Locality-sensitive Hashing without False Negatives

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IT University of Copenhagen

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European Research Council
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SCALABLE SIMILARITY SEARCH

1-2 post-doc positions available starting fall 2016 - contact me for details!

Talk outline

- Hamming similarity search
- Approximate similarity search using LSH
- Recent developments
- New result: Avoiding false negatives

Hamming similarity search

vanilla version

- Build data structure for set $S \subseteq \{0,1\}^d$ s.t. given query vector q and radius r, can decide

$$\exists x \in S : ||x - q|| \leq r$$

where $||x - q||$ = Hamming distance between x and q.

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Hamming similarity search in time $n^{0.99}\, 2^{o(d)} \Longrightarrow$

k-SAT w. n variables can be solved in time α^n, $\alpha < 2$

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Strong ETH states: Not possible!

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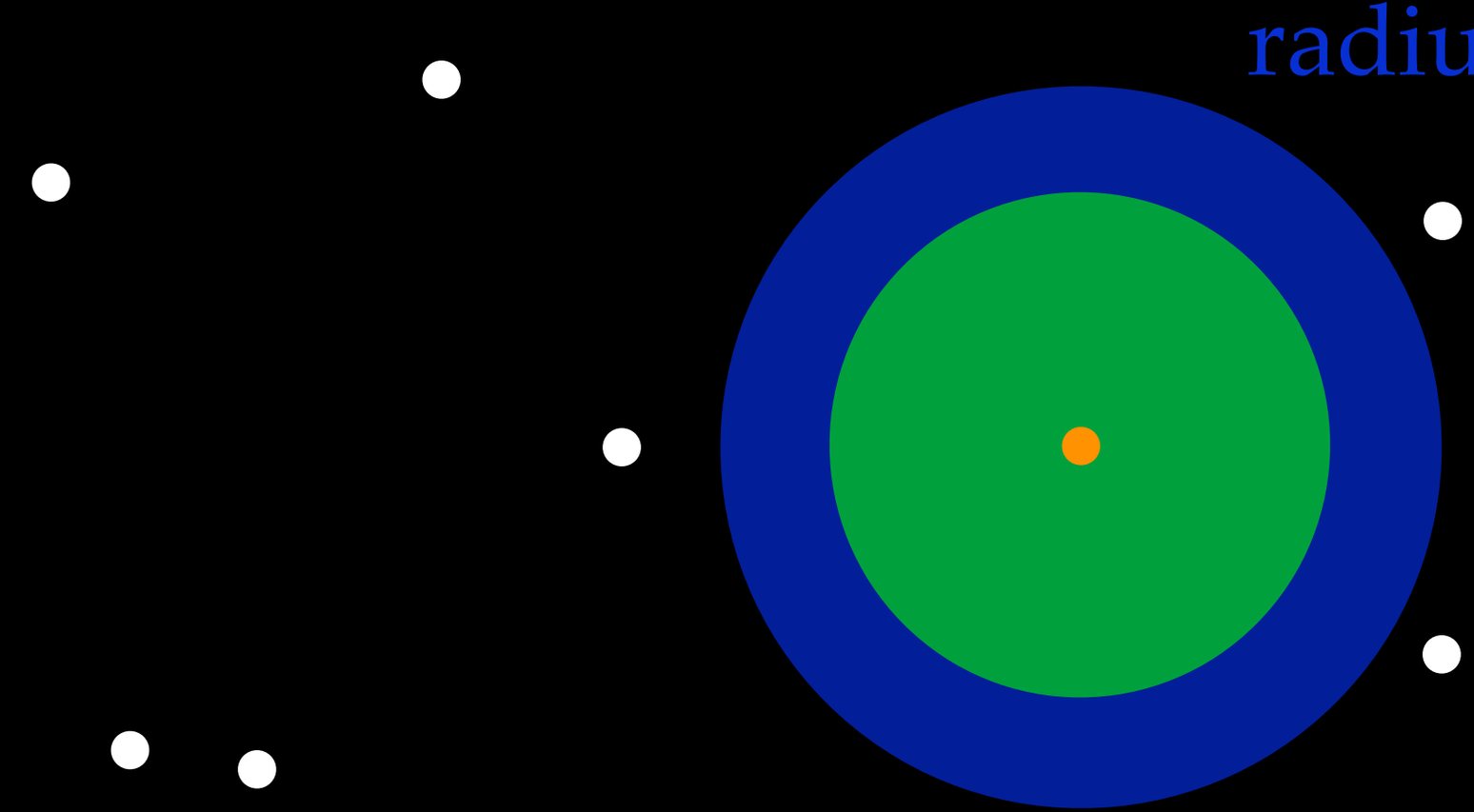
Approximate similarity search



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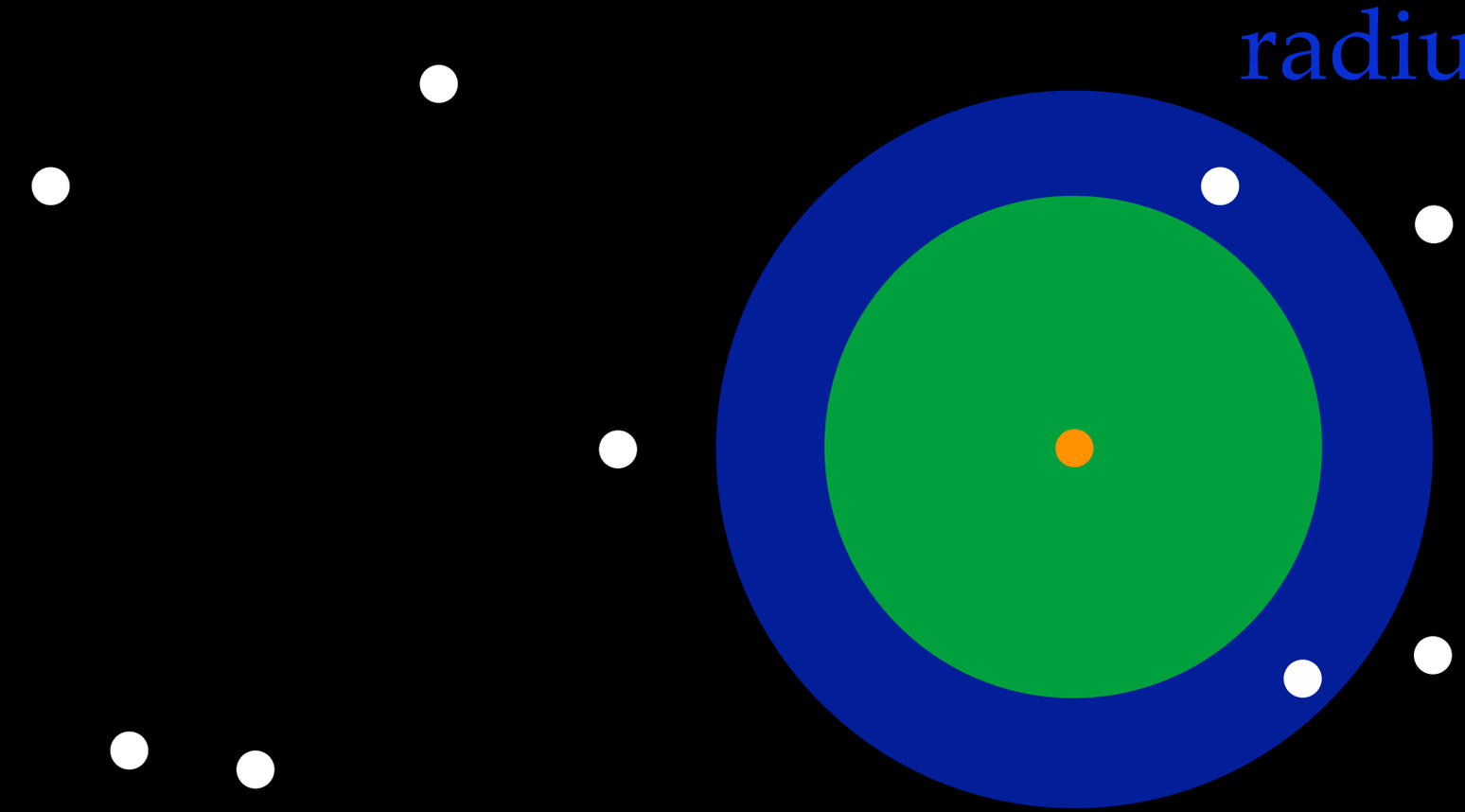

