Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 56, 1–58.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3173

Localization and number of visited valleys for a transient diffusion in random environment*

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Abstract

We consider a transient diffusion in a $(-\kappa/2)$-drifted Brownian potential W_κ with $0<\kappa<1$. We prove its localization at time t in the neighborhood of some random points depending only on the environment, which are the positive h_t-minima of the environment, for h_t a bit smaller than $\log t$. We also prove an Aging phenomenon for the diffusion, a renewal theorem for the hitting time of the farthest visited valley, and provide a central limit theorem for the number of valleys visited up to time t.

The proof relies on a decomposition of the trajectory of W_κ in the neighborhood of h_t-minima, with the help of results of Faggionato [25], and on a precise analysis of exponential functionals of W_κ and of W_κ Doob-conditioned to stay positive.

Keywords: Diffusion in a random potential ; Localization ; Aging ; Renewal theorem.
AMS MSC 2010: 60F05 ; 60K05 ; 60K37 ; 60J60 ; 82D30.
Submitted to EJP on November 29, 2013, final version accepted on March 18, 2015.
Supersedes arXiv:1311.6332v2.
Supersedes HAL:hal-00908626.

1 Introduction and notation

1.1 Presentation of the model

We are interested in a diffusion $(X(t),\ t\geq 0)$ in a random càdlàg potential $(V(x),\ x\in\mathbb{R})$. It is defined informally by $X(0)=0$ and

$$\mathrm{d}X(t)=d\beta(t)-\frac{1}{2}V'(X(t))\mathrm{d}t,$$

where $(\beta(t),\ t\geq 0)$ is a Brownian motion independent of V. More rigorously, X is a diffusion process, starting from 0, and whose conditional generator given V is

$$\frac{1}{2}e^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

*This research was supported by the french ANR projects MEMEMO 2 (2010 BLAN 0125 02) and Randymeca.

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These diffusions in random potentials are considered as continuous time analogues of random walks in random environment (RWRE) (see e.g. P. Révész [38], B.D. Hughes [31], Z. Shi [43] and O. Zeitouni [51] for reviews on RWRE).

The study of such a process starts with a choice for V. A classic one, originally introduced by S. Schumacher [41] and T. Brox [10], is to choose V as a Lévy process. In fact only a few papers deal with the discontinuous case, see for example P. Carmona [11] or A. Singh [46, 45]. Most of the results concern diffusions in a continuous Lévy potential V, that is,

$$V(x) = W_\kappa(x) := W(x) - \frac{\kappa}{2}x, \qquad x \in \mathbb{R},$$

where $\kappa \in \mathbb{R}$ and $(W(x),\ x \in \mathbb{R})$ is a two sided Brownian motion. We denote by P the probability measure associated to W_κ. The probability conditionally on the potential W_κ is denoted by $\mathbb{P}^{W_\kappa}$ and is called the *quenched probability*. We also define the *annealed probability* as

$$\mathbb{P}(.) := \int \mathbb{P}^{W_\kappa}(.)P(W_\kappa \in \mathrm{d}\omega).$$

We denote respectively by $\mathbb{E}^{W_\kappa}$, $\mathbb{E}$, and E the expectations with regard to $\mathbb{P}^{W_\kappa}$, $\mathbb{P}$ and P.

In the case $\kappa = 0$, the diffusion X is a.s. recurrent. More precisely, T. Brox [10] shows that it is sub-diffusive with asymptotic behavior in $(\log t)^2$, and that it is localized, at time t, in the neighborhood of a random point $b_{\log t}$ depending only on t and W, similarly as Sinai's walk (see Ya. G. Sinai [44]). More precisely, this result can be written:

Theorem 1.1 (Brox [10]). *Assume $\kappa = 0$. Then, for all $\varepsilon > 0$,*

$$\lim_{t\to+\infty} \mathbb{P}\left[X(t) \in [b_{\log t} - \varepsilon(\log t)^2, b_{\log t} + \varepsilon(\log t)^2]\right] = 1. \tag{1.1}$$

The limit law of $b_{\log t}/(\log t)^2$ and therefore of $X(t)/(\log t)^2$ was made explicit independently by H. Kesten [33] and A. O. Golosov [29]. For recent results for this recurrent case, see for example P. Andreoletti et al. [3], and R. Diel [18].

In the case $\kappa \neq 0$, the diffusion X is a.s. transient, with a wide range of limiting behaviors, depending on the value of κ. It was first studied by K. Kawazu and H. Tanaka. Let us denote by $H(r)$ the hitting time of $r \in \mathbb{R}$ by X:

$$H(r) := \inf\{s > 0,\ X(s) = r\}.$$

Kawazu et al. [32] proved in particular that under the annealed probability $\mathbb{P}$, $H(r)/r^{1/\kappa}$ converges in law to a stable distribution when $0 < \kappa < 1$, whereas $H(r)/(r \log r)$ converges in probability to 4 when $\kappa = 1$, and $H(r)/r$ converges almost surely to $4/(\kappa - 1)$ if $k > 1$ (see also Y. Hu et al. [30], and H. Tanaka [49]). More recently we mention the results for large and moderate deviations, by M. Taleb ([47] and [48]), A. Devulder [14] and G. Faraud [26].

In this paper we study the case $0 < \kappa < 1$. We follow a different approach from Y. Hu et al. [30] and K. Kawazu et al. [32]. Indeed we focus on a quenched study, which has attracted much interest for transient RWRE in the last few years, see for example the works of N. Enriquez et al. [21], [22], [23], [24], D. Dolgopyat et al. [19], and J. Peterson et al. [35], [36], [37]. Heuristically, the diffusion X goes to locations where the potential is low, hence it goes to $+\infty$, but it is slowed by "valleys" of the potential, which trap the diffusion for some time. The diffusion even spends most of its time in these valleys. We will prove this more in details in the present paper.

1.2 Main results

The goals of this paper are to localize the diffusion X, when $0 < \kappa < 1$, in some valleys of the potential W_κ, to understand the differences with Brox's result given by

(1.1), and to prove an Aging phenomenon, corresponding to results obtained by Enriquez et al. in their papers [22], [23] and [24] for transient zero-speed RWRE. We moreover obtain a central limit theorem for the number of valleys visited up to time t. We also prove some intermediate results, which we think will be useful for obtaining new results about the maximum local time of X, as explained later in this introduction.

Let $t \mapsto \phi(t)$ be a positive increasing function, such that $\phi(t) = o(\log t)$ and $\log\log t = o(\phi(t))$ as $t \to +\infty$, where $f(t) = o(g(t))$ means $\lim_{t\to+\infty} f(t)/g(t) = 0$. We prove the following aging phenomenon:

Proposition 1.2 (aging)**.** *Assume* $0 < \kappa < 1$*. For all* $\alpha > 1$*, we have*

$$\lim_{t\to+\infty} \mathbb{P}\Big(|X(\alpha t) - X(t)| \leq \phi(t)\Big) = \frac{\sin(\kappa\pi)}{\pi} \int_0^{1/\alpha} u^{\kappa-1}(1-u)^{-\kappa}\mathrm{d}u.$$

More generally, *aging* usually denotes dynamical out-of-equilibrium physical phenomenons, which appear in some disordered systems. It refers to the existence of a limit for a given two-time correlation function of the system as both times diverge but keep a fixed ratio between them. This subject has received a considerable attention in physics. For a physical or a mathematical point of view on aging, see e.g. respectively Bouchaud et al. [8] and Zindy [52], and references therein.

Proposition 1.2 is actually a consequence of Theorem 1.3. Before stating it, we first recall the notion of h-*extrema*, which was first introduced by J. Neveu et al. [34], and studied in the case of drifted Brownian motion by A. Faggionato [25]. For $h > 0$, we say that $x \in \mathbb{R}$ is an h-*minimum* for a given function f, $\mathbb{R} \to \mathbb{R}$, if there exist $u < x < v$ such that $f(x) = \inf_{y\in[u,v]} f(y)$, $f(u) \geq f(x) + h$ and $f(v) \geq f(x) + h$. Moreover, x is an h-*maximum* for f iff x is an h-minimum for $-f$. Finally, x is an h-*extremum* for f iff it is an h-maximum or an h-minimum for f.

Since we want to study the diffusion X until time $t > 0$, we are more especially interested in the h_t-extrema of W_κ, where

$$h_t := \log t - \phi(t).$$

It is known (see [25]) that almost surely, the h_t-extrema of W_κ form a sequence indexed by $\mathbb{Z}$, unbounded from below and above, and that the h_t-minima and h_t-maxima alternate. We denote respectively by $(m_j,\ j \in \mathbb{Z})$ and $(M_j,\ j \in \mathbb{Z})$ the increasing sequences of h_t-minima and of h_t-maxima of W_κ, such that $m_0 \leq 0 < m_1$ and $m_j < M_j < m_{j+1}$ for every $j \in \mathbb{Z}$. These h_t-minima m_i, $i \in \mathbb{Z}$, can be considered as the bottoms of some valleys of the potential W_κ, of height at least h_t, that will be defined more precisely in Section 2. We also introduce

$$N_t := \max\Big\{k \in \mathbb{N},\ \sup_{0\leq s\leq t} X(s) \geq m_k\Big\},$$

so that m_{N_t} is the largest h_t-minimum visited by X until time t if $N_t > 0$ (and so that $\lim_{t\to+\infty} \mathbb{P}(N_t > 0) = 1$ as proved later, see Proposition 1.6 or Lemma 5.2). The main result of this paper concerns the localization of the diffusion. It is stated as follows:

Theorem 1.3 (localization)**.** *Assume* $0 < \kappa < 1$*. There exists a constant* $\mathcal{C}_1 > 0$*, such that*

$$\lim_{t\to+\infty} \mathbb{P}\Big(|X(t) - m_{N_t}| \leq \mathcal{C}_1\phi(t)\Big) = 1.$$

We first recall that $X(t)$ is asymptotically of order t^κ (see Kawazu et al. [32]), and that the typical distance between two h_t-minima of W_κ is asymptotically of order $e^{\kappa h_t} = t^\kappa e^{-\kappa\phi(t)}$ (see Faggionato [25] Prop. 1, partly recalled in our Fact 2.2 below). So, the size $2\mathcal{C}_1\phi(t)$ of the intervals in which X is localized in Theorem 1.3 is very small, since it is $o(\log t)$ and can be as small as, for example, $(\log\log t)^{1+\varepsilon}$, $\varepsilon > 0$. Notice that it depends on the minimum height h_t of our valleys. We could not say however if it the best interval size that can be obtained.

The main difference with the result of Brox (1.1) is the appearance of the (random) integer N_t, which is the number of typical valleys of height h_t visited before time t. In the recurrent case of Brox, the diffusion X is, with a large probability, localized near the bottom of a unique valley of the potential, whereas in our transient case, the diffusion is localized near the bottom of one among several valleys of the potential. This, and the absence of scaling for the potential in the case $0 < \kappa < 1$, contrarily to the case $\kappa = 0$, makes the study much more involved technically.

We also prove a renewal theorem for hitting time of the bottom m_{N_t} of the last valley visited by X before t:

Proposition 1.4. *Assume $0 < \kappa < 1$. We have the following convergence in law under the annealed probability $\mathbb{P}$,*

$$\left(\frac{H(m_{N_t})}{t}, \frac{H(m_{N_t+1})}{t}\right) \overset{\mathcal{L}}{\to}_{t\to+\infty} \frac{\kappa\sin(\pi\kappa)}{\pi}(y-x)^{-\kappa-1}x^{\kappa-1}\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[1,\infty)}(y)\mathrm{d}x\mathrm{d}y.$$

This unables us to get the following results, which are useful for the proofs of Proposition 1.2 and Theorem 1.3:

Corollary 1.5. *Assume $0 < \kappa < 1$ and let $0 \le r < s \le 1$ and $v \ge 0$. Then,*

$$\lim_{t\to+\infty}\mathbb{P}\left(1-s \le \frac{H(m_{N_t})}{t} \le 1-r\right) = \frac{\sin(\pi\kappa)}{\pi}\int_{1-s}^{1-r} x^{\kappa-1}(1-x)^{-\kappa}\mathrm{d}x, \qquad (1.2)$$

$$\lim_{t\to+\infty}\mathbb{P}\left(\frac{H(m_{N_t+1})}{t} \ge 1+v\right) = \frac{\sin(\pi\kappa)}{\pi}\int_{v}^{+\infty}(1+x)^{-1}x^{-\kappa}\mathrm{d}x. \qquad (1.3)$$

Moreover, the total time spent in the last valley of height at least h_t visited before time t renormalized by t, that is $[H(m_{N_t+1}) - H(m_{N_t})]/t$, converges in law under $\mathbb{P}$ to a r.v. with density $\sin(\pi\kappa)\pi^{-1}x^{-\kappa-1}[(1-(1-x)^{\kappa})\mathbb{1}_{[0,1]}(x) + \mathbb{1}_{(1,+\infty)}(x)]$.

Let $\lfloor x\rfloor$ denote the integer part of x, for any $x \in \mathbb{R}$. We introduce $0 < \delta < 1$ and

$$n_t := \left\lfloor e^{\kappa\phi(t)(1+\delta)}\right\rfloor. \qquad (1.4)$$

We will see in Section 4 and 5 that Proposition 1.4 is a consequence of the fact that for any integer $1 \le k \le n_t$, the hitting time $H(m_k)$ can approximated by a sum of i.i.d. random variables having the law of a r.v. $\mathbf{U}$. This r.v. $\mathbf{U}$ is an approximation of the time the diffusion X spends in a typical valley of height at least h_t before escaping this valley.

These results are in accordance with those obtained by Enriquez et al. in their three papers [22], [23] and [24] for transient RWRE. Compared to their study, we have the advantage of being able to use some powerful stochastic tools. However, some other technical difficulties appear in continuous time: for example local time and excursions are more complicated to deal with in continuous time than in discrete time. The present paper is self contained, in particular we provide in this same paper the technical study of the Laplace transform of the first exit time $\mathbf{U}$ of a typical valley. The study of the environment mainly requires continuous arguments of stochastic calculus, starting by a decomposition of the trajectory of W_κ near its h_t-minima, which mainly comes from results of A. Faggionato [25].

The number N_t of valleys having height at least h_t, visited before time t by the diffusion X, goes to $+\infty$ as $t \to +\infty$. However, we prove that $\mathbb{P}(N_t \le n_t) \to_{t\to+\infty} 1$, which explains why we study the potential $W_\kappa(x)$ only for $x \le m_{n_t}$, and the hitting times $H(m_k)$ only for $k \le n_t$. More precisely, we prove the following central limit theorem for N_t, with renormalization $e^{\kappa\phi(t)}$:

Proposition 1.6 (number of visited valleys)**.** *Assume* $0 < \kappa < 1$. *Then* $N_t e^{-\kappa\phi(t)} \to_{t\to+\infty} \mathcal{N}$ *in law under the annealed law* $\mathbb{P}$*; the law of* $\mathcal{N}$ *is determined by its Laplace transform:*

$$\forall u > 0, \qquad \mathbb{E}\left(e^{-u\mathcal{N}}\right) = \sum_{j=0}^{+\infty} \frac{1}{\Gamma(\kappa j+1)} \left(\frac{-u}{C_\kappa}\right)^j, \tag{1.5}$$

where $C_\kappa > 0$ *is explicitly known (see Proposition 4.1). This r.v.* $\mathcal{N}$ *has then a Mittag-Leffler distribution of order* κ.

Moreover we expect that the results of this paper will be useful to study other properties for the diffusion. In particular, let $(\mathcal{L}_X(t,x),\ t \geq 0,\ x \in \mathbb{R})$ be a bicontinuous version of the local time of X. It is known that the maximum local time of X at time t, that is $\mathcal{L}_X^*(t) := \max_{x\in\mathbb{R}} \mathcal{L}_X(t,x)$, satisfies $\limsup_{t\to+\infty} \mathcal{L}_X^*(t)/t = +\infty$ a.s. in the cases $\kappa = 0$ (see Z. Shi [42] and R. Diel [18]) and even in the transient case $0 < \kappa < 1$ (see A. Devulder [16]). Hence the maximum local time of X exhibits very interesting properties, that contrast with those of the maximum local time of RWRE at time t, which is naturally bounded by $t/2$.

We especially think that the better understanding of the localization of X and some intermediate results provided in this paper will be useful to prove new results about $\mathcal{L}_X^*$. Indeed, in a work in progress with G. Vechambre [2], we use the methods and results of the present paper to study the local time of the diffusion. In particular we expect to obtain the limit law of the maximal local time after suitable renormalization in the case $0 < \kappa < 1$. Note that the local time plays a crucial role for estimation problems for random walk in random environment, recently studied e.g. in [1], [4] and [13]. In particular in [13], the limit law of the local time process in the neighborhood of the minima, obtained previously by [28], is used. In the same way a better understanding of the local time of X may be useful to study estimation problems for diffusions in a random potential.

1.3 Sketch of the proof and organization of the paper

We now give a general idea of the proof and provide, at the same time, the organization of the paper. This subsection contains some non rigorous heuristics which will be made rigorous and explained in details in the following sections.

First, in Section 2, we build some valleys $\left(\tilde{L}_i^-, \tilde{m}_i, \tilde{L}_i\right)$ of the potential, of height at least h_t (see Figure 1 page 11). The i-th valley is the potential W_κ, restricted to some interval $\left[\tilde{L}_i^-, \tilde{L}_i\right]$; the minimum of W_κ in this interval is attained at a location called $\tilde{m}_i$. The height of this valley is at least h_t, and more precisely, $W_\kappa\left(\tilde{L}_i\right) - W_\kappa\left(\tilde{m}_i\right) \geq h_t$, $W_\kappa\left(\tilde{L}_i^-\right) - W_\kappa\left(\tilde{m}_i\right) \geq (1+\kappa+2\delta)h_t$, These valleys, when recentered at $\tilde{m}_i$, are i.i.d. We prove in Lemma 2.3 that with probability nearly 1, $\tilde{m}_i = m_i$ for $1 \leq i \leq n_t$, that is, the $\tilde{m}_i$, $1 \leq i \leq n_t$ are in fact the n_t first positive h_t-minima. We also provide in this Section 2 different tools to study the law of W_κ near $\tilde{m}_i$ or m_i, and in particular drifted Brownian motions Doob-conditioned to stay positive.

Then in Section 3, we prove (see Lemma 3.2) that with probability nearly 1, after hitting the bottom $\tilde{m}_i$ of the each valley $\left(\tilde{L}_i^-, \tilde{m}_i, \tilde{L}_i\right)$ the diffusion X leaves this valley on the right, that is, on $\tilde{L}_i$. Moreover, we prove (see Lemma 3.3) that with probability nearly 1, the diffusion X visits successively the valleys $\left(\tilde{L}_1^-, \tilde{m}_1, \tilde{L}_1\right)$, $\left(\tilde{L}_2^-, \tilde{m}_2, \tilde{L}_2\right)$, $\dots$, $\left(\tilde{L}_i^-, \tilde{m}_i, \tilde{L}_i\right)$, $\dots$, and does not come back to previously visited valleys. Moreover, we show in Lemma 3.7 that the time spent outside these valleys up to time t is negligible compared to t. That is, the hitting time $H(\tilde{m}_n)$ of the bottom $\tilde{m}_n$ of each valley number $n \leq n_t$ can be approximated as

$$H(\tilde{m}_n) \approx U_1 + U_2 + \dots + U_{n-1}, \tag{1.6}$$

where U_i can be seen as the time spent in valley number i. We also prove in Lemma 3.4 that these r.v. U_i are "nearly" independent under the annealed and quenched probabilities $\mathbb{P}$ and $\mathbb{P}^{W_\kappa}$. Moreover, we prove that they have (under the annealed probability $\mathbb{P}$) the same law as some random variable $\mathbf{U}$, defined by

$$\mathbf{U} := \int_{\tilde{L}_2^-}^{\tilde{L}_2} e^{-[W_\kappa(u)-W_\kappa(\tilde{m}_2)]}\mathcal{L}_B\big[\tau^B\big(\tilde{A}_2\big(\tilde{L}_2\big)\big), \tilde{A}_2(u)\big]\mathrm{d}u,$$

where $\tilde{A}_2(z) := \int_{\tilde{m}_2}^z e^{W_\kappa(x)-W_\kappa(\tilde{m}_2)}\mathrm{d}x$, $z\in\mathbb{R}$, and $\mathcal{L}_B$ and τ^B denote respectively the local time and hitting time of some Brownian motion B independent of W_κ. This r.v. $\mathbf{U}$, which depends on t, can be seen as the time spent by X in a typical valley, which X leaves on its right.

