

Local Torelli theorem for non-singular complete intersections

By Sampei USUI

(Received February 9, 1976)

§ 0. Introduction

After studying periods of integrals of algebraic manifolds ([2]), P. A. Griffiths summarized the main results and open problems ([3]). In his article [3], he presented a problem: "(7.1). PROBLEM*. *Find methods to treat the local Torelli theorem. In particular, decide whether it is true or false for simply-connected canonical surfaces (i.e. surfaces with ample canonical bundle)*". The local Torelli problem is the question whether a period map separates infinitely near points (see § 1). The purpose of the present paper is to answer the above problem in the case of non-singular complete intersections with ample canonical line bundle.

In § 1 we recall period map briefly in order to formulate the local Torelli problem. In § 2 we give a proof of the local Torelli theorem for non-singular complete intersections with ample canonical line bundle.

Griffiths examined the local Torelli problem in some examples such as curves, special complex manifolds (i.e. complex Kähler manifolds whose canonical line bundles are trivial), surfaces in P^3 , and surfaces on abelian varieties, and also gave some counter-examples (II, section 3 in [2]). In the investigation of hypersurfaces, Griffiths used the Macaulay theorem on a polynomial ring (suggested by D. Mumford) as an essential tool. By using the same theorem, C. Peters succeeded in giving a proof of the local Torelli theorem for cyclic branched coverings of P^n , and of $P^1 \times P^1$ ([6]). As for cyclic branched coverings of P^n , since they can be seen as weighted hypersurfaces, we can go on the same route as for hypersurfaces (for "weighted complete intersection" see S. Mori [5]). In the present paper the writer shows that, for complete intersections, the Macaulay theorem can be taken the place of by vanishing of some cohomology.

Throughout this paper, we understand by a variety one which is defined over the complex number field C .

The writer wishes to thank C. Peters at Leiden University for the stimulating communication.

§ 1. Period map and local Torelli problem

In this section we will recall the period map briefly and formulate the local Torelli problem after Griffiths (II of [2]).

Let $\{X_s\}_{s \in S}$ be a complex-analytic family of m -dimensional Kähler manifolds. Put $X = X_0$ for some fixed point 0 on S . Choose and fix a C^∞ -trivialisation g of the family

$$\begin{array}{ccc} X \times S & \xrightarrow{g} & \mathfrak{X} \\ & \searrow & \downarrow \\ & & S \end{array}$$

Then g induces a C^∞ -isomorphism $g_s: X \rightarrow X_s$ for each point $s \in S$, and g_s induces an isomorphism of the complex cohomology groups $g_s^*: H^r(X_s, \mathbb{C}) \rightarrow H^r(X, \mathbb{C})$. Let $H^r(X, \mathbb{C}) = \sum_{p+q=r} H^{p,q}(X)$ be the Hodge decomposition. Put $H^r = H^r(X, \mathbb{C})$, $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$, and $f^i = h^{r,0} + \dots + h^{r-i,i}$ ($0 \leq i \leq r$). Note that the Hodge numbers $h^{p,q}$ and hence the numbers f^i are deformation-invariants. Let $Fl(H^r; f^0, \dots, f^r)$ be the flag manifold parametrizing filtrations of H^r by \mathbb{C} -vector subspaces. We see that the mapping

$$\Phi^r: S \rightarrow Fl(H^r; f^0, \dots, f^r)$$

given by

$$\begin{aligned} \Phi^r(s) = & [g_s^* H^{r,0}(X_s) \subset g_s^*(H^{r,0}(X_s) + H^{r-1,1}(X_s)) \subset \dots \\ & \subset g_s^*(H^{r,0}(X_s) + \dots + H^{0,r}(X_s))] \end{aligned}$$

is well-defined. Griffiths showed that this map Φ^r is holomorphic and called it the period map in r -th cohomology.

On the tangent space level, we get the following factorization of $\Phi^r_*(0)$ via Kodaira-Spencer infinitesimal deformation map ρ :

$$\begin{array}{ccc} T_S(0) & \xrightarrow{\Phi^r_*(0)} & T_{Fl}(\Phi^r(0)) \\ \rho \downarrow & \nearrow \varphi^r & \\ H^1(T_X) & & \end{array}$$

In the case that $\{X_s\}_{s \in S}$ is the Kuranishi family of deformations of X , ρ is an isomorphism and hence it is natural to call the following the local Torelli problem.

Local Torelli problem in r -th cohomology: Is the map φ^r injective?

