, since X is locally compact; and indeed we can take G large enough to be a neighborhood of x. Also $F\subseteq K\cap(X-x)$, so that there exists $U\in\mathfrak{U}_0$, such that $F\subseteq G-U$. Hence, if $\mathfrak{K}'$ is the set of pairs $(G, G-U)$, where $G\in\mathfrak{U}_c$, $U\in\mathfrak{U}_0$, then $\mathfrak{K}'$ cof $\mathfrak{K}$, and so

(i) $$H_n\Omega(X, X-x) \approx \text{Dlim }\{H_n\Omega(G, G-U), \pi\}, \ (G, G-U) \in \mathfrak{K}'.$$

But G is closed, and therefore if it is so small that Interior $(X-G)\neq 0$, the inclusion $(G, G-U)\subseteq(X, X-U)$ induces an isomorphism

(ii) $$\eta_{G,U}: H_n\Omega(G, G-U) \approx H_n\Omega(X, X-U) = B^U;$$

this is by the fact (whose proof we omit) that $H\Omega$ satisfies the Excision Axiom. Since η commutes with injections, we therefore obtain from (i) and (ii)

$$H_n\Omega(X, X-x) \approx \text{Dlim }\{B, b\},$$

as required.

(11) We use Wilder's notation $p^n(x)$, but remark that our $H_n\Omega$ is his H^n (Cech groups).

REMARK. If $\mathfrak{T}$ denotes the system of all open sets of X directed by inclusion, then $\{B, b\}_{\mathfrak{T}}$ defines a pre-sheaf, and hence a sheaf $\mathcal{S}$, on X (see Cartan [2]), and the above shows that the "stalk" at each point x is $H_n\Omega(X, X-x)$. Hence, if coefficients are in the field $\mathfrak{F}$, and at all $x\in X$, $p^j(x)=\delta_{nj}$ (Kronecker delta) where dim $X=n$, then X is a generalized manifold in the sense of Wilder, while $\mathcal{S}$ is the simple faisceau $X\times\mathfrak{F}$ (provided X is orientable) because all the stalks are isomorphic to $\mathfrak{F}$ (by 4.11 (ii)). This suggests a new direction in which to generalize Wilder's work (which we shall not here pursue).

4.12. The first new local group we introduce is the following. Fixing n, define for each $P\in\mathfrak{U}(x)$

$$\Omega_P = H_n\Omega(P - x),$$

and if $P\supseteq Q$, let $\omega_P^Q\colon \Omega_Q\to\Omega_P$ be the injection. Then (Ω, ω) is an inverse system over $\mathfrak{U}$, with limit

$$H_n\Omega(x) = \operatorname{Ilim}\,(\Omega, \omega). \tag{i}$$

Also, for each pair $(P, Q)\in\mathfrak{U}^2$,

$$\Omega_{PQ} = H_n\Omega(Q - x\,|\,P - x)$$

in the notation of 1.1 (iv) and 2.3 (iii); so that if $(\bar{\Omega}, \bar{\omega})$ is stable, then in the sense of [LTI, Definition 6.1], a group $C_v^n(x)$ exists at x, and is isomorphic to $H_n\Omega(x)$. But of course the converse may not hold. For example, in the coordinate plane R^2, let Z_n be the circle $(x-1/2^n)^2+y^2=(1/2^{n+1})^2$. Let $z=(0, 0)$ and let $Z=\bigcup_n Z_n$. Then for any neighborhoods P, Q of z in Z, with $P\supseteq Q$, we have—using integer coefficients—that $H_1\Omega(Q-z\,|\,P-z)$ is the free group A on $\aleph_0$ generators, and so $C_v^1(z)$ exists. On the other hand $H_1\Omega(z)=0\neq A$, but the group is not stable. However, in many cases, the two local groups coincide, as is shown by some of the theorems of LTI and CTM. All the latter depend on remarks of this kind:

(a) if $L\subseteq M\subseteq N$ are subsets of X, then as in 1.3 (iii) there is an epimorphism $q\colon H_n\Omega(L\,|\,M)\to H_n\Omega(L\,|\,N)$, and if both groups are known to be isomorphic to the same finitely generated Abelian group, then q is an isomorphism (it is therefore important to know when the various groups *are* finitely generated Abelian and this is why we proved 2.32, 3.14, above);

(b) as in 1.3 (iv), $H_n\Omega(L\,|\,N)\subseteq H_n\Omega(M\,|\,N)$, so that if certain topological conditions like 4.51(a) below (see e.g. [CTM, 4.3]) are satisfied then the inclusion is an equality. Similar remarks apply when we use the singular and homotopy functors.

The upshot of all this is, that in all "reasonable" cases where a "C" group exists and is finitely generated, it is equal to the corresponding limit group which is also stable (we do not propose to make here a detailed study of the pathology of the question). In view of the greater harmony of the "limit" theory, we shall now drop the "C" invariants and concentrate on their usurpers, the "limit" groups. This does not of course answer all the technical questions of the "C" theory: we merely explain them away, feeling that it is more important to get on with the nonpathological theory. Incidentally, an example of the aforementioned harmony is this: in [LTI, 6.8], we wonder whether in the definition of the C groups we should use groups of the form $H_n\Omega(P-x|Q-x)$ or $H_n\Omega(\overline{P}-x|\overline{Q}-x)$, $(P, Q\in\mathfrak{U}_0)$. But by 4.1 (i), it is immaterial for stability whether we use a $(P, Q)\in\mathfrak{U}_0^2$ or $(S, T)\in\mathfrak{U}_c^2$, because of 1.21 and 1.23. And this, one feels, is the way things should be.

4.2. **Local connectivity.** In addition to the types of local connectivity considered in 2.3, there is the homotopy form:

4.21 X is $q-LC$ *at* x *whenever, given* $P\in\mathfrak{U}(x)$, *there exists* $Q\in\mathfrak{U}(x)$ *such that* $P\supseteq Q$ *and*(12) $\pi_q(Q|P, x)=0$. X *is* LC^r [*at* x] *whenever it is* $q-LC$ *everywhere* [*at* x], $0\leqq q\leqq r$.

Put

$$Q = \Lambda_q(P).$$

4.22. THEOREM. *If* X *is* LC^q *at* x, *it is* lc_s^q *at* x (*over the integers*).

Proof. (a) $q=0$. The proof for this case is straightforward, easy, and omitted. We remark however that it holds for all coefficients.