So, this reduces the study of $H(\tilde{m}_n)$ to the study of a sum $U_1+U_2+\dots+U_{n-1}$ of independent random variables having the same law as $\mathbf{U}$ under $\mathbb{P}$.

The main goal of Section 4 is to study the asymptotics of the Laplace transform of $\mathbf{U}$. More precisely, we prove in Proposition 4.1 that for all $\lambda>0$,

$$\mathbb{E}\big(e^{-\lambda\mathbf{U}/t}\big) = 1 - C_\kappa\lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)}) \tag{1.7}$$

as $t\to+\infty$. To this aim, we first approximate $\mathbf{U}$ by a more tractable expression, which is done in Proposition 4.4 and Lemma 4.7. First, by scaling, $\mathbf{U}$ is equal in law to

$$\int_{\tilde{L}_2^-}^{\tilde{L}_2} e^{-[W_\kappa(u)-W_\kappa(\tilde{m}_2)]}\tilde{A}_2\big(\tilde{L}_2\big)\mathcal{L}_B\big[\tau^B(1), \tilde{A}_2(u)/\tilde{A}_2\big(\tilde{L}_2\big)\big]\mathrm{d}u. \tag{1.8}$$

Loosely speaking, for u "close" to $\tilde{m}_2$, $\tilde{A}_2(u)/\tilde{A}_2(\tilde{L}_2)\approx 0$, and so (using our Lemma 4.3) we have $\mathcal{L}_B\big[\tau^B(1), \tilde{A}_2(u)/\tilde{A}_2(\tilde{L}_2)\big]\approx\mathcal{L}_B[\tau^B(1),0] =: \mathbf{e_1}$, which is by the first Ray Knight theorem an exponential variable of mean 2, and is independent of W_κ.

One the other hand, $\tilde{m}_2$ is the minimum of the potential W_κ in $\big[\tilde{L}_2^-, \tilde{L}_2\big]$ so for u "far" from $\tilde{m}_2$ we have $e^{-[W_\kappa(u)-W_\kappa(\tilde{m}_2)]}\approx 0$. Moreover, $\tilde{A}_2(\tilde{L}_2)\approx\int_{\tilde{\tau}_2(h_t/2)}^{\tilde{L}_2} e^{W_\kappa(x)-W_\kappa(\tilde{m}_2)}\mathrm{d}x$ where $\tilde{\tau}_2(h)$ is the first hitting time of $h>0$ by $W_\kappa - W_\kappa(\tilde{m}_2)$ after $\tilde{m}_2$, because $e^{W_\kappa(x)-W_\kappa(\tilde{m}_2)}$ is negligible compared to $\tilde{A}_2\big(\tilde{L}_2\big)$ for $\tilde{m}_2\le x\le\tilde{\tau}_2(h_t/2)\le\tilde{\tau}_2(h_t)<\tilde{L}_2$. All this leads to the approximation

$$\begin{aligned}\mathbf{U} &\approx \left(\int_{\tilde{L}_2^-}^{\tilde{m}_2} + \int_{\tilde{m}_2}^{\tilde{\tau}_2(h_t/2)} e^{-[W_\kappa(u)-W_\kappa(m_2)]}\mathrm{d}u\right)\left(\int_{\tilde{\tau}_2(h_t/2)}^{\tilde{\tau}_2(h_t)} + \int_{\tilde{\tau}_2(h_t)}^{\tilde{L}_2} e^{W_\kappa(x)-W_\kappa(\tilde{m}_2)}\mathrm{d}x\right)\mathbf{e_1}\\ &\approx: (\mathcal{I}_1^- + \mathcal{I}_2^-)(\mathcal{I}_1^+ + \mathcal{I}_2^+)\mathbf{e_1},\end{aligned}$$

which is the product of 5 independent random variables, the first 4 depending only on the potential W_κ, and $\mathbf{e_1}$ being independent of W_κ. This approximation, the asymptotics of the Laplace transforms of $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$ provided by Lemma 4.2, and some technical calculations help us to prove (1.7) as claimed in our Proposition 4.1.

Section 5 is devoted to the proofs of the main results of this paper. First, using the asymptotics (1.7) of the Laplace transform of $\mathbf{U}$ as well as (1.6), we prove the renewal results Proposition 1.4 and Corollary 1.5, using the same kind of techniques as in Enriquez et al. [24], inspired by Feller [27]. We also show Proposition 1.6 about the number of visited valleys with a similar method. We prove before, in Lemma 5.2, that with probability nearly 1, the number N_t of valleys visited by X up to time t is less than n_t, which explains why we only consider the first n_t valleys.

We then turn to the proof of the localization, that is, our Theorem 1.3. To this aim, using the previous renewal results, we prove that with probability nearly 1, at time t, the diffusion X has already spent a quite large amount of time in the last valley visited, that is, on $[\tilde{L}^-_{N_t}, \tilde{L}_{N_t}]$, and that $X(t)$ still belongs to this interval. This allows us to prove that, knowing $N(t) = j$, the quenched law of $X(t)$ is nearly equal to the invariant probability $\tilde{\mu}_j$ of a diffusion Y_j in the potential W_κ, starting inside $[\tilde{L}^-_j, \tilde{L}_j]$ and reflected at $\tilde{L}^-_j$ and $\tilde{L}_j$. This is a kind of convergence to the invariant probability measure, which we prove as in Brox [10], by using a coupling between Y_j starting from $Y_j(0)$ distributed as its invariant measure $\tilde{\mu}_j$, and $X(. + H(\tilde{m}_j))$. Since $\tilde{\mu}_j$ is proportional to $\exp(-[W_\kappa(x) - W_\kappa(\tilde{m}_j))\mathbb{1}_{[\tilde{L}^-_j, \tilde{L}_j]}(x)\mathrm{d}x$, it is highly concentrated on a small neighborhood of $\tilde{m}_j$, which leads to the localization of X at time t in this small neighborhood of $\tilde{m}_j$.

We then prove the Aging, that is, Proposition 1.2. For this, we apply the localization (Theorem 1.3), first at time t with function ϕ, and second at time αt but with another function ϕ_α defined by $\log(\alpha t) - \phi_\alpha(\alpha t) = \log t - \phi(t)$, so that the r.v. m_i are the same in both cases.

Our hypothesis $0 < \kappa < 1$ is used many times in this paper. We recall that the typical distance between two h_t-minima of W_κ, or valleys of depth at least h_t, is asymptotically of order $e^{\kappa h_t} = t^\kappa e^{-\kappa\phi(t)}$. Moreover, the time spent by the diffusion X in such valleys is approximatively proportional to the exponential of the depth of such valleys. So X is trapped a long time, of order at least $te^{-\phi(t)}$, in such valleys, and then is slowed by these valleys if (and only if) it hits some of them.

In particular, the first h_t-minimum, or valley, appears at a distance of order $e^{\kappa h_t} = t^\kappa e^{-\kappa\phi(t)}$ which is much smaller than t if $0 < \kappa < 1$, and much larger than t if $\kappa > 1$. Heuristically, if $0 < \kappa < 1$, there are many h_t-minima in $[0, t^{\kappa+\varepsilon}]$, $0 < \varepsilon < 1 - \kappa$, which trap and slow the diffusion X, which explains why X is zero-speed. However if $\kappa > 1$, loosely speaking, the first positive h_t-minimum is very far from the origin, at a distance much large than t, and then at time t the diffusion X has not yet reached it; X has then not been trapped nor slowed by such deep valleys, and in this case X has a positive speed.

In all the paper, $0 < \kappa < 1$ is fixed, and C_+ and c_+ (resp. C_- and c_-) denote positive constants that may increase (resp. decrease) from line to line and may only depend on our fixed constant κ. Moreover some events are denoted by $\mathcal{E}_{\mathrm{i}}^{\mathrm{j.k}}$ for some i, j, k; for example $\mathcal{E}_3^{4.7}$ is the event number 3 introduced in the proof of Lemma 4.7.

2 Standard valleys and path decomposition of the potential

2.1 3-dimensional drifted Bessel processes

In this subsection, we introduce 3-dimensional drifted Bessel processes as drifted Brownian motions conditioned to stay positive. These processes are helpful to describe the law of the potential W_κ near the h_t-minima m_i and then to estimate relevant quantities depending on this potential W_κ, mainly with formulas (2.3) and (2.4) below. For any process $(U(t),\ t \geq 0)$ and any $a \in \mathbb{R}$, we denote the hitting time of a by U as

$$\tau^U(a) := \inf\{t > 0,\ U(t) = a\},$$

with the convention $\inf \emptyset = +\infty$. We denote by $(\mathcal{L}_U(t,x),\ t \geq 0,\ x \in \mathbb{R})$ the bicontinuous version of the local time of U when it exists, which is the case for X and for Brownian motions. We also denote by U^a the process U starting from a, with the notation $U = U^0$. We sometimes write $P^a(U \in .) := P(U^a \in .)$. In particular, for $x \in \mathbb{R}$, $\zeta \neq 0$, W_ζ^x is a $(-\zeta/2)$-drifted Brownian motion starting from x.

Let $\zeta \neq 0$. We recall the definition of a $(-\zeta/2)$-*drifted Brownian motion* W_ζ *Doob-conditioned to stay positive* (see [25], 5. p. 1783, or [6], Chapter VII.3 and references therein for more details). We consider the σ-fields $\mathcal{F}_t$ defined on $C([0,\infty), \mathbb{R})$ by $\mathcal{F}_t := \sigma(Y(s),\ 0 \leq s \leq t)$, $t \geq 0$, and $\mathcal{F}_\infty := \sigma(Y(s),\ s \geq 0)$, for a generic element $(Y(s), s \geq 0)$ of the path space $C([0,\infty), \mathbb{R})$. Following ([25], p. 1783), for $z > 0$, the probability measure $P_z^{\zeta/2,\uparrow}$ is defined on $C([0,+\infty),\mathbb{R})$ by

$$P_z^{\zeta/2,\uparrow}(\Lambda) := \frac{1}{1-e^{\zeta z}} E\Big[\big[1-\exp(\zeta W_\zeta^z(t))\big],\, W_\zeta^z \in \Lambda,\, t < \tau^{W_\zeta^z}(0)\Big], \qquad \Lambda \in \mathcal{F}_t,\ t \geq 0. \quad (2.1)$$

This induces a unique probability measure $P_z^{\zeta/2,\uparrow}$ on $(C([0,\infty),\mathbb{R}),\mathcal{F}_\infty)$. Moreover, $P_z^{\zeta/2,\uparrow}$ converges weakly as $z \to 0^+$, in the space of Skorokhod $D([0,\infty),\mathbb{R})$ (see [6] VII.3 Prop. 14 and comments below) and in $C([0,\infty),\mathbb{R})$ (see [25] p. 1784) to a probability measure on $C([0,\infty),\mathbb{R})$ denoted by $P_0^{\zeta/2,\uparrow}$. The canonical process, which we denote by $(R(s),\ s\geq 0)$, takes values in $\mathbb{R}_+$ under $P_z^{\zeta/2,\uparrow}$ for $z \geq 0$. It is a Feller process for the family $\big(P_z^{\zeta/2,\uparrow},\ z \geq 0\big)$, and then it is strong Markov. Its infinitesimal generator is given for every $x > 0$ by

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\zeta}{2}\coth\left(\frac{\zeta}{2}x\right)\frac{\mathrm{d}}{\mathrm{d}x}. \quad (2.2)$$

This infinitesimal generator is given by Lemma 6 of [25], which is true in the case of positive or negative drift $\zeta/2$.

This process R can be thought of as a $(-\zeta/2)$-drifted Brownian motion W_ζ Doob-conditioned to stay positive, with the terminology of [6], which is called Doob conditioned to reach $+\infty$ before 0 in [25]. We notice in particular that, by (2.2), or by ([25], eq. (5.4)), or directly by (2.1) combined with Girsanov theorem, the law of R is the same if ζ is replaced by $-\zeta$. That is, W_ζ Doob-conditioned to stay positive has the same law as $W_{-\zeta}$ Doob-conditioned to stay positive. This is the case in particular in ([25], Thm. 2) and then also in ([25], eq. (1.1)), where the sign should be a $+$ in every case.

This process R is also shown in Rogers et al. ([40], Thm. 3 and eq. (13)) to be equal in law to the euclidian norm of a 3-dimensional drifted Brownian motion, with drift $\zeta/2$ in some direction given by a unit vector of $\mathbb{R}^3$. We do not use this result in this paper, but for this reason, in the rest of the paper, we call for $z \geq 0$ the process R with law $P_z^{\zeta/2,\uparrow}$ a 3-dimensional $|\zeta/2|$-drifted Bessel process starting from z. As in [40], its law is denoted by $\mathrm{BES}^z(3,|\zeta/2|)$, and by $\mathrm{BES}(3,|\zeta/2|) = \mathrm{BES}^0(3,|\zeta/2|)$ if it starts from $z=0$.

In the rest of the paper, it is often useful to consider a process $(R(s),\ s \geq 0)$ with law $\mathrm{BES}^z(3,\kappa/2)$ for some $z \geq 0$. We have by the previous remark, when R starts from $z > 0$,

$$P^z(R\in\Lambda) = P_z^{\kappa/2,\uparrow}(\Lambda) = P_z^{-\kappa/2,\uparrow}(\Lambda) = P\big(W_{-\kappa}^z \in \Lambda | \tau^{W_{-\kappa}^z}(0) = \infty\big), \qquad z>0,\ \Lambda \in \mathcal{F}_\infty, \quad (2.3)$$

where the last equality is noticed in [25] just before its Lemma 6 for $\Lambda \in \mathcal{F}_t$, $t \geq 0$ since $-\kappa/2 < 0$ and then $W_{-\kappa}$ has a positive drift $\kappa/2$, and so this is true for all $\Lambda \in \mathcal{F}_\infty$. As a consequence, when R starts from 0, that is, when the law of $(R(s),\ s\geq 0)$ is $\mathrm{BES}(3,\kappa/2)$, we have for all $\Lambda \in \mathcal{F}_\infty$ such that $P(R \in \partial\Lambda) \neq 0$,

$$P(R\in\Lambda) = P_0^{\kappa/2,\uparrow}(\Lambda) = P_0^{-\kappa/2,\uparrow}(\Lambda) = \lim_{z\downarrow 0} P_z^{-\kappa/2,\uparrow}(\Lambda) = \lim_{z\downarrow 0} P\big(W_{-\kappa}^z \in \Lambda | \tau^{W_{-\kappa}^z}(0) = \infty\big). \quad (2.4)$$

2.2 Path decomposition of the potential W_κ in the neighborhood of the h_t-minima m_i

The point of view of h-extrema has been used recently in some studies of random walks or diffusions in random environment in the recurrent case, see e.g. Cheliotis [12], Bovier et al. [9] and Devulder [17]. We now recall some results for h-extrema of W_κ. Let

$$V^{(i)}(x) := W_\kappa(x) - W_\kappa(m_i), \qquad x \in \mathbb{R},\ i \in \mathbb{N}^*,$$

which is the potential W_κ translated so that it is 0 at the local minimum m_i. We also define

$$\tau_i^-(h) := \sup\{s < m_i,\ V^{(i)}(x) = h\}, \qquad \tau_i(h) := \inf\{s > m_i,\ V^{(i)}(x) = h\}, \qquad h > 0.$$

The following result has been proved by Faggionato [25] for **(i)** and **(ii)**.

Fact 2.1 (path decomposition of W_κ around the h_t-minima m_i)**.**
(i) *The truncated trajectories* $\big(V^{(i)}(m_i - s),\ 0 \le s \le m_i - \tau_i^-(h_t)\big)$, $\big(V^{(i)}(m_i + s),\ 0 \le s \le \tau_i(h_t) - m_i\big)$, $i \ge 1$ *are independent.*
(ii) *Let* $(R(s),\ s \ge 0)$ *be a process with law* $BES(3, \kappa/2)$. *All the truncated trajectories* $\big(V^{(i)}(m_i - s),\ 0 \le s \le m_i - \tau_i^-(h_t)\big)$ *for* $i \ge 2$ *and* $\big(V^{(j)}(m_j + s),\ 0 \le s \le \tau_j(h_t) - m_j\big)$ *for* $j \ge 1$ *are equal in law to* $\big(R(s),\ 0 \le s \le \tau^R(h_t)\big)$.
(iii) *For* $i \ge 1$, *the truncated trajectory* $\big(V^{(i)}(s + \tau_i(h_t)),\ s \ge 0\big)$ *is independent of* $\big(W_\kappa(s),\ s \le \tau_i(h_t)\big)$ *and is equal in law to* $\big(W_\kappa^{h_t}(s),\ s \ge 0\big)$, *that is, to a* $(-\kappa/2)$*-drifted Brownian motion starting from* h_t.

We point out that for reasons linked to renewal theory, the first trajectory in **(ii)** for $i = 1$ has a different law, which we will not use in this paper.

Proof of Fact 2.1. Notice that $M_{i-1} < \tau_i^-(h_t) < m_i < \tau_i(h_t) < M_i$, $i \in \mathbb{Z}$. Moreover the h_t-extrema of W_κ are the r.v. m_i and M_i, $i \in \mathbb{Z}$. So **(i)** follows from the independence of the truncated trajectories between consecutive h_t-extrema proved by Faggionato ([25], Theorem 1; notice in particular the comment about independence just before its equation (2.26)). Result **(ii)** is proved in Faggionato ([25], Theorem 2 p. 1785), since, as explained in the paragraph after (2.2) of the present paper, a Brownian motion starting at 0 with drift $\kappa/2$ Doob conditioned to reach $+\infty$ before 0 has the same law as a Brownian motion starting at 0 with drift $-\kappa/2$ Doob conditioned to reach $+\infty$ before 0, and the same law as $BES(3, \kappa/2)$.

