LOCALIZATION AND COMPLETION THEOREMS FOR MU-MODULE SPECTRA

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Abstract. Let G be a finite extension of a torus. Working with highly structured ring and module spectra, let M be any module over MU; examples include all of the standard homotopical MU-modules, such as the Brown-Peterson and Morava K-theory spectra. We shall prove localization and completion theorems for the computation of $M_*(BG)$ and $M^*(BG)$. The G-spectrum MU_G that represents stabilized equivariant complex cobordism is an algebra over the equivariant sphere spectrum S_G, and there is an MU_G-module M_G whose underlying MU-module is M. This allows the use of topological analogues of constructions in commutative algebra. The computation of $M_*(BG)$ and $M^*(BG)$ is expressed in terms of spectral sequences whose respective E_2 terms are computable in terms of local cohomology and local homology groups that are constructed from the coefficient ring MU_*^G and its module M_*^G. The central feature of the proof is a new norm map in equivariant stable homotopy theory, the construction of which involves the new concept of a global $\mathscr{I}_*$-functor with smash product.

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1. Introduction and statements of results

Completion theorems relate the nonequivariant cohomology of classifying spaces to algebraic completions of associated equivariant cohomology theories. They are at the heart of equivariant stable homotopy theory and its nonequivariant applications.

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The most celebrated result of this kind is the Atiyah-Segal completion theorem [1]. For any compact Lie group G, it computes $K(BG)$ as the completion of the representation ring $R(G)$ at its augmentation ideal. A more recent such result is the Segal conjecture [3]. For any finite group G, it computes the cohomotopy $\pi^*(BG)$ as the completion of the equivariant cohomotopy π^*_G at the augmentation ideal of the Burnside ring $A(G)$. Unlike the Atiyah-Segal completion theorem, in which the representation ring is under good algebraic control, the Segal conjecture relates two sequences of groups that are largely unknown and difficult to compute.

Shortly after the Atiyah-Segal completion theorem appeared, Landweber [31] and others raised the problem of whether an analog might hold for complex cobordism. It was seen almost immediately that the appropriate equivariant form of complex cobordism to consider was the stabilized version, MU^*_G, introduced by tom Dieck [8]. Shortly after the question was raised, Löffler [36] sketched a proof of the following result. A complete argument has been given by Comezaña and May [6].

Theorem 1.1 (Löffler)**.** *If G is a compact Abelian Lie group, then*

$$(MU^*_G)\hat{_{J_G}} \cong MU^*(BG),$$

where J_G is the augmentation ideal of MU^*_G.

Here $MU^*(BG)$ is completely understood [30, 35, 36], and the result is not difficult because the Euler classes of the irreducible complex representations of G, which of course are all 1-dimensional, are under good control. There has been no further progress in over twenty years. In fact, in his 1992 survey of equivariant stable homotopy theory [4], Carlsson stated the problem as follows:

"Formulate a conjecture about $MU^*(BG)$, for G a finite group."

Landweber [31] had noted that the problem of studying $MU_*(BG)$ seemed to be even harder than the problem of studying $MU^*(BG)$.

In [15], the first author introduced a new approach to the Atiyah-Segal completion theorem (for finite groups), in which he deduced it from what we now understand to be a kind of localization theorem giving a computation of $K_*(BG)$ in terms of local cohomology. When such a localization theorem holds in homology, it is a considerably stronger result than the implied completion theorem in cohomology. For example, the localization theorem for stable homotopy theory is false, although the completion theorem for stable cohomotopy is true. We refer the reader to [21, §§6-8] and [22] for a general discussion of localization theorems in equivariant homology and completion theorems in equivariant cohomology. We shall here prove theorems of this kind for stabilized equivariant complex cobordism. Our results were announced in [9], and an outline of the proofs has appeared in [23].

To make sense of the approach of [15], one must work in a sufficiently precise context of highly structured ring and module spectra that one can mimic constructions in commutative algebra topologically. The theory developed by Elmendorf, Kriz, Mandell, and the second author [11] provides these essential foundations. That paper was written nonequivariantly but, as stated in a metatheorem in its introduction and explained in more detail in [12], all of its theory applies verbatim to G-spectra for any compact Lie group G; see also [10, 21, 13]. In the language of [11], stabilized equivariant cobordism is represented by a commutative algebra MU_G over the equivariant sphere G-spectrum S_G. The underlying nonequivariant S-algebra of MU_G is MU. In earlier language, this means that MU_G is an

E_∞ ring G-spectrum with underlying nonequivariant E_∞ ring spectrum MU. We understand S_G-algebras to be commutative from now on.

A considerable virtue of the kind of localization theorem that we have in mind is that, when it applies to an S_G-algebra R_G with underlying nonequivariant S-algebra R, it automatically implies localization and completion theorems for the computation of $M_*(BG)$ and $M^*(BG)$ for the underlying R-module M of any split R_G-module M_G. (The notion of a split G-spectrum is defined and discussed in [34, II.84], [19, §0], and [21, §3].) This is an especially happy feature of our work since MU_G is split and every MU-module M is the underlying nonequivariant spectrum of a certain split MU_G-module $M_G = MU_G \wedge_{MU} M$ [38]. Therefore, by [11, V§4], our work applies to all of the standard MU-modules that are constructed from MU by quotienting out the ideal generated by a regular sequence of elements of MU_* and localizing by inverting some other elements. In particular, it applies to the Brown-Peterson spectra BP, the Morava K-theory spectra $K(n)$, and the Johnson-Wilson spectra $E(n)$. There is a long and extensive history of explicit calculations of groups $M^*(BG)$ and $M_*(BG)$ in special cases. Some of the relevant authors are: Landweber; Johnson, Wilson, and Yan; Tezuka and Yagita; Bahri, Bendersky, Davis, and Gilkey; Hopkins, Kuhn, and Ravenel; Hunton; and Kriz. See [31, 30, 32, 2] for MU, [28, 29, 40, 41, 42, 43] for BP, and [26, 27, 25] for $K(n)$. Our theorem gives a general conceptual framework into which all such computations must fit.

As we shall make precise shortly, the theorem shows that these nonequivariant homology and cohomology groups are isomorphic to the equivariant homotopy groups of certain homotopical J_G-power torsion MU_G-modules $\Gamma_{J_G}(M_G)$ and homotopical completion MU_G-modules $(M_G)^\wedge_{J_G}$, where J_G is the augmentation ideal of MU^G_*. There result spectral sequences for the computation of these homotopy groups in terms of "local cohomology groups" and "local homology groups" that can be computed from knowledge of the ring MU^G_* and its module M^G_*. Thus the theorem establishes a close connection between the geometrically defined equivariant cobordism groups and the homology and cohomology of classifying spaces with coefficients in MU-modules.

This is entirely satisfactory on a conceptual level. However, like the Segal conjecture, our theorem relates two sequences of groups that are largely unknown and difficult to compute. Thus, on the computational level, it merely points the direction towards further study. Explicit computations will require better understanding of MU^G_* than is now available. We recall an old and probably false conjecture.

Conjecture 1.2. MU^G_* is MU_*-free on generators of even degree.

The conjecture is true when G is Abelian, as was announced by Löffler [35, 36] and proven in detail by Comezaña [5]. Little is known for non-Abelian groups.

Since our work is based on the importation of techniques of commutative algebra into equivariant stable homotopy theory, we briefly recall the relevant algebraic constructions; see [20] for details and discussion. Let R be a graded commutative ring and let $I = (\alpha_1, \ldots, \alpha_n)$ be a finitely generated ideal in R. Define $K^\bullet(I)$ to be the tensor product of the graded cochain complexes

$$K^\bullet(\alpha_i) = (R \to R[1/\alpha_i]),$$

where R and $R[1/\alpha_i]$ lie in homological degrees 0 and 1. Up to quasi-isomorphism, $K^\bullet(I)$ depends only on the radical of I. For a graded R-module M, define

$$H_I^{s,t}(R;M) = H^{s,t}(K^\bullet(I)\otimes M),$$

where s indicates the homological degree and t the internal grading. Such "local cohomology groups" were first defined by Grothendieck [24]. It is easy to see that $H_I^0(R;M)$ is the submodule

$$\Gamma_I(M) = \{m\in M\,|\,I^N m = 0 \text{ for some } N\}$$

of I-power torsion elements of M. If R is Noetherian it is not hard to prove directly that the functor $H_I^*(R;-)$ is effaceable and hence that local cohomology calculates the right derived functors of $\Gamma_I(-)$ [24]. It is clear that the local cohomology groups vanish above degree n, but in the Noetherian case Grothendieck's vanishing theorem shows the powerful fact that they are zero above the Krull dimension of R. One key fact that we shall use is that if $\beta\in I$ then $H_I^*(R;M)[1/\beta]=0$; this is a restatement of the easily proven fact that $K^\bullet(I)[1/\beta]$ is exact [20, 1.1]. We abbreviate

$$H_I^*(R) = H_I^*(R;R).$$

These algebraic local cohomology groups are relevant to topological homology groups.

There are dual "local homology groups" which, to the best of our knowledge, were first introduced in [17, 18]. Replacing $K^\bullet(I)$ by a quasi-isomorphic R-free chain complex $K'^\bullet(I)$, define

$$H^I_{s,t}(R;M) = H_{s,t}(Hom(K'^\bullet(R),M)).$$

There is a tri-graded universal coefficient spectral sequence that converges to these groups; ignoring the internal grading t, which is unchanged by the differentials, it converges in total degree $s=-(p+q)$ and satisfies

$$E_2^{p,q} = Ext_R^p(H_I^{-q}(R),M) \quad\text{and}\quad d_r : E_r^{p,q}\to E_r^{p+r,q-r+1}.$$

There is a natural epimorphism $H_0^I(R;M)\to M_{\hat{I}}$ whose kernel is a certain lim^1 group. It is not hard to check from the definition that if R is Noetherian and M is free or finitely generated then $H_0^I(R;M)\cong M_I^\wedge$, and one may also prove that in these cases the higher local homology groups are zero. It follows that, at least when R is Noetherian, $H_*^I(R;M)$ calculates the left derived functors of the (not necessarily right exact) I-adic completion functor. These algebraic local homology groups are relevant to topological cohomology groups.

Now, returning to topology, let R_G be an S_G-algebra and M_G be an R_G-module; we always understand algebras and modules in the highly structured sense of [11]. We understand G-spectra to be indexed on a complete G-universe U, which implies that our equivariant homology and cohomology theories are $RO(G)$-graded. However, we restrict attention to integer degrees except where explicitly stated otherwise. We write $E_n^G = E_G^{-n}$ for the nth homotopy group $\pi_n^G(E) = [S_G,E_G]_n^G$ of a G-spectrum E_G.

For $\alpha\in R_k^G$, let $R_G[1/\alpha]$ be the telescope of iterates

$$R_G\to\Sigma^{-k}R_G\to\Sigma^{-2k}R_G\to\cdots$$

of multiplication by α and let $K(\alpha)$ be the fiber of the canonical map $R_G\to R_G[1/\alpha]$. For a finitely generated ideal $I=(\alpha_1,\ldots,\alpha_n)$ in R_*^G, let $K(I)$ be the smash product over R_G of the R_G-modules $K(\alpha_i)$. Up to equivalence of R_G-modules, $K(I)$ depends only on the radical of I. Define

$$\Gamma_I(M_G) = K(I) \wedge_{R_G} M_G$$

and

$$(M_G)^{\wedge}_I = F_{R_G}(K(I), M_G).$$

There is a spectral sequence converging to $\Gamma_I(M_G)^G_*$ (in total degree $p+q$), with

$$E^2_{p,q} = H_I^{-p,-q}(R^G_*; M^G_*) \quad \text{and} \quad d^r : E^r_{p,q} \to E^r_{p-r,q+r-1},$$

and there is a spectral sequence converging to $((M_G)^{\wedge}_I)^*_G$ (in total degree $p+q$) with

$$E_2^{p,q} = H^I_{-p,-q}(R^*_G; M^*_G) \quad \text{and} \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

Now take I to be a finitely generated ideal contained in the augmentation ideal $J_G = Ker(R^G_* \to R_*)$. Note that the ring R^G_* need not be Noetherian and the augmentation ideal need not be finitely generated. In particular, MU^G_* is not Noetherian and its augmentation ideal is not finitely generated, even when G is finite.