(b) $q\geqq 1$. Define a chain $R=Q_0\subseteq \cdots \subseteq Q_{q+1}=P$, where

$$Q_r = \Lambda_r(Q_{r+1}), \qquad 0 \leqq r \leqq q, \qquad (P, Q_r, R \in \mathfrak{U}_c)$$

so that

$$\pi_r(Q_r\,|\,Q_{r+1}, x) = 0.$$

We then have the commutative diagram

$$\begin{tikzcd}
H_qS(R) \arrow[r, "i'"] & H_qS(Q_q) \arrow[r, "i"] & H_qS(P) \\
H_qS_q(R) \arrow[u, "\lambda"] \arrow[r, "k'"'] & H_qS_q(Q_q) \arrow[u, "\mu"'] \arrow[r, "k"'] \arrow[from=ul, "\pi"] & H_qS_q(P) \arrow[u, "\nu"'] \\
& \pi_q(Q_q) \arrow[u, "\nu_1"] \arrow[r, "j"'] & \pi_q(P) \arrow[u, "\nu_2"']
\end{tikzcd}$$

where all arrows except π, ν_1 and ν_2 denote injections, ν_1, ν_2 are the homomorphisms of Eilenberg [3, p. 443] and π is the "q" of 3.13, diagram (i). (The restriction $n>1$ in 3.13 does not apply to that diagram.) Then X will be q-lc$_s$ at x if we can prove $H_qS(R|P)=0$. But

(12) In dimension zero, this is to be interpreted: every pair of points in Q can be joined by a path in P.

$$\begin{aligned} H_qS(R\,|\,P) &= ii'\Gamma, \qquad\qquad (\Gamma = H_qS(R)) \\ &= i\mu\pi\Gamma = \nu k\pi\Gamma \subseteq \nu kH_qS_q(Q_q) \\ &= \nu k\nu_1\pi_q(Q_q) \end{aligned}$$

because ν_1 is always onto (when $n=1$ or $n>1$; see Eilenberg [3, p. 443]). But $k\nu_1=\nu_2 j$, and $j\pi_q(Q_q)=0$ because $Q_q=\Lambda_q(Q_{q+1})$. Therefore $H_qS(R|P)=0$, as required. This completes the proof[13].

4.23. THEOREM[14]. *If X is at x both lc^q_s (over the integers) and $1-LC$, then it is LC^q at x.*

Proof. If $q=0$, this follows from 4.22; and hence if $q=1$, there is nothing to prove. Suppose then that $q>1$, and assume inductively that we have already proved X to be LC^{q-1} at x. Then given $P\in\mathfrak{U}(x)$, there is in $\mathfrak{U}$ a chain

$$Q = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_q = P$$

where

$$A_r = \Lambda_r(A_{r+1}), \qquad r = 0, \cdots, q-1,$$

and so 3.13 applies with $X=Q$, $Y=P$, $n=q$. Define R to be $\lambda^s_q(Q)$, λ^s_q defined in 2.3 (x). There results a commutative diagram

$$\begin{tikzcd} H_qS(R) \arrow[r, "i'"] & H_qS(Q) \arrow[r, "i"] \arrow[dr, "\pi"] & H_qS(P) \\ H_qS_q(R) \arrow[r, "k'"'] \arrow[u, "\nu"] & H_qS_q(Q) \arrow[r, "k"'] \arrow[u, "\lambda"] & H_qS_q(P) \arrow[u, "\mu"'] \\ \pi_q(R) \arrow[r, "j'"'] \arrow[u, "\nu_0"] & \pi_q(Q) \arrow[r, "j"'] \arrow[u, "\nu_2"] & \pi_q(P) \arrow[u, "\nu_1"'] \end{tikzcd}$$

where all arrows except ν_0, ν_1, ν_2 and π are injections, these ν's are the isomorphisms of Eilenberg [3, p. 443], and π is the "q" of 3.13, diagram (i). X will be q-LC at x if we can prove $\pi_q(R|P)=0$, i.e. $jj'\pi_q(R)=0$. But $\nu_1 jj' = kk'\nu_0=\pi\lambda k'\nu_0=\pi i'\nu\nu_0$; and since $R=\lambda^s_q(Q)$, $i'=0$. Hence $\nu_1 jj'=0$, and so, because $q>1$ implies ν_1 univalent, $jj'=0$. Therefore $\pi_q(R|P)=0$ as required. Thus X is q-LC and LC^{q-1} at x, i.e. X is LC^q at x, and the proof is complete.

It is well-known that if X is locally compact metric, then X is 0-lc$_v$ if and only if X is 0-LC. Hence by 4.22, the 0-lc$_v$, 0-lc$_s$ and 0-LC properties coincide. But in dimension 1, things are different as the following example shows. For

[13] The referee points out that the result is a strengthening of Theorem X of Lefschetz, Duke Math. J. vol. 1 (1935) p. 15, in that we have assumed here only LC^1 at x.

[14] Cf. Hurewicz [10].

each $n>0$, let P_n be a Poincaré space in R^7, and let all the P_n be united at a single point p, (but otherwise disjoint) to form a single space P_* in which

$$\text{diam } P_n \to 0 \quad \text{as} \quad n \to \infty;$$

then by Griffiths [9, p. 477], P_* is lc_v^1 but not lc_s^1 over the integers, (and therefore not LC^1, by 4.23). On the other hand for any ring of coefficients (commutative and with unit):

4.24. LEMMA(15). *If X is lc_s^1 it is lc_v^1.*

Proof. By the remarks above and the result of Begle quoted in 2.3, it suffices to prove X to be $(V, 1)$-lc; and by 2.31 we can assume X to be $(S, 1)$-lc. Let then $x \in X$, and $P \in \mathfrak{U}_c(x)$; let $\epsilon > 0$ be given. In the notation of 2.3 (x), let

$$Q = \overset{s}{\lambda}_1(P), \qquad \delta = \overset{s}{\lambda}_1(P, \epsilon),$$

and let R be a compact neighborhood of x such that

(i) $$\text{dist } (R, X - Q) = \xi > 0.$$

Then since R is compact and X is 0-LC, there is a function $\mu(\alpha, \beta)$ such that any pair of points in R whose distance apart is $<\mu(\alpha, \beta)$, can be joined by a path of diameter $<\beta$, in the α-neighborhood of R. Put $\nu = \mu(\xi, \delta)$.

To show that X is $(V, 1)$-lc at x, it suffices to show that every 1-cycle in $C_1\Omega(R, \nu)$ bounds in $C_1\Omega(P, \epsilon)$. Let $\tau: \Delta_1 \to R$ be any 1-cell in $C_1\Omega(Q, \nu)$; thus dist $(\tau d^0, \tau d^1) < \nu$, and so by definition of $\mu(\xi, \delta)$ above, there is in the ξ-neighborhood of R (and therefore in Q, by (i)) a path—that is, a singular 1-cell—$T: \Delta_1 \to Q$, of diameter $<\delta$, such that

(ii) $$Td^0 = \tau d^0, \qquad Td^1 = \tau d^1.$$

Hence, distinguishing the appropriate boundary operators, we have

(iii) $$\partial_s T = T^{(0)} - T^{(1)} = \tau^{(0)} - \tau^{(1)} = \partial_v \tau.$$

Using the map $\sigma: HS \to H\Omega$ of 2.2 (v), equations (ii) become

$$\sigma T = \tau;$$

so that if by linearity we extend the correspondence $\tau \to T$ to be a homomorphism θ: $C_1\Omega(R, \nu) \to C_1S(Q, \delta)$, we get

(iv) $$\sigma\theta = 1,$$

while by (iii), $\partial_s\theta = \partial_v$. Hence, if γ is a 1-cycle in $C_1\Omega(R, \nu)$, then

$$0 = \partial_v\gamma = \partial_s\theta\gamma$$

(15) Lefschetz, in Duke Math. J. vol. 2 (1936) p. 439, asserts that lc_s^n implies lc_v^n using "say, rational coefficients." No proof has appeared. [*Added in proof*, September 1958: for paracompact Hausdorff spaces, the assertion follows from the result of Mardešić, cited after 2.32.]

and so $\theta\gamma$ is a 1-cycle in $C_1S(Q, \delta)$. By definition of (Q, δ), there is a 2-chain $\Gamma \in C_2S(P, \epsilon)$ such that $\theta\gamma = \partial_s\Gamma$. Hence by (iv)

$$\gamma = \sigma\theta\gamma = \sigma\partial_s\Gamma = \partial_v\sigma\Gamma$$

and $\sigma\Gamma \in C_2\Omega(P, \epsilon)$. Thus γ bounds in $C_1\Omega(P, \epsilon)$, as required, and the proof is complete.