We modify the above problem a little. Recall that

$$T_{Fl}(\Phi^r(0)) \approx \prod_{1 \leq i \leq r} \text{Hom}(H^{r-i+1, i-1}(X), H^{r-i, i}(X) + \dots + H^{0, r}(X)),$$

and, in fact, the map φ^r factors as follows (see Griffiths II of [2]):

$$\begin{array}{ccc}
 H^1(T_X) & \xrightarrow{\varphi^r} & T_{Fl}(\Phi^r(0)) \\
 & \searrow & \uparrow \\
 & & \prod_{1 \leq i \leq r} \text{Hom}(H^{r-i+1, i-1}(X), H^{r-i, i}(X)).
 \end{array}$$

We also denote the above map $H^1(T_X) \rightarrow \prod_{1 \leq i \leq r} \text{Hom}(H^{r-i+1, i-1}(X), H^{r-i, i}(X))$ by the same notation φ^r . This map φ^r is nothing but the map induced by the pairing $T_X \otimes \Omega_X^{r-i+1} \rightarrow \Omega_X^{r-i}$ ($1 \leq i \leq r$). Therefore the local Torelli problem in r -th cohomology is formulated as:

(P₀) Let $H^1(T_X) \otimes H^{r-i+1, i-1}(X) \rightarrow H^{r-i, i}(X)$ be the pairing induced by $T_X \otimes \Omega_X^{r-i+1} \rightarrow \Omega_X^{r-i}$ ($1 \leq i \leq r$). Is $\tau \in H^1(T_X)$ zero if $\tau \cdot \alpha_i$ is zero in $H^{r-i, i}(X)$ for every $\alpha_i \in H^{r-i+1, i-1}(X)$ ($1 \leq i \leq r$)?

§ 2. Local Torelli theorem for non-singular complete intersections

Let $R = C[X_0, \dots, X_n]$ be a polynomial ring over the complex number field. For each non-negative integer a , R_a denotes a module of homogeneous forms of degree a . Let a_j be integers with $a_j > 1$ ($1 \leq j \leq c$) and let $F_j \in R_{a_j}$ ($1 \leq j \leq c$). Let I be the ideal of R generated by F_1, \dots, F_c . Suppose $X = V_+(F_1, \dots, F_c)$ is a non-singular complete intersection. Put $m = \dim X = n - c$ and assume that $m \geq 2$. Let K_X be the canonical line bundle of X and put $k = \deg K_X = \sum_{j=1}^c a_j - (n + 1)$. Assume that $k > 0$.

Now we reformulate local Torelli problem for non-singular complete intersection X . Let $T_X \otimes K_X \simeq \Omega_X^{m-1}$ be the canonical isomorphism, where T_X is the tangent bundle of X and Ω_X^{m-1} the $(m-1)$ -th exterior product of the cotangent bundle of X . From this we get a homomorphism

$$(1) \quad H^1(T_X) \otimes H^0(K_X) \longrightarrow H^1(\Omega_X^{m-1}).$$

In order to prove the local Torelli theorem for X , it is enough to show the following:

(P₁) The above pairing (1) is non-degenerate in the first factor.

From the well-known exact sequence for complete intersection

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus(n+1)} \longrightarrow T_P|X \longrightarrow 0,$$

we get the exact sequence of cohomology groups:

$$H^1(\mathcal{O}_X(1))^{\oplus(n+1)} \longrightarrow H^1(T_P|X) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow H^2(\mathcal{O}_X(1))^{\oplus(n+1)}.$$

(P₃) Let Φ_j be a homogeneous polynomial in R ($1 \leq j \leq c$). Suppose, for each i ($0 \leq i \leq n$), there exist homogeneous polynomials A_{i_0}, \dots, A_{i_n} in R such that $\deg A_{i_\nu} \leq k+1$ ($0 \leq i, \nu \leq n$) and that

$$(H_i) \quad X_i \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} A_{i_0} \\ \vdots \\ A_{i_n} \end{bmatrix} \pmod{I}.$$

Then there exist homogeneous polynomials B_0, \dots, B_n in R such that

$$\begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} B_0 \\ \vdots \\ B_n \end{bmatrix} \pmod{I}.$$

Before proving (P₃) we prove the following lemmas.

LEMMA 1 (See Proposition 1.3 in [1]). $H^0(\Omega_X^{n-1}(\ell)) = 0$ ($\ell \leq 1$).