Since $R[1/\alpha]$ is nonequivariantly contractible for $\alpha \in J_G$, the canonical map $K(I) \to R_G$ is an equivalence of underlying spectra and so induces an equivalence upon smashing with EG_+, where EG_+ is the union of EG and a G-fixed disjoint basepoint. Inverting this equivalence and using the projection $EG_+ \longrightarrow S^0$, we obtain a canonical map of R_G-modules

$$\kappa : EG_+ \wedge R_G \to K(I).$$

For an R_G-module M_G, κ induces maps of R_G-modules

$$EG_+ \wedge M_G \simeq EG_+ \wedge R_G \wedge_{R_G} M_G \to K(I) \wedge_{R_G} M_G = \Gamma_I(M_G)$$

and

$$(M_G)^{\wedge}_I = F_{R_G}(K(I), M_G) \to F_{R_G}(EG_+ \wedge R_G, M_G) \simeq F(EG_+, M_G),$$

both of which will be equivalences for all R_G-modules M_G if κ is an equivalence. We can now state our completion theorem for modules over MU_G.

Theorem 1.3. *Let G be finite or a finite extension of a torus. Then, for any sufficiently large finitely generated ideal $I \subset J_G$, $\kappa : EG_+ \wedge MU_G \to K(I)$ is an equivalence. Therefore,*

$$EG_+ \wedge M_G \to \Gamma_I(M_G) \quad \textit{and} \quad (M_G)^{\wedge}_I \to F(EG_+, M_G)$$

are equivalences for any MU_G-module M_G.

It is reasonable to define $K(J_G)$ to be $K(I)$ for any sufficiently large I and to define $\Gamma_{J_G}(M_G)$ and $(M_G)^{\wedge}_{J_G}$ similarly. The theorem implies that these MU_G-modules are independent of the choice of I.

Our main interest is in finite groups. However, the fact that the result holds for a finite extension of a torus and therefore for the normalizer of a maximal torus in an arbitrary compact Lie group suggests the following generalization. There should be an appropriate transfer argument, but we have not succeeded in finding one.

Conjecture 1.4. The theorem remains true for any compact Lie group G.

It is valuable to obtain a completion theorem about $EG_+ \wedge_G X$ for a general based G-space X, obtaining the motivating result about BG (which referred to unreduced homology and cohomology) by taking X to be S^0. For this purpose, we replace M_G by $M_G \wedge X$ in the first equivalence and by $F(X, M_G)$ in the second. We write M_* and M^* for the reduced homology and cohomology of based spaces. If M_G is split with underlying nonequivariant MU-module M, then

$$M_*(EG_+ \wedge_G X) \cong \pi_*^G((EG_+ \wedge X) \wedge \Sigma^{-Ad(G)} M_G)$$

and

$$M^*(EG_+ \wedge_G X) \cong \pi_{-*}^G(F(EG_+ \wedge X, M_G)),$$

where $Ad(G)$ is the adjoint representation of G [34, II.8.4]. Thus the theorem has the following immediate consequence.

Theorem 1.5. *Assume the hypotheses of the theorem and assume that M_G is split. Then*

$$M_*(EG_+ \wedge_G X) \cong \Gamma_I(\Sigma^{-Ad(G)} M_G \wedge X)_*^G$$

and

$$M^*(EG_+ \wedge_G X) \cong (F(X, M_G))_I^{\wedge})_G^*$$

for any based G-space X.

Implicitly replacing X by its suspension G-spectrum, we are entitled to the following spectral sequences.

Corollary 1.6. *There is a homological spectral sequence that converges from*

$$E^2_{p,q} = H_I^{-p,-q}(MU_*^G; M_*^G(\Sigma^{-Ad(G)} X))$$

to $M_*(EG_+ \wedge_G X)$. There is a cohomological spectral sequence that converges from

$$E_2^{p,q} = H^I_{-p,-q}(MU_G^*; M_G^*(X))$$

to $M^*(EG_+ \wedge_G X)$.

Combining Löffler's theorem with ours, we see that the topology forces the following algebraic conclusion. A direct proof would be out of reach at present.

Corollary 1.7. *If G is a compact Abelian Lie group and I is sufficiently large, then*

$$H_0^I(MU_G^*) \cong ((MU_G)_{\hat{I}})_G^* \cong (MU_G^*)_{\hat{I}}$$

and

$$H_p^I(MU_G^*) = 0 \quad \text{if} \quad p \neq 0.$$

2. The strategy of proof

For clarity, we shall emphasize the general strategy of proof, focusing on MU_G only where necessary. Let G be a compact Lie group and let S_G be the sphere G-spectrum. We assume given a commutative S_G-algebra R_G with underlying nonequivariant commutative S-algebra R.

For a (closed) subgroup H of G, let res_H^G denote the restriction

$$R_G^* = R_G^*(S^0) \longrightarrow R_G^*(G/H_+) = R_H^*;$$

it is induced by the projection $G/H_+ \longrightarrow S^0$. Let J_H denote the augmentation ideal in R_H^*, namely the kernel of $res_1^H : R_H^* \longrightarrow R^*$. In [21, 7.5], we explained a general localization and completion theorem for the calculation of $M_*^G(EG_+ \wedge X)$

and $M_G^*(EG_+ \wedge X)$ for any split R_G-module M_G and G-spectrum X. In that theorem, we assumed that G is finite, each $RO(G)$-graded theory $R_H^\#$ has Thom isomorphisms, and each R_H^* is Noetherian and satisfies

$$\sqrt{res_H^G(J_G)} = \sqrt{J_H}.$$

In fact, we did not discuss the verification of this last property in [21], and we remarked that its verification "can be the the main technical obstruction to the implementation of our strategy when we work more generally with compact Lie groups and non-Noetherian coefficient rings". See [16] for a discussion of this point in the Noetherian case. The algebraic and topological constructions in our general approach demand that we work with finitely generated subideals of J_G, and the last property then makes little sense. Thus we need to modify our strategy of proof. We begin work by describing our modified strategy.

Thom isomorphisms and Euler classes are essential to the strategy, and we begin with these. Here we must consider $RO(G)$-graded cohomology groups, and we use the notation $R_G^\#(X)$ for the $RO(G)$-graded cohomology of a G-spectrum X to distinguish it from the $\mathbb{Z}$-graded part $R_G^*(X)$. In particular, we write $R_G^\#$ for the $RO(G)$-graded coefficient ring. For a real representation V of G. The inclusion

$$e_V : S^0 \longrightarrow S^V$$

induces a map

$$e_V^\# : R_G^\#(S^V) \longrightarrow R_G^\#(S^0) = R_G^\#.$$

Let $1 \in R_G^0$ be the identity element; it is represented by the unit $\eta : S_G \longrightarrow R_G$. Its suspension $\Sigma^V 1$ is an element of $R_G^V(S^V)$, and we define

$$e(V) = e_V^\#(\Sigma^V 1) \in R_G^V(S^0) = R_G^V.$$

We need to be able to shift these classes into integer degrees.

Definition 2.1. The theory $R_G^\#$ has Thom isomorphisms if, for each complex representation V of G, there is a natural isomorphism

$$\phi_V : R_G^\#(X \wedge S^{|V|}) \longrightarrow R_G^\#(X \wedge S^V)$$

of $R_G^\#$-modules, where $|V|$ is the real dimension of V. We may view ϕ_V as giving isomorphisms

$$\phi_V : R_G^{W-|V|}(X) \longrightarrow R_G^{W-V}(X)$$

for $W \in RO(G)$. Taking $W = V + |V|$ and $X = S^0$, we define

$$\chi(V) = \phi_V(e(V)) \in R_G^{|V|}.$$

Remark 2.2. Let $\mu(V) = \phi_V(1) \in R_G^{|V|-V}$. Since ϕ_V is an isomorphism of $R_G^\#$-modules, $\mu(V)$ is a unit in $R_G^\#$, and we may as well insist that $\phi_V(x) = x \cdot \mu(V)$ for all $x \in R^\#(X \wedge S^{|V|})$ and all G-spectra X. That is, we take our Thom isomorphisms to be given by right multiplication by Thom classes. In particular, $\chi(V) = e(V)\mu(V)$.

Remark 2.3. If V contains a trivial representation, so that $V^G \neq 0$, then e_V is null homotopic and therefore $e(V) = 0$ and $\chi(V) = 0$.

Now let I be a given finitely generated subideal of J_G. For $H \subseteq G$, let $r_H^G(I)$ denote the resulting subideal $res_H^G(I) \cdot R_H^*$ of J_H.

Definition 2.4. Assume that each $R_H^{\#}$ has Thom isomorphisms. The ideal I in R_G^* is sufficiently large at H if there is a non-zero complex representation V of H such that $V^H = 0$ and the Euler class $\chi(V) \in R_H^{|V|}$ is in the radical $\sqrt{r_H^G(I)}$. The ideal I is sufficiently large if it is sufficiently large at all $H \subseteq G$.

As explained in the introduction, we have a canonical map of R_G-modules

$$\kappa : EG_+ \wedge R_G \longrightarrow K(I),$$

and our goal is to prove that it is an equivalence. The essential point of our strategy is the following result, which reduces the problem to the construction of a sufficiently large finitely generated subideal I of J_G.

Theorem 2.5. *Assume that $R_H^{\#}$ has Thom isomorphisms for all $H \subseteq G$. If I is a sufficiently large finitely generated subideal of J_G, then*

$$\kappa : EG_+ \wedge R_G \longrightarrow K(I)$$

is an equivalence. Therefore,

$$EG_+ \wedge M_G \to \Gamma_I(M_G) \text{ and } (M_G)^{\wedge}_I \to F(EG_+, M_G)$$

are equivalences for any R_G-module M_G.