If we try to use this procedure in the next dimension, we cannot obtain a map θ of $C_2\Omega(R, \nu)$ satisfying 4.24 (iv), unless we assume by analogy that X is 1-LC also. However, no example is known of a space which is lc_s^1 but not LC^1 (see [9] for a further discussion) and every locally compact metric LC^n space is(16) lc_v^n for all n.

4.3. **The local cut-point groups.** The "C" groups of LTI generalized Wilder's notion of a "local noncut point," [LTI, p. 356]. We agreed in 4.1 to jettison the "C" groups in favor of stable groups like $H_n\Omega(x)$ in 4.12 (i), and so we shall call these latter the "local (G) Cut-point groups," where G refers to the particular functor under consideration. Thus, the singular analogue of $H_n\Omega(x)$ is $H_nS(x)$. With our usual fixed point $x \in X$, let $\mathfrak{U}_0(x)$ be as in 4.1 (i); thus for each $P \in \mathfrak{U}_0$, $P - x$ is also open. Hence, if X is lc_v^q and lc_s^q so is $P - x$, and therefore by 2.32 we have isomorphisms

$$\sigma_P \colon H_rS(P - x) \approx H_r\Omega(P - x)$$

which commute with injections. Therefore since $\mathfrak{U}_0$ cof $\mathfrak{U}$, we have, on taking inverse limits, the following "local" analogue of 2.32:

4.31. Theorem. *If X is locally compact metric, lc_s^q and lc_v^q, then*

$$\sigma_\infty \colon H_rS(x) \approx H_r\Omega(x), \qquad 0 \leqq r \leqq q,$$

and if one group is stable, so is the other.

In order to define the local homotopy cut-point groups, we have to assume that x in X satisfies:

4.32. $\mathfrak{U}$ *has a cofinal subset* $\mathfrak{U}_\gamma$ *such that for each* $P \in \mathfrak{U}_\gamma$ *both* P *and* $P - x$ *are path-wise connected.*

Simple topological conditions on the pair (X, x) ensure that 4.32 is satisfied: see, for example, [LTI, 4.3]. Not all "reasonable" spaces satisfy 4.32; for example, with a double cone, 4.32 fails at the vertex—yet if the "upper" half of the cone is bent over so that one of its generators lies along a generator of the "lower" half, the resulting space satisfies 4.32 everywhere.

4.33. Assuming then that X satisfies 4.32 at x, we shall now define a group $\pi_q(x)$, in a manner analogous to the definition of $H_q\Omega(x)$. We choose a fixed

(16) Hurewicz [10]. Strictly, Hurewicz proves this for a compact space, but only trivial changes are required in his proof. Nontrivial changes are needed for the other half of the theorem: See Newman [13].

path λ from x to some point $y \neq x$. This λ will exist if $\mathfrak{U}_\gamma$ contains a U with at least two distinct points, because U is path-wise connected. Let

$$\mathfrak{U}_\gamma' = \{U \mid U \in \mathfrak{U}_\gamma \,\&\, y \notin U\};$$

thus, for each $U \in \mathfrak{U}_\gamma'$, λ meets U (in x), and $X - U$ (in y), and so λ meets Frontier (U) in a "first" point, travelling from x—say $f(U)$. Now X is metric and so we can assume $\mathfrak{U}_\gamma'$ is countable—say

$$\mathfrak{U}_\gamma' = \{U_1, U_2, \cdots, U_n, \cdots\}$$

where

$$U_n \supseteq \overline{U}_{n+1}.$$

Let

$$f_n = f(U_{n+1});$$

then the portion λ_n of λ from x to f_n lies wholly in $\overline{U}_{n+1} \subseteq U_n$. Moreover, if $U_n \supseteq U_m$, the path

$$\lambda_{mn} = \lambda_n - \lambda_m \tag{i}$$

lies wholly in U_n. Hence, fixing q, λ_{mn} induces an isomorphism

$$\lambda_{mn}: \pi_q(U_n - x, f_m) \to \pi_q(U_n - x, f_n),$$

and there is an injection

$$i_{mn}: \pi_q(U_m - x, f_m) \to \pi_q(U_n - x, f_m).$$

Next, define $P_n = \pi_q(U_n - x, f_n)$ and if $n \leqq m$,

$$\overset{m}{p}_n = \begin{cases} \text{identity on } P_n, & n = m, \\ \lambda_{mn}\, i_{mn}, & n < m. \end{cases}$$

Then, if $n < m < j$, the diagram

$$\begin{array}{ccccccc}
\pi_q(U_j - x, f_j) & \xrightarrow{i_{jm}} & \pi_q(U_m - x, f_j) & \xrightarrow{\lambda_{jm}} & \pi_q(U_m - x, f_m) & \xrightarrow{i_{mn}} & \pi_q(U_n - x, f_m) \\
{\scriptstyle i_{jn}}\downarrow & & & & & & \downarrow{\scriptstyle \lambda_{mn}} \\
\pi_q(U_n - x, f_j) & & & \xrightarrow[\lambda_{jn}]{\hspace{6em}} & & & \pi_q(U_n - x, f_n)
\end{array}$$

is commutative because

$$\lambda_{jn} = \lambda_n - \lambda_j = (\lambda_n - \lambda_m) + (\lambda_m - \lambda_j) = \lambda_{mn} + \lambda_{jm}.$$

Therefore

$$\overset{n}{p}_j = \overset{n}{p}_m \overset{m}{p}_j.$$

Hence, if J denotes the set of integers >0, directed by the natural ordering

$\leqq$, then it can be verified that (P, p) is an inverse system over J, in that (i) and (ii) of 1.1 hold. We define

$$\text{(i)} \qquad \pi_q(x) = \operatorname{Ilim} (P, p)_J.$$

Of course, this definition depends on the choice of the points f_n, and the path λ; and corresponding to all possible choices we obtain a transitive system of groups in the sense of [E-S, p. 17], whose inverse limit is what "ought" to be called $\pi_q(x)$. It is less complex, for our purposes, to take $\pi_q(x)$ as defined in (i), with a fixed λ and sequence $\{f_n\}$ throughout.