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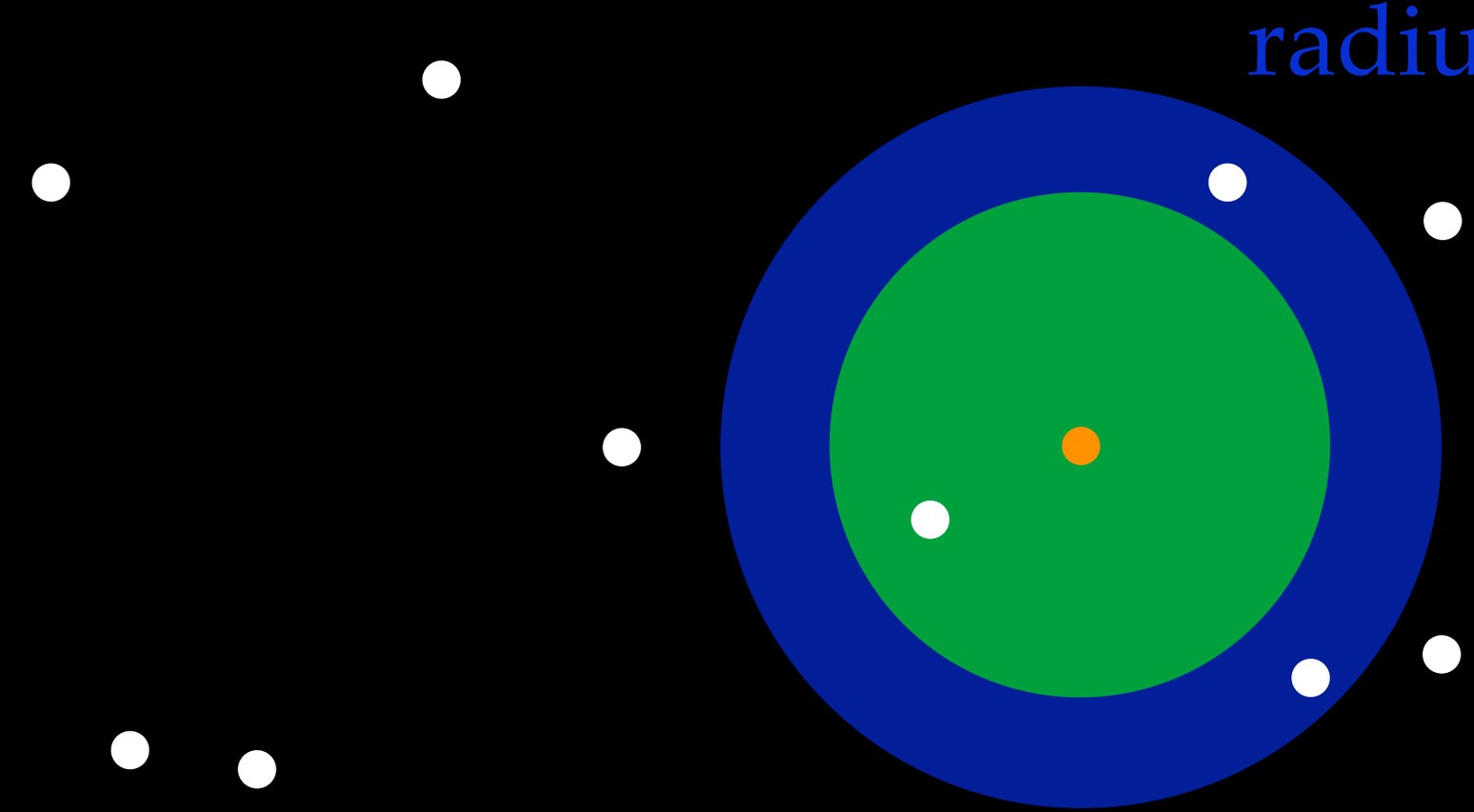

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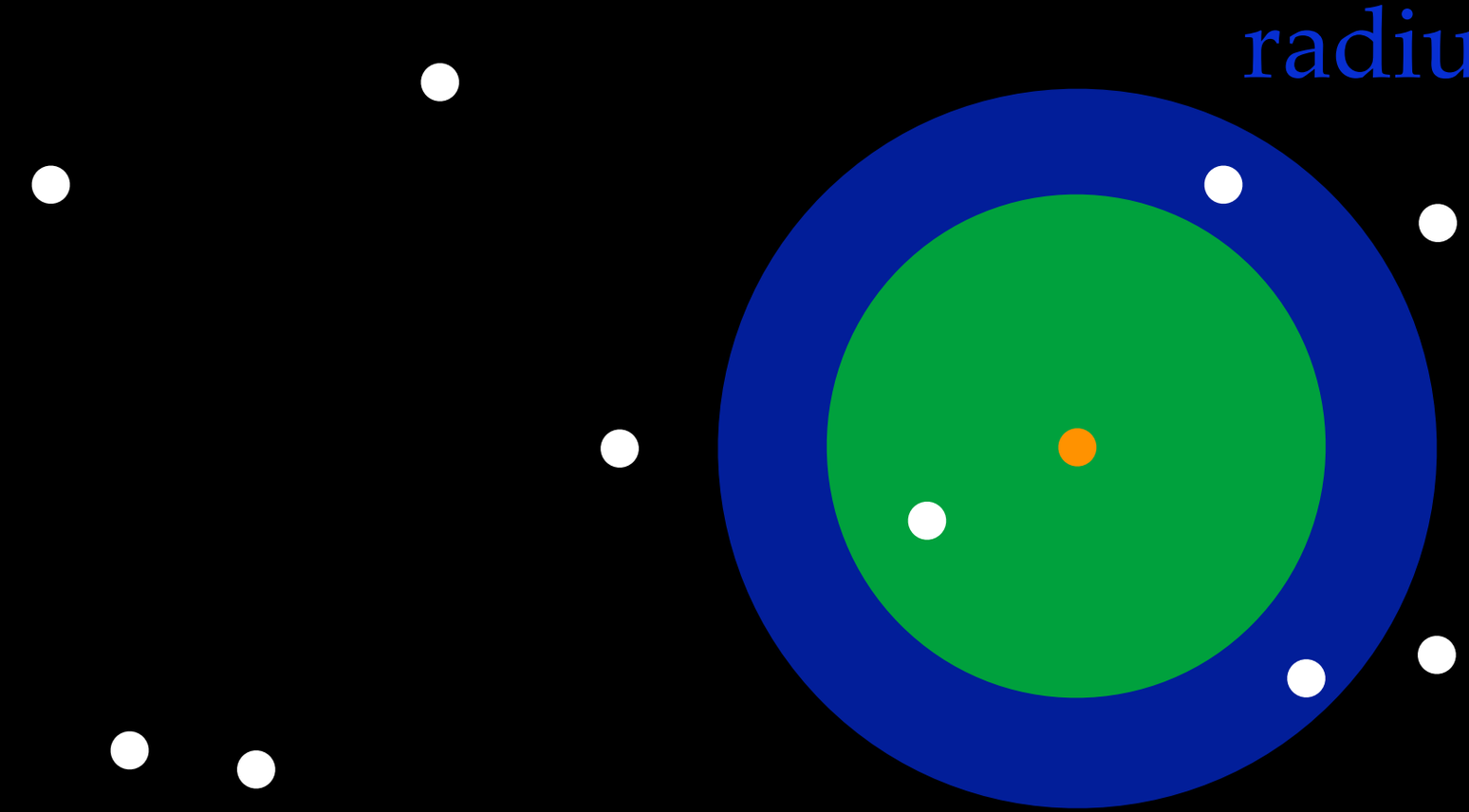