Finally, let $i \ge 1$. For every $x \ge 0$, $\tau_i(h_t) \le x$ if and only if the function $s \mapsto W_\kappa(s)\mathbb{1}_{(-\infty,x]}(s) + W_\kappa(x)\mathbb{1}_{(x,+\infty)}(s)$ has at least i h_t-minima in $(0, x]$. Consequently, $\tau_i(h_t)$ is a stopping time for the σ-field $\sigma(W_\kappa(s), s \in (-\infty, x])$, $x \ge 0$. Hence the strong Markov property gives **(iii)**. □

We now introduce

$$\tau_1^*(h) := \inf\{u \ge 0,\ W_\kappa(u) - \inf_{[0,u]} W_\kappa \ge h\}, \qquad h \ge 0. \tag{2.5}$$

We gather here some other results proved by Faggionato [25] that will be useful in the following:

Fact 2.2 (Faggionato [25])**.** *The random variables* $W_\kappa(M_i) - W_\kappa(m_{i+1}) - h_t$ *and* $W_\kappa(M_i) - W_\kappa(m_i) - h_t$, $i \ge 1$, *are independent. They are exponential variables of mean respectively* $(2/\kappa)\sinh(\kappa h_t/2)e^{\kappa h_t/2}$ *and* $(2/\kappa)\sinh(\kappa h_t/2)e^{-\kappa h_t/2}$. *Moreover for* $i \ge 1$,

$$E\big(e^{-\alpha(m_{i+1}-M_i)}\big) = \frac{e^{-\kappa h_t/2}\sqrt{2\alpha+\kappa^2/4}}{\sqrt{2\alpha+\kappa^2/4}\cosh(h_t\sqrt{2\alpha+\kappa^2/4}) - \frac{\kappa}{2}\sinh(h_t\sqrt{2\alpha+\kappa^2/4})}, \quad \alpha > 0, \tag{2.6}$$

$$E\left(e^{-\alpha(\tau_i(h)-m_i)}\right) = \frac{2\sqrt{2\alpha+\kappa^2/4}\sinh(\kappa h/2)}{\kappa\sinh(h\sqrt{2\alpha+\kappa^2/4})}, \qquad 0 < h \le h_t, \qquad \alpha > -\kappa^2/8, \tag{2.7}$$

$$P(0 < M_0 < m_1) = P(0 \in [m_0, M_0)) \sim_{t\to+\infty} \kappa h_t e^{-\kappa h_t}. \tag{2.8}$$

Also, the random variable $-\inf_{[0,\tau_1^*(h)]} W_\kappa$ *is, for* $h > 0$, *exponentially distributed with mean* $2\kappa^{-1}\sinh(\kappa h/2)e^{\kappa h/2}$. *So if* $0 < \kappa < 1$, *for large* h *for all* $x > 0$,

$$P\big(\inf_{[0,\tau_1^*(h)]} W_\kappa \ge -x\big) \le e^{-\kappa h}x. \tag{2.9}$$

Finally,

$$E\big(\tau_1^*(h)\big) \leq C_+ e^{\kappa h}, \qquad h>0, \tag{2.10}$$

Proof of Fact 2.2. All this is proved in Faggionato [25] with $\mu=\kappa/2$, so we just explain where these results are stated in this paper [25].

Thanks to ([25] Thm. 1 and the remark before its (2.26)), $(W_\kappa(M_i+x)-W_\kappa(M_i),\ 0\leq x\leq m_{i+1}-M_i)$ and $(W_\kappa(m_i+x)-W_\kappa(m_i),\ 0\leq x\leq M_i-m_i)$, $i\geq 1$, are independent, and their laws are respectively $P_-^{\kappa/2}$ and $P_+^{\kappa/2}$, which are defined in ([25] p. 1769), with $h=h_t$ in our case. In particular, the lengths $m_{i+1}-M_i$ and M_i-m_i of these truncated and translated trajectories, called h_t-slopes in [25], and their excess heights $W_\kappa(M_i)-W_\kappa(m_{i+1})-h_t$ and $W_\kappa(M_i)-W_\kappa(m_i)-h_t$ are denoted respectively by ℓ_-, ℓ_+, ζ_- and ζ_+ in ([25], Prop. 1, formulas (2.11), and (2.14) with $\lambda=0$, see also (2.1)), which provides their law, and in particular this proves the present fact up to (2.6).

For $0<h\leq h_t$, m_i is a h-minimum, so for $i\geq 1$, by ([25], Thm. 1), $(W_\kappa(x+m_i)-W_\kappa(m_i),\ 0\leq x\leq \tau_i(h)-m_i)$ has the law defined as $P_+^{\kappa/2}$ for this h in ([25], eq. (2.9)). Hence $\tau_i(h)-m_i$ has the same law as the r.v. called $\tau-\sigma$ in ([25] eq. (2.9), see also (2.2)). Its Laplace transform is given in ([25] eq. (2.3) of Lem. 1). This gives (2.7).

Now, $0<M_0<m_1$ if and only if $0\in[m_0,M_0)$, that is, the translated trajectory between the two consecutive h_t-extrema surrounding 0 belongs to the set of slopes starting at an h_t-minimum and ending at an h_t-maximum, denoted by $\mathcal{W}_+$ in ([25] after eq. (2.10)). The probability of this event is given in ([25], Thm. 1 and eq. (2.25) in the case $\mathcal{W}_+$, $h=h_t$), which leads to (2.8).

We turn to (2.10). For $h>0$, $\tau_1^*(h)$ is denoted by τ in ([25], eq (2.2)). Let ℓ_- and ℓ_+ be as in ([25], Prop. 1), that is, $\ell_+\overset{\mathcal{L}}{=}\tau-\sigma+\sigma'\geq\tau-\sigma$ by ([25], eq. (2.9)) and $\ell_-\overset{\mathcal{L}}{=}\tau'-\sigma'+\sigma\geq\sigma$ by ([25], eq. (2.10)), where the values of $\sigma'\geq 0$ and $\tau'-\sigma'\geq 0$ are not important here. Applying ([25], eq. (2.17)) gives $E\big(\tau_1^*(h)\big)=E(\tau)=E[(\tau-\sigma)+\sigma]\leq E(\ell_-+\ell_+)\leq 2\kappa^{-2}e^{\kappa h}$.

Finally, $\inf_{[0,\tau_1^*(h)]}W_\kappa$ is denoted by β in ([25], eq. (2.2)); its law is given by ([25], eq. (2.4) in its Lem. 1), which leads to our (2.9). □

2.3 Definition of valleys, and standard h_t-minima $\tilde{m}_i$, $i\in\mathbb{N}^*$

We would like to consider some valleys of the potential around the h_t-minima m_i, $i\in\mathbb{N}^*$, which would be intervals containing at least $[\tau_i^-((1+\kappa)h_t),M_i]$. However, these valleys could intersect. In order to define valleys which are well separated and i.i.d., we introduce the following notation. This notation is later used to define valleys of the potential around some $\tilde{m}_i$, which are shown in Lemma 2.3 to be equal to the m_i for $1\leq i\leq n_t$ with large probability.
Let

$$h_t^+:=(1+\kappa+2\delta)h_t.$$

We define $\tilde{L}_0^+:=0$, $\tilde{m}_0:=0$, and recursively for $i\geq 1$ (see Figure 1),

$$\begin{aligned}
\tilde{L}_i^\sharp &:= \inf\{x>\tilde{L}_{i-1}^+,\ W_\kappa(x)\leq W_\kappa(\tilde{L}_{i-1}^+)-h_t^+\},\\
\tilde{\tau}_i(h_t) &:= \inf\big\{x\geq\tilde{L}_i^\sharp,\ W_\kappa(x)-\inf\nolimits_{[\tilde{L}_i^\sharp,x]}W_\kappa\geq h_t\big\},\\
\tilde{m}_i &:= \inf\big\{x\geq\tilde{L}_i^\sharp,\ W_\kappa(x)=\inf\nolimits_{[\tilde{L}_i^\sharp,\tilde{\tau}_i(h_t)]}W_\kappa\big\},\\
\tilde{L}_i^+ &:= \inf\{x>\tilde{\tau}_i(h_t),\ W_\kappa(x)\leq W_\kappa(\tilde{\tau}_i(h_t))-h_t-h_t^+\}.
\end{aligned} \tag{2.11}$$

We also introduce the following random variables for $i\in\mathbb{N}^*$:

$$\begin{aligned}
\tilde{M}_i &:= \inf\{s > \tilde{m}_i,\ W_\kappa(s) = \max_{\tilde{m}_i \le u \le \tilde{L}_i^+} W_\kappa(u)\},\\
\tilde{L}_i &:= \inf\{x > \tilde{\tau}_i(h_t),\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = h_t/2\}, && (2.12)\\
\tilde{\tau}_i(h) &:= \inf\{s > \tilde{m}_i,\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \qquad h > 0, && (2.13)\\
\tilde{\tau}_i^-(h) &:= \sup\{s < \tilde{m}_i,\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \qquad h > 0,\\
\tilde{L}_i^- &:= \tilde{\tau}_i^-(h_t^+).
\end{aligned}$$

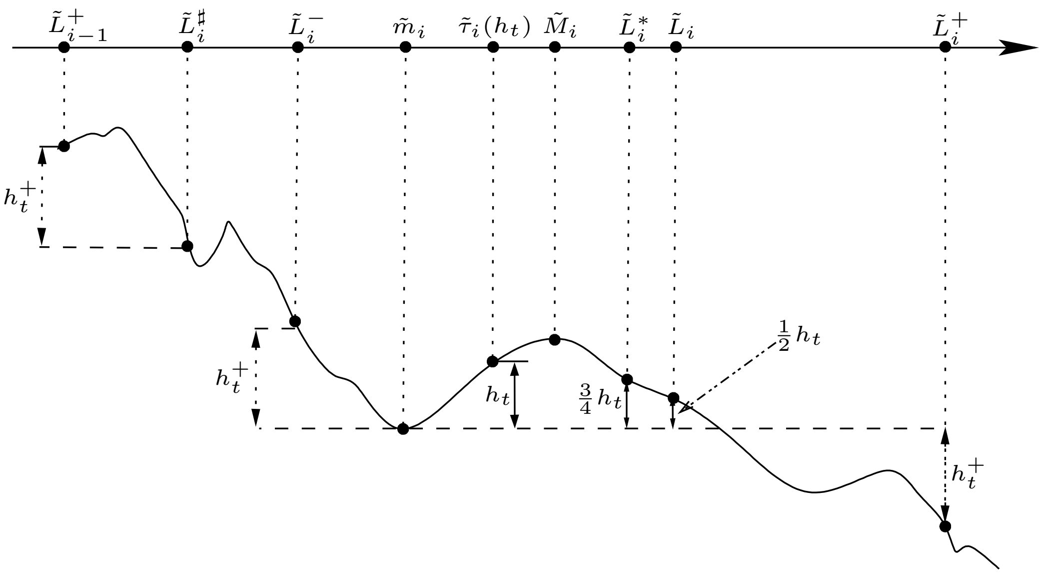

Figure 1: Schema of the potential between $\tilde{L}_{i-1}^+$ and $\tilde{L}_i^+$, in the case $\tilde{L}_i^\sharp < \tilde{L}_i^-$

We stress that these r.v. depend on t, which we do not write as a subscript to simplify the notation. Notice also that $\tilde{\tau}_i(h_t)$ is the same in definitions (2.11) and (2.13) with $h = h_t$. Moreover by continuity of W_κ, $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$. So, the $\tilde{m}_i$, $i \in \mathbb{N}^*$, are h_t-minima, since $W_\kappa(\tilde{m}_i) = \inf_{[\tilde{L}_{i-1}^+, \tilde{\tau}_i(h_t)]} W_\kappa$, $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$ and $W_\kappa(\tilde{L}_{i-1}^+) \ge W_\kappa(\tilde{m}_i) + h_t$. Moreover,

$$\tilde{L}_{i-1}^+ < \tilde{L}_i^\sharp \le \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{L}_i < \tilde{L}_i^+, \qquad i \in \mathbb{N}^*, \tag{2.14}$$

$$\tilde{L}_{i-1}^+ \le \tilde{L}_i^- < \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{M}_i < \tilde{L}_i^+, \qquad i \in \mathbb{N}^*. \tag{2.15}$$

Furthermore by induction the r.v. $\tilde{L}_i^\sharp$, $\tilde{\tau}_i(h_t)$ and $\tilde{L}_i^+$, $i \in \mathbb{N}^*$ are stopping times for the natural filtration of $(W_\kappa(x),\ x \ge 0)$, and so $\tilde{L}_i$, $i \in \mathbb{N}^*$, are also stopping times. Also by induction,

$$W_\kappa(\tilde{L}_i^\sharp) = \inf_{[0,\tilde{L}_i^\sharp]} W_\kappa, \qquad W_\kappa(\tilde{m}_i) = \inf_{[0,\tilde{\tau}_i(h_t)]} W_\kappa, \qquad W_\kappa(\tilde{L}_i^+) = \inf_{[0,\tilde{L}_i^+]} W_\kappa = W_\kappa(\tilde{m}_i) - h_t^+, \tag{2.16}$$

for $i \in \mathbb{N}^*$. We also introduce the analogue of $V^{(i)}$ for $\tilde{m}_i$ as follows:

$$\tilde{V}^{(i)}(x) := W_\kappa(x) - W_\kappa(\tilde{m}_i), \qquad x \in \mathbb{R},\ i \in \mathbb{N}^*.$$

We call i *th valley* the translated truncated potential $\left(\tilde{V}^{(i)}(x),\ \tilde{L}_i^- \le x \le \tilde{L}_i\right)$, for $i \ge 1$.

We show in the following lemma that, with an overwhelming high probability, the first $n_t + 1$ positive h_t-minima m_i, $1 \le i \le n_t + 1$, coincide with the r.v. $\tilde{m}_i$, $1 \le i \le n_t + 1$. We introduce the corresponding event $\mathcal{V}_t := \cap_{i=1}^{n_t+1} \{m_i = \tilde{m}_i\}$.

Lemma 2.3. *Assume* $0<\delta<1$. *For* t *large enough,*

$$P\left(\overline{\mathcal{V}}_t\right)\leq C_+ n_t e^{-\kappa h_t/2}=e^{[-\kappa/2+o(1)]h_t}.$$

Moreover, the sequence $\Big(\big(\tilde V^{(i)}(x+\tilde L_{i-1}^+),\ 0\leq x\leq \tilde L_i^+-\tilde L_{i-1}^+\big),\ i\geq 1\Big)$*, is i.i.d.*

We notice that consequently, the valleys $\big(\tilde V^{(i)}(x),\ \tilde L_i^-\leq x\leq \tilde L_i\big)$, $i\geq 1$, are also i.i.d.

Proof of Lemma 2.3. Since $\tilde m_i$ is an h_t-minimum for W_κ for every $i\geq 1$, we have $\{\tilde m_i,\ i\in\mathbb{N}^*\}\subset\{m_i, i\in\mathbb{N}^*\}$. We now assume that we are on the complement $\overline{\mathcal{V}}_t$ of $\mathcal{V}_t$. So the smallest $j\geq 1$ such that $m_j\neq\tilde m_j$ satisfies $1\leq j\leq n_t+1$. Due to the previous inclusion and since $\tilde m_0=0$, we have $\tilde m_{j-1}<m_j<\tilde m_j$. If for this j, $\tilde L_j^\sharp<m_j<\tilde m_j$, there would be a $v>m_j>\tilde L_j^\sharp$ such that

$$\begin{aligned}W_\kappa(m_j) &= \inf_{[m_j,v]}W_\kappa\geq\inf_{[\tilde L_j^\sharp,v]}W_\kappa,\\ W_\kappa(v) &\geq W_\kappa(m_j)+h_t\geq \inf_{[\tilde L_j^\sharp,v]}W_\kappa+h_t,\end{aligned}\tag{2.17}$$

since m_j is an h_t-minimum. The definition of $\tilde\tau_j(h_t)$ and (2.17) would give $\tilde\tau_j(h_t)\leq v$, and then $\tilde L_j^\sharp<m_j<\tilde m_j\leq\tilde\tau_j(h_t)\leq v$, by definition of $\tilde m_j$. Then $W_\kappa(m_j)=\inf_{[m_j,v]}W_\kappa\leq W_\kappa(\tilde m_j)$, which contradicts the definition of $\tilde m_j$.

Hence, $\tilde m_{j-1}<m_j\leq\tilde L_j^\sharp$. Thus by (2.16), $W_\kappa(m_j)\geq W_\kappa\big(\tilde L_j^\sharp\big)=W_\kappa\big(\tilde L_{j-1}^+\big)-h_t^+=W_\kappa(\tilde m_{j-1})-2h_t^+$ if $j\geq 2$, whereas $W_\kappa(m_j)\geq W_\kappa\big(\tilde L_1^\sharp\big)=-h_t^+$ if $j=1$. Consequently,

$$\overline{\mathcal{V}}_t\subset\{W_\kappa(m_1)\geq -h_t^+\}\cup\cup_{j=2}^{n_t+1}\{W_\kappa(m_j)\geq W_\kappa(m_{j-1})-2h_t^+\}.\tag{2.18}$$

For $j\geq 2$, we have by the start of Fact 2.2,

$$\begin{aligned}&P[W_\kappa(m_j)\geq W_\kappa(m_{j-1})-2h_t^+]\\ \leq\ & P\big[W_\kappa(M_{j-1})-W_\kappa(m_j)\leq e^{\kappa h_t/2}\big]+P\big[W_\kappa(M_{j-1})-W_\kappa(m_{j-1})>e^{\kappa h_t/2}-2h_t^+\big]\\ \leq\ & \frac{C_+}{e^{\kappa h_t/2}}.\end{aligned}\tag{2.19}$$

For $j=1$, notice that either $0<M_0<m_1$, which has probability

$$P(0<M_0<m_1)\leq 2h_t e^{-\kappa h_t}\tag{2.20}$$

for large t by (2.8), either there is no h_t-maximum in $(0,m_1]$, and so $M_0\leq 0$.

In this last case we show that $m_1\leq\tau_1^*(h_t)$, which we defined in (2.5). Indeed, assume $\tau_1^*(h_t)<m_1$. We consider $u=\inf\big\{x\geq 0,\ W_\kappa(x)=\inf_{[0,\tau_1^*(h_t)]}W_\kappa\big\}$ and $y=\inf\{x\geq 0,\ W_\kappa(x)=\sup_{[\tau_1^*(h_t),m_1]}W_\kappa\}$. It follows from the definition of h_t-extrema that $W_\kappa(m_1)=\inf_{[M_0,m_1]}W_\kappa$. Since $M_0\leq 0\leq u<\tau_1^*(h_t)<m_1$, this would give $W_\kappa(m_1)\leq W_\kappa(u)=W_\kappa[\tau_1^*(h_t)]-h_t\leq W_\kappa(y)-h_t$, and $W_\kappa(y)=\sup_{[u,m_1]}W_\kappa$, so y is an h_t-maximum and belongs to $(0,m_1]$, which contradicts this case. So $m_1\leq\tau_1^*(h_t)$.

Hence in this case, there exists $v>m_1$ such that $W_\kappa(v)\geq W_\kappa(m_1)+h_t$ and $W_\kappa(m_1)=\inf_{[m_1,v]}W_\kappa$, so $\tau_1^*(h_t)\leq v$. Thus, $W_\kappa(m_1)=\inf_{[m_1,\tau_1^*(h_t)]}W_\kappa$. This, $M_0\leq 0$ and $W_\kappa(m_1)=\inf_{[M_0,m_1]}W_\kappa$ yield $W_\kappa(m_1)=\inf_{[0,\tau_1^*(h_t)]}W_\kappa$. So, (2.9) and (2.20) give for large t,

$$P[W_\kappa(m_1)\geq -h_t^+]\leq 2h_te^{-\kappa h_t}+P\big[\inf_{[0,\tau_1^*(h_t)]}W_\kappa\geq -h_t^+\big]\leq C_+e^{-\kappa h_t/2}.$$

Hence, using (2.18) and (2.19), we get $P(\overline{\mathcal{V}}_t)\leq C_+n_te^{-\kappa h_t/2}$ for large t.

Finally, the fact that the sequence $\big(\tilde V^{(i)}(x+\tilde L_{i-1}^+),\ 0\leq x\leq\tilde L_i^+-\tilde L_{i-1}^+\big)$, $i\geq 1$, is i.i.d. follows from the strong Markov property applied at times $\tilde L_{i-1}^+$, which are stopping times. □

The following remark will be useful in the rest of the paper.