4.34. An important case is when the system $(\overline{P}, \bar{p})_{\bar{J}}$ is stable, and $\pi_q(x) = 0$; we shall then write

$$\pi_q(x) \equiv 0.$$

Now $P_{(n,m)}(n<m)$ is $\lambda_{mn}\pi_q(U_m-x| U_n-x, f_m)$ and so is zero if and only if $\pi_q(U_m-x| U_n-x, f_m)=0$, because λ_{mn} is an isomorphism. By 1.26, there exist sequences A, B cof J, such that $(\overline{P}, \bar{p})$ is stable rel $A \circ B$. Hence, given $j \in J$ there is a first $a_* = a^q(j) \in A$, $(a_* > j)$ such that if $a \geqq a_*$ and $a \in A$, there is a first $b_* = b^q(j, a) \in B$ such that $b_* > a$ and for all $b \in B$ with $b \geqq b_*$, $\overline{P}_{(a;b)} = 0$, i.e.

$$\pi_q(U_b - x \mid U_a - x, f_b) = 0.$$

The "local" analogue of Hurewicz's global theorem is now:

4.35. Theorem. *If* $q>1$ *and* $\pi_r(x) \equiv 0$, $0 \leqq r < q$, *then* $\pi_q(x) \approx H_q S(x)$ (*integer coefficients*) *and if one group is stable, so is the other.*

Proof. Define a map $g: J \to J$ as follows. With a^r, b^r as in 4.34, let $j \in J$ and put

$$j_1 = b^{q-1}(j, a^{q-1}(j)), \qquad j_2 = b^{q-2}(j_1, a^{q-2}(j_1)) \cdots,$$
$$j_q = b^0(j_{q-1}, a^0(j_{q-1})).$$

Define $g(j)$ to be j_q; $g(j) > j$ because $a^r(j) > j$, and $b^r(j, a) > a$ when $a > j$. Hence the set

$$\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n \cdots, \}$$

where

$$\gamma_{n+1} = g(\gamma_n), \qquad \gamma_1 = g(1)$$

is cofinal in J. By construction and 4.34 (i) we have, in the notation of 3.13, that for each j, $k \in \Gamma$ with $j < k$,

$$U_k - x <_{q-1} U_j - x.$$

Hence there is a commutative diagram of the form given by 3.13:

(i)
$$\begin{tikzcd} \pi_q(U_k - x, f_k) \arrow[r, "p"] \arrow[d, "h_k"'] & \pi_q(U_j - x, f_j) \arrow[d, "h_j"] \\ H_qS(U_k - x) \arrow[r, "s"'] \arrow[ur, "\pi"] & H_qS(U_j - x, f_j) \end{tikzcd}$$

where $s = s_j^k$ is the injection, $p = p_j^k$, and h_k, h_j are the Hurewicz homomorphisms. For brevity write

$$S_j = H_qS(U_j - x);$$

then the system (S, s) is an inverse system over J. Let (P, p) be the inverse system defined in 4.33, so that the Hurewicz homomorphisms h_k define a homomorphism

$$h: (P, p) \to (Q, q).$$

The existence of diagram (i), with $k = j+1$, enables us to apply 1.42, to assert that

$$h_\infty: \text{Ilim } (P, p) \approx \text{Ilim } (S, s),$$

i.e.

$$h_\infty: \pi_q(x) \approx H_qS(x).$$

To prove the stability part of the theorem, we apply 1.43 with $(Q, q) = (S, s)$, $(M, \leqq) = (\Gamma, <)$, in view of diagram (i) above. Thus we have immediately: if $\pi_q(x)$ is stable, so is $H_qS(x)$, and conversely. This completes the proof.

In dimension 1 we have

4.36. THEOREM. *If $\pi_1(x)$ is stable, so is $H_1S(x)$, and $H_1S(x)$ is $\pi_1(x)$ made Abelian.*

Proof. By 4.32, each $U \in \mathfrak{U}_\gamma$ is such that $U - x$ is pathwise connected. Hence the natural homomorphism

$$\nu_U: \pi_1(U - x, f(U)) \to H_1S(U),$$

where $f(U)$ is defined in 4.33, is onto and its kernel is the commutator of $\pi_1(U - x, f(U))$. Therefore 1.51 applies with $M = \mathfrak{U}_\gamma'$ and the theorem follows at once.

4.37. The converse of 4.36 (and so of 1.51) is false, as the following example shows. For each n, let Z_n be a Poincaré space in Euclidean R^7, such that, E^1 being the unit segment in R^7, $Z_n \cap E^1$ is the point $1/n \in E^1$, the Z_n are all mutually disjoint and diam $Z_n \to 0$ when $n \to \infty$. Let $Z = E^1 \cup \bigcup_{m=0}^{\infty} Z_m$. Then the origin z of R^7 has in Z a basis of neighborhoods of the form

(i) $$W_m = l_m \cup \bigcup_{n=m}^{\infty} Z_m,$$

where l_m is the segment of E^1 from z to $1/m$. But $H_1S(Z_m)=0$, so that $H_1S(W_m-z)=0$ because H_1S has compact carriers. Then $H_1S(z)\equiv 0$; whereas $\pi_1(x)$ is easily seen to be zero but unstable.

4.4. **Some pathology.** The last example is useful for studying pathological behavior of the singular homology functor. We recall from 2.3 the groups

$$L_qS(x), \qquad L_q\Omega(x),$$

and use the machinery of 4.33 to define the homotopy analogue

$$L_q\pi(x) = \text{Ilim}\,\{\pi_q(U_n, f_n), l_n^m\}$$

where $U_n \in \mathfrak{U}_\gamma$, and l_n^m is defined as follows. The paths λ_{mn} of 4.33 induce isomorphisms

$$\nu_{mn}\colon \pi_q(U_n, f_m) \to \pi_q(U_n, f_n),$$

and there is an injection

$$j_{mn}\colon \pi_q(U_m, f_m) \to \pi_q(U_n, f_m);$$

so we define l_n^m to be the identity on $\pi_q(U_n, f_n)$, if $m=n$, and otherwise to be $\nu_{mn}j_{mn}$. We assert that

(i) *$L_q\Omega(x)$ and $L_q\pi(x)$ are always zero*[17].

A sketch of the proof is as follows. By (iv) and (v) of 1.1, if K cof J, then

$$\text{Ilim}\,(P, p)_K \approx \text{Ilim}\,(\overline{P}, \bar{p})_{\overline{K}} \approx \text{Ilim}\,(Q, q)_K$$

where

$$Q_n = \bigcap_{m \in K} P_{nm},$$

and $q_n^j\colon Q_j \to Q_n$ is defined by $q_n^j = p_{n,1}^{j,1} | Q_j$. Now take $P_{nm} = \pi_q(U_m | U_n, f_m)$. Then every loop λ in an element $[\lambda] \in Q_n$ has the property that given U_r however small (and so $r > n$) there is a loop λ' on U_r such that $\lambda \simeq \lambda'$ rel f_r on U_n. If, moreover, $[\lambda'] \in Q_r$ and $q_r^s[\lambda''] = [\lambda']$ where $[\lambda''] \in Q_s$ and $s > r$, then $\lambda' \simeq \lambda''$ rel f_s on U_r; and so on, inductively. By piecing these homotopies together in the obvious way we obtain a homotopy $\lambda \simeq x$ rel f_n on U_n, and so $Q_n = 0 = \pi_q(x)$. That $L_q\Omega(x)=0$ follows because the Vietoris and Cech theories coincide and the latter satisfies the Axiom of Continuity; thus

$$L_q\Omega(x) = \text{Ilim}_{x \in U} H_q\Omega(U - x) = H_q\Omega(\bigcap (U - x))$$
$$= 0.$$

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(17) Compare Wilder [15, VI 6.13, p. 192].