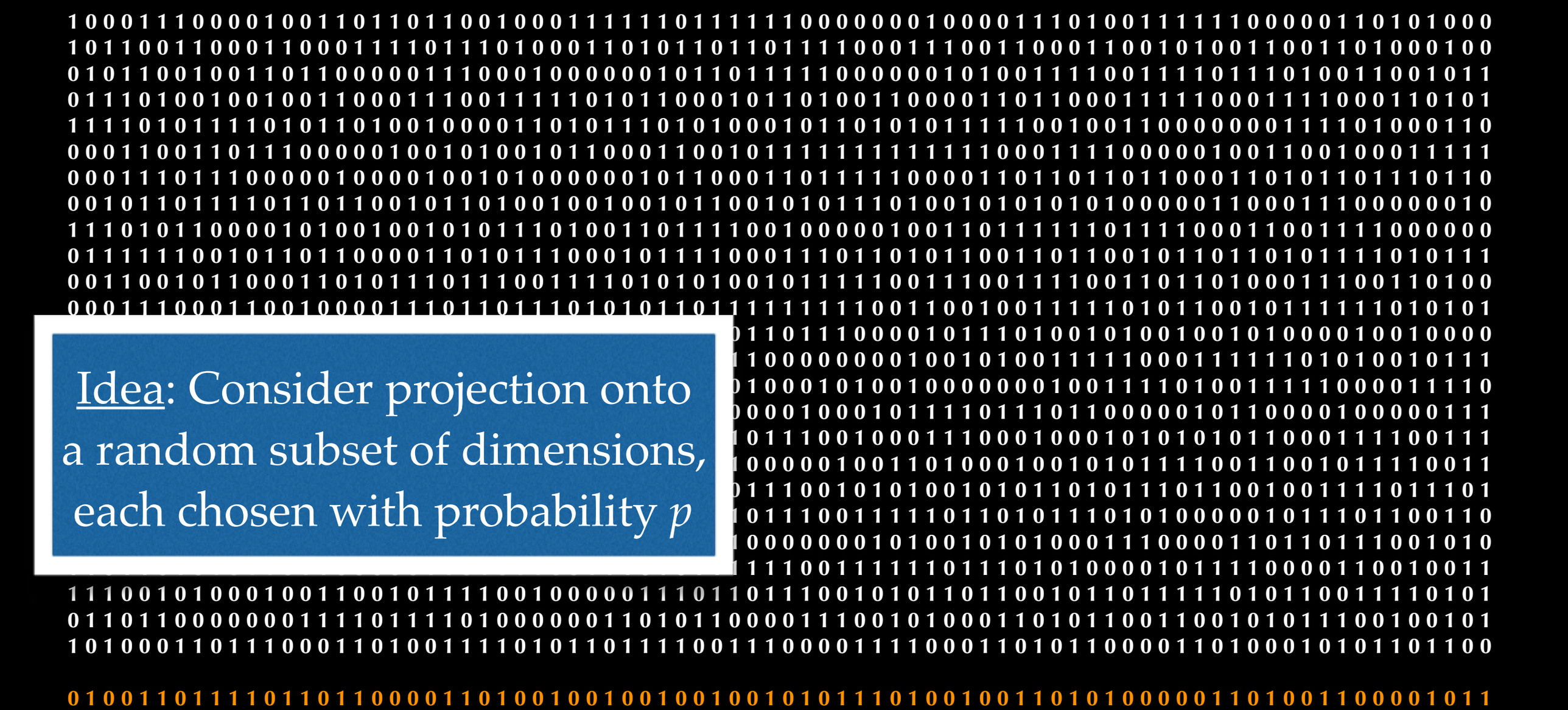
[Indyk & Motwani, STOC '98]:
Time $O(dn^{1/c})$, space $O(n^{1+1/c} + dn)$

[Indyk & Motwani ’98]

Locality-sensitive hashing

[Indyk & Motwani '98]