Proof. Let $\tilde{E}G$ be the cofiber of the projection $EG_+ \longrightarrow S^0$. Then the cofiber of κ is equivalent to $\tilde{E}G \wedge K(I)$, and we must prove that this is contractible. Using the transitivity of restriction maps to see that $r_H^G(I)$ is a large enough subideal of R_H^*, we see that the hypotheses of the theorem are inherited by any subgroup. Using the fact that there is no infinite descending chain of compact Lie groups, we see that we may assume that the theorem holds for $H \in \mathscr{P}$, where $\mathscr{P}$ is the family of proper subgroups of G. Thus $\tilde{E}H \wedge K(r_H^G(I))$ is contractible for $H \in \mathscr{P}$, and clearly

$$(\tilde{E}G \wedge K(I))|_H = \tilde{E}H \wedge K(r_H^G(I)).$$

From here, the proof is just like that of [20, 7.5]. We have that

$$G/H_+ \wedge \tilde{E}G \wedge K(I)$$

is contractible for $H \in \mathscr{P}$. We take $\tilde{E}\mathscr{P}$ to be the colimit of spheres S^V, where V runs through the complex representations such that $V^G = \{0\}$ in a complete complex universe U. Since $\tilde{E}\mathscr{P}/S^0$ is triangulable as a G-CW complex whose cells have proper orbit type, the induction hypothesis implies that

$$(\tilde{E}\mathscr{P}/S^0) \wedge \tilde{E}G \wedge K(I)$$

is contractible. By the cofiber sequence $S^0 \longrightarrow \tilde{E}\mathscr{P} \longrightarrow \tilde{E}\mathscr{P}/S^0$ and the fact that $\tilde{E}\mathscr{P} \wedge S^0 \longrightarrow \tilde{E}\mathscr{P} \wedge \tilde{E}G$ is an equivalence, it suffices to show that $\tilde{E}\mathscr{P} \wedge K(I)$ is contractible. For any R_G-module M_G, the construction of $\tilde{E}\mathscr{P}$ and our translation of Euler classes to integer gradings imply directly that $\pi_*^G(\tilde{E}\mathscr{P} \wedge M_G)$ is the localization $\pi_*^G(M_G)[\{\chi(V)^{-1}\}]$ obtained by inverting the Euler classes $\chi(V)$ (see [20, 3.20]). With $M_G = K(I)$, we have a spectral sequence that converges from the local cohomology groups $H_I^*(R_*^G)$ to $\pi_*^G(K(I))$. Localizing by inverting the $\chi(V)$, we obtain a spectral sequence that converges from the localization $H_I^*(R_*^G)[\{\chi(V)^{-1}\}]$ to $\pi_*^G(\tilde{E}\mathscr{P} \wedge K(I))$. As we pointed out earlier, the local cohomology of a ring at an ideal vanishes when it is localized by inverting an element in that ideal. Thus our assumption that I is sufficiently large at G ensures that the E^2-term of our spectral sequence is zero. □

3. Constructing sufficiently large finitely generated ideals

The idea is to obtain enough elements of J_G to give a good approximation to it. For those groups G that act freely on a finite product of unit spheres of representations, such as the finite p-groups, there are enough representations that we can simply use finitely many Euler classes $\chi(V)$. However, even for general finite groups, the ideal generated by the $\chi(V)$ is usually not sufficiently large. We need to add in other elements, and we shall do so by exploiting norm, or "multiplicative transfer", maps that are analogous to Evens' norm maps in the cohomology of groups [14, Ch. 5]. This is a new construction in equivariant stable homotopy theory and should have other applications.

However, we shall state three theorems and two lemmas that explain our strategy of proof before specifying in Definition 3.6 what it means for a theory to have norm maps. Similarly, the theorems are stated in terms of "natural" Thom isomorphisms, and we shall specify the relevant naturality conditions in Definition 3.7.

We assume given a toral group G, namely an extension of the form

$$1 \longrightarrow T \longrightarrow G \longrightarrow F \longrightarrow 1,$$

where T is a torus and F is a finite group. Most of our work will be necessary even when T is trivial and we are dealing only with the finite group F.

Theorem 3.1. *Let G be toral. If, for each $H \subseteq G$, $R_H^{\#}$ has natural Thom isomorphisms and R_H^* has norm maps, then J_G contains a sufficiently large finitely generated subideal.*

We shall define the notion of a global $\mathscr{I}_*$ functor with smash product, abbreviated $\mathscr{G}\mathscr{I}_*$-FSP, in Section 5. A $\mathscr{G}\mathscr{I}_*$-FSP T has an associated S_G-algebra $R(T)_G$ for every compact Lie group G; regarded as an S_H-algebra for $H \subseteq G$, $R(T)_G$ is canonically isomorphic to $R(T)_H$. The real work in this paper is the proof of the following theorem, which is the subject of Sections 6–9.

Theorem 3.2. *Let T be a $\mathscr{G}\mathscr{I}_*$-FSP T such that $R(T)_G^{\#}$ has natural Thom isomorphisms for every compact Lie group G. Then every $R(T)_G^*$ has norm maps.*

The application to Thom spectra is justified by the following result, which is proven in Example 5.8 and Section 10.

Theorem 3.3. *There is a $\mathscr{G}\mathscr{I}_*$-FSP TU such that $R(TU)_G = MU_G$ for every compact Lie group G, and every $MU_G^{\#}$ has natural Thom isomorphisms.*

The previous two results show that Theorem 3.1 applies to MU_G.

The proof of Theorem 3.1 depends on two lemmas. The first, whose proof will be deferred to Section 11, is an exercise in the representation theory of Lie groups that has nothing to do with the hypotheses on our theories. For $H \subseteq G$, we write

$$res_H^G : R(G) \longrightarrow R(H)$$

for the restriction homomorphism. When H has finite index in G, we write

$$ind_H^G : R(H) \longrightarrow R(G)$$

for the induction homomorphism. Recall that $ind_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

Lemma 3.4. *There are non-zero complex representations $V_1, \cdots, V_s$ of T such that T acts freely on the product of the unit spheres of the representations $res^G_T ind^G_T V_i$.*

Lemma 3.5. *Assume the hypotheses of Theorem 3.1 and let F' be a subgroup of F with inverse image G' in G. Then there is an element $\xi(F')$ of J_G such that*

$$res^G_{G'}(\xi(F')) = \chi(V')^{w'},$$

where V' is the reduced regular complex representation of F' regarded by pullback as a representation of G' and w' is the order of $WG' = NG'/G'$.

The proof will be given at the end of the section.

Proof of Theorem 3.1. We claim that the ideal

$$I = (\chi(ind^G_T V_1), \cdots, \chi(ind^G_T V_s)) + (\xi(F')|F' \subseteq F)$$

is sufficiently large.

If H is a subgroup of G that intersects T non-trivially, then, by Lemma 3.4, $(res^G_T ind^G_T V_i)^{H\cap T} = \{0\}$ for some i and therefore $(ind^G_T V_i)^H = \{0\}$. Since

$$\chi(res^G_H ind^G_T V_i) = res^G_H(\chi(ind^G_T V_i)) \in r^G_H(I),$$

this shows that I is sufficiently large at H in this case.

If H is a subgroup of G that intersects T trivially, as is always the case when G is finite, then H maps isomorphically to its image F' in F. If G' is the inverse image of F' in G and V' is the reduced regular complex representation of F' regarded as a representation of G', then $res^{G'}_H(V')$ is the reduced regular complex representation of H and $(res^{G'}_H(V'))^H = 0$. By Lemma 3.5, we have $res^G_{G'}(\xi(F')) = \chi(V')^{w'}$ and therefore

$$\chi(res^{G'}_H(V'))^{w'} = res^{G'}_H(\chi(V')^{w'}) = res^{G'}_H res^G_{G'}(\xi(F')) = res^G_H(\xi(F')) \in r^G_H(I).$$

This shows that I is sufficiently large at H in this case. □

We need conjugation isomorphisms to describe the properties of norm maps and to prove the lemmas. For $g \in G$ and $H \subseteq G$, let ${}^gH = gHg^{-1}$ and $c_g : {}^gH \longrightarrow H$ be the conjugation isomorphism. For a representation V of H, let ${}^gV$ be the pullback of V along c_g. For $H \subseteq G$, we have a natural restriction homomorphism

$$res^G_H : R^G_*(X) \longrightarrow R^H_*(X)$$

on based G-spaces X. For $g \in G$, we also have a natural isomorphism

$$c_g : R^H_*(X) \longrightarrow R^{{}^gH}_*({}^gX),$$

where X is a based H-space and ${}^gX$ denotes X regarded as a ${}^gH$-space by pullback along c_g. To give content to the proof of Lemma 3.5, we must explain our hypothesis that R^G_* also has norm maps. We give a crude and perhaps unilluminating definition that prescribes exactly what we shall use in the proof. A description closer to the motivating example of group cohomology will be given in the next section.

Definition 3.6. We say that R^G_* has norm maps if, for a subgroup H of finite index n in G and an element $y \in R^H_{-r}$, where $r \geq 0$ is even, there is an element

$$\overline{norm}^G_H(1+y) \in \sum_{i=0}^{n} R^G_{-ri}$$

that satisfies the following properties; here $1 = 1_H \in R_0^H$ denotes the identity element.

(i) $\overline{norm}_G^G(1+y) = 1+y$.
(ii) $\overline{norm}_H^G(1) = 1$.
(iii) [The double coset formula]

$$res_K^G\, \overline{norm}_H^G\,(1+y) = \prod_g \overline{norm}_{^gH\cap K}^K\, res_{^gH\cap K}^{^gH}\, c_g(1+y),$$

where K is any subgroup of G and g runs through a set of double coset representatives for $K\backslash G/H$.

We must also explain what it means for Thom isomorphisms to be "natural".

Definition 3.7. Let $R_H^{\#}$ have Thom isomorphisms ϕ_{V_H} given by right multiplication by Thom classes $\mu(V_H)$ for all $H \subset G$ and all complex representations V_H of H. We say that the Thom isomorphisms are natural if the following three conditions hold.

(i) Compatibility under restriction: $\mu(V_G)|_H = \mu(V_G|_H)$.
(ii) Compatibility under conjugation: $\mu(^gV_H) = c_g(\mu(V_H))$ in $R_H^{|V_H|-{}^gV_H}$ for $g \in G$.
(iii) Multiplicativity: $\mu(V_H)\mu(V'_H) = \mu(V_H \oplus V'_H)$.

Proof of Lemma 3.5. Since the restriction of the reduced regular representation of F' to any proper subgroup contains a trivial representation, the restriction of $\chi(V') \in R_{G'}^*$ to a subgroup that maps to a proper subgroup of F' is zero. In $R_{G'}^*$, the double coset formula gives

$$res_{G'}^G\, \overline{norm}_{G'}^G\,(1+\chi(V')) = \prod_g \overline{norm}_{^gG'\cap G'}^{G'}\, res_{^gG'\cap G'}^{^gG'}\, c_g(1+\chi(V')), \tag{3.8}$$

where g runs through a set of double coset representatives for $G'\backslash G/G'$. Taking V' as in the statement of the lemma, compatibility under conjugation gives that

$$c_g(1+\chi(V')) = 1+\chi(^gV').$$

Here $^gV'$ is the reduced regular representation of $^gG'$. Clearly $^gG'\cap G'$ is the inverse image in G of $^gF'\cap F'$. If $^gF'\cap F'$ is a proper subgroup of F', then the restriction of $\chi(V')$ to $^gG'\cap G'$ is zero. Therefore all terms in the product on the right side of (3.8) are 1 except for those that are indexed on elements $g \in NG'$. There is one such g for each element of $WG' = NG'/G'$, and the term in the product that is indexed by each such g is just $1+\chi(V')$. Therefore (3.8) reduces to

$$res_{G'}^G\, \overline{norm}_{G'}^G(1+\chi(V')) = (1+\chi(V'))^{w'}. \tag{3.9}$$

If V' has real dimension r, then the summand of $(1+\chi(V'))^{w'}$ in degree rw' is $\chi(V')^{w'}$. Since $res_{G'}^G$ preserves the grading, we may take $\xi(F')$ to be the summand of degree rw' in $\overline{norm}_{G'}^G(1+\chi(V'))$. □

4. The idea and properties of norm maps

We give an intuitive idea of the construction, leaving details and rigor to later sections. Let H be a subgroup of finite index n in a compact Lie group G. For based H-spaces X, we can give the smash power X^n an action of G. Intuitively, this is done in exactly the same way that one induces up a representation of H to a representation of G, and the analogy will guide much of our work.