This proves (i). Of course the groups are not necessarily stable: when they are we have local connectivity.

Now let us consider the group $L_1S(z)$ (over the integers) in the space Z of 4.37. We shall show that it is nonzero and stable. If W_m is as in 4.37 (i), then

$$W_m = W_{m+1} \cup V_m, \qquad V_m = Z_m \cup \langle 1/m + 1, 1/m \rangle,$$

and $W_{m+1} \cap V_m$ is the point $1/m+1$ of E^1. Thus, if $j_m: H_1S(W_{m+1}) \to H_1S(W_m)$, $k_m: H_1S(V_m) \to H_1S(W_m)$ are the injections, then since each Z_m is everywhere LC^1, it follows[18] that j_m, k_m are univalent and

$$\begin{aligned} H_1S(W_m) &= j_mH_1S(W_{m+1}) + k_mH_1S(V_m) \\ &= j_mH_1S(W_{m+1}) \end{aligned}$$

because $H_1S(V_m)=0$, Z_m being a Poincaré space. Therefore

(ii) $$j_m: H_1S(W_{m+1}) \approx H_1S(W_m),$$

and so

(iii) $$L_1S(z) \approx \operatorname{Ilim}\,(H_1S(W_m), \overset{n}{k_m}) \approx H_1S(W_1)$$

where

(iv) $$\overset{n}{k_m} = j_mj_{m+1} \cdots j_{n-1}.$$

But $W_1=Z$, and Z is of the same homotopy type as a space of the form $T=\bigcup_{m=1}^{\infty} (Z_m \cup p_m)$; where the Z_m are all disjoint as in Z, p_m meets Z_m just once and joins it to z, being otherwise disjoint from all other Z_n or p_n, and diam $p_m \to 0$ when $m \to \infty$. By Griffiths [9, p. 470], $H_1S(T)$ is infinite and therefore so is $H_1S(Z)$. Hence, by (ii), (iii) and (iv), $L_1S(z)$ is stable and infinite, in contrast to (i).

4.5. **Relative groups.** In this section, we link the classical local Betti number with the local homology cut-point groups, by using the results of 1.6. We also obtain analogues for the other functors and thereby show that introduction of relative groups does not, in general, lead to new invariants. First let $\mathfrak{U}_0'(x)$ be the subset of $\mathfrak{U}_0$ in 4.1 (i) consisting of all U with $\overline{U} \in \mathfrak{U}_c$. We recall from [6, 4.3] the following result.

4.51. LEMMA. *Let X be both (V, r)-lc and $(V, r+1)$-lc at x. If U, U_1, U_2, $W \in \mathfrak{U}_0'(x)$ such that*[19] $U_1 \subseteq \lambda_{r+1}^*(U)$, $U_2 \subseteq \lambda_r^*(U_1)$, $\overline{W} \subseteq U_2$, *then*

(a) *The inclusion*[20] $H_r\Omega(\mathrm{F}W | \overline{U}_1 - W) \subseteq H_r\Omega(\overline{U}_2 - W | \overline{U}_1 - W)$ *is an equality*;

(b) *the boundary homomorphism*

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[18] See Griffiths [8, 2.5].
[19] λ_r^* was defined in 2.3(x).
[20] $\mathrm{F}W$ = Frontier of W.

$$\partial: H_{r+1}\Omega(\overline{U}_1, \overline{U}_2 - W) \to H_r\Omega(\overline{U}_2 - W)$$

is onto.

(In [CTM], (b) is not proved explicitly; but an indication of the proof is given in the remark at the top of p. 73, *op. cit.*) Now W above can be as small as desired, and $H\Omega$ has compact carriers. Hence, taking the direct limit over all $W \in \mathfrak{U}'(x)$, we have the following statement, more suited to our purposes; in it, U_0 and Q can be taken to be respectively $\lambda_r^\bullet(\lambda_{r+1}^\bullet(U))$ and $\overline{U}_2$ (in the notation of 4.51).

4.52. Lemma. *If X is both (V, r)-lc and $(V, r+1)$-lc at x, then given $U \in \mathfrak{U}_c$ there exists $U_0 \in \mathfrak{U}_c$ such that for all $Q \subseteq U_0$ in $\mathfrak{U}_c$, the homomorphism*

$$\partial: H_{r+1}\Omega(U, Q - x) \to H_r\Omega(Q - x)$$

is onto.

It will be convenient to define the "first" suitable Q above to be

$$Q_r(U).$$

4.53. We now provide an example of the situation of 1.6: take $(M, \leqq)$ to be $\mathfrak{U}_c(x)$, and for each $U \in \mathfrak{U}_c$, define

$$R_{(U,V)} = H_{r+1}\Omega(U, V - x), \qquad (U \supseteq V)$$
$$A_U = H_r\Omega(U - x),$$

and satisfy 1.6 (i) by taking $r_{U,V}^{P,Q}$, a_U^V, d_U there to be the injections and the boundary operator, respectively. If X satisfies the conditions of 4.52, 1.6 (ii) holds, with $p: M \to M$ taken as the function $Q^r(U)$ of (ii) above. To show that under the same conditions, 1.6 (iii) holds, it suffices to prove: given pairs $(S, T) \supseteq (U, V) \supseteq (P, Q)$ in $\mathfrak{U}_c^2$, then in the diagram

$$\text{(i)} \qquad \begin{array}{ccccc} R_{(P,Q)} & \xrightarrow{r} & R_{(U,V)} & \xrightarrow{r'} & R_{(S,T)} \\ \downarrow & & \downarrow \partial & & \\ A_Q & \longrightarrow & A_V & & \end{array}$$

we have $\operatorname{Ker}(\partial r) \subseteq \operatorname{Ker}(r'r)$, whenever $U \subseteq Q_r(S)$. (Thus, the function $q: M \to M$ of 1.6 (iii) can be taken to be $Q_r: \mathfrak{U}_c \to \mathfrak{U}_c$). The proof depends on the fact that the sequence of 2.2(i) is exact, the unmarked arrows denoting injections; we leave the details to the reader.