Remark 2.4. On $\mathcal{V}_t$, we have for every $1 \le i \le n_t$, $m_i = \tilde{m}_i$, and as a consequence, $\tilde{V}^{(i)}(x) = V^{(i)}(x)$, $x \in \mathbb{R}$, $\tau_i^-(h) = \tilde{\tau}_i^-(h)$ and $\tau_i(h) = \tilde{\tau}_i(h)$ for $h > 0$. Moreover, $\tilde{M}_i = M_i$. Indeed, $\tilde{M}_i$ is an h_t-maximum for W_κ, which belongs to $[\tilde{m}_i, \tilde{m}_{i+1}] = [m_i, m_{i+1}]$ on $\mathcal{V}_t$, and there is exactly one h_t-maximum in this interval since the h_t-maxima and minima alternate, which we defined as M_i, so $\tilde{M}_i = M_i$. So in the following, on $\mathcal{V}_t$, we can write these r.v. with or without tilde.

2.4 Some technical estimates related to the potential

We first provide estimates for the distance between some points of a given valley:

Lemma 2.5. *Assume $0 < \delta < 1/2$. For large t, for all $1 \le i \le n_t$,*

$$P\big[\tilde{m}_{i+1} - \tilde{M}_i \le e^{\kappa(1-\delta)h_t}, \mathcal{V}_t\big] \le P\big(m_{i+1} - M_i \le e^{\kappa(1-\delta)h_t}\big) \le C_+ e^{-\kappa\delta h_t}, \quad (2.21)$$

$$P\big[\tilde{\tau}_i(h) - \tilde{m}_i \ge 8h/\kappa\big] \le C_+ e^{-\kappa h/(2\sqrt{2})}, \qquad 0 \le h \le h_t, \quad (2.22)$$

$$P\big[\tilde{m}_i - \tilde{\tau}_i^-(h) \ge 8h/\kappa\big] \le C_+ e^{-\kappa h/(2\sqrt{2})}, \qquad 0 \le h \le h_t. \quad (2.23)$$

Proof of Lemma 2.5. First, it follows from Remark 2.4 that $\{\tilde{m}_{i+1} - \tilde{M}_i \le e^{\kappa(1-\delta)h_t}\} \cap \mathcal{V}_t \subset \{m_{i+1} - M_i \le e^{\kappa(1-\delta)h_t}\}$ for $1 \le i \le n_t$. This gives the first inequality of (2.21), whereas for the second one we can use the results of Faggionato [25] that we have gathered in Facts 2.1 and 2.2, and then BES$(3, \kappa/2)$ processes. This technic is used several times in this proof and later in the paper.
Let $i \ge 1$. We apply (2.6) with $\alpha(t) = e^{-\kappa(1-\delta)h_t}$ so that $E(e^{-\alpha(t)(m_{i+1}-M_i)}) \sim_{t\to+\infty} \kappa^2 e^{-\kappa\delta h_t}/2$. This and Markov inequality yield for large t,

$$P\big(m_{i+1} - M_i \le e^{\kappa(1-\delta)h_t}\big) = P\big(e^{-\alpha(t)(m_{i+1}-M_i)} \ge e^{-1}\big) \le C_+ e^{-\kappa\delta h_t}.$$

This ends the proof of (2.21).

Applying (2.7) with $\alpha = -\kappa^2/16$ gives

$$E\left(e^{\kappa^2(\tau_i(h)-m_i)/16}\right) = \frac{\sinh(\kappa h/2)}{\sqrt{2}\sinh(\kappa h/\sqrt{8})} \sim_{h\to+\infty} \frac{1}{\sqrt{2}} e^{\kappa(1-1/\sqrt{2})h/2}. \quad (2.24)$$

So, choosing C_+ large enough, LHS of (2.24) $\le C_+ e^{\kappa(1-1/\sqrt{2})h/2}$ for all $0 \le h \le h_t$, where LHS means left hand side. Thus,

$$P\big(\tau_i(h) - m_i \ge 8h/\kappa\big) = P\big(e^{(\kappa^2/16)[\tau_i(h)-m_i]} \ge e^{\kappa h/2}\big) \le C_+ e^{-\kappa h/(2\sqrt{2})}, \quad (2.25)$$

for all $0 \le h \le h_t$ by Markov inequality. Since $\tilde{\tau}_i(h) - \tilde{m}_i = \tau_i(h) - m_i$ on $\mathcal{V}_t$ by Remark 2.4, and $P\big(\overline{\mathcal{V}_t}\big) \le e^{-\kappa h_t/(2\sqrt{2})} \le e^{-\kappa h/(2\sqrt{2})}$ for large t for all $0 \le h \le h_t$ by Lemma 2.3,

$$P\big(\tilde{\tau}_i(h) - \tilde{m}_i \ge 8h/\kappa\big) \le P\big(\tau_i(h) - m_i \ge 8h/\kappa, \mathcal{V}_t\big) + P\big(\overline{\mathcal{V}_t}\big) \le C_+ e^{-\kappa h/(2\sqrt{2})}, \qquad 0 \le h \le h_t,$$

which gives (2.22). According to Fact 2.1 **(ii)**, $m_i - \tau_i^-(h)$ and $\tau_i(h) - m_i$ are equal in law for $i \ge 2$ and $0 \le h \le h_t$. So we can replace $\tau_i(h) - m_i$ by $m_i - \tau_i^-(h)$ in (2.25), which similarly as before gives (2.23) for $i \ge 2$. Finally, $\tilde{m}_1 - \tilde{\tau}_1^-(h)$ has the same law as $\tilde{m}_2 - \tilde{\tau}_2^-(h)$ for $0 \le h \le h_t$ by Lemma 2.3, so the case $i = 1$ of (2.23) follows from the case $i = 2$. □

We also recall that a scale function of W_ζ $(x\mapsto W(x)-\zeta x/2)$ is for $\zeta\neq 0$ (see e.g. [25], (5.1)),

$$\begin{aligned} s_\zeta(u) &:= e^{\zeta u}-1=2e^{\zeta u/2}\sinh(\zeta u/2), \qquad u\in\mathbb{R},\\ P\big[\tau^{W_\zeta^y}(z)<\tau^{W_\zeta^y}(x)\big] &= s_\zeta(y-x)/s_\zeta(z-x), \qquad x<y<z, \end{aligned} \tag{2.26}$$

for which we remind that W_ζ^y denotes a $(-\zeta/2)$-drifted Brownian motion starting from y.

We also need some estimates on hitting times by W_κ and by a process R with law $\text{BES}(3,\kappa/2)$ (recall that in (2.27) below, $P^{\alpha h}$ denotes the law of R starting from αh, that is, with law $\text{BES}^{\alpha h}(3,\kappa/2)$):

Lemma 2.6. *Let $0<\gamma<\alpha<\omega$, and R having law $\text{BES}(3,\kappa/2)$. For all h large enough,*

$$P^{\alpha h}\left(\tau^R(\gamma h)<\tau^R(\omega h)\right) \leq 2\exp[-\kappa(\alpha-\gamma)h], \tag{2.27}$$

$$P\left(\tau^R(\omega h)-\tau^R(\alpha h)\leq 1\right) \leq 4\exp\big[-[(\omega-\alpha)h]^2/3\big], \tag{2.28}$$

$$P\bigg(\int_0^{\tau^R(h)}e^{R(u)}\mathrm{d}u\geq e^{(1-\alpha)h}\bigg) \geq 1-3\exp[-\kappa\alpha h/2], \qquad 0<\alpha<1, \tag{2.29}$$

$$P\big(\tau^R(h)>8h/\kappa\big) \leq C_+\exp\big[-\kappa h/(2\sqrt{2})\big]. \tag{2.30}$$

Proof of Lemma 2.6. We recall that R has the same law as the $(\kappa/2)$-drifted Brownian motion $W_{-\kappa}$ Doob-conditioned to stay positive. In particular, this proof relies on (2.3) and (2.4). First, using (2.3) and (2.26), we have for $0<\gamma<\alpha<\omega$,

$$\begin{aligned} &\text{LHS of (2.27)}\\ =\ & P\big[\tau^{W_{-\kappa}^{\alpha h}}(\gamma h)<\tau^{W_{-\kappa}^{\alpha h}}(\omega h)\mid \tau^{W_{-\kappa}^{\alpha h}}(\infty)<\tau^{W_{-\kappa}^{\alpha h}}(0)\big]\\ =\ & P\big[\tau^{W_{-\kappa}^{\alpha h}}(\gamma h)<\tau^{W_{-\kappa}^{\alpha h}}(\omega h)\big]P\big[\tau^{W_{-\kappa}^{\gamma h}}(\infty)<\tau^{W_{-\kappa}^{\gamma h}}(0)\big]/P\big[\tau^{W_{-\kappa}^{\alpha h}}(\infty)<\tau^{W_{-\kappa}^{\alpha h}}(0)\big]\\ =\ & \left(1-\frac{s_{-\kappa}[(\alpha-\gamma)h]}{s_{-\kappa}[(\omega-\gamma)h]}\right)\frac{s_{-\kappa}(\gamma h)}{s_{-\kappa}(\alpha h)}\\ =\ & \frac{\sinh[\kappa(\omega-\alpha)h/2]\sinh(\kappa\gamma h/2)}{\sinh[\kappa(\omega-\gamma)h/2]\sinh(\kappa\alpha h/2)}, \end{aligned} \tag{2.31}$$

This gives (2.27) for large h.

Now, we notice that the left hand side of (2.28) is thanks to (2.4) for $0<\alpha<\omega$,

$$\begin{aligned} &\lim_{z\downarrow 0}P\big[\tau^{W_{-\kappa}^z}(\omega h)-\tau^{W_{-\kappa}^z}(\alpha h)\leq 1\mid \tau^{W_{-\kappa}^z}(0)=\infty\big]\\ =\ & \lim_{z\downarrow 0}P\big[\tau^{W_{-\kappa}^z}(\alpha h)<\tau^{W_{-\kappa}^z}(0)\big]P\big[\tau^{W_{-\kappa}^{\alpha h}}(\omega h)\leq 1,\tau^{W_{-\kappa}^{\alpha h}}(0)=\infty\big]/P\big[\tau^{W_{-\kappa}^z}(0)=\infty\big], \end{aligned}$$

where we applied the strong Markov property at time $\tau^{W_{-\kappa}^z}(\alpha h)$. Moreover for large h,

$$\begin{aligned} P\big[\tau^{W_{-\kappa}^{\alpha h}}(\omega h)\leq 1\big] &= P\bigg(\sup_{x\in[0,1]}\big(W(x)+\alpha h+\kappa x/2\big)\geq\omega h\bigg)\leq P\bigg(\sup_{[0,1]}W\geq(\omega-\alpha)h-\frac{\kappa}{2}\bigg)\\ &\leq 2\exp\big[-[(\omega-\alpha)h]^2/3\big], \end{aligned}$$

because $\sup_{[0,1]}W\overset{\mathcal{L}}{=}|W(1)|$, where $\overset{\mathcal{L}}{=}$ denotes equality in law, and $P(|W(1)|\geq x)\leq 2e^{-x^2/2}$ for $x\geq 1$. Since $\lim_{z\downarrow 0}P\big[\tau^{W_{-\kappa}^z}(\alpha h)<\tau^{W_{-\kappa}^z}(0)\big]/P\big[\tau^{W_{-\kappa}^z}(0)=\infty\big]=(1-e^{-\kappa\alpha h})^{-1}\leq 2$ for large h by (2.26), we get (2.28).

To prove (2.29), let $0<\alpha<1$. Notice that $\big\{\inf_{\tau^R[(1-\alpha/2)h]\leq u\leq\tau^R(h)} R(u)\geq(1-\alpha)h\big\}\cap\big\{\tau^R(h)-\tau^R[(1-\alpha/2)h]\geq 1\big\}$ has probability at least $1-3e^{-\kappa\alpha h/2}$ for large h by (2.27) and (2.28). Moreover, we have on this event,

$$\int_0^{\tau^R(h)} e^{R(u)}\mathrm{d}u\geq\int_{\tau^R[(1-\alpha/2)h]}^{\tau^R(h)} e^{R(u)}\mathrm{d}u\geq\big[\tau^R(h)-\tau^R[(1-\alpha/2)h]\big]e^{(1-\alpha)h}\geq e^{(1-\alpha)h},$$

which proves (2.29). Finally for $0\leq h\leq h_t$, $\tau^R(h)$ is equal in law to $\tau_1(h)-m_1$ by Fact 2.1 **(ii)**, so (2.30) follows from (2.25) □

We also provide some probabilities concerning $\big(\tilde V^{(i)}(x),\ \tilde\tau_i^-(h_t^+)\leq x\leq\tilde\tau_i^-(h_t)\big)$ in the following lemma. Unfortunately, they cannot be evaluated directly with the help of Fact 2.1 and $\mathrm{BES}(3,\kappa/2)$ processes, so we evaluate them with more elementary technics. The proof of this lemma is deferred to Section 6.

Lemma 2.7. *With probability P at least $1-e^{-\kappa h_t/8}$ for large t, we have for every $1\leq i\leq n_t$,*

$$\tilde m_i-\tilde L_i^-\leq\tilde L_i^+-\tilde L_i^-=\tilde L_i^+-\tilde\tau_i^-(h_t^+)\quad\leq\quad 40h_t^+/\kappa,\tag{2.32}$$

$$\int_{\tilde L_i^-}^{\tilde m_i} e^{\tilde V^{(i)}(u)}\mathrm{d}u\quad\geq\quad e^{h_t^+-\kappa h_t/2},\tag{2.33}$$

$$\inf_{[\tilde\tau_i^-(h_t^+),\tilde\tau_i^-(h_t)]}\tilde V^{(i)}\quad\geq\quad h_t/2.\tag{2.34}$$

We end this section with the following basic result and its proof:

Lemma 2.8. *Let $0<\alpha<\omega$. We have for large h,*

$$P\big[\tau^{W_\kappa}(-\alpha h)\geq 2\omega h/\kappa\big]\leq\exp\big[-\kappa(\omega-\alpha)^2h/(4\omega)\big].\tag{2.35}$$

Proof of Lemma 2.8. We have,

$$P\big[\tau^{W_\kappa}(-\alpha h)\geq 2\omega h/\kappa\big]\leq P\big[W(2\omega h/\kappa)\geq(\omega-\alpha)h\big]=P\big[W(1)\geq\sqrt{\kappa h}(\omega-\alpha)/\sqrt{2\omega}\big].$$

Since $P(W(1)\geq x)\leq e^{-x^2/2}$ for $x\geq 1$, this proves the lemma. □

3 Quasi-Independence in the trajectories of X

We now assume that $0<\kappa<1$. In this section we provide some information on the typical trajectories of X. We also show that the times spent in the different valleys are, asymptotically in t, nearly independent under the annealed probability $\mathbb{P}$ (see Proposition 3.4). Then we prove that the time spent by X between the valleys is negligible.

We start with some classical formulas about the diffusion X, its hitting times and local times, which will be important in the rest of the paper.

3.1 Some formulas related to the diffusion X

We first introduce

$$A(r):=\int_0^r e^{W_\kappa(x)}\mathrm{d}x,\qquad r\in\mathbb{R},\qquad A_\infty:=\int_0^\infty e^{W_\kappa(x)}\mathrm{d}x<\infty\ a.s.\tag{3.1}$$

We recall that A is a scale function of X under $\mathbb{P}^{W_\kappa}$ (see e.g. [43] formula (2.2)), that is,

$$\mathbb{P}_y^{W_\kappa}\big[H(z)<H(x)\big]\quad=\quad[A(y)-A(x)]/[A(z)-A(x)],\qquad x<y<z.\tag{3.2}$$

Here $\mathbb{P}_y^{W_\kappa}$ denotes the law of the diffusion X in the potential W_κ, starting from y instead of 0, conditionally on W_κ, and $\mathbb{E}_y^{W_\kappa}$ is the corresponding expectation. As in (Brox [10], eq. (1.1) or Shi [42], eq. (2.2)), there exists a Brownian motion $(B(s),\ s\geq 0)$, independent of W_κ, such that $X(t)=A^{-1}[B(T^{-1}(t))]$ for every $t\geq 0$, where

$$T(r):=\int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\}\mathrm{d}s,\qquad 0\leq r<\tau^B(A_\infty). \tag{3.3}$$

As a consequence, with this notation, we have (see e.g. Shi [43] eq. (4.5) and (4.6)),

$$H(r)=T[\tau^B(A(r))]=\int_{-\infty}^r \exp[-W_\kappa(u)]\mathcal{L}_B[\tau^B(A(r)),A(u)]\mathrm{d}u,\qquad r\geq 0. \tag{3.4}$$

Moreover the local time of X is $\mathcal{L}_X(t,x)=e^{-W_\kappa(x)}\mathcal{L}_B[T^{-1}(t),A(x)]$, $t\geq 0$, $x\in\mathbb{R}$, as proved in Shi ([42], eq. (2.5)). So,

$$\mathcal{L}_X(H(r),x)=e^{-W_\kappa(x)}\mathcal{L}_B[\tau^B(A(r)),A(x)]\qquad r\geq 0,\ x\in\mathbb{R}. \tag{3.5}$$

It will sometimes be useful to notice that $H(r)=H_-(r)+H_+(r)$, where

$$H_-(r):=\int_{-\infty}^0 \mathcal{L}_X(H(r),x)\mathrm{d}x,\qquad H_+(r):=\int_0^r \mathcal{L}_X(H(r),x)\mathrm{d}x,\qquad r\geq 0, \tag{3.6}$$

are the time spent by X respectively in $\mathbb{R}_-$ and in $\mathbb{R}_+$ before it first hits r. We will sometimes need the following result (see Dufresne [20], or Borodin et al. [7] IV.48 p. 78):

Fact 3.1 (Dufresne)**.** *The r.v.* $2/A_\infty$ *is a gamma variable of parameter* $(\kappa,1)$*, and so has a density equal to* $e^{-x}x^{\kappa-1}\mathbb{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$*.*

3.2 Probability that the diffusion X leaves the valleys on the right

In this subsection, we prove that for most environments, with a large quenched probability, the diffusion X, after first hitting $\tilde{m}_i$, leaves the valley $[\tilde{L}_i^-,\tilde{L}_i]$ on its right, for every $1\leq i\leq n_t$. More precisely, we introduce $H_{x\to y}:=\inf\{s>H(x),\ X(s)=y\}-H(x)$ for $(x,y)\in\mathbb{R}_+^2$, which is equal to $H(y)-H(x)$ if $x<y$. We also introduce

$$U_i:=H(\tilde{L}_i)-H(\tilde{m}_i)=H_{\tilde{m}_i\to\tilde{L}_i},\qquad \mathcal{E}_i:=\big\{U_i<H_{\tilde{m}_i\to\tilde{L}_i^-}\big\},\qquad i\geq 1. \tag{3.7}$$

We have (notice that $n_te^{-\kappa\delta h_t}=o(1)$ since $\phi(t)=o(\log t)$),

Lemma 3.2. *Assume* $0<\delta<1/8$*. We have, for large* t*,*

$$P\left[\bigcap_{i=1}^{n_t}\Big\{\mathbb{P}_{\tilde{m}_i}^{W_\kappa}\big[H\big(\tilde{L}_i\big)>H\big(\tilde{L}_i^-\big)\big]=\mathbb{P}^{W_\kappa}\big(\overline{\mathcal{E}_i}\big)\leq e^{-\kappa h_t/2}\Big\}\right]\geq 1-C_+n_te^{-\kappa\delta h_t}.$$