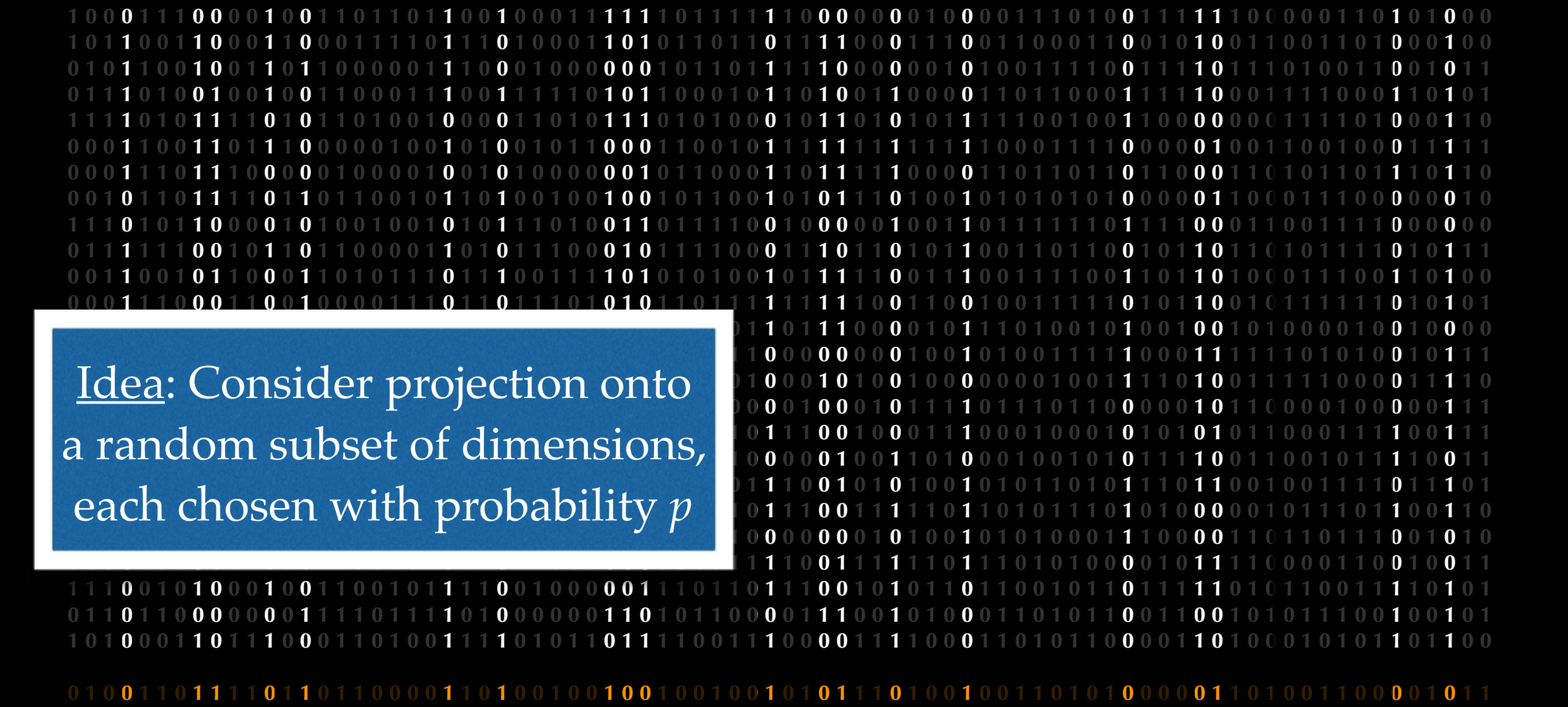
Locality-sensitive hashing



Idea: Consider projection onto a random subset of dimensions, each chosen with probability p

[Indyk & Motwani ’98]

Locality-sensitive hashing



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Locality-sensitive hashing

$$h(x) = x \wedge a$$

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Candidate match

[Indyk & Motwani '98]

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$t = n^{1/c}$ repetitions ensure constant success probability

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Locality-sensitive hashing

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Candidate match

Possibility of *false negatives*

$$t = n^{1/c}$$

repetitions ensure constant success probability

Some recent developments*

Reference	Exponent, search time	Comment
Linear search	1	
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* Focus on subquadratic space; lower order terms ignored.

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Alman & Williams FOCS '15	$1-\tilde{\Omega}(\log(n)/d)$	Batched, c=1
Karppa, Kaski & Kohonen SODA '16	$\frac{2\omega-3}{3}$	Batched [c>1, random data, $\omega > 2.25$]

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Possibility of *false negatives*

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Similarity search without false negatives

Reference	Exponent, search time	Comment
Arasu, Ganti & Kaushik VLDB '06	$\approx 3/c$ (analysis not in their paper)	
This paper	$\ln(4)/c$	
This paper	$1/c$	For $cr = \log(n)$

[Arasu, Ganti & Kaushik '06]

Basic correlated LSH

partitioning

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h_1

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h_2

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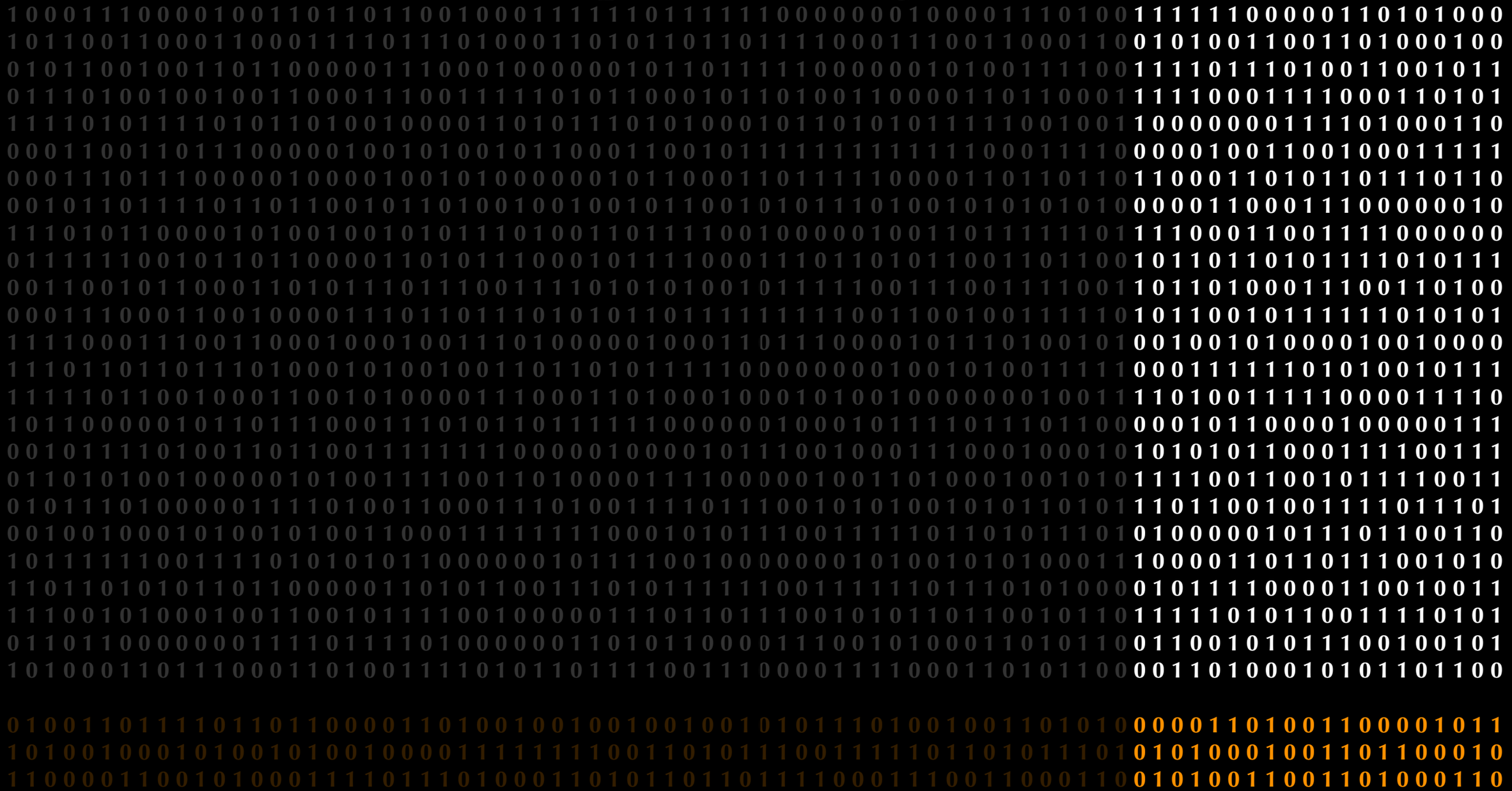

h_4

[Arasu, Ganti & Kaushik '06]

Basic correlated LSH
partitioning

For Hamming distance ≤ 3,
a collision is guaranteed!