4.54. It now follows that under the conditions of 4.52, we can apply 1.61 directly. It remains to interpret the groups Ilim $(\overline{A}, \bar{a})$, Ilim $(\overline{S}, \bar{s})$ which occur there. From the definition of (A, a), from 1.1(xi) and 4.12(i) we know that

$$\text{(i)} \qquad \operatorname{Ilim}(\overline{A}, \bar{a}) \approx H_r\Omega(x),$$

while from 1.61, the group S_U is $R_{(U,U)}$, i.e. $H_{r+1}\Omega(U, U-X)$. But U is compact, so that $G=X-U$ is open and $\overline{G}\subseteq \operatorname{Int}(X-x)$. Hence by the Excision Axiom

$$\text{(ii)} \qquad \eta_U\colon H_{r+1}\Omega(U, U-x) \approx H_{r+1}\Omega(X, X-x),$$

η_U being the injection, and so we obtain from the commutative diagram, when $U\supseteq V$:

$$\begin{tikzcd} H_{r+1}\Omega(U, U-x) \arrow[r, "\eta_U"] & H_{r+1}\Omega(X, X-x) \\ H_{r+1}\Omega(V, V-x) \arrow[u, "s"] \arrow[ur, "\eta_V"'] & \end{tikzcd}$$

that $s=s_U^V\colon H_{r+1}\Omega(V, V-x)\approx H_{r+1}\Omega(U, U-x)$. Therefore, *with no local connectivity assumptions,*

(iii) *the system* (S, s) *is itself stable rel* $\mathfrak{U}$ *and*

$$\operatorname{Ilim}(S, s) \approx H_{r+1}\Omega(X, X-x).$$

Thus, using 4.52, we can apply 1.61 to assert

(iv) *Under the assumptions of* 4.52, $H_r\Omega(x)$ *is stable and*

$$H_r\Omega(x) \approx H_{r+1}\Omega(X, X-x);$$

and so by 4.1(ii), if the coefficient group is a field,

$$\text{(v)} \qquad p^{r+1}(x) \text{ finite}\cdot\Rightarrow\cdot\dim H_r\Omega(x) = p^{r+1}(x).$$

We should like to prove a converse of (iv), for general coefficients, but have to restrict ourselves to the following result, with coefficients in a commutative ring with unit. We recall from 4.1 the system $\{B, b\}$ over $\mathfrak{U}_0$.

4.55. Lemma. *If the locally compact metric space* X *is* lc_v^r, *and is at* x $(V, r+1)$*-lc, and if* $H_r\Omega(x)$ *is finitely generated, then the system* $\{\overline{B}, \bar{b}\}$ *is stable in dimension* $r+1$, *and its* Dlim *is naturally isomorphic to* $H_r\Omega(x)$.

Proof. By the result of Begle quoted in 2.3, X is (V, r)-lc because it is lc_v^r. Hence the hypotheses of the lemma allow us to assert 4.54(iii), that $H_r\Omega(x)$ is stable. Thus there exist subsystems $\mathfrak{U}_1$, $\mathfrak{U}_2$ cof $\mathfrak{U}_c(x)$ such that for every (U_1, U_2) with $U_i\in\mathfrak{U}_i$ and $U_2\subseteq U_1$ we have

$$\text{(i)} \qquad H_r\Omega(U_2 - x \mid U_1 - x) \approx H_r\Omega(x).$$

Since $H_r\Omega(x)$ is finitely generated, and $H\Omega$ has compact carriers, it follows as in [CTM, 3.4], that there exists $V\in\mathfrak{U}_0$, $V\subseteq U_2$, such that the inclusion

$$\text{(ii)} \qquad H_r\Omega(U_2 - V \mid U_1 - x) \subseteq H_r\Omega(U_2 - x \mid U_1 - x)$$

is an equality. If $W\in\mathfrak{U}_0$ and $\overline{W}\subset V$, then, since X is lc_v^r, the group,

$$H_r\Omega(U_2 - V \mid U_1 - W)$$

is finitely generated, by Newman [13, Theorem 1]. It now follows, as in [CTM, 3.3], that there exist neighborhood functions $\delta^r(U_1, U_2)$, $\delta^r(U_1, U_2, U_3)$ such that given $U_3 \subseteq \delta^r(U_1, U_2)$, $U_4 \subseteq \delta^r(U_1, U_2, U_3)$ in $\mathfrak{U}_0$, then

$$\text{(iii)} \qquad H_r\Omega(U_2 - U_3 \mid U_1 - U_4) \approx H_r\Omega(x).$$

We can obviously assume the δ's to be monotone functions of their variables (e.g. $U_2' \subseteq U_2$ implies $\delta^r(U_1, U_2') \subseteq \delta^r(U_1, U_2)$). The stability of $H_r\Omega(x)$ is then quickly seen to imply

(v) *if $U_2' \subseteq U_2$ in $\mathfrak{U}_2$, $U_3 \subseteq \delta^r(U_1, U_2')$, $U_4 \subseteq \delta^r(U_1, U_2', U_3)$, then the inclusion $H_r\Omega(U_2' - U_3 \mid U_1 - U_4) \subseteq H_r\Omega(U_2 - U_3 \mid U_1 - U_4)$ is an equality*;

(vi) *if $U_1' \subseteq U_1$ in $\mathfrak{U}_1$, $U_2 \subseteq U_1'$, $U_3 \subseteq \delta^r(U_1', U_2)$, $U_4 \subseteq \delta^r(U_1, U_2, U_3)$ then the epimorphism $H_r\Omega(U_2 - U_3 \mid U_1' - U_4) \to H_r\Omega(U_2 - U_3 \mid U_1 - U_4)$ is univalent.*

Since X is (V, s)-lc $(s = r, r+1)$ there is a function $Q_r(U)$ of the sort following 4.52; we shall now show that given neighborhoods U, A, B, C, P, Q *of* x, satisfying $U, C \in \mathfrak{U}_1$, $A \in \mathfrak{U}_2$, $P, Q \in \mathfrak{U}_0$

$$\text{(vii)} \qquad U \supseteq A \supseteq Q_r(A) \supseteq B,$$

$$\text{(viii)} \qquad U \supseteq Q_r(U) \supseteq C \supseteq A,$$

$$\text{(ix)} \qquad Q \subseteq \delta^r(U, B) \cap \delta^r(C, A),\ P \subseteq \delta^r(U, B, Q) \cap \delta^r(C, A, Q) (\subseteq Q),$$

then the boundary homomorphism induces an isomorphism

$$\text{(x)} \qquad \partial_0 \colon H_{r+1}\Omega(A, A - Q \mid U, U - P) \approx H_r\Omega(A - Q \mid U - P),$$

where the left-hand group is the image of the injection

$$H_{r+1}\Omega(A, A - Q) \to H_{r+1}\Omega(U, U - P).$$

To prove (x), we look at (a) and (b) of the proof of 1.61. In (a) we ignore the set Δ, and simply interpret the diagram there. We take

$$\mu = U, \qquad \alpha = A, \qquad p = Q_q, \qquad \beta = B,$$

$$R_{(\alpha,\beta)} = H_{r+1}\Omega(A, B - Q), \quad S_\alpha = H_{r+1}\Omega(A, A - Q), \quad S_\mu = H_{r+1}\Omega(U, U - P),$$

$$A_\beta = H_r\Omega(B - Q), \qquad A_\alpha = H_r\Omega(A - Q), \qquad A_\mu = H_r\Omega(U - P);$$

the b's and a's in the first diagram of 1.61(a) are taken to be injections, and the d's to be boundary homomorphisms. By (v) and (ix) above,

$$H_r\Omega(A - Q \mid U - P) = H_r\Omega(B - Q \mid U - P),$$

so that, in the notation of 1.61(a), $A_{\mu\beta} = A_{\mu\alpha}$. By (vii) and 4.51(b), the homomorphism $\partial \colon R_{(\alpha,\beta)} \to A_\beta$ is onto. Hence all the hypotheses of 1.61(a) hold, and so in (x) the homomorphism ∂_0 is onto. A similar interpretation of the proof of 1.61(b), with β there put equal to C, proves that ∂_0 is univalent; we use (vi) and (ix) above to assert

$$H_r\Omega(A - Q \mid C - P) \approx H_r\Omega(A - Q \mid U - P),$$

which in the notation of 1.61(b) says: $A_{\beta\alpha} \approx A_{\mu\alpha}$.