Proof of Lemma 3.2. By the strong Markov property and then by (3.2), we have for all $1\leq i\leq n_t$,

$$\mathbb{P}^{W_\kappa}(\overline{\mathcal{E}_i})=\mathbb{P}_{\tilde{m}_i}^{W_\kappa}\Big(H(\tilde{L}_i)>H(\tilde{L}_i^-)\Big)=\Big(\int_{\tilde{m}_i}^{\tilde{L}_i}e^{\tilde{V}^{(i)}(x)}\mathrm{d}x\Big)\Big(\int_{\tilde{L}_i^-}^{\tilde{L}_i}e^{\tilde{V}^{(i)}(x)}\mathrm{d}x\Big)^{-1}\leq Q_i/D_i, \tag{3.8}$$

where, since $\sup_{[\tilde{m}_i,\tilde{L}_i]}\tilde{V}^{(i)}\leq\sup_{[\tilde{m}_i,\tilde{L}_i^+]}\tilde{V}^{(i)}=\tilde{V}^{(i)}(\tilde{M}_i)$ and $\tilde{L}_i^-<\tilde{m}_i$,

$$Q_i:=(\tilde{L}_i-\tilde{L}_i^-)e^{\tilde{V}^{(i)}(\tilde{M}_i)},\qquad D_i:=\int_{\tilde{L}_i^-}^{\tilde{m}_i}e^{\tilde{V}^{(i)}(x)}\mathrm{d}x.$$

We start with the denominator D_i. By (2.33), we have for all $1 \leq i \leq n_t$, since $\delta < 1/8$,

$$P\big[D_i \geq e^{h_t^+ - \kappa h_t/2}\big] \geq 1 - e^{-\kappa h_t/8} \geq 1 - e^{-\kappa\delta h_t}. \tag{3.9}$$

We now consider the numerator Q_i for some $1 \leq i \leq n_t$. First by (2.32) and since $\tilde{L}_i < \tilde{L}_i^+$, we have $P(\tilde{L}_i - \tilde{L}_i^- > 40h_t^+/\kappa) \leq e^{-\kappa h_t/8}$. Moreover, since $\tilde{\tau}_i(h_t)$ is a stopping time, $(V^{(i)}[x + \tilde{\tau}_i(h_t)] - h_t,\ x \geq 0)$ is equal in law to $(W_\kappa(x),\ x \geq 0)$, so

$$\begin{aligned} P\big[\tilde{V}^{(i)}(\tilde{M}_i) > h_t(1+\delta)\big] &= P\bigg(\sup_{[\tilde{\tau}_i(h_t), \tilde{L}_i^+]} \big(\tilde{V}^{(i)} - h_t\big) \geq \delta h_t\bigg) \\ &\leq P\left(\sup_{s\geq 0} W_\kappa(s) > \delta h_t\right) \leq e^{-\delta\kappa h_t}, \end{aligned}$$

since $P[\sup_{[0,\infty)} W_\kappa \geq x] = P[\inf_{[0,\infty)} W_{-\kappa} \leq -x] = e^{-\kappa x}$, $x \geq 0$, e.g. by (2.26). Finally

$$P\big[Q_i \leq 40h_t^+\kappa^{-1}e^{(1+\delta)h_t}\big] \geq 1 - 2e^{-\delta\kappa h_t}$$

for $\delta < 1/8$ and t large enough. This combined with (3.8) and (3.9) gives for large t,

$$P\big[\mathbb{P}^{W_\kappa}(\overline{\mathcal{E}_i}) \leq C_+ h_t e^{(1+\delta)h_t - (h_t^+ - \kappa h_t/2)} \leq e^{-(\kappa/2)h_t}\big] \geq 1 - 3e^{-\kappa\delta h_t}. \tag{3.10}$$

This proves the lemma. □

3.3 Probability that the diffusion X goes back to $\tilde{m}_i$ after leaving the i-th valley

We introduce for $i \in \mathbb{N}^*$ (see Figure 1),

$$\tilde{L}_i^* := \inf\{x \geq \tilde{\tau}_i(h_t),\ W_\kappa(x) - W_\kappa(\tilde{m}_i) \leq 3h_t/4\} < \tilde{L}_i. \tag{3.11}$$

In the next lemma, we show that with a large probability, after hitting $\tilde{L}_i$, X hits $\tilde{m}_{i+1}$ before (maybe) going back to $\tilde{L}_i^*$ for $1 \leq i < n_t$. This is helpful in the proof of Lemma 3.7 and in Subsection 5.2.

Lemma 3.3. *We have for large t,*

$$P\left[\mathbb{P}^{W_\kappa}\big(\cap_{i=1}^{n_t-1}\big\{H_{\tilde{L}_i\to\tilde{m}_{i+1}} < H_{\tilde{L}_i\to\tilde{L}_i^*}\big\}\big) \geq 1 - 2n_t e^{-h_t/8}\right] \geq 1 - C_+ n_t e^{-\kappa h_t/16}. \tag{3.12}$$

This, combined with Lemma 3.2, gives a picture of the typical trajectories of X. In particular, with large probability, X visits successively the different valleys $[\tilde{L}_i^-, \tilde{L}_i]$, $1 \leq i < n_t$; it exits each one on its right $\tilde{L}_i$, then does not go back to $\tilde{L}_i^*$ and then also does not go back to $\tilde{m}_i$, but goes to $\tilde{m}_{i+1}$. That is, with a large probability, the diffusion X hits successively each h_t-minimum m_i, $1 \leq i \leq n_t$ and does not come back to the previously visited m_j.

Proof of Lemma 3.3. We define $A_\infty^x := \int_x^\infty e^{W_\kappa(u) - W_\kappa(x)}\mathrm{d}u$, $x \in \mathbb{R}$. Let

$$\begin{aligned} \mathcal{E}_1^{3.3} &:= \cap_{i=1}^{n_t-1}\{H_{\tilde{L}_i\to\tilde{\tau}_{i+1}(h_t)} < H_{\tilde{L}_i\to\tilde{L}_i^*}\} \subset \cap_{i=1}^{n_t-1}\big\{H_{\tilde{L}_i\to\tilde{m}_{i+1}} < H_{\tilde{L}_i\to\tilde{L}_i^*}\big\}, \quad (3.13) \\ \mathcal{E}_2^{3.3} &:= \cap_{i=1}^{n_t-1}\big\{A_\infty^{\tilde{L}_i} \leq e^{h_t/16},\ A_\infty^{\tilde{\tau}_{i+1}(h_t)} \leq e^{h_t/16},\ A_\infty^{\tilde{L}_i^*} \geq e^{-h_t/16}\big\}. \end{aligned}$$

Applying Fact 3.1 (Dufresne) we have $P(A_\infty \geq y) \leq C_+ y^{-\kappa}$ for $y > 0$, and $P(A_\infty \leq y) \leq e^{-1/y}$ for small $y > 0$. Moreover, since $\tilde{L}_i$, $\tilde{\tau}_{i+1}(h_t)$ and $\tilde{L}_i^*$ are stopping times for the natural filtration of W_κ, the r.v. $A_\infty^{\tilde{L}_i}$, $A_\infty^{\tilde{\tau}_{i+1}(h_t)}$ and $A_\infty^{\tilde{L}_i^*}$ have the same law as A_∞ under P by the strong Markov property. Consequently for large t,

$$P\big(\overline{\mathcal{E}_2^{3.3}}\big) \leq n_t\big(2C_+ e^{-\kappa h_t/16} + e^{-e^{h_t/16}}\big) \leq C_+ n_t e^{-\kappa h_t/16}. \tag{3.14}$$

Moreover, applying the strong Markov property and (3.2), we have on $\mathcal{E}_2^{3.3}$,

$$\mathbb{P}^{W_\kappa}\big[H_{\tilde{L}_i\to\tilde{\tau}_{i+1}(h_t)} > H_{\tilde{L}_i\to\tilde{L}_i^*}\big] = \mathbb{P}^{W_\kappa}_{\tilde{L}_i}\big[H(\tilde{\tau}_{i+1}(h_t)) > H(\tilde{L}_i^*)\big] = Q_i^*/D_i^*, \qquad 1\le i\le n_t-1,$$

where, recalling that $W_\kappa(\tilde{L}_i) = W_\kappa(\tilde{m}_i)+h_t/2$ and $W_\kappa(\tilde{L}_i^*) = W_\kappa(\tilde{m}_i)+3h_t/4$,

$$Q_i^* := \int_{\tilde{L}_i}^{\tilde{\tau}_{i+1}(h_t)} e^{W_\kappa(x)}\mathrm{d}x \le e^{W_\kappa(\tilde{L}_i)}A_\infty^{\tilde{L}_i} \le \exp(W_\kappa(\tilde{m}_i)+h_t/2+h_t/16), \quad (3.15)$$

$$D_i^* := \int_{\tilde{L}_i^*}^{\tilde{\tau}_{i+1}(h_t)} e^{W_\kappa(x)}\mathrm{d}x = e^{W_\kappa(\tilde{L}_i^*)}A_\infty^{\tilde{L}_i^*} - e^{W_\kappa(\tilde{\tau}_{i+1}(h_t))}A_\infty^{\tilde{\tau}_{i+1}(h_t)}$$

on $\mathcal{E}_2^{3.3}$. Moreover, $W_\kappa(\tilde{m}_{i+1})\le W_\kappa(\tilde{L}_{i+1}^\sharp)\le W_\kappa(\tilde{L}_i^+)\le W_\kappa(\tilde{m}_i)-h_t$, so on $\mathcal{E}_2^{3.3}$ for large t,

$$D_i^* \ge e^{W_\kappa(\tilde{m}_i)+11h_t/16} - e^{W_\kappa(\tilde{m}_{i+1})+h_t}e^{h_t/16} \ge e^{W_\kappa(\tilde{m}_i)}\big[e^{11h_t/16}-e^{h_t/16}\big] \ge e^{W_\kappa(\tilde{m}_i)+11h_t/16}/2$$

for all $1\le i<n_t$, so $Q_i^*/D_i^*\le 2e^{-h_t/8}$. Thus $P^{W_\kappa}\big(\overline{\mathcal{E}_1^{3.3}}\big)\mathbb{1}_{\mathcal{E}_2^{3.3}}\le 2n_te^{-h_t/8}$. This and (3.14) give

$$\mathbb{P}\big(\overline{\mathcal{E}_1^{3.3}}\big) \le \mathbb{P}\big(\overline{\mathcal{E}_1^{3.3}}\cap\mathcal{E}_2^{3.3}\big) + P\big(\overline{\mathcal{E}_2^{3.3}}\big) \le C_+n_te^{-\kappa h_t/16}, \quad (3.16)$$

which we need in the proof of Lemma 3.7. Moreover, we get

$$\text{LHS of (3.12)} \ge P\big(\mathbb{P}^{W_\kappa}(\mathcal{E}_1^{3.3})\ge 1-2n_te^{-h_t/8}\big) \ge P\big(\mathcal{E}_2^{3.3}\big) \ge 1-C_+n_te^{-\kappa h_t/16}.$$

This proves (3.12). □

3.4 Independence in a trajectory of X

We are interested in the law of $\mathbf{U}$, defined as follows:

$$\tilde{A}_i(z) := \int_{\tilde{m}_i}^{z} e^{\tilde{V}^{(i)}(u)}\mathrm{d}u, \qquad z\in\mathbb{R},\ i\in\mathbb{N}^*, \quad (3.17)$$

$$\mathbf{U} := \int_{\tilde{L}_2^-}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)}\mathcal{L}_B\big[\tau^B(\tilde{A}_2(\tilde{L}_2)),\tilde{A}_2(u)\big]\mathrm{d}u, \quad (3.18)$$

where $(B(s),\ s\ge 0)$ is a Brownian motion independent of W_κ and then of $\tilde{V}^{(2)}$. As explained below in (3.23), $\mathbf{U}$ is equal in law to the exit time of X from the valley $\big[\tilde{L}_2^-,\tilde{L}_2\big]$ under $\mathbb{P}^{W_\kappa}_{\tilde{m}_2}$ if X leaves this valley on its right. Notice that we have chosen the second valley in the definition of $\mathbf{U}$ because Fact 2.1 provides the law of $V^{(i)}$ near m_i for $i\ge 2$ but not for $i=1$. Moreover we stress that $\mathbf{U}$, as well as the r.v. U_i, $i\ge 1$, depend on t, since the $\tilde{m}_i$, $\tilde{L}_i^-$ and $\tilde{L}_i$ depend on h_t.

We now prove that the law of the sum $U_1+\dots+U_n$ (defined in (3.7)) can be approximated by the law of the sum of n independent copies of $\mathbf{U}$, in the following sense:

Proposition 3.4. *Assume $0<\delta<1/8$. There exists a constant $C_2>0$ such that for large t,*

$$\forall\lambda>0,\ \forall 1\le n\le n_t, \qquad \Big|\mathbb{E}\left(e^{-\lambda\sum_{i=1}^n U_i}\right) - \left[\mathbb{E}\left(e^{-\lambda\mathbf{U}}\right)\right]^n\Big| \le C_2n_te^{-\delta\kappa h_t}, \quad (3.19)$$

where $n_te^{-\delta\kappa h_t}=o(1)$ since $\phi(t)=o(\log t)$. Moreover for all $2\le n\le n_t$, $[a,b]\subset[0,1]$ and $\alpha>0$,

$$\left|\mathbb{P}\Big(\sum_{i=1}^{n-1}\frac{U_i}{t}\in[a,b],\sum_{i=1}^{n}\frac{U_i}{t}\ge\alpha\Big) - \int_a^b\mathbb{P}\Big(\sum_{i=1}^{n-1}\frac{U_i}{t}\in\mathrm{d}x\Big)\mathbb{P}(\mathbf{U}/t\ge\alpha-x)\right| \le C_+n_te^{-\kappa\delta h_t}. \quad (3.20)$$

Similarly for $n=1$, $|\mathbb{P}(U_1/t\ge\alpha)-\mathbb{P}(\mathbf{U}/t\ge\alpha)|\le C_+n_te^{-\kappa\delta h_t}$.

Proof of Proposition 3.4. We fix $0 < \delta < 1/8$ and $\lambda > 0$. We also introduce $\mathcal{G}_s := \sigma(X(u),\ 0 \leq u \leq s,\ W_\kappa(x),\ x \in \mathbb{R})$ for $s \geq 0$. For $n \in \mathbb{N}^*$ and $1 \leq i < n$, we have $\tilde{L}_i^- < \tilde{m}_i < \tilde{L}_i < \tilde{m}_n$ by (2.14) and (2.15), so U_i and $\mathbb{1}_{\mathcal{E}_i}$ (defined in (3.7)) are $\mathcal{G}_{H(\tilde{m}_n)}$-measurable. Hence for t and n such that $2 \leq n \leq n_t$,

$$\mathbb{E}^{W_\kappa}\left(e^{-\lambda \sum_{i=1}^n U_i} \prod_{i=1}^n \mathbb{1}_{\mathcal{E}_i}\right) = \mathbb{E}^{W_\kappa}\left(\mathbb{E}^{W_\kappa}\left(e^{-\lambda U_n} \mathbb{1}_{\mathcal{E}_n} \Big| \mathcal{G}_{H(\tilde{m}_n)}\right) e^{-\lambda \sum_{i=1}^{n-1} U_i} \prod_{i=1}^{n-1} \mathbb{1}_{\mathcal{E}_i}\right)$$
$$= \mathbb{E}^{W_\kappa}\left(\mathbb{E}^{W_\kappa}_{\tilde{m}_n}\left(e^{-\lambda H(\tilde{L}_n)} \mathbb{1}_{\{H(\tilde{L}_n) < H(\tilde{L}_n^-)\}}\right) e^{-\lambda \sum_{i=1}^{n-1} U_i} \prod_{i=1}^{n-1} \mathbb{1}_{\mathcal{E}_i}\right),$$

by the strong Markov property, applied at time $H(\tilde{m}_n)$ to X which is a Markov process under the quenched probability $\mathbb{P}^{W_\kappa}$. Hence we obtain by induction on n,

$$\mathbb{E}^{W_\kappa}\left(e^{-\lambda \sum_{i=1}^n U_i} \prod_{i=1}^n \mathbb{1}_{\mathcal{E}_i}\right) = \prod_{i=1}^n \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\left(e^{-\lambda H(\tilde{L}_i)} \mathbb{1}_{\{H(\tilde{L}_i) < H(\tilde{L}_i^-)\}}\right). \tag{3.21}$$

Under $\mathbb{P}^{W_\kappa}_{\tilde{m}_i}$, $(X(u) - \tilde{m}_i,\ u \geq 0)$ is a diffusion in the potential $W_\kappa(x + \tilde{m}_i) - W_\kappa(\tilde{m}_i) = \tilde{V}^{(i)}(x + \tilde{m}_i)$, $x \in \mathbb{R}$, and starting from 0. So, applying (3.1) and (3.4) with $\tilde{A}_i(. + \tilde{m}_i)$ instead of $A(.)$, there exists a Brownian motion $\big(\tilde{B}_i(s),\ s \geq 0\big)$, independent of $\tilde{V}^{(i)}$, such that the hitting time of $r \geq 0$ by $X - \tilde{m}_i$ is under $\mathbb{P}^{W_\kappa}_{\tilde{m}_i}$,

$$\int_{-\infty}^{r} e^{-\tilde{V}^{(i)}(x+\tilde{m}_i)} \mathcal{L}_{\tilde{B}_i}\big[\tau^{\tilde{B}_i}\big(\tilde{A}_i(r + \tilde{m}_i)\big), \tilde{A}_i(x + \tilde{m}_i)\big] \mathrm{d}x, \tag{3.22}$$

and is in fact $H(r+\tilde{m}_i)$, hitting time of $r+\tilde{m}_i$ by X. So under $\mathbb{P}^{W_\kappa}_{\tilde{m}_i}$ on $\big\{H\big(\tilde{L}_i\big) < H\big(\tilde{L}_i^-\big)\big\}$,

$$H\big(\tilde{L}_i\big) = \mathbf{U}_i := \int_{\tilde{L}_i^-}^{\tilde{L}_i} e^{-\tilde{V}^{(i)}(u)} \mathcal{L}_{\tilde{B}_i}\big[\tau^{\tilde{B}_i}\big(\tilde{A}_i(\tilde{L}_i)\big), \tilde{A}_i(u)\big] \mathrm{d}u, \tag{3.23}$$

where we replaced the born $-\infty$ in the integral by $\tilde{L}_i^-$ because by (3.5), under $\mathbb{P}^{W_\kappa}_{\tilde{m}_i}$,

$$e^{-\tilde{V}^{(i)}(u)} \mathcal{L}_{\tilde{B}_i}\big[\tau^{\tilde{B}_i}\big(\tilde{A}_i\big(\tilde{L}_i\big)\big), \tilde{A}_i(u)\big] = \mathcal{L}_X\big[H\big(\tilde{L}_i\big), u\big], \qquad u \in \mathbb{R}, \tag{3.24}$$

which is 0 for $u < \tilde{L}_i^-$ on $\big\{H\big(\tilde{L}_i\big) < H\big(\tilde{L}_i^-\big)\big\}$. Due to the same local time formula, we also have, a.s. under $\mathbb{P}^{W_\kappa}_{\tilde{m}_i}$, that is, whether $H\big(\tilde{L}_i\big) < H\big(\tilde{L}_i^-\big)$ or not,

$$\mathbf{U}_i = \mathcal{L}_X\big(H\big(\tilde{L}_i\big), \big[\tilde{L}_i^-, \tilde{L}_i\big]\big), \tag{3.25}$$

where $\mathcal{L}_X(s, \Delta) := \int_\Delta \mathcal{L}_X(s, x)\mathrm{d}x$, $\Delta \subset \mathbb{R}$ is the total time spent by X in Δ until time s. Also, let $\mathcal{L}_X([u, v], \Delta) := \mathcal{L}_X(v, \Delta) - \mathcal{L}_X(u, \Delta)$ for $0 \leq u < v$. We have by (3.23),

$$(3.21) = \prod_{i=1}^n \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\left(\exp(-\lambda \mathbf{U}_i) \mathbb{1}_{\{H(\tilde{L}_i) < H(\tilde{L}_i^-)\}}\right) \leq \prod_{i=1}^n \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\left(\exp(-\lambda \mathbf{U}_i)\right). \tag{3.26}$$