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Basic correlated LSH
partitioning

For Hamming distance ≤ 3,
a collision is guaranteed!

[Arasu et al. '06]:
To bound probability of collision for distance > 3 *randomly permute* the dimensions

[Arasu, Ganti & Kaushik '06]

Basic correlated LSH

enumeration

h_{12}

[Arasu, Ganti & Kaushik '06]

Basic correlated LSH

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h_{13}

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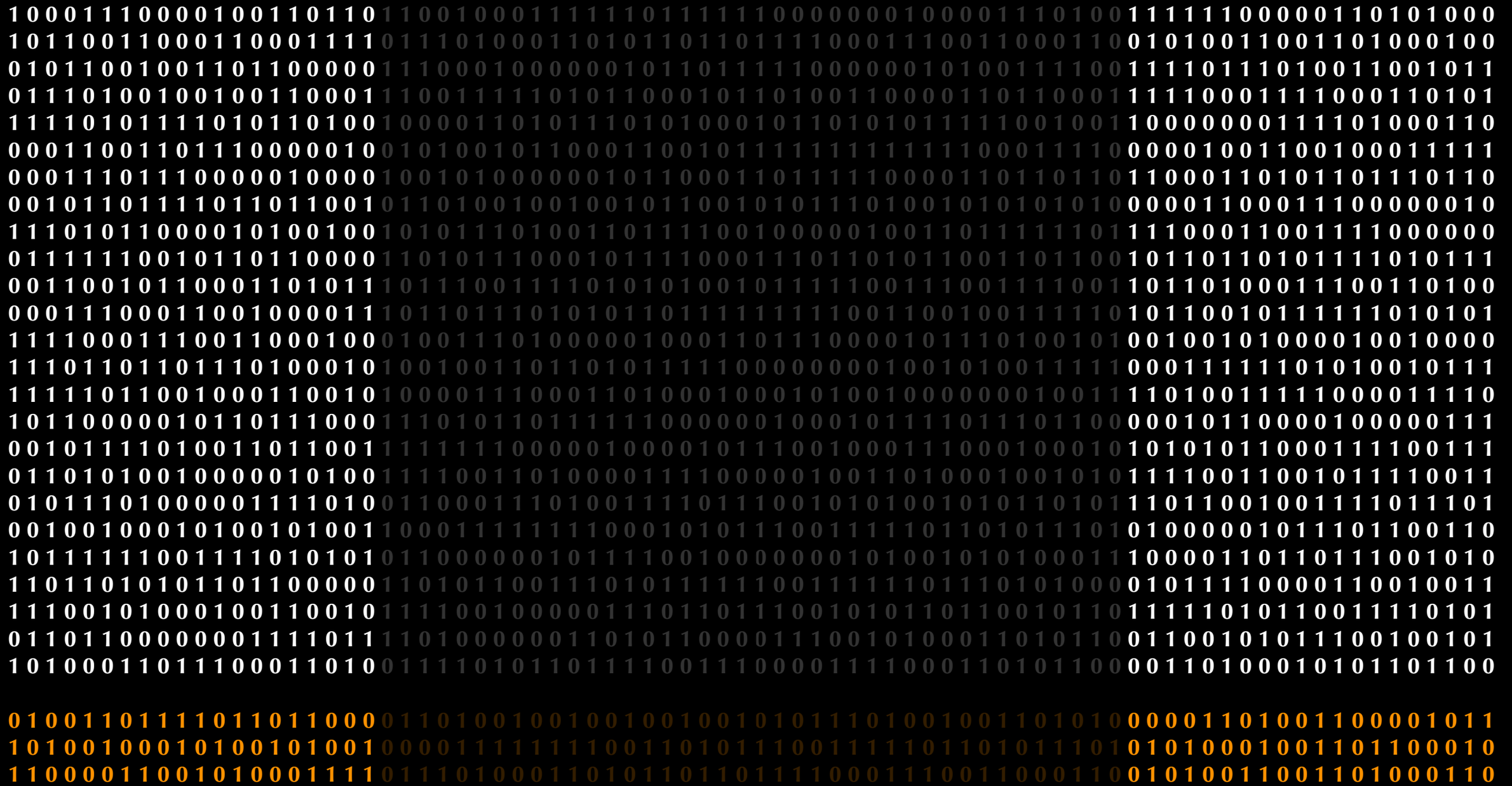

h_{14}

[Arasu, Ganti & Kaushik '06]

Basic correlated LSH
enumeration

For Hamming distance ≤ 2,
a collision is guaranteed in

$h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}$

h_{14}

[Arasu, Ganti & Kaushik ’06]

Basic correlated LSH
enumeration

For Hamming distance ≤ 2, a collision is guaranteed in

$h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}$

Probabilistic argument suggests that it is possible to do much better. But how?

h_{14}

Mathematical question

- A (*p,r*)-***covering*** matrix of dim. *d* satisfies:
  - Has (1-*p*)*d* zeros and *pd* ones in each row;
  - for every set of *r* columns there exists a row with 0s in all of them.

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- <u>Question</u>:
  How few rows can such a matrix have?

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$p = 4/7$
$r = 2$

Mathematical question

- A (p,r)-***covering*** matrix of dim. d satisfies:
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In the mathematics literature such a construction is referred to as a *covering design*

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[Gordon, Patashnik, Kuperberg '95]

A covering design

- Parameters: $d = 2^{r+1}\text{-}1$, $pd = 2^r\text{+}1 \approx d/2$

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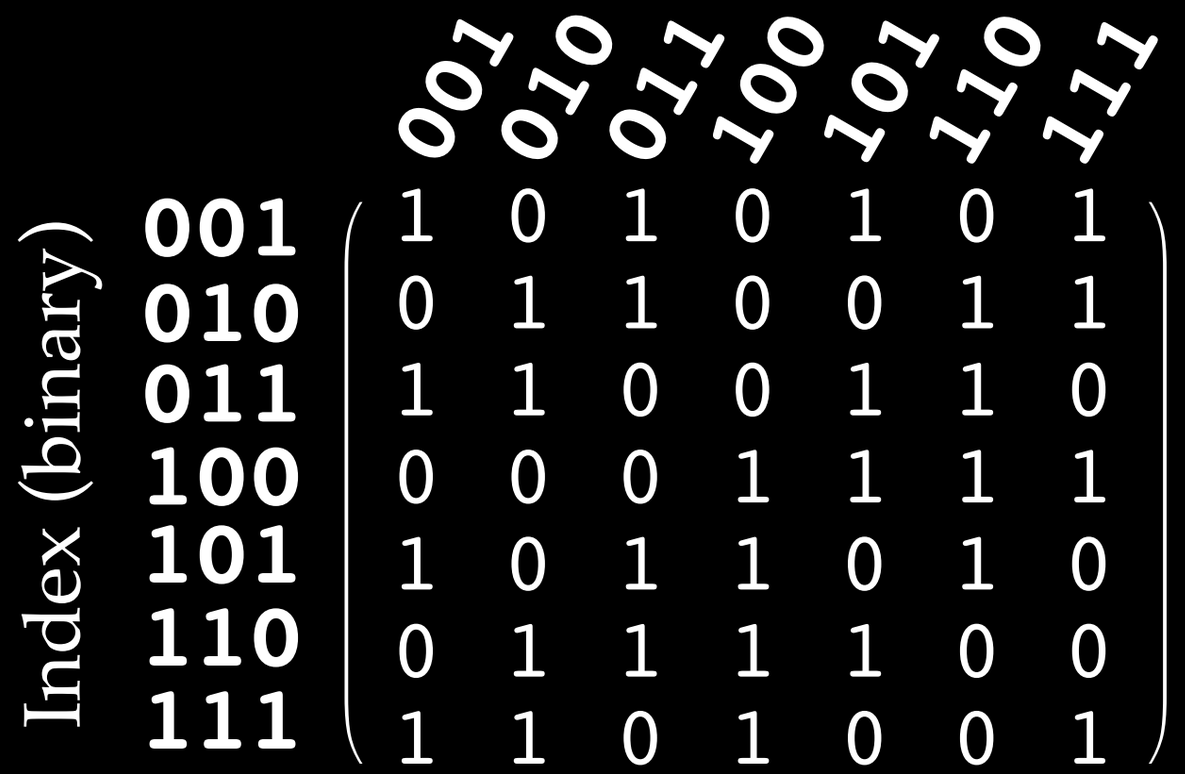