PROOF. Put $E = \bigoplus_{j=1}^c \mathcal{O}_P(-a_j)$. Let b_μ be non-negative integers which are not all zero. Then we have

$$(2) \quad H^q \left(\bigotimes_{\mu} \left(\bigwedge^{b_\mu} E \right) \otimes \Omega_P^p \otimes \mathcal{O}_P(\ell) \right) = 0 \quad \text{for } p+q < n \text{ and } \ell \leq 1.$$

We want to show

$$(3) \quad \begin{aligned} H^q(\Omega_P^p \otimes \mathcal{O}_P(\ell)) &\xrightarrow{\sim} H^q(\Omega_X^p \otimes \mathcal{O}_P(\ell)) && \text{for } p+q < m = n-c, \quad \text{and} \\ H^q(\Omega_P^p \otimes \mathcal{O}_P(\ell)) &\hookrightarrow H^q(\Omega_X^p \otimes \mathcal{O}_P(\ell)) && \text{for } p+q = m = n-c \end{aligned}$$

by induction on p . When $p=0$, we have $H^0(\mathcal{O}_P(\ell)) \xrightarrow{\sim} H^0(\mathcal{O}_X(\ell))$, $H^q(\mathcal{O}_P(\ell)) = 0$ for $0 < q \leq m$, and $H^q(\mathcal{O}_X(\ell)) = 0$ for $0 < q < m$. Now suppose that $p > 0$ and that (3) is verified for $p' < p$. The exact sequence

$$0 \longrightarrow \check{N}_{X/P} \otimes \mathcal{O}_P(\ell) \longrightarrow \Omega_P^1 | X \otimes \mathcal{O}_P(\ell) \longrightarrow \Omega_X^1 \otimes \mathcal{O}_P(\ell) \longrightarrow 0$$

induces a descending filtration of $\Omega_P^p | X \otimes \mathcal{O}_P(\ell)$ whose successive quotients are given by

$$Gr^b(\Omega_P^p | X \otimes \mathcal{O}_P(\ell)) \approx \bigwedge^b E \otimes \Omega_X^{p-b}.$$

The above filtration defines a spectral sequence

$$E_1^{b, q-b} = H^q \left(\bigwedge^b E \otimes \Omega_X^{p-b} \otimes \mathcal{O}_P(\ell) \right) \implies H^q(\Omega_P^p | X \otimes \mathcal{O}_P(\ell)).$$

By the induction hypothesis, we have

$$H^q\left(\bigwedge^b E \otimes \Omega_X^{2-b} \otimes \mathcal{O}_P(\ell)\right) = 0 \quad \text{for } b > 0 \text{ and } p+q < m+b.$$

Hence we get

$$(4) \quad \begin{aligned} H^q(\Omega_P^2 | X \otimes \mathcal{O}_P(\ell)) &\xrightarrow{\sim} H^q(\Omega_X^2 \otimes \mathcal{O}_P(\ell)) && \text{for } p+q < m, \quad \text{and} \\ H^q(\Omega_P^2 | X \otimes \mathcal{O}_P(\ell)) &\hookrightarrow H^q(\Omega_X^2 \otimes \mathcal{O}_P(\ell)) && \text{for } p+q = m. \end{aligned}$$

On the other hand, since X is a complete intersection, the Koszul complex of \mathcal{O}_X gives a resolution of $\Omega_P^2 | X \otimes \mathcal{O}_P(\ell)$ consisting of $\bigwedge^b E \otimes \Omega_P^2 \otimes \mathcal{O}_P(\ell)$. This resolution defines a spectral sequence

$$E_1^{-b,q} = H^q\left(\bigwedge^b E \otimes \Omega_P^2 \otimes \mathcal{O}_P(\ell)\right) \implies H^{q-b}(\Omega_P^2 | X \otimes \mathcal{O}_P(\ell)).$$

Hence by (2) we get

$$(5) \quad \begin{aligned} H^q(\Omega_P^2 \otimes \mathcal{O}_P(\ell)) &\xrightarrow{\sim} H^q(\Omega_P^2 | X \otimes \mathcal{O}_P(\ell)) && \text{for } p+q < m, \quad \text{and} \\ H^q(\Omega_P^2 \otimes \mathcal{O}_P(\ell)) &\hookrightarrow H^q(\Omega_P^2 | X \otimes \mathcal{O}_P(\ell)) && \text{for } p+q = m. \end{aligned}$$

Combining (4) and (5), we get (3). Applying (3) for $p=m-1$ and $q=0$, we have

$$0 = H^0(\Omega_P^{m-1} \otimes \mathcal{O}_P(\ell)) \xrightarrow{\sim} H^0(\Omega_X^{m-1} \otimes \mathcal{O}_P(\ell)) \quad (\ell \leq 1). \quad \text{Q.E.D.}$$