We notice that on $\big\{H\big(\tilde{L}_i\big) > H\big(\tilde{L}_i^-\big)\big\}$ under $\mathbb{P}^{W_\kappa}_{\tilde{m}_i}$, we have thanks to (3.25), $\mathbf{U}_i \geq \mathcal{L}_X\big([H(\tilde{L}_i^-) + H_{\tilde{L}_i^- \to \tilde{m}_i}, H(\tilde{L}_i)], [\tilde{L}_i^-, \tilde{L}_i]\big)$, which is the time spent in $[\tilde{L}_i^-, \tilde{L}_i]$ by X between times $H(\tilde{L}_i^-) + H_{\tilde{L}_i^- \to \tilde{m}_i}$ and $H(\tilde{L}_i)$. So, we get

$$\begin{aligned} & \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\left(e^{-\lambda \mathbf{U}_i} \mathbb{1}_{\{H(\tilde{L}_i) > H(\tilde{L}_i^-)\}}\right) \\ \leq\ & \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\Big[\mathbb{1}_{\{H(\tilde{L}_i) > H(\tilde{L}_i^-)\}} \\ & \qquad \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\left(e^{-\lambda \mathcal{L}_X([H(\tilde{L}_i^-) + H_{\tilde{L}_i^- \to \tilde{m}_i}, H(\tilde{L}_i)], [\tilde{L}_i^-, \tilde{L}_i])} \Big| \mathcal{G}_{[H(\tilde{L}_i^-) + H_{\tilde{L}_i^- \to \tilde{m}_i}] \wedge H(\tilde{L}_i)}\right)\Big] \\ =\ & P^{W_\kappa}_{\tilde{m}_i}\big[H\big(\tilde{L}_i\big) > H\big(\tilde{L}_i^-\big)\big] \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\big(e^{-\lambda \mathbf{U}_i}\big) = P^{W_\kappa}\big(\overline{\mathcal{E}_i}\big) \mathbb{E}^{W_\kappa}_{\tilde{m}_i}\big(e^{-\lambda \mathbf{U}_i}\big) \end{aligned}$$

by the strong Markov property. This and (3.26) yield

$$\begin{aligned}(3.21) &= \prod_{i=1}^{n}\Big[\mathbb{E}_{\tilde{m}_i}^{W_\kappa}\left(e^{-\lambda \mathbf{U}_i}\right) - \mathbb{E}_{\tilde{m}_i}^{W_\kappa}\left(e^{-\lambda \mathbf{U}_i}\mathbb{1}_{H(\tilde{L}_i)>H(\tilde{L}_i^-)}\right)\Big]\\ &\geq \prod_{i=1}^{n}\big[1-P^{W_\kappa}(\overline{\mathcal{E}_i})\big]\mathbb{E}_{\tilde{m}_i}^{W_\kappa}\left(e^{-\lambda \mathbf{U}_i}\right).\end{aligned}$$

Consequently by Lemma 3.2, for $1 \leq n \leq n_t$,

$$\mathbb{E}\left(e^{-\lambda\sum_{i=1}^n U_i}\right) \geq \mathbb{E}\big[(3.21)\big] \geq \Big[1-e^{-\kappa h_t/2}\Big]^n E\Big[\prod_{i=1}^{n}\mathbb{E}_{\tilde{m}_i}^{W_\kappa}\left(e^{-\lambda \mathbf{U}_i}\right)\Big] - \frac{C_+ n_t}{e^{\kappa\delta h_t}}.$$

Since $\big(\tilde{V}^{(i)}(x),\ \tilde{L}_i^- \leq x \leq \tilde{L}_i\big)$, $i \geq 1$ are i.i.d. by Lemma 2.3, the r.v. $\mathbb{E}_{\tilde{m}_i}^{W_\kappa}\left(e^{-\lambda \mathbf{U}_i}\right)$ are also i.i.d, and $\mathbf{U}_n \overset{\mathcal{L}}{=} \mathbf{U}$ under $\mathbb{P}$. Moreover, $(1-x)^n \geq 1-xn$ for $0 \leq x \leq 1$, so this gives for large t since $\delta < 1/8$,

$$\mathbb{E}\left(e^{-\lambda\sum_{i=1}^n U_i}\right) \geq \Big[\mathbb{E}\big(e^{-\lambda \mathbf{U}}\big)\Big]^n - C_+ n_t e^{-\kappa\delta h_t}, \tag{3.27}$$

for all $1 \leq n \leq n_t$. Similarly, using (3.26) and Lemma 3.2,

$$\mathbb{E}\left(e^{-\lambda\sum_{i=1}^n U_i}\right) \leq \mathbb{E}\big[(3.21)\big] + \mathbb{P}\big[\cup_{i=1}^n \overline{\mathcal{E}_i}\big] \leq \Big[\mathbb{E}\big(e^{-\lambda \mathbf{U}}\big)\Big]^n + C_+ n_t e^{-\kappa\delta h_t},$$

for every $1 \leq n \leq n_t$. This together with (3.27) proves (3.19).

For (3.20), we obtain for $n \geq 2$, $[a,b] \subset [0,1]$ and $\alpha > 0$,

$$\begin{aligned}&\mathbb{P}\Big(\sum_{i=1}^{n-1}\frac{U_i}{t} \in [a,b], \sum_{i=1}^{n}\frac{U_i}{t} \geq \alpha,\ \mathcal{E}_n\Big)\\ =\ &\mathbb{E}\Big[\mathbb{1}_{\sum_{i=1}^{n-1}\frac{U_i}{t}\in[a,b]}\mathbb{E}^{W_\kappa}\Big(\mathbb{1}_{\frac{U_n}{t}\geq\alpha-\sum_{i=1}^{n-1}\frac{U_i}{t}}\mathbb{1}_{\mathcal{E}_n}\Big|\mathcal{G}_{H(\tilde{m}_n)}\Big)\Big].\end{aligned}$$

Since U_i is, for $1 \leq i \leq n-1$, $\mathcal{G}_{H(\tilde{m}_n)}$-measurable, whereas U_n and $\mathbb{1}_{\mathcal{E}_n}$ are under $\mathbb{P}^{W_\kappa}$ independent of $\mathcal{G}_{H(\tilde{m}_n)}$ by strong Markov property, this is equal to

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\sum_{i=1}^{n-1}\frac{U_i}{t}\in[a,b]}\Big(\mathbb{E}^{W_\kappa}\Big(\mathbb{1}_{\frac{U_n}{t}\geq\alpha-x}\mathbb{1}_{\mathcal{E}_n}\Big)_{\big|x=\sum_{i=1}^{n-1}\frac{U_i}{t}}\Big)\Big]\\ =\ &\int_a^b \mathbb{P}\Big(\sum_{i=1}^{n-1}\frac{U_i}{t}\in \mathrm{d}x\Big)\mathbb{P}(U_n/t \geq \alpha - x, \mathcal{E}_n),\end{aligned}$$

since $\sum_{i=1}^n U_i/t$ is measurable with respect to $\sigma\big(W_\kappa(v),\ v \leq \tilde{L}_{n-1},\ X(u),\ u \leq H\big(\tilde{L}_{n-1}\big)\big)$, and so is independent of $\mathbb{E}^{W_\kappa}\big(\mathbb{1}_{U_n/t\geq\alpha-x}\mathbb{1}_{\mathcal{E}_n}\big)$ which is for every $x \in \mathbb{R}$ measurable with respect to $\sigma\big(W_\kappa\big(v+\tilde{L}_{n-1}^+\big) - W_\kappa\big(\tilde{L}_{n-1}^+\big),\ v \geq 0\big)$ with $\tilde{L}_{n-1} \leq \tilde{L}_{n-1}^+ \leq \tilde{L}_n^-$ by (2.14).

By (3.23), for $n \geq 1$, $U_n = H(\tilde{L}_n) = \mathbf{U}_n$ on $\mathcal{E}_n$ under $\mathbb{P}_{\tilde{m}_n}^{W_\kappa}$, and $\mathbf{U}_n \overset{\mathcal{L}}{=} \mathbf{U}$ under $\mathbb{P}$ by Lemma 2.3. So,

$$\mathbb{P}(\mathbf{U}/t \geq \alpha - x) - \mathbb{P}(\overline{\mathcal{E}_n}) \leq \mathbb{P}(U_n/t \geq \alpha - x, \mathcal{E}_n) \leq \mathbb{P}(\mathbf{U}/t \geq \alpha - x), \qquad x \in \mathbb{R}. \tag{3.28}$$

Since $\mathbb{P}(\overline{\mathcal{E}_i}) \leq C_+ n_t e^{-\kappa\delta h_t}$ for $1 \leq i \leq n_t$ by Lemma 3.2, we get (3.20) for $2 \leq n \leq n_t$. Finally the case $n = 1$ follows from (3.28). ☐

3.5 Negligible parts in the trajectory of X

We now prove that the total time spent by the diffusion X between the first n_t valleys is negligible compared to t.

We first give some estimates about hitting times. To this aim, we recall the notation of Subsection 3.1 and in particular $H_-(r)$ and $H_+(r)$, which are defined in (3.6) and (3.5). We start with an estimate concerning the total time spent by X in $\mathbb{R}_-$, that is, $H_-(+\infty) := \lim_{r\to+\infty} H_-(r)$.

Lemma 3.5. *For z large enough (this lemma is in fact true for every $\kappa \in (0,\infty)$),*

$$\mathbb{P}(H_-(+\infty) > z) \quad \leq \quad C_+[(\log z)/z]^{\kappa/(\kappa+2)}. \tag{3.29}$$

Proof of Lemma 3.5. For $a > 0$, $\alpha > 0$ and $b > 0$, let

$$\mathcal{E}_1^{3.5} := \left\{ \sup_{x<0} e^{-W_\kappa(x)} \leq a \right\}, \quad \mathcal{E}_2^{3.5} := \{A_\infty \leq \alpha\}, \quad \mathcal{E}_3^{3.5} := \left\{ \sup_{y<0} \mathcal{L}_B[\tau^B(\alpha), y] \leq b \right\},$$
$$\mathcal{L}_X^{*-}(+\infty) := \sup_{r\geq 0} \sup_{x<0} \mathcal{L}_X(H(r), x).$$

We first prove an inequality with regards to $\mathcal{L}_X^{*-}(+\infty)$. We notice that by (3.5),

$$\mathcal{L}_X^{*-}(+\infty) = \sup_{r\geq 0} \sup_{x<0} \left\{ e^{-W_\kappa(x)} \mathcal{L}_B\left[\tau^B(A(r)), A(x)\right] \right\} \leq \left(\sup_{x<0} e^{-W_\kappa(x)} \right) \sup_{y<0} \mathcal{L}_B[\tau^B(A_\infty), y].$$

By the first Ray–Knight theorem (see e.g. Revuz and Yor [39], chap. XI), there exist two Bessel processes R_2 and R_0, of dimensions 2 and 0 respectively, starting from 0 and $R_2(\alpha)$, such that $\mathcal{L}_B(\tau^B(\alpha), x)$ is equal to $R_2^2(\alpha - x)$ for $x \in [0, \alpha]$ and to $R_0^2(-x)$ for $x < 0$. Hence, for $\alpha \leq b$,

$$\mathbb{P}\big(\overline{\mathcal{E}_3^{3.5}}\big) = \mathbb{P}\big(R_2^2(\alpha) > b\big) + \int_0^b \mathbb{P}\left(\sup_{y>0} R_0^2(y) > b \big| R_0^2(0) = x \right) \mathbb{P}\big(R_2^2(\alpha) \in [x, x + \mathrm{d}x]\big).$$

Since R_2 is equal in law to the euclidian norm of a 2-dimensional Brownian motion and so $R_2^2(\alpha)$ is exponentially distributed with mean 2α, since $\mathbb{P}(\sup_{\mathbb{R}_+} R_0^2 > b | R_0^2(0) = x) = x/b$, $0 \leq x \leq b$ (a scale function of R_0^2 being $x \mapsto x$, see e.g. Revuz and Yor, [39] p. 442 with $\nu = -1$), and since $e^{-x} \leq 1/x$, $x > 0$, this gives by scaling

$$\mathbb{P}(\overline{\mathcal{E}_3^{3.5}}) = \exp\left(-\frac{b}{2\alpha} \right) + \mathbb{E}\left(\frac{R_2^2(\alpha)}{b} \mathbf{1}_{\{R_2^2(\alpha)\leq b\}} \right) \leq \left[2 + \mathbb{E}\left(R_2^2(1)\right)\right] \frac{\alpha}{b} = \frac{4\alpha}{b}. \tag{3.30}$$

Now, let $z > 0$, $a := z^{\frac{1}{\kappa+2}}$, $\alpha := z^{\frac{1}{\kappa+2}}$ and $b := z^{\frac{\kappa+1}{\kappa+2}}$. Notice that on $\mathcal{E}_1^{3.5} \cap \mathcal{E}_2^{3.5} \cap \mathcal{E}_3^{3.5}$, $\mathcal{L}_X^{*-}(+\infty) \leq z$. Moreover, $P[\sup_{[0,\infty)} W_\kappa \geq x] = e^{-\kappa x}$, $x \geq 0$ by (2.26) as in the proof of Lemma 3.2. So, using Fact 3.1 (Dufresne) for $\mathbb{P}\big(\overline{\mathcal{E}_2^{3.5}}\big)$, we get for z large enough,

$$\mathbb{P}[\mathcal{L}_X^{*-}(+\infty) > z] \leq \mathbb{P}(\overline{\mathcal{E}_1^{3.5}}) + \mathbb{P}(\overline{\mathcal{E}_2^{3.5}}) + \mathbb{P}(\overline{\mathcal{E}_3^{3.5}}) \leq a^{-\kappa} + [2/\alpha]^\kappa/(\kappa\Gamma(\kappa)) + 4\alpha/b \leq C_+ z^{-\frac{\kappa}{\kappa+2}}. \tag{3.31}$$

We now turn back to $H_-(+\infty)$. Define for $c > 0$,

$$\mathcal{E}_4^{3.5} := \left\{ \min_{0\leq s\leq \tau^B(\alpha)} B(s) > -\alpha z^{\frac{\kappa+1}{\kappa+2}} \right\}, \qquad \mathcal{E}_5^{3.5} := \left\{ \left|A^{-1}(-z)\right| \leq c \log z \right\}.$$

On $\cap_{i=1}^5 \mathcal{E}_i^{3.5}$, notice that for $r \geq 0$, $\mathcal{L}_X(H(r), x) = 0$ if $A(x) \leq \min_{0\leq s\leq \tau^B(A(r))} B(s)$ due to (3.5), and this is the case in particular if $A(x) < -\alpha z^{\frac{\kappa+1}{\kappa+2}} = -z$ since $\alpha \geq A_\infty \geq A(r)$. Consequently,

$$H_-(r) = \int_{A^{-1}(-z)}^0 \mathcal{L}_X(H(r), x)\mathrm{d}x \leq \int_{A^{-1}(-z)}^0 \mathcal{L}_X^{*-}(+\infty)\mathrm{d}x.$$

This gives on $\cap_{i=1}^5\mathcal{E}_i^{3.5}$, since $\mathcal{L}_X^{*-}(+\infty)\le z$ on this event,

$$H_-(+\infty)\le \left|A^{-1}(-z)\right|\mathcal{L}_X^{*-}(+\infty)\le cz\log z. \tag{3.32}$$

Moreover, for $c>2/\kappa$, small $\varepsilon>0$, and all large z, using $(W(-u),\ u\in\mathbb{R})\overset{\mathcal{L}}{=}(W(u),\ u\in\mathbb{R})$,

$$\begin{aligned}\mathbb{P}\big(\overline{\mathcal{E}_5^{3.5}}\big) &= \mathbb{P}\big[-z<A(-c\log z)\big]=\mathbb{P}\left(z>\int_0^{c\log z}e^{W(-u)+\kappa u/2}\mathrm{d}u\right)\\ &\le \mathbb{P}\left[z>\exp\left(\inf_{[0,c\log z]}W\right)\frac{2}{\kappa}(z^{\frac{\kappa c}{2}}-1)\right]\\ &\le \mathbb{P}\left[\exp\left(\inf_{[0,c\log z]}W\right)<z^{1-\frac{\kappa c}{2}+\varepsilon}\right]\le 2z^{-\frac{1}{2c}\left(\frac{\kappa c}{2}-1-\varepsilon\right)^2}, \end{aligned}\tag{3.33}$$

since $\inf_{[0,c\log z]}W\overset{\mathcal{L}}{=}-\sqrt{c\log z}|W(1)|$ and $\mathbb{P}(W(1)\ge x)\le e^{-x^2/2}$ for large x. Moreover, $\mathbb{P}(\overline{\mathcal{E}_4^{3.5}})=\alpha/[\alpha+\alpha z^{\frac{\kappa+1}{\kappa+2}}]\le z^{-\frac{\kappa+1}{\kappa+2}}$. Choosing c large enough, this, together with (3.31), (3.32) and (3.33) gives $\mathbb{P}[H_-(+\infty)>cz\log z]\le C_+z^{-\kappa/(\kappa+2)}$, which proves (3.29). ☐

Lemma 3.6. *There exists a constant $C_3>0$ such that for every $h>0$,*

$$\mathbb{E}\big[H_+(\tau_1^*(h))\big]\le C_3e^h, \tag{3.34}$$

where $\tau_1^*(h)=\inf\left\{u\ge 0,\ W_\kappa(u)-\inf_{[0,u]}W_\kappa\ge h\right\}$ as in (2.5). Moreover,

$$\mathbb{P}[H_-(\tilde m_1)\ge t/\log h_t]\le C_+[(\log t)^2/t]^{\kappa/(\kappa+2)}. \tag{3.35}$$

Proof of Lemma 3.6. First, (3.35) comes directly from Lemma 3.5 since $\log h_t\sim_{t\to+\infty}\log\log t$.