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- Parameters: $d = 2^{r+1}\text{-}1$, $pd = 2^r\text{+}1 \approx d/2$

Idea: Entry is dot product (mod 2) of row / column ID vectors (Hadamard code)

Index (binary)	001	010	011	100	101	110	111
001	1	0	1	0	1	0	1
010	0	1	1	0	0	1	1
011	1	1	0	0	1	1	0
100	0	0	0	1	1	1	1
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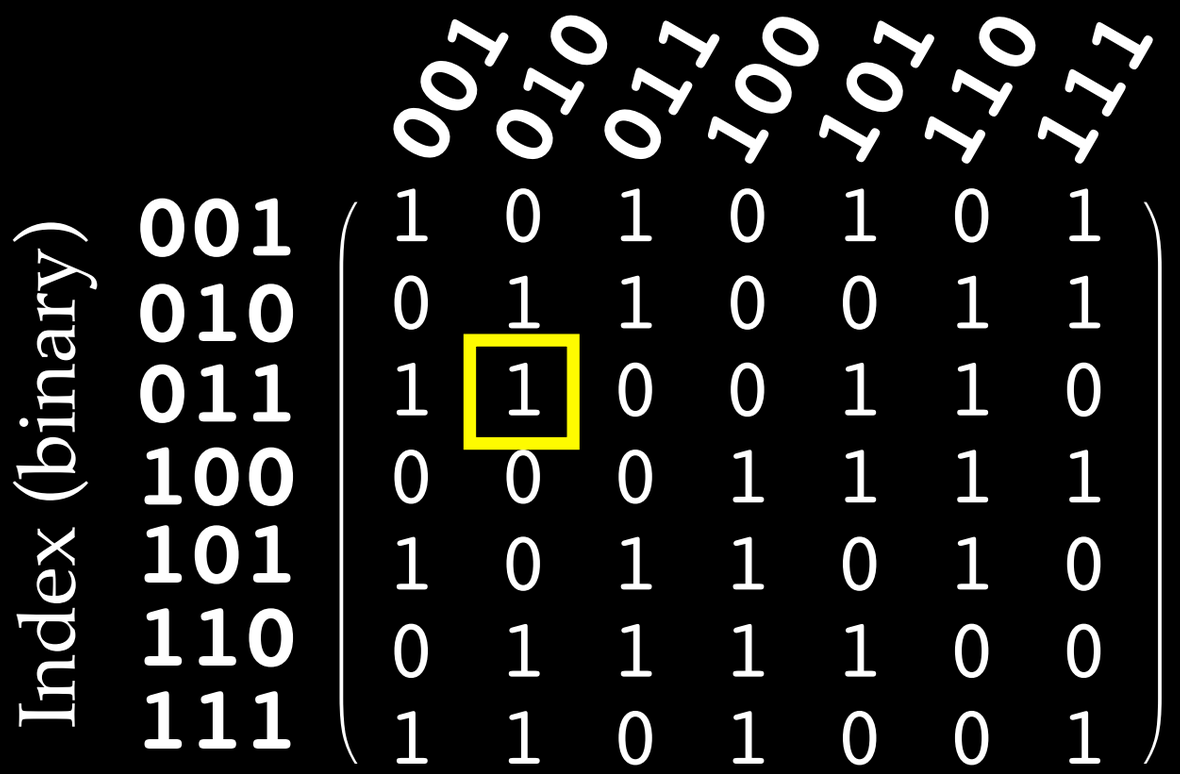

$(0,1,1) \cdot (0,1,0)=1$

[Gordon, Patashnik, Kuperberg '95]

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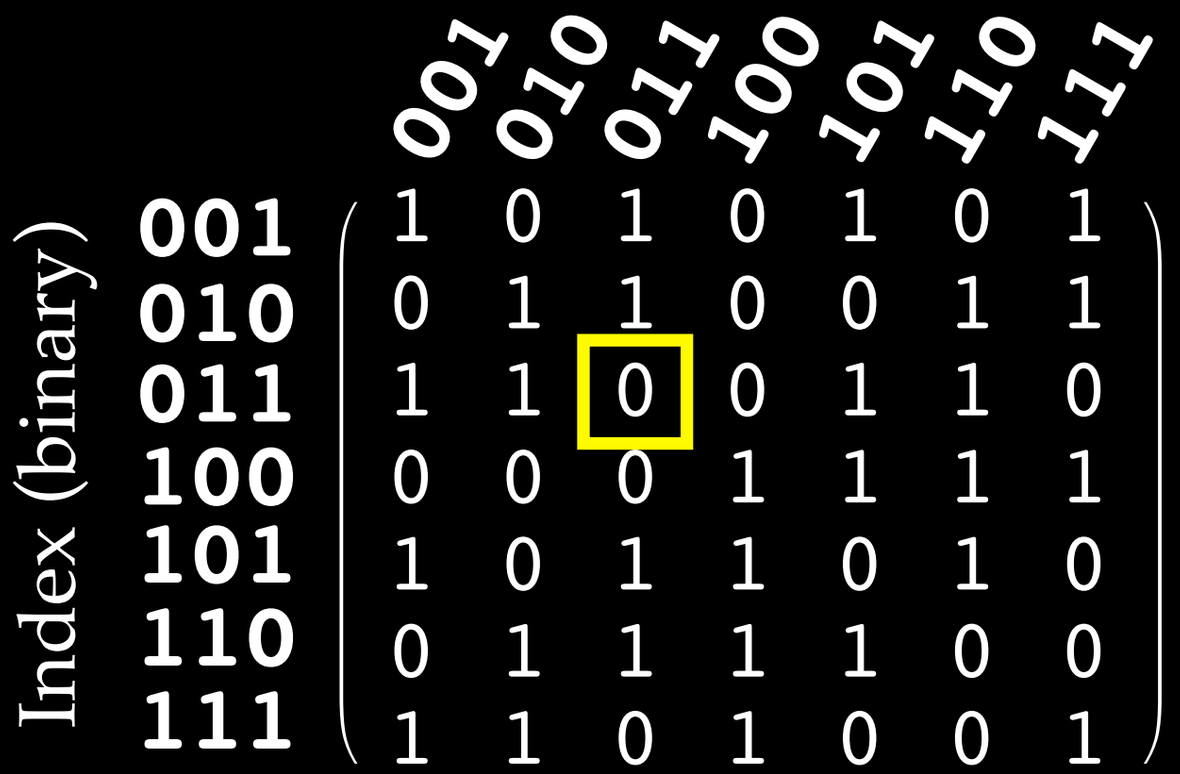