To begin with, the norm map will be a natural function

$$norm^G_H : R^H_0(X) \longrightarrow R^G_0(X^n). \tag{4.1}$$

Norm maps $\overline{norm}^G_H$ in the sense of Definition 3.6 will be obtained by taking X to be the wedge $S^0 \vee S^r$, studying the decomposition of X^n into wedge summands of G-spaces described in terms of representations, and using Thom isomorphisms to translate the result to integer gradings. The norm map $norm^G_H$ will satisfy the following properties.

$$norm^G_G \text{ is the identity function.} \tag{4.2}$$

$$norm^G_H(1_H) = 1_G, \text{ where } 1_H \in R^H_0(S^0) \text{ is the identity element.} \tag{4.3}$$

$$norm^G_H(xy) = norm^G_H(x) norm^G_H(y) \text{ if } x \in R^H_0(X) \text{ and } y \in R^H_0(Y). \tag{4.4}$$

Here the product xy on the left is defined by use of the evident map

$$R^H_0(X) \otimes R^H_0(Y) \longrightarrow R^H_0(X \wedge Y) \tag{4.5}$$

and similarly on the right, where we must also use the isomorphism

$$R^G_0(X^n \wedge Y^n) \cong R^G_0((X \wedge Y)^n).$$

The most important property of the norm map will be the double coset formula

$$res^G_K \, norm^G_H(x) = \prod_g norm^K_{{}^gH \cap K} \, res^{{}^gH}_{{}^gH \cap K} \, c_g(x), \tag{4.6}$$

where K is any subgroup of G and g runs through a set of double coset representatives for $K \backslash G/H$. Here, if ${}^gH \cap K$ has index $n(g)$ in ${}^gH$, then $n = \sum n(g)$ and the product on the right is defined by use of the evident map

$$\bigotimes_g R^K_0(X^{n(g)}) \longrightarrow R^K_0(X^n). \tag{4.7}$$

An element of $R^H_0(X)$ is represented by an H-map $x : S_G \longrightarrow R_G \wedge X$. There is no difficulty in using the product on R_G to produce an H-map

$$S_G \cong (S_G)^n \xrightarrow{x^n} (R_G \wedge X)^n \cong (R_G)^n \wedge X^n \longrightarrow R_G \wedge X^n.$$

The essential point of our construction is that this may be done in such a way as to produce a G-map: this will be $norm^G_H(x)$. Here and later, all powers are understood to be smash powers.

This is the basic idea, but carrying it out entails several difficulties. Since our group actions involve permutations of smash powers, we cannot hope to control equivariance unless we are using a smash product that is strictly associative and commutative and a multiplication on R_G that is strictly associative and commutative. Moreover, our G-actions come by restriction of actions of wreath products $\Sigma_n \int H$, and it turns out to be essential to work with $(\Sigma_n \int H)$-spectra. If we start just with a G-spectrum R_G, then it is not clear how to proceed. Similarly, to make our spectra precise, we must specify appropriate universes on which to index them, and we find that the norm map acts nontrivially on universes.

We explain why the last phenomenon occurs and at the same time explain the perhaps surprising restriction of our initial description of the norm map to degree zero. The point is that, for $q \geq 0$ at least, the norm gives a map

$$R^H_{-q}(X) = R^H_0(X \wedge S^q) \longrightarrow R^G_0((X \wedge S^q)^n) \cong R^G_0(X^n \wedge (S^q)^n).$$

However, the sphere $(S^q)^n$ is twisted: in fact, it is the sphere S^V associated to the representation $V = ind^G_H(q)$, where q denotes the q-dimensional trivial representation of H. Thus the target is $R^G_{-V}(X^n)$, which is not in the integer graded part of the theory. Of course, if q is even and R^*_G has Thom isomorphisms, then we can use them to translate to integer degrees and thus to obtain the translated norm map

$$R^H_{-q}(X) \longrightarrow R^G_{-nq}(X^n).$$

An elaboration of this idea to sums of elements will give the modified norm maps $\overline{norm}^G_H$. We shall explain this elaboration of the definition after making sense of the geometric construction of the norm map $norm^G_H$ and proving its double coset formula.

5. Global $\mathscr{I}_*$-functors with smash product

To deal with the difficulties that we have indicated, we shall assume that R_G arises from a $\mathscr{G}\mathscr{I}_*$-FSP, where FSP stands for functor with smash product. This is a global version of the notion of an $\mathscr{I}_*$-prefunctor that was introduced in [37, IV.2.1]. The earlier notion was defined nonequivariantly but transcribes directly to a definition in which we restrict attention to a given compact Lie group G acting on everything in sight. The adjective "global" means that we allow G to range through all compact Lie groups G (or through all G in some suitably restricted class). Let $\mathscr{G}$ denote the category of compact Lie groups and their homomorphisms; for the purposes of the present theory, it would suffice to restrict attention to injective homomorphisms, but our examples are defined on the larger category.

Definition 5.1. Define the global category $\mathscr{G}\mathscr{T}$ of equivariant based spaces to have objects (G, X), where G is a compact Lie group and X is a based G-space. The morphisms are the pairs

$$(\alpha, f) : (G, X) \longrightarrow (G', X')$$

where $\alpha : G \longrightarrow G'$ is a homomorphism of Lie groups and $f : X \longrightarrow X'$ is an α-equivariant map, in the sense that $f(gx) = \alpha(g)f(x)$ for all $x \in X$ and $g \in G$. Topologize the set of maps $(G, X) \longrightarrow (G', X')$ as a subspace of the evident product of mapping spaces and observe that composition is continuous.

Definition 5.2. Define the global category $\mathscr{G}\mathscr{I}_*$ of finite dimensional equivariant inner product spaces to have objects (G, V), where G is a compact Lie group and V is a finite dimensional inner product space with an action of G through linear isometries. The morphisms are the pairs

$$(\alpha, f) : (G, V) \longrightarrow (G', V')$$

where $\alpha : G \longrightarrow G'$ is a homomorphism and $f : V \longrightarrow V'$ is an α-equivariant linear isomorphism.

We often find it convenient to work with complex rather than real inner product spaces. Our definitions apply equally well under either interpretation.

Observe that each morphism (α, f) in $\mathscr{G}\mathscr{I}_*$ factors as a composite

$$(G, V) \xrightarrow{(\mathrm{id}, f)} (G, W) \xrightarrow{(\alpha, \mathrm{id})} (H, W),$$

where G acts through α on W. We have a similar factorization of morphisms in $\mathscr{G}\mathscr{T}$. Observe too that we have forgetful functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}$ and $\mathscr{G}\mathscr{T} \longrightarrow \mathscr{G}$. We shall be interested in functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over $\mathscr{G}$, that is, functors that preserve the group coordinate. For example, one-point compactification of inner product spaces gives such a functor, which we shall denote by $S^\bullet$. As in this example, the space coordinate of our functors will be the identity on morphisms of the form (α, id). For these reasons, we shall usually omit the group coordinate from the notation for functors.

Definition 5.3. A $\mathscr{G}\mathscr{I}_*$-functor is a continuous functor $T : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over $\mathscr{G}$, written (G, TV) on objects (G, V), such that

$$T(\alpha, \mathrm{id}) = (\alpha, \mathrm{id}) : (G, TW) \longrightarrow (H, TW)$$

for a representation W of H and a homomorphism $\alpha : G \longrightarrow H$.

The following observation will be the germ of the definition of the norm map.

Lemma 5.4. *Let $A = \mathrm{Aut}(G, V)$ be the group of automorphisms of (G, V) in the category $\mathscr{G}\mathscr{I}_*$. For any $\mathscr{G}\mathscr{I}_*$-functor T, the group $A \ltimes G$ acts on the space TV.*

Proof. An element of A is a map $(\alpha, f) : (G, V) \longrightarrow (G, V)$ such that $f(gv) = \alpha(g)v$ for all $v \in V$ and $g \in G$ and α is an isomorphism. The semi-direct product $A \ltimes G$ is the set $A \times G$ with the multiplication

$$((\alpha, f), g)((\beta, \ell), h) = ((\alpha\beta, f\ell), \beta^{-1}(g)h),$$

and G is contained in $A \ltimes G$ as the normal subgroup of elements (id, g). The action of $A \ltimes G$ on TV is specified by

$$((\alpha, f), g)x = T(\alpha, f)(gx).$$

This does define an action since functoriality and equivariance imply that

$$T(\alpha, f)(gT(\beta, \ell)(hx)) = \alpha(g)\alpha\beta(h)T(\alpha\beta, f\ell)(x) = T(\alpha\beta, f\ell)(\beta^{-1}(g)hx). \quad \square$$

Define the direct sum functor $\oplus : \mathscr{G}\mathscr{I}_* \times \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{I}_*$ by

$$(G, V) \oplus (H, W) = (G \times H, V \oplus W).$$

Define the smash product functor $\wedge : \mathscr{G}\mathscr{T} \times \mathscr{G}\mathscr{T} \longrightarrow \mathscr{G}\mathscr{T}$ by

$$(G, X) \wedge (H, Y) = (G \times H, X \wedge Y).$$

These functors lie over the functor $\times : \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$.

Definition 5.5. A $\mathscr{G}\mathscr{I}_*$-FSP is a $\mathscr{G}\mathscr{I}_*$-functor together with a continuous natural unit transformation $\eta : S^\bullet \longrightarrow T$ of functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ and a continuous natural product transformation $\omega : T \wedge T \longrightarrow T \circ \oplus$ of functors $\mathscr{G}\mathscr{I}_* \times \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ such that the composite

$$TV \cong TV \wedge S^0 \xrightarrow{\mathrm{id} \wedge \eta} TV \wedge T(0) \xrightarrow{\omega} T(V \oplus 0) \cong TV$$

is the identity map and the following unity, associativity, and commutativity diagrams commute:

$$\begin{array}{ccc} S^V \wedge S^W & \xrightarrow{\eta\wedge\eta} & TV \wedge TW \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\omega} \\ S^{V\oplus W} & \xrightarrow[\eta]{} & T(V\oplus W), \end{array}$$

$$\begin{array}{ccc} TV \wedge TW \wedge TZ & \xrightarrow{\omega\wedge \mathrm{id}} & T(V\oplus W)\wedge TZ \\ {\scriptstyle \mathrm{id}\wedge\omega}\downarrow & & \downarrow{\scriptstyle\omega} \\ TV\wedge T(W\oplus Z) & \xrightarrow[\omega]{} & T(V\oplus W\oplus Z), \end{array}$$

and

$$\begin{array}{ccc} TV \wedge TW & \xrightarrow{\omega} & T(V\oplus W) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle T(\tau)} \\ TW\wedge TV & \xrightarrow[\omega]{} & T(W\oplus V). \end{array}$$

Actually, the notion that we have just defined is that of a commutative $\mathscr{G}\mathscr{I}_*$-FSP; For the more general non-commutative notion, the commutativity diagram must be replaced by a weaker centrality of unit diagram. Observe that the space coordinate of each map $T(\alpha, f)$ is necessarily a homeomorphism since $(\alpha, f) = (\alpha, \mathrm{id}) \circ (\mathrm{id}, f)$ and f is an isomorphism. We record the following observation for later use.

Remark 5.6. For objects (H_i, V_i) of $\mathscr{G}\mathscr{I}_*$, $1 \le i \le n$, and a permutation $\sigma \in \Sigma_n$, the associativity and commutativity diagrams imply the following commutative diagram:

$$\begin{array}{ccc} TV_1\wedge\cdots\wedge TV_n & \xrightarrow{\omega} & T(V_1\oplus\cdots\oplus V_n) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle T(\sigma)} \\ TV_{\sigma^{-1}(1)}\wedge\cdots\wedge TV_{\sigma^{-1}(n)} & \xrightarrow[\omega]{} & T(V_{\sigma^{-1}(1)}\oplus\cdots\oplus V_{\sigma^{-1}(n)}). \end{array}$$

Here and in the commutativity axiom, we have suppressed the group coordinate from the notation and $T(\sigma)$ means $T(\sigma, \sigma)$: we must permute the groups in the same way that we permute the inner product spaces.