For (3.34), we use (3.6) and (3.5). We notice that by the scale property of B, recalling that $A(u)\ge 0$ for all $u\ge 0$ and A is independent of B, we have for every $r\ge 0$, which can depend on the environment W_κ,

$$\mathbb{E}^{W_\kappa}[H_+(r)]=\mathbb{E}^{W_\kappa}\left(\int_0^r e^{-W_\kappa(u)}A(r)\mathcal{L}_B\left(\tau^B(1),A(u)/A(r)\right)\mathrm{d}u\right).$$

We remind that $\mathbb{E}[\mathcal{L}_B(\tau^B(1),y)]=\mathbb{E}[R_2^2(1-y)]=2(1-y)$ for $0\le y\le 1$, by the first Ray–Knight theorem, R_2 being a 2-dimensional Bessel process starting from 0 as in the proof of Lemma 3.5. So by Fubini and due to the independence of B and W_κ,

$$\mathbb{E}^{W_\kappa}[H_+(r)]=\int_0^r e^{-W_\kappa(u)}2\left(A(r)-A(u)\right)\mathrm{d}u=2\int_0^r\int_u^r e^{W_\kappa(v)-W_\kappa(u)}\mathrm{d}v\mathrm{d}u. \tag{3.36}$$

Hence, applying this to $r=\tau_1^*(h)$ for $h>0$, we get

$$\begin{aligned}\mathbb{E}[H_+(\tau_1^*(h))] &= 2E\bigg(\int_0^{\tau_1^*(h)}\int_u^{\tau_1^*(h)}e^{W_\kappa(v)-W_\kappa(u)}\mathrm{d}v\mathrm{d}u\bigg)\\ &\le 2E\bigg(\int_0^\infty \mathbb{1}_{u\le\tau_1^*(h)}\int_u^{\tau_1^*(u,h)}e^{W_\kappa(v)-W_\kappa(u)}\mathrm{d}v\mathrm{d}u\bigg),\end{aligned}$$

where $\tau_1^*(u,h):=\inf\{x\ge u,\ W_\kappa(x)-\inf_{[u,x]}W_\kappa\ge h\}\ge\tau_1^*(h)$. Applying Fubini followed by the Markov property at time u, we get

$$\begin{aligned}\mathbb{E}[H_+(\tau_1^*(h))] &\le 2\int_0^\infty E\bigg(\mathbb{1}_{u\le\tau_1^*(h)}\int_0^{\tau_1^*(u,h)-u}e^{W_\kappa(\alpha+u)-W_\kappa(u)}\mathrm{d}\alpha\bigg)\mathrm{d}u\\ &= 2\int_0^\infty E(\mathbb{1}_{u\le\tau_1^*(h)})E\bigg(\int_0^{\tau_1^*(h)}e^{W_\kappa(\alpha)}\mathrm{d}\alpha\bigg)\mathrm{d}u=2E\big(\tau_1^*(h)\big)\beta_0(h), \end{aligned}\tag{3.37}$$

where, similarly as in Enriquez et al. ([23], Lem. 4.9),

$$\beta_0(h) := E\Big(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)}\mathrm{d}u\Big).$$

We now prove that $\beta_0(h) \leq C_+ e^{(1-\kappa)h}$. We notice that $W_\kappa(u) \leq h$ for all $0 \leq u \leq \tau_1^*(h)$ and so $\mathcal{L}_{W_\kappa}(\tau_1^*(h), x) = 0$ for all $x > h$. Consequently, by the occupation time formula and Fubini,

$$\begin{aligned}\beta_0(h) &= E\bigg(\int_{-\infty}^{h} e^x \mathcal{L}_{W_\kappa}(\tau_1^*(h), x)\mathrm{d}x\bigg)\\ &\leq E\bigg(\int_{-\infty}^{h} e^x \mathcal{L}_{W_\kappa}(\infty, x)\mathrm{d}x\bigg) = \int_{-\infty}^{h} e^x E\big[\mathcal{L}_{W_\kappa}(\infty, x)\big]\mathrm{d}x,\end{aligned}$$

where $\mathcal{L}_{W_\kappa}(\infty, x) = \lim_{u\to+\infty} \mathcal{L}_{W_\kappa}(u, x)$. Moreover, $E\big[\mathcal{L}_{W_\kappa}(\infty, 0)\big] = 2/\kappa < \infty$, since $\mathcal{L}_{W_\kappa}(\infty, 0)$ is an exponential variable of mean $2/\kappa$ (see e.g. Borodin et al. [7], p. 90 at the end of paragraph V.11). Furthermore by the strong Markov property for every $x \in \mathbb{R}$,

$$E\big[\mathcal{L}_{W_\kappa}(\infty, x)\big] = P\big[\tau^{W_\kappa}(x) < \infty\big] E\big[\mathcal{L}_{W_\kappa}(\infty, 0)\big] = \big[\mathbb{1}_{(-\infty,0)}(x) + e^{-\kappa x}\mathbb{1}_{(0,\infty)}(x)\big]2/\kappa,$$

since $P\big[\tau^{W_\kappa}(x) < \infty\big] = e^{-\kappa x}$ for $x > 0$ by (2.26), and is 1 for $x \leq 0$ since $\lim_{+\infty} W_\kappa = -\infty$ a.s. So for large h,

$$\beta_0(h) \leq \int_{-\infty}^{0} \frac{2}{\kappa} e^x \mathrm{d}x + \int_0^h \frac{2}{\kappa} e^{(1-\kappa)x}\mathrm{d}x \leq C_+ e^{(1-\kappa)h}. \tag{3.38}$$

This, together with (3.37) and $E\big(\tau_1^*(h)\big) \leq C_+ e^{\kappa h}$ provided in Fact 2.2 gives (3.34). ☐

We now have all the tools needed to bound the time spent by X between the valleys. We recall that $U_i = H(\tilde{L}_i) - H(\tilde{m}_i)$ for $i \geq 1$. More precisely, we prove the following lemma:

Lemma 3.7. *Assume $0 < \delta < 2^{-3/2}$ and $(1+2\delta)\kappa < 1$. For t large enough,*

$$\mathbb{P}\left(H(\tilde{m}_1) \leq \frac{2t}{\log h_t}\right) \geq \mathbb{P}\left(\bigcap_{k=1}^{n_t}\left\{0 \leq H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i \leq \frac{2t}{\log h_t}\right\}\right) \geq 1 - C_+ n_t \frac{(\log h_t)}{e^{\phi(t)}},$$

where $\sum_{i=1}^{0} \cdots = 0$ by convention. Notice that $n_t \frac{(\log h_t)}{e^{\phi(t)}} \leq (\log\log t) e^{[(1+\delta)\kappa - 1]\phi(t)} = o(1)$ as $t \to +\infty$ since $\log\log t = o(\phi(t))$.

Proof of Lemma 3.7. We use the notation $\tilde{L}_i^*$ defined in (3.11). We introduce

$$X_i(u) := X\big(u + H\big(\tilde{L}_i\big)\big), \qquad X_i^*(u) := X\big(u + H\big(\tilde{L}_i^*\big)\big), \qquad u \geq 0, \tag{3.39}$$

which are diffusions in the environment W_κ, starting respectively from $\tilde{L}_i$ and $\tilde{L}_i^*$, by the strong Markov property. We also denote by $H_{X_i}(r)$ the hitting time of r by X_i for $r \in \mathbb{R}$.

We first notice that since $U_i = H(\tilde{L}_i) - H(\tilde{m}_i)$, $i \in \mathbb{N}^*$,

$$H(\tilde{m}_k) = H(\tilde{m}_1) + \sum_{i=1}^{k-1} U_i + \sum_{i=1}^{k-1}\Big(H(\tilde{m}_{i+1}) - H(\tilde{L}_i)\Big), \qquad 1 \leq k \leq n_t, \tag{3.40}$$

and $H(\tilde{m}_{i+1}) - H(\tilde{L}_i) \geq 0$ since $\tilde{m}_{i+1} > \tilde{L}_i$ by (2.14). So, we just have to prove that $H(\tilde{m}_1) + \sum_{i=1}^{n_t-1}\big(H(\tilde{m}_{i+1}) - H(\tilde{L}_i)\big) \leq 2t/\log h_t$ with large probability.

The idea of the proof is to use Lemma 3.3, which says that on some large event $\mathcal{E}_1^{3.3}$, the diffusion X_i starting from $\tilde{L}_i$ hits $\tilde{m}_{i+1}$ before $\tilde{L}_i^*$. This allows us to write (see step 2) $E^{W_\kappa}[H_{X_i}(\tilde{m}_{i+1})\mathbb{1}_{\mathcal{E}_1^{3.3}}] \leq E^{W_\kappa}_{\tilde{L}_i^*}[H_+(\tilde{\tau}_{i+1}(h_t))]$. Thanks to some large event studied in Step 1, we can compare the expectation of this last quantity with $\mathbb{E}[H_+(\tilde{\tau}_1^*(h_t))]$, which we can bound by Lemma 3.6.

Step 1: In this step, we prove that $P(\overline{\mathcal{E}_2^{3.7}}) \leq C_+ n_t e^{-\kappa h_t/2}$, where,

$$\begin{aligned} \tilde{\tau}^*_{i+1}(h_t) &:= \inf\{u \geq \tilde{L}_i^*,\ W_\kappa(u) - \inf_{[\tilde{L}_i^*,u]} W_\kappa \geq h_t\} \leq \tilde{\tau}_{i+1}(h_t), \qquad i \geq 1, \\ \mathcal{E}_2^{3.7} &:= \cap_{i=1}^{n_t-1}\big\{\tilde{\tau}^*_{i+1}(h_t) = \tilde{\tau}_{i+1}(h_t)\big\}, \end{aligned}$$

where we used, for the inequality, $\tilde{L}_i^* < \tilde{L}_i < \tilde{L}_{i+1}^\sharp$, coming from (3.11) and (2.14). By definition of $\tilde{\tau}_{i+1}(h_t)$ and (2.16), we observe that $\big\{\tilde{\tau}^*_{i+1}(h_t) \neq \tilde{\tau}_{i+1}(h_t)\big\} = \big\{\tilde{\tau}^*_{i+1}(h_t) \leq \tilde{L}^\sharp_{i+1}\big\} = \big\{\inf_{[\tilde{L}_i^*,\tilde{\tau}^*_{i+1}(h_t)]} W_\kappa - W_\kappa(\tilde{L}_i^*) \geq -2h_t^+ - 3h_t/4\big\}$. So, applying the strong Markov property at stopping time $\tilde{L}_i^*$ yields

$$P[\tilde{\tau}^*_{i+1}(h_t) \neq \tilde{\tau}_{i+1}(h_t)] = \mathbb{P}\big(\inf_{[0,\tau_1^*(h_t)]} W_\kappa \geq -2h_t^+ - 3h_t/4\big) \leq C_+ h_t e^{-\kappa h_t} \tag{3.41}$$

by (2.9). Then $P\big(\overline{\mathcal{E}_2^{3.7}}\big) \leq C_+ n_t h_t e^{-\kappa h_t} \leq C_+ n_t e^{-\kappa h_t/2}$.

Step 2: On $\mathcal{E}_2^{3.7}$, $H(\tilde{m}_{i+1}) - H\big(\tilde{L}_i\big) = H_{X_i}(\tilde{m}_{i+1}) \leq H_{X_i}(\tilde{\tau}_{i+1}(h_t)) = H_{X_i}(\tilde{\tau}^*_{i+1}(h_t))$, which is, on $\mathcal{E}_1^{3.3}$ (see (3.13)), the total time spent by X_i in $[\tilde{L}_i^*,+\infty)$ before hitting $\tilde{\tau}_{i+1}(h_t) = \tilde{\tau}^*_{i+1}(h_t)$. This last quantity is less than or equal to the total time spent in $[\tilde{L}_i^*,+\infty)$ by X_i^* before hitting $\tilde{\tau}^*_{i+1}(h_t)$. This is the total time spent in $[0,+\infty)$ by $X_i^* - \tilde{L}_i^*$ before it first hits $\tilde{\tau}^*_{i+1}(h_t) - \tilde{L}_i^*$, which has the same law as $H_+(\tau_1^*(h_t))$ under the annealed probability $\mathbb{P}$, since $X_i^* - \tilde{L}_i^*$ is a diffusion in the environment $(W_\kappa(\tilde{L}_i^* + x) - W_\kappa(\tilde{L}_i^*),\ x \in \mathbb{R})$, which has on $[0,+\infty)$ the same law as $(W_\kappa(x),\ x \geq 0)$ because $\tilde{L}_i^*$ is a stopping time for W_κ. Consequently,

$$\mathbb{E}\big[\big(H(\tilde{m}_{i+1}) - H\big(\tilde{L}_i\big)\big)1_{\mathcal{E}_1^{3.3}\cap\mathcal{E}_2^{3.7}}\big] \leq \mathbb{E}[H_+(\tau_1^*(h_t))], \qquad 1 \leq i \leq n_t - 1. \tag{3.42}$$

A Markov inequality, this last inequality (3.42), and then Lemma 3.6 lead to

$$\begin{aligned} &\mathbb{P}\bigg(H_+(\tilde{m}_1) + \sum_{i=1}^{n_t-1}\Big(H(\tilde{m}_{i+1}) - H\big(\tilde{L}_i\big)\Big) \geq \frac{t}{\log h_t},\ \mathcal{E}_1^{3.3},\ \mathcal{E}_2^{3.7},\ \mathcal{E}_3^{3.7},\ \mathcal{V}_t\bigg) \qquad (3.43) \\ &\leq \frac{\log h_t}{t}\Big[\mathbb{E}\big[H_+(m_1)\mathbb{1}_{\mathcal{E}_3^{3.7}}\big] + (n_t - 1)\mathbb{E}\big[H_+(\tau_1^*(h_t))\big]\Big] \leq \frac{\log h_t}{t} n_t C_3 e^{h_t}, \end{aligned}$$

where $\mathcal{E}_3^{3.7} := \{m_1 \leq \tau_1^*(h_t)\}$ and since $\tilde{m}_1 = m_1$ on $\mathcal{V}_t$. Recall that $\phi(t) = o(\log t)$ and $\log\log t = o(\phi(t))$ as $t \to +\infty$, and then $\log h_t \sim_{t\to+\infty} \log\log t$. This and (3.40) lead to

$$\begin{aligned} &\mathbb{P}\bigg(H(\tilde{m}_{n_t}) - \sum_{i=1}^{n_t-1} U_i \geq \frac{2t}{\log h_t}\bigg) = \mathbb{P}\bigg(H(\tilde{m}_1) + \sum_{i=1}^{n_t-1}\Big(H(\tilde{m}_{i+1}) - H(\tilde{L}_i)\Big) \geq \frac{2t}{\log h_t}\bigg) \\ \leq\ & \mathbb{P}\big(H_-(\tilde{m}_1) \geq t/\log h_t\big) + (3.43) + \mathbb{P}\big(\overline{\mathcal{E}_1^{3.3}}\big) + \mathbb{P}\big(\overline{\mathcal{E}_2^{3.7}}\big) + \mathbb{P}\big(\overline{\mathcal{E}_3^{3.7}}\big) + P\big(\overline{\mathcal{V}_t}\big) \\ \leq\ & C_+ n_t (\log h_t) e^{-\phi(t)}. \end{aligned} \tag{3.44}$$

Indeed, we used in the last inequality (3.35), (3.16), Step 1, Lemma 2.3, and the fact that $\mathbb{P}(\overline{\mathcal{E}_3^{3.7}}) \leq \mathbb{P}(0 < M_0 < m_1) \leq 2h_t e^{-\kappa h_t}$ by (2.20) and the definition of h_t-maximum and M_0. As explained after (3.40), this concludes the proof. □

4 Time spent in a standard valley

The aim of this section is to prove the following proposition, which gives the second order of the Laplace transform of $\mathbf{U}$, which is defined in (3.18) and is useful because of Proposition 3.4:

Proposition 4.1 (second order of the Laplace transform of $\mathbf{U}$). *Assume $\kappa \in (0,1)$ and $0 < \delta < \inf(2/27, \kappa^2/2)$. Let $\lambda > 0$. As $t \to +\infty$,*

$$e^{\kappa\phi(t)}\left(1 - \mathbb{E}\left(e^{-\lambda \mathbf{U}/t}\right)\right) = C_\kappa \lambda^\kappa + o(1),$$

where $C_\kappa := 8^\kappa(C_0 + |\Upsilon_0|) > 0$, with $C_0 := \Gamma(1-\kappa)\Gamma(\kappa+2)/(1+\kappa)^\kappa$ and

$$\Upsilon_0 := \kappa \int_0^\infty \left[\frac{y^\kappa}{[\Gamma(\kappa+1)I_\kappa(2\sqrt{y})]^2} - \left(1 + \frac{y}{\kappa+1}\right)^{-2}\right] y^{-\kappa-1}\mathrm{d}y < 0, \tag{4.1}$$

I_κ being the modified Bessel function of the first kind.

Before proving this in Subsection 4.3, we need additional estimates given below.

4.1 Some technical estimates

We recall that $(R(s),\ s \geq 0)$ is a process with law BES$(3, \kappa/2)$, and that for $a < b$, $\left(W_\kappa^b(s),\ 0 \leq s \leq \tau^{W_\kappa^b}(a)\right)$ is a $(-\kappa/2)$-drifted Brownian motion starting from b and killed when it first hits a. We now introduce

$$F^\pm(x) := \int_0^{\tau^R(x)} \exp(\pm R(s))ds, \ \ x > 0, \qquad G^\pm(a,b) := \int_0^{\tau^{W_\kappa^b}(a)} \exp\left(\pm W_\kappa^b(s)\right)ds, \quad a < b. \tag{4.2}$$

The following technical lemma is useful to estimate the Laplace transform appearing in Proposition 4.1:

Lemma 4.2. *Assume $0 < \kappa < 1$. There exists $C_4 > 0$, $M > 0$ and $\eta_1 \in (0,1)$ such that $\forall y > M, \forall \gamma \in (0, \eta_1]$,*

$$\left|E\left(e^{-\gamma F^-(y)}\right) - [1 + 2\gamma/(\kappa+1)]^{-1}\right| \leq C_4 \max(e^{-\kappa y}, \gamma^{3/2}), \tag{4.3}$$

$$\left|E\left(e^{-\gamma F^+(y)/e^y}\right) - [1 - 2\gamma/(\kappa+1)]\right| \leq C_4 \max(e^{-\kappa y}, \gamma^{3/2}), \tag{4.4}$$

$$\left|E\left(e^{-\gamma G^+(y/2,y)/e^y}\right) - [1 - \Gamma(1-\kappa)(2\gamma)^\kappa/\Gamma(1+\kappa)]\right| \leq C_4 \max(\gamma^\kappa e^{-\kappa y/2}, \gamma). \tag{4.5}$$

Moreover, there exists $c_1 > 0$, such that for all $y > 0$, $E[F^-(y)] = E\left[F^+(y)/e^y\right] \leq c_1$. Finally,

$$\lim_{x \to +\infty} E\left(e^{-\gamma F^-(x)}\right) = \frac{(2\gamma)^{\kappa/2}}{\kappa\Gamma(\kappa)I_\kappa(2\sqrt{2\gamma})} \qquad \gamma > 0. \tag{4.6}$$

The proof of this lemma is deferred to Section 6.

Before proving Proposition 4.1, we also need to introduce the following technical lemma, which is useful to approximate $\mathbf{U}$, and in particular the local time appearing in its expression (3.18):

Lemma 4.3. $(B(t),\ t \in \mathbb{R})$ *being a standard two-sided Brownian motion, there exists a* *constant* c_2 *such that for every* $0<\varepsilon<1$, $0<\eta<1$ *and* $x>0$,

$$\mathbb{P}\Big(\sup_{u\in[-\eta,\eta]}\Big|\mathcal{L}_B\big(\tau^B(1),u\big)-\mathcal{L}_B\big(\tau^B(1),0\big)\Big|>\varepsilon\mathcal{L}_B\big(\tau^B(1),0\big)\Big)\leq\ c_2\frac{\eta^{1/6}}{\varepsilon^{2/5}},\tag{4.7}$$

$$\mathbb{P}\Big(\sup_{u\in[0,1]}\mathcal{L}_B\big(\tau^B(1),u\big)\geq x\Big)\leq\ 4e^{-x/2},\tag{4.8}$$

$$\mathbb{P}\Big(\sup_{u\leq 0}\mathcal{L}_B\big(\tau^B(1),u\big)\geq x\Big)\leq\ \frac{4}{x}.\tag{4.9}$$

Proof of Lemma 4.3. First, (4.9) is the particular case $\alpha=1$ of (3.30), which we proved in Lemma 3.5.

Second, by the first Ray-Knight theorem (see e.g. Revuz and Yor [39], chap. XI), $\mathcal{L}_B(\tau^B(1),u)=R_2^2(1-u)$ for $u\in[0,1]$, where R_2^2 is a 2-dimensional squared Bessel process starting from 0, so (4.8) follows directly from Diel ([18] Lem. 2.3 (iii)).