$(0,1,1) \cdot (0,1,1)=0$

[Gordon, Patashnik, Kuperberg '95]

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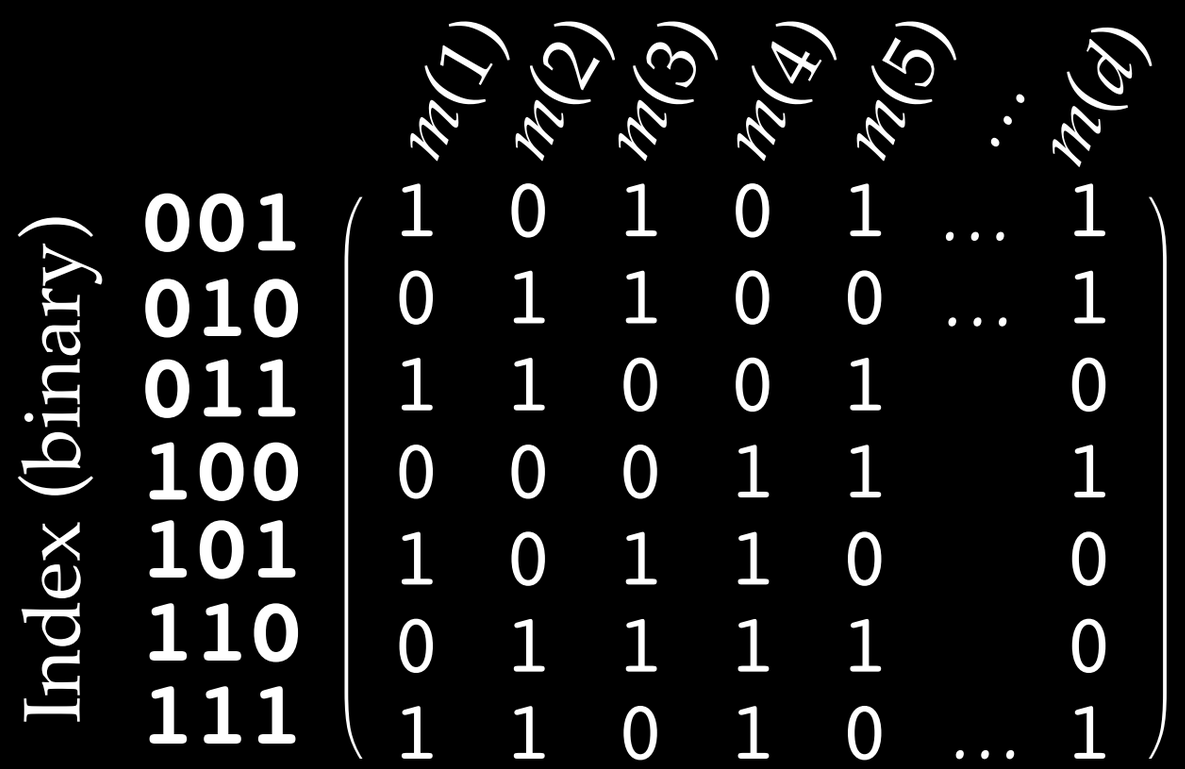
Lemma:
For every set of r vectors in $\{0,1\}^{r+1}$ there exists a nonzero vector that is orthogonal to all

Sampling the covering design

- Parameters: $d = 2^{r+1}\text{-}1$, $pd = 2^r + 1 \approx d/2$

Idea: Choose columns of covering design using a hash function m: $\{1,\ldots,d\} \longrightarrow \{0,1\}^{r+1}$.



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Index (binary)	$m(1)$	$m(2)$	$m(3)$	$m(4)$	$m(5)$	$\ldots$	$m(d)$
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Lemma:

$\Pr[x \wedge a_i = y \wedge a_i] \leq 2^{-||x-y||}$

Implementation in Python

```
def buildCovering(d,r):
    for v in xrange(1,2**(r+1)): A[v] = 0
    for i in xrange(d):
        m = randint(1, 2**(r+1)-1)
        for v in xrange(1,2**(r+1)):
            A[v] = A[v] + (1<<i) * (popcnt(m & v) % 2)

def buildDataStructure(S,r):
    D = {}
    for x in S:
        for v in xrange(1,2**(r+1)):
            if not (x & A[v]) in D: D[x & A[v]] = Set()
            D[x & A[v]].add(x)
    return D

def nearestNeighbor(D,r,y):
    best, nn = infinity, None
    for v in xrange(1,2**(r+1)):
        if (y & A[v]) in D:
            for x in D[y & A[v]]:
                if dist(x,y) < best:
                    best, nn = dist(x,y), x
        if best <= floor(log(v+1,2)): return nn
    return None
```

Too many collisions?

- Can map vectors from $\{0,1\}^d$ to $\{0,1\}^{td}$, increasing all distances by an integer factor t.