Example 5.7. The sphere functor $S^\bullet$ is a $\mathscr{G}\mathscr{I}_*$-FSP with unit given by the identity maps of the S^V and product given by the isomorphisms $S^V \wedge S^W \cong S^{V\oplus W}$. For any $\mathscr{G}\mathscr{I}_*$-FSP T, the unit $\eta : S^\bullet \longrightarrow T$ is a map of $\mathscr{G}\mathscr{I}_*$-FSP's.

Example 5.8. Let $\dim V = n$ and define TV to be the one-point compactification of the canonical n-plane bundle EV over the Grassmann manifold $Gr_n(V\oplus V)$. An action of G on V induces an action of G that makes EV a G-bundle and TV a based G-space. Take $V = V \oplus \{0\}$ as a canonical basepoint in $Gr_n(V\oplus V)$. The inclusion of the fiber over the basepoint induces a map $\eta : S^V \longrightarrow TV$. The canonical bundle map $EV \oplus EW \longrightarrow E(V\oplus W)$ induces a map $\omega : TV \wedge TW \longrightarrow T(V\oplus W)$. With the evident definition of T on morphisms, T is a $\mathscr{G}\mathscr{I}_*$-functor. Actually, there are two variants: we write TO when dealing with real inner product spaces and TU when dealing with complex inner product spaces.

6. The passage to spectra

It is useful to regard a $\mathscr{G}\mathscr{I}_*$-FSP as a $\mathscr{G}\mathscr{I}_*$-prespectrum with additional structure.

Definition 6.1. A $\mathscr{G}\mathscr{I}_*$-prespectrum is a $\mathscr{G}\mathscr{I}_*$-functor $T : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ together with a continuous natural transformation $\sigma : T \wedge S^\bullet \longrightarrow T \circ \oplus$ of functors $\mathscr{G}\mathscr{I}_* \times \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{T}$ such that the composites

$$TV \cong TV \wedge S^0 \xrightarrow{\sigma} T(V \oplus 0) \cong TV$$

are identity maps and each of the following diagrams commutes:

$$\begin{array}{ccc} TV \wedge S^W \wedge S^Z & \xrightarrow{\sigma\wedge\mathrm{id}} & T(V \oplus W) \wedge S^Z \\ \cong\downarrow & & \downarrow\sigma \\ TV \wedge S^{W\oplus Z} & \xrightarrow[\sigma]{} & T(V \oplus W \oplus Z). \end{array}$$

We say that a $\mathscr{G}\mathscr{I}_*$-prespectrum is an inclusion $\mathscr{G}\mathscr{I}_*$-prespectrum if each adjoint map

$$\tilde{\sigma} : TV \longrightarrow F(S^W, T(V \oplus W))$$

is an inclusion.

Lemma 6.2. *If T is a $\mathscr{G}\mathscr{I}_*$-FSP, then T is a $\mathscr{G}\mathscr{I}_*$-prespectrum with respect to the composite maps*

$$\sigma : TV \wedge S^W \xrightarrow{\mathrm{id}\wedge\eta} TV \wedge TW \xrightarrow{\omega} T(V \oplus W).$$

Now fix a group G and a G-universe U, namely a countably infinite dimensional inner product space that contains a trivial representation and contains each of its finite dimensional representations infinitely often. We say that the universe U is complete if it contains all irreducible representations of G. A G-prespectrum indexed on U consists of based G-spaces TV for finite dimensional inner product spaces $V \subset U$ and a transitive system of structure G-maps $\sigma : \Sigma^{W-V}TV \longrightarrow TW$ for $V \subset W$, where $W - V$ is the orthogonal complement of V in W. A spectrum E is a prespectrum whose adjoint structure maps $EV \longrightarrow \Omega^{W-V}EW$ are homeomorphisms. There is a spectrification functor L from prespectra to spectra that is left adjoint to the evident forgetful functor. See [34, 21, 39] for the development of equivariant stable homotopy theory from this starting point. It is evident that a $\mathscr{G}\mathscr{I}_*$-prespectrum restricts to a G-prespectrum indexed on U for every G and U.

Notations 6.3. Let $T_{(G,U)}$ denote the G-prespectrum indexed on U associated to a $\mathscr{G}\mathscr{I}_*$-FSP T. Write $R(T)_{(G,U)}$ for the spectrum $LT_{(G,U)}$ associated to $T_{(G,U)}$.

Let $\mathscr{L}(j)$ be the G-space of linear isometries $U^j \longrightarrow U$, with G acting by conjugation. As discussed in the cited sources and [10], $\mathscr{L}$ is a G-operad and is an E_∞ G-operad when U is complete. There is a notion of an $\mathscr{L}$-prespectrum [37, IV.1.1] (amended slightly in [34, VII.2.4-2.6]). Exactly as in [37, IV.2.2], $T_{(G,U)}$ is an $\mathscr{L}$-prespectrum. The essential point is that if $f : U^j \longrightarrow U$ is a linear isometry and V_i are indexing spaces in U, then we have maps

$$\xi_j(f) : TV_1 \wedge \cdots TV_j \xrightarrow{\omega} T(V_1 \oplus \cdots \oplus V_j) \xrightarrow{Tf} Tf(V_1 \oplus \cdots \oplus V_j).$$

The notion of an $\mathscr{L}$-prespectrum is defined in terms of just such maps.

The notion of an $\mathscr{L}$-spectrum E is defined more conceptually in terms of maps

$$\mathscr{L}(j) \ltimes E^j \longrightarrow E.$$

(In fact, the twisted half-smash product $\ltimes$ was not known when [37] was written.) However, by [37, IV.1.6] and, in current terms, [34, VII§2], the functor L converts $\mathscr{L}$-prespectra to $\mathscr{L}$-spectra. We conclude that, for every G and every complete G-universe U, $R(T)_{(G,U)}$ is an $\mathscr{L}$-spectrum and thus an E_∞ ring G-spectrum. Moreover, as explained in [11, 10], $\mathscr{L}$-spectra functorially determine weakly equivalent commutative S_G-algebras.

While the constructions of [20, 21] on which our work is based depend on the fact that the G-spectra we are working with admit S_G-algebra structures, we will not need to make explicit use of these structures in our study of the norm map.

Remark 6.4. Let T be a global $\mathscr{G}\mathscr{I}_*$-functor, choose a complete G-universe U_G for each G, and consider the G-spectra $R(T)_{(G,U_G)}$ as G varies. It is a routine exercise to verify that the collection $\{R(T)_{(G,U_G)}\}$ defines a $\mathscr{G}$-spectrum, in the sense defined in [34, II.8.5]. In particular, it follows immediately from [34, II.8.6 and II.8.7] that $R(T)_{(G,U_G)}$ is a split G-spectrum for each G. This shows that our Thom G-spectra are split, as stated in the introduction.

Remark 6.5. If U is a complete complex G-universe, we may regard it as a complete real G-universe by neglect of structure. This gives two variants of all definitions in sight, one in which we restrict attention to complex finite dimensional inner product spaces V in U and the other in which we allow all real finite dimensional inner product spaces. The resulting categories of G-spectra are canonically equivalent [34, I.2.4] because the adjoint structure maps of G-spectra are homeomorphisms and complex inner product spaces are cofinal among real ones.

7. Wreath products and the definition of the norm map

Recall that the wreath product $\Sigma_n \int H$ is the semi-direct product $\Sigma_n \ltimes H^n$, where Σ_n acts by permutations on H^n; explicitly, $\Sigma_n \int H$ is the set $\Sigma_n \times H^n$ with the product

$$(\sigma, h_1, \ldots, h_n)(\tau, k_1, \ldots, k_n) = (\sigma\tau, h_{\tau 1}k_1, \cdots, h_{\tau n}k_n). \tag{7.1}$$

We have the following evident actions of this group. We display them explicitly because of their centrality in our work.

Lemma 7.2. *If V is a representation of H, then the sum V^n of n copies of V is a representation of $\Sigma_n \int H$ with action given by*

$$(\sigma, h_1, \ldots, h_n)(v_1, \ldots, v_n) = (h_{\sigma^{-1}(1)}v_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n)}v_{\sigma^{-1}(n)}).$$

Lemma 7.3. *If X is a based H-space, then the smash power X^n is a $(\Sigma_n \int H)$-space with action given by*

$$(\sigma, h_1, \ldots, h_n)(x_1 \wedge \ldots \wedge x_n) = h_{\sigma^{-1}(1)}x_{\sigma^{-1}(1)} \wedge \ldots \wedge h_{\sigma^{-1}(n)}x_{\sigma^{-1}(n)}.$$

This leads to the following crucial observation.

Proposition 7.4. *Let T be a $\mathscr{G}\mathscr{I}_*$-FSP. For an H-representation V, $(TV)^n$ and $T(V^n)$ are $\Sigma_n \int H$-spaces and the map*

$$\omega : (TV)^n \longrightarrow T(V^n)$$

is $(\Sigma_n \int H)$-equivariant. If U is an H-universe, then U^n is a $(\Sigma_n \int H)$-universe and the maps ω define a map of $(\Sigma_n \int H)$-prespectra indexed on U^n

$$\omega : (T_{(H,U)})^n \longrightarrow T_{(\Sigma_n \int H, U^n)},$$

where $(T_{(H,U)})^n$ is the nth external smash power of $T_{(H,U)}$. If $T = S^\bullet$, then ω is an isomorphism of prespectra. If $n = \sum n(i)$, where $n(i) \geq 1$ and $1 \leq i \leq m$, then the following diagram of prespectra commutes:

$$\begin{array}{ccc} \bigwedge_{i=1}^m (T_{(H,U)})^{n(i)} & \xrightarrow{\wedge_i \omega} & \bigwedge_{i=1}^m T_{(\Sigma_{n(i)} \int H, U^{n(i)})} \\ \| & & \downarrow \omega \\ (T_{(H,U)})^n & \xrightarrow[\omega]{} & T_{(\Sigma_n \int H, U^n)}. \end{array}$$

Proof. For $\sigma \in \Sigma_n$, (σ, σ) is an automorphism of (H^n, V^n) and thus Σ_n maps to $A = \mathrm{Aut}(H^n, V^n)$. This induces a map from $\Sigma_n \int H$ to $A \ltimes H^n$ (which is an injection unless $V = \{0\}$). Now Lemma 5.4 restricts to give the action of $\Sigma_n \int H$ on $T(V^n)$. We see that ω is $(\Sigma_n \int H)$-equivariant by taking each (H_i, V_i) to be (H, V) in the diagram of Remark 5.6. We may index our prespectra on the cofinal family of indexing spaces of the form V^n in U^n , and the external smash product has V^nth space $(TV)^n$. The prespectrum level statements are now easily verified from the definition of a $\mathscr{G}\mathscr{I}_*$-FSP. □

We shall use the proposition to define the norm map, but we first need a bit of algebra. For the rest of the section, assume given a subgroup H of finite index n in a compact Lie group G. Choose coset representatives $t_1, t_2, \cdots, t_n$ for H in G, taking $t_1 = e$, and define the monomial representation

(7.5) $$\alpha : G \longrightarrow \Sigma_n \int H$$

by the formula

(7.6) $$\alpha(\gamma) = (\sigma(\gamma), h_1(\gamma), \ldots, h_n(\gamma)),$$

where $\sigma(\gamma)$ and $h_i(\gamma)$ are defined implicitly by the formula

(7.7) $$\gamma t_i = t_{\sigma(\gamma)(i)} h_i(\gamma).$$

Lemma 7.8. *The map α is a homomorphism of groups. If α' is defined with respect to a second choice of coset representatives $\{t_i'\}$, then α and α' differ by a conjugation in $\Sigma_n \int H$.*

Proof. The first statement holds by the definition (7.1) of the product in $\Sigma_n \int H$ and the observation that

$$(\gamma\delta)t_i = \gamma t_{\sigma(\delta)(i)} h_i(\delta) = t_{\sigma(\gamma)(\sigma(\delta)(i))} h_{\sigma(\delta)(i)}(\gamma) h_i(\delta).$$

For the second statement, if $t_i' = t_i k_i$, then

$$\gamma t_i' = t_{\sigma(\gamma)(i)} h_i(\gamma) k_i = t'_{\sigma(\gamma)(i)} k^{-1}_{\sigma(\gamma)(i)} h_i(\gamma) k_i$$

and therefore

$$\alpha'(\gamma) = (1, k_1, \ldots, k_n)^{-1} \alpha(\gamma)(1, k_1, \ldots, k_n). \quad \square$$

The homomorphism α is implicitly central to induction in representation theory, as the following lemma explains. Write $\alpha^* W$ for a representation W of $\Sigma_n \int H$ regarded as a representation of G by pullback along α.