Next, we have a commutative diagram

$$\begin{array}{ccc} H_{r+1}\Omega(A, A - Q) & \xrightarrow{\eta} & H_{r+1}\Omega(X, X - Q) = B^Q \\ i \downarrow & & \downarrow b \\ H_{r+1}\Omega(U, U - P) & \xrightarrow[\eta']{} & H_{r+1}\Omega(X, X - P) = B^P \end{array}$$

where i, b are injections, and η, η' are excision isomorphisms as in 4.11(ii). Hence η' induces an isomorphism

(xi) η_0: $H_{r+1}\Omega(A, A - Q \mid U, U - P) \approx H_{r+1}\Omega(X, X - Q \mid X, X - P) = B^{PQ}$.

Now fix U, A, and consider the diagram

$$\begin{array}{ccccc} H_{r+1}\Omega(X, X-Q \mid X, X-P) & \xleftarrow{\eta_0} & H_{r+1}\Omega(A, A-Q \mid U, U-P) & \xrightarrow{\partial_0} & H_r\Omega(A-Q \mid U-P) \\ \uparrow \lambda & & \uparrow \mu & & \uparrow \nu \\ H_{r+1}\Omega(X, X-T \mid X, X-S) & \xleftarrow[j]{} & H_{r+1}\Omega(A, A-T \mid U, U-S) & \xrightarrow[d]{} & H_r\Omega(A-T \mid U-S) \end{array}$$

where $(P, Q) \supseteq (S, T)$ in $\mathfrak{U}_0^2$, and $S \subseteq \delta^r(U, A)$, $T \subseteq \delta^r(U, A, S)$, j, d correspond to η_0, ∂_0, and λ, μ, ν are injections. Since $H_r\Omega(x)$ is finitely generated, arguments like those for (v) and (vi) show that ν is an isomorphism; hence by commutativity so is μ (since ∂_0, d are), and hence again by commutativity, so is λ (since η_0, j are). In the notation of 4.1, λ is a homomorphism $\bar{b}$; hence we have shown $\{\overline{B}, \bar{b}\}$ to be stable rel Λ, where Λ is the set of all (S, T) satisfying $S \subseteq \delta^r(U, A)$, $T \subseteq \delta^r(U, A, S)$. But clearly, Λ cof $\mathfrak{U}_0^2$, and therefore $\{\overline{B}, \bar{b}\}$ is stable, as required. Further, since ν is an isomorphism in the last diagram, it follows from (iii) that Dlim $\{\overline{B}, \bar{b}\}$ is naturally isomorphic to $H_r\Omega(x)$. This completes the proof.

COROLLARY. *If coefficients are in a field, then under the conditions of* 4.55,

$$p^{r+1}(x) = \dim H_r\Omega(x).$$

(This follows from 4.11, on combining (iii), (x) and (xi) above).

4.56. Similar results hold for the singular functor HS, because the Excision Axiom is satisfied. Thus we replace $R_{(U,V)}$, A_U in 4.53 by their singular analogues, and replace the hypotheses of 4.52 by

(i) *X is both r-lc_s and $(r+1)$-lc_s at x.*

The exactness of the singular sequence then gives quick proofs of the singular analogues([21]) of 4.52(b) and 4.53(i). Therefore the conditions of 1.6 are satis-

([21]) By [CTM, §2], all the Vietoris groups coincide with their Cech analogues.

fied with $p=\lambda_s$, $q=\lambda_s^{r+1}$ (defined in 2.3(x)). Hence, if X satisfies (i), the analogues of 4.54, (i)–(v) all immediately follow. To obtain the analogue of 4.55, we need to assume that X is $(r+1)$-lc$_s$ at x, and everywhere both lc$_s^r$ and lc$_s^r$; then we can use 2.33 where its analogue, Newman [13, Theorem 1], was used in the proof of 4.55.

4.57. For the homotopy functor, only partial results can be obtained, because homotopy does not in general satisfy the Excision axiom. To obtain a "local" homotopy theory we first have to assume that X satisfies 4.32, and then we replace $R_{(U,V)}$, A_U in 4.53 by their homotopy analogues. We replace the assumptions of 4.52 by

(i) *X is both r-LC and $(r+1)$-LC,*

and the exactness of the homotopy sequence gives proofs (formally identical with their singular counterparts) of the analogues of 4.52(b) and 4.53(i). Therefore the conditions of 1.6 are satisfied, with $p=\Lambda_r$, $q=\Lambda_{r+1}$ (functions defined in 4.21). By cofinality, the limit of the corresponding system (S, s) is a purely local concept, whereas the invariant $\mathfrak{R}^r(x)$ of [LTI] shows that $\pi_{r+1}(X, X-x)$ and the corresponding system $\{B, b\}$, are not; hence the system $\{B, b\}$ has no place in "local" homotopy theory. The natural homotopy counterpart of 4.5(x) is based on the analogue of the relative groups

$$H_k\Omega(A, A-Q \mid U, U-P),$$

but to define these analogues, we need to suppose that in addition to 4.32, X satisfies:

(ii) *There is a system $\mathfrak{U}_\delta$ cof $\mathfrak{U}$ such that given $U\in\mathfrak{U}_\gamma$ and $V\in\mathfrak{U}_\delta$ with $V\subseteq U$, then $U-V$ is path-wise connected.* Simple conditions such as in [5, 4.3] ensure that X does satisfy (ii). Then, with the obvious treatment for base-points of homotopy groups, we can get the analogue of the proof of 4.55(x); and under the following assumptions we can deduce the homotopy analogues of (iii), (v) and (vi) of 4.5:

(iii) $\pi_r(x)$ is stable (to get analogues of 4.5(i), (v) and (vi));

(iv) $\pi_r(U_2-V\mid U_1-W)$ is finitely generated abelian (to get analogues of 4.5(ii) and (iii)). To ensure the "abelian" part of (iv), we need $r>1$, even though we have 3.15; and for $r>1$ we can apply 3.14 provided the right conditions hold. Thus we have on combining the analogues of 4.5(iii) and (x), that