$q^t =$ 110**0**10111**0**10111**0**10111**0**101

$x^t =$ 110**1**10111**1**10111**1**10111**1**101

- Try to "hit" sweet spot $tr = \log(n)/c$

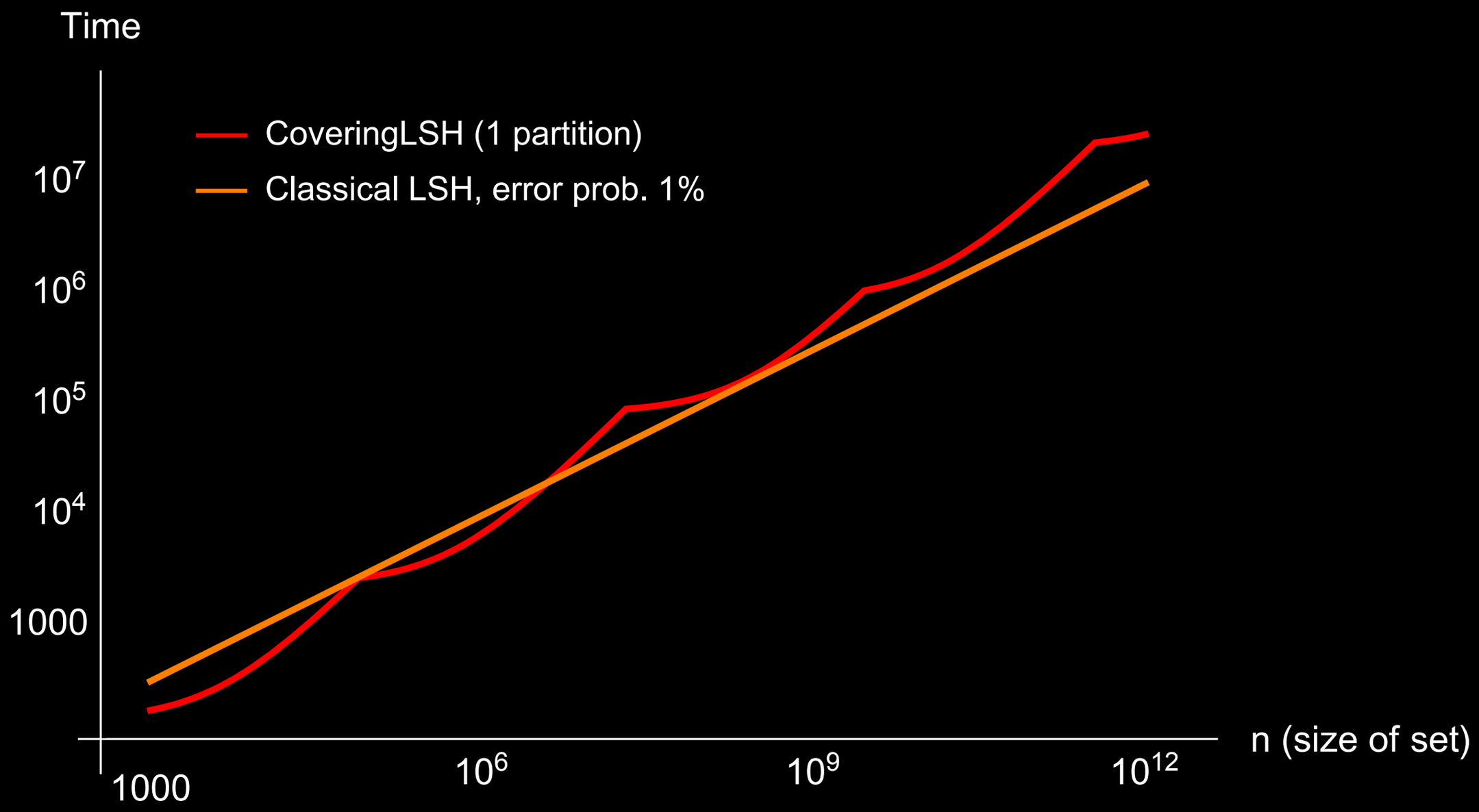
Too many collisions?

- Can map vectors from $\{0,1\}^d$ to $\{0,1\}^{td}$, increasing all distances by an integer factor t.

- For $cr=o(\log n/\log\log n)$, not in SODA version:
  - Use mod p (rather than mod 2) linear algebra to produce denser coverings.
  - Matches Indyk-Matwani up to polylog(n) factor.

Example

r=6, c=2



Time 2^r too much?

- Partition dimensions (randomly):

Time 2^r too much?

- Partition dimensions (randomly):

- Distance in *some* part will be $\leq r/s$
  - Use CoveringLSH with radius r/s on each part

$$\begin{pmatrix}
1&0&1&0&1&0&1&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&1&0&0&0&0&0&0&0&0\\
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0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
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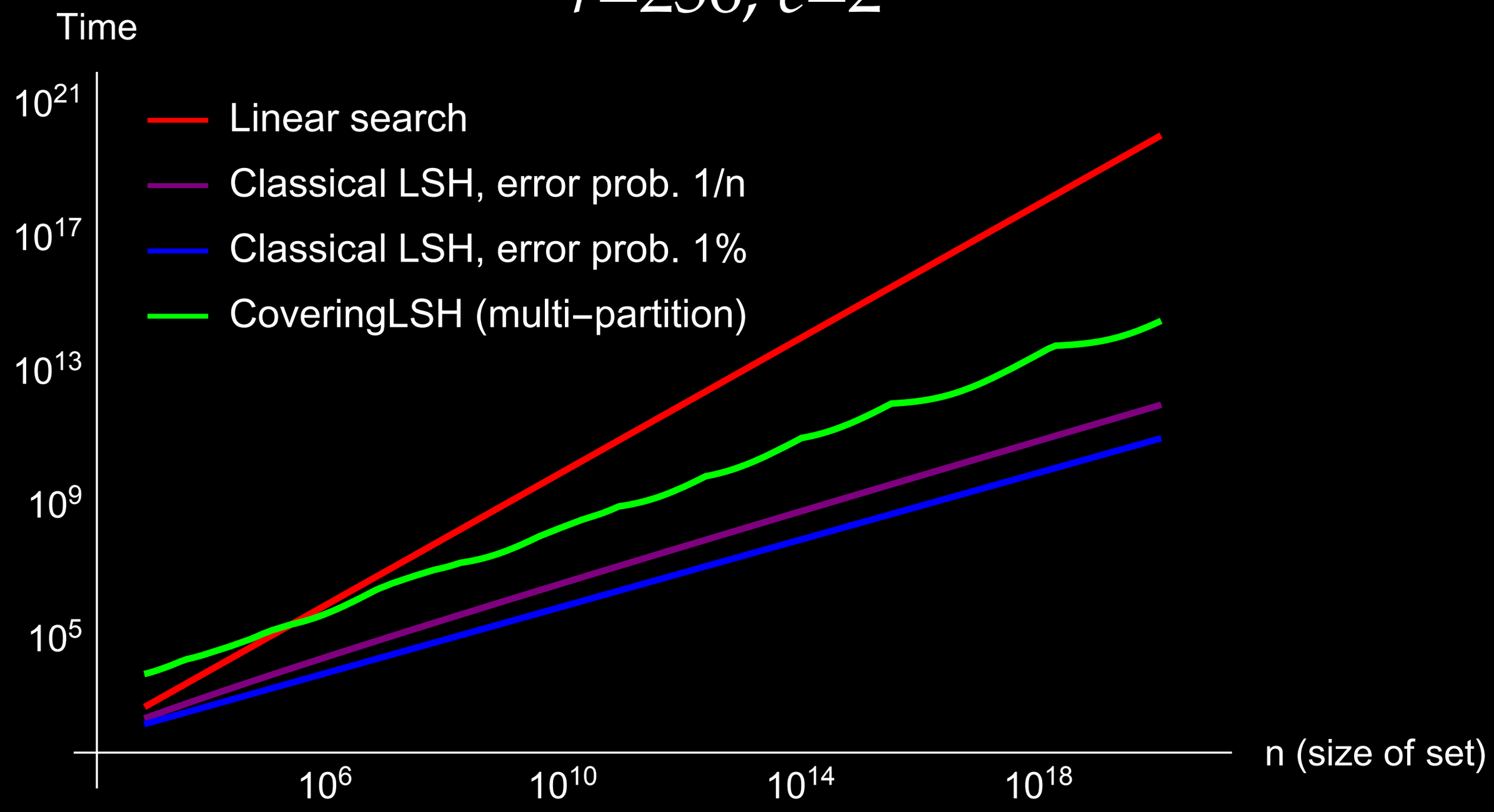
0&0&0&0&0&0&0&1&1&0&1&0&0&1
\end{pmatrix}$$

$s = 2$

$r = 5$

Example

r=256, c=2



Some open questions

Match performance of classical (or data dep.) LSH without false neg.?

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Linear space near neighbor search without false negatives: Matching known bounds?

Some open questions

Match performance of classical (or data dep.) LSH without false neg.?

Linear space near neighbor search without false negatives: Matching known bounds?

Conditional lower bounds for *approximate* similarity search? (Based on SETH, 3SUM,…)

Thank you for your attention!

Acknowledgement of economic support:

European Research Council

Established by the European Commission