Lemma 7.9. *If V is a representation of H, then $\alpha^* V^n \cong ind_H^G V$.*

Proof. Recall that $ind_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. The isomorphism is given by mapping $t_i \otimes v$ on the left to v in the ith summand on the right, as is dictated by the case $i = 1$ and equivariance. □

Analogously, for a based $\Sigma_n \int H$-space Y, write $\alpha^* Y$ for Y regarded as a G-space by pullback along α. In particular, $\alpha^*((S^V)^n) \cong S^{ind_H^G V}$.

Here, finally, is the definition of the norm map. Recall Notations 6.3.

Definition 7.10. Let T be a $\mathscr{G}\mathscr{I}_*$-FSP, let X be a based H-space, and let U be a complete H-universe. An element $x \in R(T)_0^H(X)$ is given by a map of H-spectra $x : S_{(H,U)} \longrightarrow R(T)_{(H,U)} \wedge X$. Define the norm of x to be the element of $R(T)_0^G(\alpha^* X^n)$ given by the pullback along α of the composite map of $\Sigma_n \int H$-spectra indexed on U^n displayed in the commutative diagram:

$$\begin{array}{ccccc} S_{(\Sigma_n \int H, U^n)} & \xrightarrow{\omega^{-1}} & (S_{(H,U)})^n & \xrightarrow{x^n} & (R(T)_{(H,U)} \wedge X)^n \\ \downarrow & & & & \downarrow \cong \\ R(T)_{(\Sigma_n \int H, U^n)} \wedge X^n & & \xleftarrow[\omega \wedge \mathrm{id}]{} & & (R(T)_{(H,U)})^n \wedge X^n. \end{array}$$

If we take it as understood that G acts on U^n through α, then the composite defining $norm_H^G(x)$ may be rewritten more simply as

(7.11)
$$\begin{array}{ccccc} S_{(G,U^n)} & \xrightarrow{\omega^{-1}} & (S_{(H,U)})^n & \xrightarrow{x^n} & (R(T)_{(H,U)} \wedge X)^n \\ {\scriptstyle norm_H^G(x)} \downarrow & & & & \downarrow \cong \\ R(T)_{(G,U^n)} \wedge X^n & & \xleftarrow[\omega \wedge \mathrm{id}]{} & & (R(T)_{(H,U)})^n \wedge X^n. \end{array}$$

Observe that the G-universe U^n is complete; for example, if G is finite, this holds because the regular representation of H induces up to the regular representation of G. Strictly speaking, if we start with H-spectra defined in fixed complete H-universes U_H for all H, then we must choose an isomorphism $U_G \cong U_H^n$ to transfer the norm to a map of spectra indexed on U_G. This is a standard procedure that must be applied to various of the maps that we shall construct; compare Remark 6.4.

Property (4.2) of the norm is obvious. Property (4.3) is an easy consequence of the unity and associativity diagrams in the definition of a $\mathscr{G}\mathscr{I}_*$-FSP. Property (4.4) also follows easily from the definition of a $\mathscr{G}\mathscr{I}_*$-FSP, once we make precise how to interpret the product (4.5). Thus suppose given H-universes U and U' and maps $x : S_{(H,U)} \longrightarrow R(T)_{(H,U)}$ and $y : S_{(H,U')} \longrightarrow R(T)_{(H,U')}$. For the present purpose, U and U' could be the same, but we will want to allow them to be different in the next section. We then define xy to be the composite displayed in the diagram

(7.12)
$$\begin{array}{ccccc} S_{(G,U \oplus U')} & \xrightarrow{\omega^{-1}} & S_{(G,U)} \wedge S_{(G,U')} & \xrightarrow{x \wedge y} & R(T)_{(G,U)} \wedge X \wedge R(T)_{(G,U')} \wedge Y \\ {\scriptstyle xy} \downarrow & & & & \downarrow \cong \\ R(T)_{(G,U \oplus U')} \wedge X \wedge Y & & \xleftarrow[\omega \wedge \mathrm{id}]{} & & R(T)_{(G,U)} \wedge R(T)_{(G,U')} \wedge X \wedge Y \end{array}$$

Of course, when $U = U'$, we can internalize this external multiplication by use of a linear isometry $f : U \oplus U \longrightarrow U$; it is then obvious that it agrees with the standard homotopical definition of such a product. The external form makes the verification of (4.4) transparent.

8. The proof of the double coset formula

The proof of the double coset formula requires a precise combinatorial discussion of double cosets. We again suppose given a subgroup H of finite index n in a compact Lie group G. We fix coset representatives t_j and use them to define the monomial representation α, as in the previous section. We suppose given a second subgroup K of G and we choose representatives $g_1, \cdots, g_m$ for the double cosets $K\backslash G/H$. We shall choose the g_i's from among the t_j's, in a manner to be specified.

The g_i give a decomposition of the finite K-set G/H as

$$\coprod_{i=1}^{m} K/{}^{g_i}H \cap K \xrightarrow{\cong} \coprod_{i=1}^{m} Kg_iH = K\backslash G/H.$$

Explicitly, the ith component is the isomorphism

$$K/{}^{g_i}H \cap K \xrightarrow{\cong} Kg_iH$$

that sends $k({}^{g_i}H \cap K)$ to kg_iH. Let the ith double coset have $n(i)$ elements, define $q(i) = n(1) + \cdots + n(i-1)$, and label the g_i and t_j so that $g_i = t_{q(i)+1}$ and the $t_{q(i)+r}$, $1 \leq r \leq n(i)$, run through the coset representatives of G/H that are in the ith double coset Kg_iH. Thus

$$Kg_iH = \coprod_{r=1}^{n(i)} t_{q(i)+r}H.$$

Define

$$s_{i,r} = t_{q(i)+r}g_i^{-1} \in K. \tag{8.1}$$

Thus the $s_{i,r}$ are coset representatives for $K/{}^{g_i}H \cap K$ that map to our chosen coset representatives $t_{q(i)+r}$ in Kg_iH. With this choice, we define homomorphisms

$$\beta_i : K \longrightarrow \Sigma_{n(i)} \int {}^{g_i}H \cap K \tag{8.2}$$

by the formula

$$\beta_i(\kappa) = (\tau_i(\kappa), \ell_1(\kappa), \ldots, \ell_{n(i)}(\kappa)), \tag{8.3}$$

where $\tau_i(\kappa)$ and $\ell_r(\kappa)$ are defined implicitly by the formula

$$\kappa s_{i,r} = s_{i,\tau_i(\kappa)(r)}\ell_r(\kappa). \tag{8.4}$$

Lemma 8.5. *With these choices of α and the β_i, the permutations $\sigma(\kappa)$ for $\kappa \in K$ decompose as block sums $\sigma_1(\kappa) \oplus \cdots \oplus \sigma_m(\kappa)$ with $\sigma_i(\kappa) \in \Sigma_{n(i)}$, and*

$$\sigma_i(\kappa) = \tau_i(\kappa) \quad \text{and} \quad h_{q(i)+r}(\kappa) = g_i^{-1}\ell_r(\kappa)g_i.$$

Proof. The first statement holds since κ must permute the $t_{q(i)+1}, \cdots, t_{q(i)+n(i)}$ among themselves. For the second statement, we obtain

$$\kappa t_{q(i)+r}g_i^{-1} = t_{q(i)+\tau_i(\kappa)(r)}g_i^{-1}\ell_r(\kappa)$$

by inserting the formula (8.1) into (8.4), while (7.7) gives

$$\kappa t_{q(i)+r} = t_{q(i)+\sigma_i(\kappa)(r)}h_{q(i)+r}(\kappa). \quad \square$$

The lemma can be rewritten in terms of homomorphisms, giving a description of the restriction of α to K in terms of the β_i.

Lemma 8.6. *The restriction of α to K maps into $\prod_i \Sigma_{n(i)} \int H$, and the following diagram commutes:*

$$\begin{array}{ccc}
G & \xrightarrow{\alpha} & \Sigma_n \int H \\
\uparrow & & \uparrow \\
K & \xrightarrow{(\alpha_1,\dots,\alpha_m)} & \prod_i \Sigma_{n(i)} \int H \\
& {\scriptstyle (\beta_1,\dots,\beta_m)} \searrow \qquad \nearrow {\scriptstyle \prod_i \mathrm{id} \int c_{g_i}} & \\
& \prod_i \Sigma_{n(i)} \int {}^{g_i}H \cap K, &
\end{array}$$

where $\alpha_i(\kappa) = (\sigma_i(\kappa), h_{q(i)+1}(\kappa), \dots, h_{q(i)+n(i)}(\kappa))$ and $c_{g_i}(g_i h g_i^{-1}) = h$.

We can now prove (4.6). For a map $x : S_{(H,U)} \longrightarrow R(T)_{(H,U)} \wedge X$, $res^G_K norm^G_H(x)$ is the composite map of K-spectra indexed on U^n that is displayed in the following diagram, where K acts on U^n and X^n through the restriction of α to K:

$$\begin{array}{ccccc}
S_{(K,U^n)} & \xrightarrow{\omega^{-1}} & S^n_{(H,U)} & \xrightarrow{x^n} & (R(T)_{(H,U)} \wedge X)^n \\
\downarrow & & & & \downarrow \cong \\
R(T)_{(K,U^n)} \wedge X^n & & \xleftarrow{\omega \wedge \mathrm{id}} & & (R(T)_{(H,U)})^n \wedge X^n.
\end{array}$$

Define $\nu_i(x)$ to be the composite displayed in the following diagram, in which K acts through α_i on $U^{n(i)}$ and $X^{n(i)}$:

$$\begin{array}{ccccc}
S_{(K,U^{n(i)})} & \xrightarrow{\omega^{-1}} & (S_{(H,U)})^{n(i)} & \xrightarrow{x^{n(i)}} & (R(T)_{(H,U)} \wedge X)^{n(i)} \\
{\scriptstyle \nu_i(x)} \downarrow & & & & \downarrow \cong \\
R(T)_{(K,U^{n(i)})} \wedge X^{n(i)} & & \xleftarrow{\omega \wedge \mathrm{id}} & & (R(T)_{(H,U)})^{n(i)} \wedge X^{n(i)}.
\end{array} \tag{8.7}$$

In view of the transitivity diagram for ω given in Proposition 7.4, applied to both $R(T) = LT$ and S, we see immediately that the following diagram commutes:

$$\begin{array}{ccccc}
S_{(K,U^n)} & \xrightarrow{\omega^{-1}} & \bigwedge_{i=1}^m S_{(K,U^{n(i)})} & \xrightarrow{\wedge_i \nu_i(x)} & \bigwedge_{i=1}^m (R(T)_{(K,U^{n(i)})} \wedge X^{n(i)}) \\
{\scriptstyle res^G_K norm^G_H(x)} \downarrow & & & & \downarrow \cong \\
R(T)_{(K,U^n)} \wedge X^n & & \xleftarrow{\omega \wedge \mathrm{id}} & & (\bigwedge_{i=1}^m R(T)_{(K,U^{n(i)})}) \wedge (\bigwedge_{i=1}^m X^{n(i)}).
\end{array}$$