(v) *if X is lc$_s^r$ and lc$_v^r$, if it is r-LC and $(r+1)$-LC at x, if $r>1$ and $\pi_j(x)\equiv 0$, $0\leqq j<r$, then for suitable neighborhoods A, Q, U, P of x,*

$$\pi_{r+1}(A, A-Q \mid U, U-P)\approx\pi_r(A-Q\mid U-P)\approx\pi_r(x).$$

Groups of the sorts considered in 4.55(x) lead us to make the following definitions for functors G_p, K_p which have the formal properties of absolute and relative homotopy groups respectively; note the analogy with the "D" groups of CTM. First, if $\mathfrak{U}_i$ cof $\mathfrak{U}(x)$, $1\leqq i\leqq 4$, define $\mathfrak{V}$ to consist of all quadruples (U_1, U_2, U_3, U_4) with $U_i \in \mathfrak{U}_i$ and $U_i \supseteq U_{i+1}$; and define $(V_1, V_2, V_3, V_4) \leqq (U_1, U_2, U_3, U_4)$ in $\mathfrak{B}$ to mean that $V_i \supseteq U_i$, $1 \leqq i \leqq 4$. Then we shall say

4.58. *G_p is D-stable $[K_p$ is B-stable$]$ at x, if and only if there exist* $\mathfrak{U}_1$, $\mathfrak{U}_2$, $\mathfrak{U}_3$, $\mathfrak{U}_4$ cof $\mathfrak{U}(x)$, *such that given $(V_1, V_2, V_3, V_4) \leqq (U_1, U_2, U_3, U_4)$ in $\mathfrak{B}$ satisfying*

$$\text{(i)} \qquad U_2 - U_3 \subseteq V_2 - V_3, \qquad U_1 - U_4 \subseteq V_1 - V_4,$$

then the injection

$$\gamma: G_p(U_2 - U_3 \mid U_1 - U_4) \to G_p(V_2 - V_3 \mid V_1 - V_4)$$
$$[\kappa: K_p(U_2, U_2 - U_3 \mid U_1, U_1 - U_4) \to K_p(V_2, V_2 - V_3 \mid V_1, V_1 - V_4)]$$

is an isomorphism. (*In the homotopy case we require that* $\mathfrak{U}_2 \subseteq \mathfrak{U}_\gamma$, $W_3 \subseteq \mathfrak{U}_\delta$.)

For brevity denote (U_1, U_2, U_3, U_4), (V_1, V_2, V_3, V_4) above by u, v and write the injections as

$$\gamma = g_u^v: G_p^u \to G_p^v, \qquad \kappa = k_u^v: K_p^u \to K_p^v.$$

Write $u \prec v$ whenever $u \leqq v$ in $\mathfrak{B}$ and 4.58(i) holds. Then $\mathfrak{B}$ is not necessarily directed by $\prec$, but still (G_p, g), (K_p, k) are inverse systems over $(\mathfrak{B}, \prec)$, in the sense that 1.1(i) and (ii) still hold. It is now easily shown that if G has compact carriers, and if $G_r(x)$ is the G-analogue of $\pi_r(x)$ then

(ii) *G_r is D-stable at $x \cdot\Rightarrow\cdot G_r(x)$ is stable and isomorphic to* Ilim (G_r, g).

By Newman [13, Theorem 1] and its singular analogue 2.33, we have

(iii) *If X is lc_v^n $[lc_v^n$ and $lc_s^n]$ and the Vietoris [singular] G_n is D-stable at x, then* Ilim (G_n, g) *is finitely generated. Similarly if X is LC^1, with the homotopy functor G_1 (by* 3.15).

Next, if X satisfies the G-analogue of 4.52, it is clear that the analogue of our passage from 4.5(iii) to (x) is still valid, step by step, in the (G, K)-theory provided G is abelian; and by a similar argument using (b) of the proof of 1.61 instead of (a), we therefore have—for any X with a $\mathfrak{U}_\gamma(x)$ and $\mathfrak{U}_\delta(x)$—

(iv) *G_p is D-stable at $x \cdot\Leftrightarrow\cdot K_{p+1}$ is B-stable at x. In either case,* Ilim $(G_p, g) \approx$ Ilim (K_{p+1}, k). With the same conditions on X, 1.61 applies, so that if $K_p(x)$ denotes the K_p-analogue of the limit of (S, s) in 1.61, we have

(v) *$G_p(x)$ stable $\cdot\Leftrightarrow\cdot K_{p+1}(x)$ stable. In either case $G_p(x) \approx K_{p+1}(x)$.*

We can now sum up the homology situation, by collecting the above results and using 4.55 in the statement:

4.59. (a) THEOREM. *If the locally compact metric space X is everywhere lc_v^n and $(V, n+1)$-lc at x [everywhere lc_v^n and lc_s^n, and $(n+1)$-lc_s at x] then the local relative and absolute cut-point homology groups at x are stable; and they coincide, in the sense that with the appropriate interpretations,*

$$\text{Ilim } (S, s) \approx G_r(x) \approx K_{r+1}(x) \approx \text{Ilim } (K_{r+1}, k) \approx \text{Ilim } (G_r, g) \approx \text{Dlim } \{B, b\}$$

and each is isomorphic to $H_{r+1}\Omega(X, X-x)\,[H_{r+1}S(X, X-x)]$, $r=0, 1, \cdots, n$.

In the homotopy theory we cannot expect to be able to apply 4.57 in all dimensions and therefore have the more restricted result (using 4.57(v)):

4.59(b). THEOREM. *If the locally compact metric* lc_s^n *and* lc_v^n *space* X *has a* $\mathfrak{U}_\gamma(x)$ *and* $\mathfrak{U}_\delta(x)$, *and is* n-*LC and* $(n+1)$-*LC at* x, *then if* G_r *is D-stable at* x, *all the homotopy analogues of the groups of* 4.59 *are stable and coincide, except* Dlim $\{B, b\}$ *and* $\pi_{r+1}(X, X-x)$, $0 \leqq r \leqq n$.

Thus if we use the local groups to define "manifolds" as in [CTM], we see from 4.59 that with Vietoris or Singular homology, whatever type of group is used leads to the same (Vietoris or Singular) definition([21]); that with homotopy, if we use the (G_r, g) systems the resulting manifolds include all those defined using the other homotopy groups; and by 4.35 and its obvious modification for the (G_r, g) system, the homotopy manifolds on any definition are integer homology manifolds. Obviously, locally Euclidean space is a manifold, under all the definitions. A converse of 4.58(ii) remains to be proved (or disproved) in homotopy theory; if proved, it will presumably show that the above homotopy manifolds will be identical with the manifolds using the groups $\pi_r(x)$.

4.6. **Mappings.** If $f\colon X \to Y$ is a map, it is desirable that f should induce homomorphisms of the local groups. But, if $y \in Y$, then $F=f^{-1}(y)$ will in general be a closed set, not necessarily a point, and so we have a homomorphism

$$\text{(i)} \qquad f_r\colon \text{Ilim } (H_r\Omega(\dot{G} - F), j) \to H_r\Omega(y)$$

where G runs through all neighborhoods of F, and j denotes injections. However, in the special case that F is a single point x, this gives us

$$\text{(ii)} \qquad f_r\colon H_r\Omega(x) \to H_r\Omega(y),$$

and similarly for the Singular and homotopy functors. The "C" and "D" groups of [LTI] and [CTN] had not got this property. Considerations of the sort given in Griffiths [7] enable a concept of "local homotopy type" to be defined, in order to investigate circumstances under which (ii) is an isomorphism. We have not studied systems of the sort $(H_r\Omega(G-F), j)$ in (i), when F is a fixed set with more than one point.

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