LEMMA 2. Let A_0, \dots, A_n be homogeneous polynomials in R with $\deg A_i \leq k+2$ ($0 \leq i \leq n$). Suppose that

$$\begin{bmatrix} \partial_0 F_1 & \cdots & \partial_n F_1 \\ \vdots & \cdots & \vdots \\ \partial_0 F_c & \cdots & \partial_n F_c \end{bmatrix} \begin{bmatrix} A_0 \\ \vdots \\ A_n \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \pmod{I}.$$

Then we can find a homogeneous polynomial B in R , which is independent of i , such that

$$A_i \equiv X_i B \pmod{I} \quad (0 \leq i \leq n).$$

PROOF. We consider the following diagram :

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & H^0(\mathcal{O}_X) & & \\
 & & & & \downarrow & & \\
 & & & & H^0(\mathcal{O}_X(1))^{\oplus(n+1)} & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & H^0(T_X) & \longrightarrow & H^0(T_P|X) & \longrightarrow & H^0(N_{X/P}) \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

Since the homomorphism $H^0(T_P|X) \rightarrow H^0(N_{X/P})$ is given by

$$\begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \vdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix},$$

the statement of Lemma 2 is equivalent to

$$(6) \quad H^0(T_X(\ell)) = 0 \quad (\ell \leq k+1).$$

Moreover, since $T_X \otimes K_X \cong \Omega_X^{m-1}$, that is, $T_X(k) = \Omega_X^{m-1}$, (6) is equivalent to Lemma 1. Q.E.D.

Now we prove (P_3) .

PROOF. Choose homogeneous coordinates of P^n such that $F_1, \dots, F_c, X_0, X_1, X_2$ form a regular sequence.

$$\begin{aligned}
 & X_0(H_1) - X_1(H_0): \\
 & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \vdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} X_0 A_{10} - X_1 A_{00} \\ \vdots \\ X_0 A_{1n} - X_1 A_{0n} \end{bmatrix} \pmod I.
 \end{aligned}$$

Applying Lemma 2 for $A_i = X_0 A_{1i} - X_1 A_{0i}$, we get B in R such that

$$(7) \quad X_0 A_{1i} - X_1 A_{0i} \equiv X_i B \pmod I \quad (0 \leq i \leq n).$$

Observing the above formula (7) for $i=2$ modulo $(F_1, \dots, F_c, X_0, X_1)$, we can cancel X_2 . Hence

$$B \equiv X_0 C + X_1 D \pmod I$$

for some C and D in R . Applying this to the formula (7), we have

$$X_0 A_{1i} - X_1 A_{0i} \equiv (X_0 C + X_1 D) X_i \pmod{I},$$

i.e.

$$X_0(A_{1i} - X_i C) - X_1(A_{0i} + X_i D) \equiv 0 \pmod{I}.$$

Observing this formula modulo (F_1, \dots, F_c, X_0) , we can cancel X_1 , and we have

$$(8) \quad A_{0i} + X_i D \equiv X_0 E_i \pmod{I}$$

for some E_i in R ($0 \leq i \leq n$). Applying this formula (8) to (H_0) in (P_3) , we have

$$X_0 \left\{ \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} - \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} E_0 \\ \vdots \\ E_n \end{bmatrix} \right\} \equiv - \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_n \end{bmatrix} D \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \pmod{I}$$

by Euler's formula. Observing this modulo (F_1, \dots, F_c) , we can cancel X_0 and get the required conclusion. Q.E.D.

References

- [1] P. Deligne, Cohomologie des intersections complètes, Exp. XI in S.G.A., 7 II, Springer Lecture Notes, 340 (1973).
- [2] P. A. Griffiths, Periods of integrals on algebraic manifolds I, II, III, Amer. J. Math., 90 (1968), 568–626; 805–865; Publ. Math. I.H.E.S., 38 (1970).
- [3] P. A. Griffiths, Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems, Bull. Amer. Math. Soc., 75 (1970), 228–296.
- [4] P. A. Griffiths and W. Schmid, Recent developments in Hodge theory: a discussion of techniques and results, Proc. Symp. at Bombay in 1973, Oxford Univ. Press (1975).
- [5] S. Mori, On a generalization of complete intersections, J. Math. Kyoto Univ., 15-3 (1975), 619–646.
- [6] C. Peters, The local Torelli theorem of some cyclic branched coverings, to appear.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
606 KYOTO-SHI