Comparing with (7.12), we see that precisely such a diagram makes sense of the iterated product (4.7), and we conclude that $res^G_K norm^G_H(x)$ is the product of the $\nu_i(x)$. Abbreviating notation (as in (4.6)), let $g = g_i$. To complete the proof of (4.6), we need only show that

$$\nu_i(x) = norm^K_{{}^gH \cap K}\, res^{{}^gH}_{{}^gH \cap K}\, c_g(x).$$

This means that $\nu_i(x)$ coincides with the composite displayed in the following diagram, in which ${}^gX$ denotes the H-space X regarded as a ${}^gH$-space by pullback

along $c_g : {}^gH \longrightarrow H$, ${}^gU$ denotes the H-universe U regarded as a ${}^gH$-universe by pullback along c_g, and K acts through β_i on ${}^gU^{n(i)}$ and ${}^gX^{n(i)}$:
(8.8)

$$\begin{array}{ccccc} S_{(K,{}^gU^{n(i)})} & \xrightarrow{\ \omega^{-1}\ } & (S_{({}^gH\cap K,{}^gU)})^{n(i)} & \xrightarrow{c_g(x)^{n(i)}} & (R(T)_{({}^gH\cap K,{}^gU)} \wedge {}^gX)^{n(i)} \\ \big\downarrow & & & & \big\downarrow \cong \\ R(T)_{(K,{}^gU^{n(i)})} \wedge {}^gX^{n(i)} & & \xleftarrow[\ \omega\wedge \mathrm{id}\ ]{} & & R(T)^{n(i)}_{({}^gH\cap K,{}^gU)} \wedge {}^gX^{n(i)}. \end{array}$$

It is immediate from the definition of a $\mathscr{G}\mathscr{I}_*$-functor that

$$R(T)_{({}^gH,{}^gU)} = {}^gR(T)_{(H,U)},$$

where ${}^gR(T)_{(H,U)}$ denotes $R(T)_{(H,U)}$ regarded as a ${}^gH$-spectrum by pullback along c_g. The same is true for S, and $c_g(x) = {}^gx$ is just the map x regarded as a ${}^gH$-map by pullback along c_g. By Lemma 8.6,

$$\alpha_i = c_g \circ \beta_i : K \longrightarrow \Sigma_{n(i)} \wr H.$$

Therefore $X^{n(i)}$ regarded as a K-space via α_i is identical to ${}^gX^{n(i)}$ regarded as a K-space via β_i and $U^{n(i)}$ regarded as a K-universe via α_i is identical to ${}^gU^{n(i)}$ regarded as a K-universe via β_i. Except that we have used that β_i takes values in $\Sigma_{n(i)} \wr {}^gH \cap K$ to restrict the group action in some of the terms of (8.8), we see that the diagrams (8.7) and (8.8) display one and the same map.

9. The norm map on sums and its double coset formula

Consider $norm_H^G(x+y)$, where $x \in R(T)^H_{-q}$ and $y \in R(T)^H_{-r}$ for even integers $q \geq 0$ and $r \geq 0$. Here we are considering the case $X = S^q \vee S^r$ of the norm map that we defined in Definition 7.10. For based H-spaces X and Y, we have

$$(9.1) \qquad (X \vee Y)^n \cong \textstyle\bigvee_{i=0}^n (\Sigma_n \wr H) \ltimes_{(\Sigma_i \times \Sigma_{n-i}) \wr H} X^i \wedge Y^{n-i}$$

as $\Sigma_n \wr H$-spaces. For any subgroup $K \subseteq G$, we therefore have

$$(9.2) \qquad (X \vee Y)^n \cong \bigvee_{i=0}^{n} \bigvee_{\gamma} K \ltimes_{\gamma((\Sigma_i \times \Sigma_{n-i}) \wr H) \cap K} {}^{\gamma}(X^i \wedge Y^{n-i})$$

as K-spaces, where γ runs through a set of double coset representatives for

$$K \setminus (\Sigma_n \wr H) \ / \ ((\Sigma_i \times \Sigma_{n-i}) \wr H).$$

Taking $K = G$, we see that, in general, the norm of the sum of elements of $R(T)^H_0(X)$ and $R(T)^H_0(Y)$ is an element of

$$(9.3) \qquad R(T)^G_0((X \vee Y)^n) = \textstyle\sum_{i=0}^n \sum_\gamma R(T)_0^{\gamma((\Sigma_i \times \Sigma_{n-i}) \wr H) \cap G}({}^{\gamma}(X^i \wedge Y^{n-i})).$$

Now return to our elements $x \in R(T)^H_{-q}$ and $y \in R(T)^H_{-r}$. We are thinking of $x = 1$. In order to obtain the norm of Definition 3.6, we must transform the element $norm_H^G(x+y)$ to an element of $R(T)^G_*$, and we must do so in a fashion that makes sense of and validates the double coset formula of Definition 3.6. This is where we use the assumption in Theorem 3.2 that each $R(T)^{\#}_G$ has natural Thom isomorphisms. By working on the level of $\Sigma_n \wr H$ as long as possible, we shall circumvent any need to deal with the complexities displayed in formulas (9.2) and (9.3).

Let V_n denote $\mathbb{R}^n$ with its permutation action by Σ_n and regard V_n as a $\Sigma_n \int H$-representation with trivial action by H. Thus $\alpha^* V_n \cong ind_H^G \mathbb{R}$. Writing V^q for the sum of q copies of V, we have $(S^q)^n = S^{V_n^q}$ as a $\Sigma_n \int H$-space. Therefore (9.1) gives

$$(9.4) \qquad (S^q \vee S^r)^n \cong \bigvee_{i=0}^n (\Sigma_n \textstyle\int H) \ltimes_{(\Sigma_i \times \Sigma_{n-i}) \int H} S^{V_i^q \oplus V_{n-i}^r}$$

as $\Sigma_n \int H$-spaces. Define a translated induction map

$$\psi_i : R(T)_*^{\Sigma_n \int H}((\Sigma_n \textstyle\int H) \ltimes_{(\Sigma_i \times \Sigma_{n-i}) \int H} S^{V_i^q \oplus V_{n-i}^r}) \longrightarrow R(T)_*^{\Sigma_n \int H}(S^{qi+r(n-i)})$$

by commutativity of the following diagram:

(9.5)

$$\begin{array}{ccc}
R(T)_*^{\Sigma_n \int H}((\Sigma_n \int H) \ltimes_{(\Sigma_i \times \Sigma_{n-i}) \int H} S^{V_i^q \oplus V_{n-i}^r}) & \xrightarrow{\cong} & R(T)_*^{(\Sigma_i \times \Sigma_{n-i}) \int H}(S^{V_i^q \oplus V_{n-i}^r}) \\
\Big\downarrow \psi_i & & \Big\downarrow \phi_{V_i^q \oplus V_{n-i}^r} \\
R(T)_*^{\Sigma_n \int H}(S^{qi+r(n-i)}) & \xleftarrow[ind_{(\Sigma_i \times \Sigma_{n-i}) \int H}^{\Sigma_n \int H}]{} & R(T)_*^{(\Sigma_i \times \Sigma_{n-i}) \int H}(S^{qi+r(n-i)}),
\end{array}$$

where we have written ind_H^G for the ordinary transfer homomorphism associated to $H \subseteq G$. In particular, restricting to degree zero, this gives

$$\psi_i : R(T)_0^{\Sigma_n \int H}((\Sigma_n \textstyle\int H) \ltimes_{(\Sigma_i \times \Sigma_{n-i}) \int H} S^{V_i^q \oplus V_{n-i}^r}) \longrightarrow R(T)^{\Sigma_n \int H}_{-qi-r(n-i)}.$$

We also write ψ_i for the corresponding map of G-equivariant homology groups obtained by pullback along α. Observe that V_i^q and V_{n-i}^r are not representations of G, so that we must start with (9.1) and not (9.2) in order for the Thom isomorphisms that we use here to make sense. Note too that we require Thom isomorphisms for $\Sigma_n \int H$ and its subgroups, not just for G and its subgroups.

Use of the ψ_i allows us to redefine the norm of sums in a $\mathbb{Z}$-graded form. We have

$$norm_H^G(x+y) \in R(T)_0^G(\alpha^*(S^q \vee S^r)^n).$$

It is the restriction to G of an element of

$$R(T)_0^{\Sigma_n \int H}((S^q \vee S^r)^n) \cong \textstyle\sum_{i=0}^n R(T)_0^{\Sigma_n \int H}((\Sigma_n \int H) \ltimes_{(\Sigma_i \times \Sigma_{n-i}) \int H} S^{V_i^q \oplus V_{n-i}^r}).$$

We write $norm_H^G(x+y)_i$ for the component in the ith summand and define

$$(9.6) \qquad \overline{norm}_H^G(x+y) = \sum_{i=0}^n \psi_i(norm_H^G(x+y)_i).$$

The double coset formula is still valid for these modified norm maps.

Proposition 9.7. *For elements $x \in R(T)_{-q}^H$ and $y \in R(T)_{-r}^H$,*

$$res_K^G\, \overline{norm}_H^G(x+y) = \textstyle\prod_g \overline{norm}_{gH \cap K}^K\, res_{gH \cap K}^{gH}\, c_g(x+y),$$

where K is any subgroup of G and $\{g\}$ runs through a set of double coset representatives for $K \backslash G/H$.

Proof. To simplify the notation, we restrict attention to the case $q = 0$, which is the case of interest. Since $V_i^0 = \{0\}$, Σ_i acts trivially on $S^0 = S^{V_i^0}$. Reverse the roles of i and $n - i$ in the notations above and let $\Sigma'_i \subset \Sigma_n$ be the subgroup of permutations that fix the first $n - i$ letters. Recall that if ${}^gH \cap K$ has index $n(g)$ in ${}^gH$, then $n = \sum n(g)$. Fix an ordering of the g's and write $\Sigma_{\{n(g)\}}$ for $\prod_g \Sigma_{n(g)}$ regarded as a subgroup of Σ_n. The left side of the equation in the statement is the restriction to K of an element of

$$\sum_{i=0}^{n} R(T)^{\Sigma_n \wr H}_{-ri}.$$

The right side is the product over $\{g\}$ of the restrictions to K of elements of

$$\sum_{a(g)=0}^{n(g)} R(T)^{\Sigma_{n(g)} \wr H}_{-ra(g)}.$$

In view of Lemma 8.6, we see that the relevant products are obtained by adding up restrictions to K of products

$$\bigotimes\nolimits_g R(T)_0^{\Sigma_{n(g)} \wr H}(S^{ra(g)}) \longrightarrow R(T)_0^{\Sigma_{\{n(g)\}} \wr H}(S^{ri}),$$

where $0 \le a(g) \le n(g)$ and $\sum a(g) = i$. For such a sequence $\{a(g)\}$, let $\Sigma'_{\{a(g)\}} = \prod_g \Sigma'_{a(g)}$ regarded as a subgroup of Σ'_i. The original double coset formula (4.6) made use of the restriction to K of the product

$$\bigotimes\nolimits_g R(T)_0^{\Sigma_{n(g)} \wr H}((S^0 \vee S^r)^{n(g)}) \longrightarrow R(T)_0^{\Sigma_{\{n(g)\}} \wr H}((S^0 \vee S^r)^n).$$

Under the wedge decomposition (9.1) and change of groups isomorphisms like those in the top line of (9.5), this product agrees with the sum over sequences $\{a(G)\}$ of the product maps

$$\bigotimes\nolimits_g R(T)_0^{\Sigma'_{a(g)} \wr H}(S^{V^r_{a(g)}}) \longrightarrow R(T)_0^{\Sigma'_{\{a(g)\}} \wr H}(S^{V^r_i}).$$

We claim that the following diagram commutes for each such sequence $\{a(g)\}$, where the left horizontal arrows are products and the right horizontal arrows are restrictions:

$$\begin{array}{ccccc}
\bigotimes_g R(T)_0^{\Sigma'_{a(g)} \wr H}(S^{V^r_{a(g)}}) & \longrightarrow & R(T)_0^{\Sigma'_{\{a(g)\}} \wr H}(S^{V^r_i}) & \longleftarrow & R(T)_0^{\Sigma'_i \wr H}(S^{V^r_i}) \\
\downarrow {\scriptstyle \otimes \phi_{V^r_{a(g)}}} & & \downarrow {\scriptstyle \phi_{V^r_i}} & & \downarrow {\scriptstyle \phi_{V^r_i}} \\
\bigotimes_g R(T)_0^{\Sigma'_{a(g)} \wr H}(S^{ra(g)}) & \longrightarrow & R(T)_0^{\Sigma'_{\{a(g)\}} \wr H}(S^{ri}) & \longleftarrow & R(T)_0^{\Sigma'_i \wr H}(S^{ri}) \\
\downarrow {\scriptstyle \otimes ind^{\Sigma_{n(g)} \wr H}_{\Sigma'_{a(g)} \wr H}} & & \downarrow {\scriptstyle ind^{\Sigma_{\{n(g)\}} \wr H}_{\Sigma'_{\{a(g)\}} \wr H}} & & \downarrow {\scriptstyle ind^{\Sigma_n \wr H}_{\Sigma'_i \wr H}} \\
\bigotimes_g R(T)_0^{\Sigma_{n(g)} \wr H}(S^{ra(g)}) & \longrightarrow & R(T)_0^{\Sigma_{\{n(g)\}} \wr H}(S^{ri}) & \longleftarrow & R(T)_0^{\Sigma_n \wr H}(S^{ri})
\end{array}$$

The top two squares commute since our Thom isomorphisms are multiplicative and compatible under restriction. The bottom two squares commute since transfer commutes with products and restriction by [34, IV.4.4 and IV.5.2]. It follows directly that the present version of the double coset formula follows from the original version. □

Taking $x = 1$, we obtain norm maps as specified in Definition 3.6.

10. The Thom classes of Thom spectra

In this section, we write T for the Thom $\mathscr{G}\mathscr{I}_*$-functor TU of Example 5.8. It determines a $\mathscr{G}$-prespectrum by neglect of structure. The structure maps σ specified in Lemma 6.2 are cofibrations between CW complexes, hence their adjoints are cofibrations and therefore inclusions [33]. We restrict attention to complex G-universes U and complex inner product spaces, and we think of $\mathbb{R}^\infty$ as the underlying real inner product space of $U^G \cong \mathbb{C}^\infty$; compare Remark 6.5. For each compact Lie group G and G-universe U, we obtain an inclusion G-prespectrum $T_{(G,U)}$ indexed on U. We write $MU_{(G,U)}$ for its associated G-spectrum. We have seen in Section 6 that these are E_∞ ring G-spectra and so determine weakly equivalent $S_{(G,U)}$-algebras. All that remains to complete the proof of Theorem 3.3 and thus of Theorem 1.3 is to construct Thom classes

$$\mu(V) \in MU^{2n-V}_{(G,U)} \cong MU^{2n}_{(G,U)}(S^V), \tag{10.1}$$

where V is a complex representation of complex dimension n, and prove their naturality. We take U to be a complete complex G-universe, and we may assume that V is a finite dimensional subspace of U.

Let $T'_{(G,U)}(V)$ be the Thom complex associated to the canonical complex n-plane G-bundle over the Grassmannian $Gr_n(V \oplus U)$ of n-planes in $V \oplus U$. For $V \subseteq W$, let

$$\sigma' : T'_{(G,U)}V \wedge S^{W-V} \longrightarrow T'_{(G,U)}W$$

be the map of Thom complexes induced by the evident map from the sum of the canonical n-plane bundle over $Gr_n(V \oplus U)$ and the trivial bundle $W - V$ over the point $\{W - V\}$ to the canonical q-plane bundle over $Gr_n(W \oplus U)$, where $\dim(W) = q$. The inclusion $V \oplus V \longrightarrow V \oplus U$ induces a G-map $TV \longrightarrow T'_{(G,U)}V$, and these maps together define a map of inclusion G-prespectra

$$i : T_{(G,U)} \longrightarrow T'_{(G,U)}.$$

A standard and easy comparison of colimits of homotopy groups shows that the associated map

$$MU_{(G,U)} \longrightarrow MU'_{(G,U)}$$

of G-spectra is a spacewise G-equivalence and thus a weak equivalence. We also have evident unit and product maps

$$\eta' : S^V \longrightarrow T'_{(G,U)}V$$

and

$$\omega' : T'_{(G,U)}V \wedge T'_{(G,U)}W \longrightarrow T'_{(G,U\oplus U)}(V \oplus W)$$

that are compatible with the unit and product maps of T. Of course, the advantage of $T'_{(G,U)}$ and its associated G-spectrum $MU'_{(G,U)}$ is that the Grassmannians $Gr_n(V \oplus U)$ are classifying spaces for complex n-plane G-bundles.

The inclusion of V in U may be viewed as a map of G-bundles from the trivial bundle over a point to the universal n-plane bundle over $Gr_n(\mathbb{C}^n \oplus U)$. On passage to Thom complexes, it gives a map

$$t(V) : S^V \longrightarrow T'_{(G,U)}\mathbb{C}^n.$$

Composing with the natural map to $MU'_{(G,U)}\mathbb{C}^n$, we see that $t(V)$ represents an element $\mu(V)$ as in (10.1). It is a Thom class, as is standard (e.g. [8] or [7, 2.1]) and can be verified in various ways. Perhaps the simplest is to define

$$t^{-1}(V) : S^{2n} \longrightarrow T'_{(G,U)}V$$

by reversing the roles of V and $\mathbb{C}^n$. Composing with the natural map to $MU'_{(G,U)}V$, we see that $t^{-1}(V)$ represents an element $\mu^{-1}(V) \in MU^{V-2n}_{(G,U)}$. Via ω', the smash product of $t(V)$ and $t^{-1}(V)$ induces a G-map

$$S^{V+2n} \longrightarrow T'_{(G,U\oplus U)}(V \oplus \mathbb{C}^n)$$

that is homotopic to the map obtained by including $V \oplus \mathbb{C}^n$ as the base plane. The latter map is part of the unit map η', and a standard unravelling of definitions shows that $\mu(V)$ and $\mu^{-1}(V)$ are inverse units of the $RO(G)$-graded ring $MU'^*_G \cong MU^*_G$.

This completes the verification that each $MU^{\#}_G$ has Thom classes. To show their compatibility under restriction, consider a G-space V in a G-universe U and observe that

$$t(V)|_H = t(V|_H) : S^V \longrightarrow T'_{(H,U)}\mathbb{C}^n, \quad \text{hence} \quad \mu(V)|_H = \mu(V|_H). \tag{10.2}$$

To show their compatibility under conjugation, consider an H-space V in an H-universe U and an element $g \in G$, write ${}^gU$ for the universe U regarded as a ${}^gH$ universe by pullback along $c_g : {}^gH \longrightarrow H$, and observe that

$$t({}^gV) = c_g^*(t(V)) : S^{{}^gV} \longrightarrow T'_{({}^gH,{}^gU)}\mathbb{C}^n, \quad \text{hence} \quad \mu({}^gV) = c_g(\mu(V)). \tag{10.3}$$

To prove their multiplicativity, recall the external form of our basic products displayed in (7.12). The following immediate observation gives an external multiplicativity formula from which the internal one of Definition 3.7 follows.

Lemma 10.4. *Let U be a G-universe and U' be a G'-universe, and let $V \subset U$ and $V' \subset U'$. Then the following diagram commutes:*

$$\begin{array}{ccc}
S^V \wedge S^{V'} & \xrightarrow{t(V)\wedge t(V')} & T'_{(G,U)}V \wedge T'_{(G',U')}V' \\
\cong \downarrow & & \downarrow \omega' \\
S^{V\oplus V'} & \xrightarrow[t(V\oplus V')]{} & T'_{(G\times G',U\oplus U')}(V \oplus V').
\end{array}$$

Therefore $\mu(V)\mu(V') = \mu(V \oplus V')$.

11. The proof of Lemma 3.4

Let us say that a representation V of G detects a subgroup H if $V \neq 0$ but $V^H = 0$. Then Lemma 3.4 can be interpreted as stating that there are finitely many complex representations V_i of the T such that every subgroup of T is detected by one of the induced representations $ind_T^G V_i$. Recall that F denotes the finite quotient group G/T.

Consider the irreducible representations V of T. They may be viewed as elements of $T^* = \mathrm{Hom}(T, S^1)$. Clearly V detects H unless $H \subseteq \ker(V)$. It is clear from the definition of induction that

$$res_T^G ind_T^G V = \bigoplus_{f\in F} {}^fV.$$

Thus H is detected by $ind_T^G V$ if and only if it is detected by all of the conjugate representations ${}^fV$, that is, if and only if, for all $f \in F$, H is not in the kernel of ${}^fV$. Note that $\ker({}^fV) = {}^f(\ker V)$.

Thus, for any list of irreducible representations $V_1, \cdots, V_q$, if K_i is the kernel of V_i, then each subgroup not detected by the V_i is a subgroup of

$$ {}^{f_1}K_1 \cap \cdots \cap {}^{f_q}K_q $$

for some list of elements f_i of F.

Let T have rank r. Proceeding inductively, we choose irreducible representations $V_1, \cdots, V_r$ such that, for $1 \leq q \leq r$, each displayed intersection of conjugated kernels has rank $r - q$. We begin the induction by choosing any V_1. Certainly each ${}^{f_1}K_1$ has rank $r - 1$. Assume that $V_1, \cdots, V_q$ have been chosen, where $q < r$. Thinking on the Lie algebra level, and noting that conjugations induce translations of Lie algebras, we see that it suffices to choose the kernel K_{q+1} of V_{q+1} so that none of the $(r-q)$-dimensional subspaces

$$ f_1(LK_1) \cap \cdots \cap f_q(LK_q) $$

of LT is contained in any $f_{q+1}(LK_{q+1})$. Translating by f_{q+1}^{-1}, we see that each such condition excludes an $(r-q)$-dimensional subspace of LT from lying in LK_{q+1}. Dually, consider the finitely many q-dimensional subspaces

$$ \{\alpha \,|\, \alpha(f_{q+1}^{-1}(f_1(LK_1) \cap \cdots \cap f_q(LK_q))) = 0\} \subset (LT)^*. $$

Since $q < r$, the set theoretic union of these subspaces cannot be dense in $(LT)^*$. Therefore we can choose $\beta \in (LT)^*$ such that β is in none of these subspaces and such that the kernel of β has rational basis with respect to the lattice on LT given by the kernel of the exponential. The rationality condition ensures that this kernel is the Lie algebra of the kernel K_{q+1} of a representation $V_{q+1} : T \longrightarrow S^1$ whose induced map $LT \longrightarrow \mathbb{R}$ of Lie algebras is β. At the rth stage, all of the intersections

$$ {}^{f_1}K_1 \cap \cdots \cap {}^{f_r}K_r $$

have dimension zero and are therefore finite. To detect the finitely many subgroups in these finitely many intersections, we need only detect their nonidentity elements g. However, if V_g is a representation whose kernel K_g does not contain all of the conjugates ${}^fg$, then $ind_T^G V_g$ detects g, so this is easily done.

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