We now turn to the proof of (4.7). Let $0<\varepsilon<1$, $0<\eta<1$ and

$$\mathcal{E}^{4.3}:=\Big\{\sup_{u\in[-\eta,\eta]}\Big|\mathcal{L}_B\big(\tau^B(1),u\big)-\mathcal{L}_B\big(\tau^B(1),0\big)\Big|>\varepsilon\mathcal{L}_B\big(\tau^B(1),0\big)\Big\}.$$

We have, for $\alpha>0$ and $x>0$,

$$\begin{aligned}\mathbb{P}(\mathcal{E}^{4.3}) &= \mathbb{P}\big(\mathcal{E}^{4.3}\cap\big\{\mathcal{L}_B\big(\tau^B(1),0\big)\geq\alpha\big\}\big)+\mathbb{P}\big(\mathcal{E}^{4.3}\cap\big\{\mathcal{L}_B\big(\tau^B(1),0\big)<\alpha\big\}\big)\\ &\leq \mathbb{P}\Big(\sup_{u\in[-\eta,\eta]}\Big|\mathcal{L}_B\big(\tau^B(1),u\big)-\mathcal{L}_B\big(\tau^B(1),0\big)\Big|>\varepsilon\alpha\Big)+\mathbb{P}\big[\mathcal{L}_B\big(\tau^B(1),0\big)<\alpha\big]\\ &\leq \mathbb{P}\big[\tau^B(1)\geq x\big]+\mathbb{P}\Big(\sup_{u\in[-\eta,\eta],\ 0\leq s\leq x}\Big|\mathcal{L}_B(s,u)-\mathcal{L}_B(s,0)\Big|>\varepsilon\alpha\Big)+\frac{\alpha}{2},\end{aligned}\tag{4.10}$$

since $\mathcal{L}_B(\tau^B(1),0)=R_2^2(1)$ is an exponential variable with mean 2. Now, notice that

$$\mathbb{P}\big[\tau^B(1)\geq x\big]=\mathbb{P}\Big(\sup_{0\leq u\leq x}B(u)<1\Big)=\mathbb{P}\big(|B(x)|<1\big)\leq 2/\sqrt{2\pi x}.$$

Let $0<\varepsilon_0<1/2$. The second term of (4.10) is less than or equal to

$$\begin{aligned}&\mathbb{P}\Big(\sup_{u\in[-\eta,\eta]-\{0\},\ 0\leq s\leq x}\frac{|\mathcal{L}_B(s,u)-\mathcal{L}_B(s,0)|}{|u|^{1/2-\varepsilon_0}}>\frac{\varepsilon\alpha}{\eta^{1/2-\varepsilon_0}}\Big)\\ \leq\ &\frac{\eta^{1/2-\varepsilon_0}}{\varepsilon\alpha}\mathbb{E}\Big(\sup_{a\neq b,\ 0\leq s\leq x}\frac{|\mathcal{L}_B(s,b)-\mathcal{L}_B(s,a)|}{|a-b|^{1/2-\varepsilon_0}}\Big),\end{aligned}\tag{4.11}$$

the last inequality being a consequence of Markov inequality. Now, applying Barlow and Yor ([5], (ii) p. 199 with $\gamma=1$) to the continuous local martingale $(B(t\wedge x),\ t\geq 0)$ and its jointly continuous local time $(\mathcal{L}_B(t\wedge x,a),\ t\geq 0,\ a\in\mathbb{R})$ proves that the expectation in (4.11) is less than or equal to $C_+\mathbb{E}\big[\big(\sup_{s\geq 0}|B(s\wedge x)|\big)^{1/2+\varepsilon_0}\big]\leq C_+(\sqrt{x})^{1/2+\varepsilon_0}$. Consequently, (4.10) leads to

$$\mathbb{P}\big(\mathcal{E}^{4.3}\big)\leq 2/\sqrt{2\pi x}+C_+(\sqrt{x})^{1/2+\varepsilon_0}\eta^{1/2-\varepsilon_0}(\varepsilon\alpha)^{-1}+\alpha/2.$$

Now, we choose $\alpha=\varepsilon^{-2/5}\eta^{1/5}$, $x=\varepsilon^{4/5}\eta^{-2/5}$ and $\varepsilon_0<1/36$; we obtain $\mathbb{P}(\mathcal{E}^{4.3})\leq C_+\eta^{1/6}\varepsilon^{-2/5}$, which concludes the proof. □

4.2 Approximation of the exit time from a typical valley

We now prove that the standard exit time $\mathbf{U}$, defined in (3.18), can be approximated by a product of (sums of) independent r.v, $(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathrm{e}_1$. We need this later to approximate the Laplace transform of $\mathbf{U}$ and then prove Proposition 4.1, in particular because we have estimates of the Laplace transforms of these r.v. $\mathcal{I}_1^\pm$ and $\mathcal{I}_2^\pm$ in Lemma 4.2.

Proposition 4.4 (approximation of $\mathbf{U}$). *Assume $0 < \delta < \inf(2/27, \kappa^2/2)$ and let $\varepsilon_t := 3e^{-(1-3\delta)h_t/6}$. Possibly on an enlarged probability space, there exist random variables $\mathcal{I}_1^+, \mathcal{I}_2^+, \mathcal{I}_1^-$ and $\mathcal{I}_2^-$, depending on t and e_1, such that*
(i) *$\mathcal{I}_1^+$, $\mathcal{I}_2^+$, $\mathcal{I}_1^-$, $\mathcal{I}_2^-$ and e_1 are independent;*
(ii) *e_1 is exponentially distributed with mean 2, and*

$$\mathcal{I}_1^+ \overset{\mathcal{L}}{=} F^+(h_t), \qquad \mathcal{I}_2^+ \overset{\mathcal{L}}{=} G^+(h_t/2, h_t), \qquad \mathcal{I}_1^- \overset{\mathcal{L}}{=} \mathcal{I}_2^- \overset{\mathcal{L}}{=} F^-(h_t/2),$$

where $\overset{\mathcal{L}}{=}$ denotes equality in law,
(iii) *for t large enough, $\mathbb{P}(\mathcal{A}_t) \geq 1 - C_+ e^{-(c_-)\delta h_t}$, where*

$$\mathcal{A}_t := \left\{ \left| \mathbf{U} - (\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathrm{e}_1 \right| \leq (\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathrm{e}_1 \varepsilon_t \right\}. \tag{4.12}$$

The proof of this proposition involves 3 lemmas. The first two are straightforward consequence of what we have already proved and the last one is more technical.

The expressions of $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_2^+$ and some intermediate r.v. $\mathcal{I}_0^+$ are given in the following lemma, which also provides their laws. The random variables $\mathbf{e_1}$ and $\mathcal{I}_1^+$ are defined later, respectively in Lemma 4.7, and in (4.35).

Lemma 4.5. *We have with the notations $F^\pm$ and $G^\pm$ introduced in (4.2),*

$$\mathcal{I}_0^+ := \int_{m_2}^{\tau_2(h_t)} e^{V^{(2)}(x)} \mathrm{d}x \overset{\mathcal{L}}{=} F^+(h_t), \qquad \mathcal{I}_1^- := \int_{m_2}^{\tau_2(h_t/2)} e^{-V^{(2)}(x)} \mathrm{d}x \overset{\mathcal{L}}{=} F^-(h_t/2), \tag{4.13}$$

$$\mathcal{I}_2^+ := \int_{\tau_2(h_t)}^{L_2} e^{V^{(2)}(x)} \mathrm{d}x \overset{\mathcal{L}}{=} G^+(h_t/2, h_t), \qquad \mathcal{I}_2^- := \int_{\tau_2^-(h_t/2)}^{m_2} e^{-V^{(2)}(x)} \mathrm{d}x \overset{\mathcal{L}}{=} F^-(h_t/2),$$

where $L_2 := \inf\{x > \tau_2(h_t),\ V^{(2)}(x) = h_t/2\}$ is defined similarly as $\tilde{L}_2$ in (2.12) without tilde, so that $L_2 = \tilde{L}_2$ on $\mathcal{V}_t$.

Proof of Lemma 4.5. This is a direct consequence of Fact 2.1 **(ii)** for $\mathcal{I}_0^+$, $\mathcal{I}_1^-$ and $\mathcal{I}_2^-$. and of Fact 2.1 **(iii)** for $\mathcal{I}_2^+$. □

Recall the notation $\tilde{A}_2(z) = \int_{\tilde{m}_2}^{z} e^{\tilde{V}^{(2)}(x)} \mathrm{d}x$ introduced in (3.17). We have,

Lemma 4.6. *For all $0 < \zeta \leq 1$ and $0 < \varepsilon < 1/2$, for t large enough,*

$$P\left[e^{\zeta h_t(1-\varepsilon)} \leq \tilde{A}_2(\tilde{\tau}_2(\zeta h_t)) \leq e^{\zeta h_t(1+\varepsilon)}\right] \geq 1 - 4e^{-\kappa\varepsilon\zeta h_t/2}. \tag{4.14}$$

Proof of Lemma 4.6. First, notice that we have on $\mathcal{V}_t$, by Remark 2.4, $\tilde{A}_2(\tilde{\tau}_2(\zeta h_t)) = \int_{m_2}^{\tau_2(\zeta h_t)} e^{V^{(2)}(x)} \mathrm{d}x$, which is equal in law to $F^+(\zeta h_t)$ thanks to Fact 2.1 **(ii)**. Hence by Lemma 2.3,

$$\begin{aligned} & \text{LHS of (4.14)} \\ \geq\ & P\left[e^{\zeta h_t(1-\varepsilon)} \leq F^+(\zeta h_t) \leq e^{\zeta h_t(1+\varepsilon)}\right] - P(\overline{\mathcal{V}}_t). \\ \geq\ & P\left[F^+(\zeta h_t) \geq e^{\zeta h_t(1-\varepsilon)}\right] - P\left[F^+(\zeta h_t) > e^{\zeta h_t(1+\varepsilon)}\right] - e^{[-\kappa/2+o(1)]h_t}. \end{aligned} \tag{4.15}$$

Since $F^+(\zeta h_t) \leq \tau^R(\zeta h_t)e^{\zeta h_t}$, we have by (2.30) for large t,

$$P\big[F^+(\zeta h_t) > e^{\zeta h_t(1+\varepsilon)}\big] \leq P\big[\tau^R(\zeta h_t) > e^{\varepsilon\zeta h_t}\big] \leq C_+ e^{-\kappa\zeta h_t/(2\sqrt{2})}. \tag{4.16}$$

For the lower bound, notice that by (2.29),

$$P\big[F^+(\zeta h_t) \geq e^{(1-\varepsilon)\zeta h_t}\big] \geq 1 - 3\exp(-\kappa\varepsilon\zeta h_t/2).$$

This together with (4.15) and (4.16) proves the lemma. □

In the following lemma, we give an approximation of $\mathbf{U}$ by the product $\tilde{A}_2\big(\tilde{L}_2\big)\mathcal{I}^-\mathbf{e}_1$, where $\tilde{A}_2\big(\tilde{L}_2\big)$ and $\mathcal{I}^-$ depend only on the potential W_κ, whereas $\mathbf{e}_1$ is independent of the potential.

Lemma 4.7. *For all* $0 < \varepsilon < \inf(2/27, \kappa^2/2)$*, and* t *large enough,*

$$\mathbb{P}\left(\left|\mathbf{U} - \tilde{A}_2\big(\tilde{L}_2\big)\mathcal{I}^-\mathbf{e}_1\right| \leq 2e^{-(1-3\varepsilon)h_t/6}\tilde{A}_2\big(\tilde{L}_2\big)\mathcal{I}^-\mathbf{e}_1\right) \geq 1 - C_+e^{-(c_-)\varepsilon h_t}, \tag{4.17}$$

where

$$\mathcal{I}^- := \int_{\tilde{\tau}_2^-(h_t/2)}^{\tilde{\tau}_2(h_t/2)} e^{-\tilde{V}^{(2)}(u)}\mathrm{d}u, \qquad \mathbf{e}_1 = \mathcal{L}_B\big[\tau^B\big(\tilde{A}_2\big(\tilde{L}_2\big)\big), 0\big]/\tilde{A}_2\big(\tilde{L}_2\big).$$

Moreover, $\mathbf{e}_1$ *is independent of* W_κ*, and exponentially distributed with mean* 2*.*

Proof of Lemma 4.7. Let $0 < \varepsilon < \inf(2/27, \kappa^2/2)$. We first notice that

$$\mathbf{U} = \int_{\tilde{L}_2^-}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)}\tilde{A}_2\big(\tilde{L}_2\big)\mathcal{L}_{B'}\big[\tau^{B'}(1), \tilde{A}_2(u)/\tilde{A}_2\big(\tilde{L}_2\big)\big]\mathrm{d}u,$$

where $B'(u) := B\big(\big[\tilde{A}_2\big(\tilde{L}_2\big)\big]^2u\big)/\tilde{A}_2\big(\tilde{L}_2\big)$ for $u \geq 0$, and therefore $(B'(u),\ u \geq 0)$ is by scaling, as B, a standard Brownian motion independent of W_κ, that is, of $\tilde{V}^{(2)}$. The idea of the proof is that, loosely speaking, for u close to 0, and more precisely for u between $\tilde{\tau}_2^-(h_t/2)$ and $\tilde{\tau}_2(h_t/2)$, $\mathcal{L}_{B'}\big[\tau^{B'}(1), \tilde{A}_2(u)/\tilde{A}_2\big(\tilde{L}_2\big)\big]$ is nearly $\mathcal{L}_{B'}\big[\tau^{B'}(1), 0\big] = \mathbf{e_1}$, whereas for u far from 0, that is $u \notin [\tilde{\tau}_2^-(h_t/2), \tilde{\tau}_2(h_t/2)]$, $e^{-\tilde{V}^{(2)}(x)}$ is "nearly" 0, with large probability.

We first notice that $\mathbf{e_1} = \mathcal{L}_{B'}\big(\tau^{B'}(1), 0\big)$ is an exponential r.v. with mean 2 by the first Ray-Knight theorem, and is independent of W_κ. We cut $\mathbf{U}/\tilde{A}_2\big(\tilde{L}_2\big)$ into three integrals:

$$\begin{aligned} \frac{\mathbf{U}}{\tilde{A}_2\big(\tilde{L}_2\big)} &= \int_{\tilde{L}_2^-}^{\tilde{\tau}_2^-(h_t/2)} + \int_{\tilde{\tau}_2^-(h_t/2)}^{\tilde{\tau}_2(h_t/2)} + \int_{\tilde{\tau}_2(h_t/2)}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)}\mathcal{L}_{B'}\big[\tau^{B'}(1), \tilde{A}_2(u)/\tilde{A}_2\big(\tilde{L}_2\big)\big]\mathrm{d}u \\ &=: \mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{4.18}$$

In what follows, we show that the main contribution comes from $\mathcal{J}_1$.

Step 1: study of $\tilde{A}_2(u)/\tilde{A}_2(\tilde{L}_2)$. We introduce

$$\delta_t := e^{-h_t(1-3\varepsilon)/2}, \tag{4.19}$$

$$\mathcal{E}_1^{4.7} := \big\{\delta_t\tilde{A}_2\big(\tilde{L}_2\big) \geq \tilde{A}_2(\tilde{\tau}_2(h_t/2))\big\}, \qquad \mathcal{E}_2^{4.7} := \big\{\delta_t\tilde{A}_2\big(\tilde{L}_2\big) \geq -\tilde{A}_2(\tilde{\tau}_2^-(h_t/2))\big\},$$

so that on $\mathcal{E}_1^{4.7} \cap \mathcal{E}_2^{4.7}$, $\tilde{A}_2(u)/\tilde{A}_2(\tilde{L}_2) \in [-\delta_t, \delta_t]$ for all $u \in [\tilde{\tau}_2^-(h_t/2), \tilde{\tau}_2(h_t/2)]$. We first prove that $P\big(\overline{\mathcal{E}_1^{4.7}}\big) \leq C_+e^{-\kappa\varepsilon h_t/4}$. By Lemma 4.6,

$$P\big[\tilde{A}_2(\tilde{\tau}_2(h_t/2)) \leq e^{h_t(1+\varepsilon)/2}\big] \geq 1 - C_+e^{-\kappa\varepsilon h_t/4}. \tag{4.20}$$

Notice that $\tilde{A}_2(\tilde{L}_2) = \mathcal{I}_0^+ + \mathcal{I}_2^+$ on $\mathcal{V}_t$, and that $\mathcal{I}_0^+ \stackrel{\mathcal{L}}{=} F^+(h_t)$ by Lemma 4.5. So

$$P\big[\tilde{A}_2(\tilde{L}_2) \geq e^{h_t(1-\varepsilon)}\big] \geq P\big[F^+(h_t) \geq e^{h_t(1-\varepsilon)}\big] - P(\overline{\mathcal{V}}_t) \geq 1 - 4e^{-\kappa\varepsilon h_t/2}, \tag{4.21}$$

where we used (2.29) and Lemma 2.3 in the last inequality. This, together with (4.20) gives

$$P\big(\mathcal{E}_1^{4.7}\big) \geq P\big[\delta_t \tilde{A}_2(\tilde{L}_2) \geq \delta_t e^{h_t(1-\varepsilon)} = e^{h_t(1+\varepsilon)/2} \geq \tilde{A}_2(\tilde{\tau}_2(h_t/2))\big] \geq 1 - C_+ e^{-\kappa\varepsilon h_t/4}. \tag{4.22}$$

Similarly on $\mathcal{V}_t$, $-\tilde{A}_2(\tilde{\tau}_2^-(h_t/2)) = \int_{\tau_2^-(h_t/2)}^{m_2} \exp(V^{(2)}(s))\mathrm{d}s \stackrel{\mathcal{L}}{=} F^+(h_t/2)$ by Fact 2.1, and so

$$P\big[-\tilde{A}_2(\tilde{\tau}_2^-(h_t/2)) \geq e^{h_t(1+\varepsilon)/2}\big] \leq P\big[F^+(h_t/2) \geq e^{h_t(1+\varepsilon)/2}\big] + P\big(\overline{\mathcal{V}_t}\big) \leq C_+ e^{-\kappa h_t/(4\sqrt{2})},$$

by (4.16) and Lemma 2.3. This and (4.21) give for large t,

$$P\big(\mathcal{E}_2^{4.7}\big) \geq 1 - 5e^{-\kappa\varepsilon h_t/2}.$$

Step 2: study of $\mathcal{J}_2$. We prove in this step that for t large enough,

$$\mathbb{P}\big[\mathcal{J}_2 \geq c_+ h_t^2 e^{-(1-\varepsilon)h_t/2}\big] \leq C_+ e^{-\kappa\varepsilon h_t/2}, \tag{4.23}$$

for some constants c_+ and C_+. Let $\mathcal{E}_3^{4.7} := \{\sup_{u\in[0,1]} \mathcal{L}_{B'}(\tau^{B'}(1), u) \leq h_t\}$, and define

$$\mathcal{E}_4^{4.7} := \big\{\inf_{[\tilde{\tau}_2(h_t/2), \tilde{\tau}_2(h_t)]} \tilde{V}^{(2)} > (1-\varepsilon)h_t/2\big\}, \qquad \mathcal{E}_5^{4.7} := \big\{\tilde{L}_2^+ - \tilde{L}_2^- \leq 40 h_t^+/\kappa\big\}.$$

We have on $\mathcal{E}_3^{4.7} \cap \mathcal{E}_4^{4.7} \cap \mathcal{E}_5^{4.7}$,

$$\mathcal{J}_2 \leq h_t \int_{\tilde{\tau}_2(h_t/2)}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)}\mathrm{d}u \leq h_t e^{-(1-\varepsilon)h_t/2}\big[\tilde{L}_2 - \tilde{\tau}_2(h_t/2)\big] \leq \frac{40 h_t^+ h_t}{\kappa} e^{-(1-\varepsilon)h_t/2}. \tag{4.24}$$

Now, Fact 2.1, equation (2.27) with $\alpha = 1/2$, $\gamma = (1-\varepsilon)/2$ and $\omega = 1$, and Lemma 2.3 give

$$P\big(\overline{\mathcal{E}_4^{4.7}}\big) \leq P\big[\inf_{[\tau_2(h_t/2), \tau_2(h_t)]} V^{(2)} \leq (1-\varepsilon)h_t/2, \mathcal{V}_t\big] + P(\overline{\mathcal{V}}_t) \leq 3e^{-\kappa\varepsilon h_t/2}.$$

Moreover, $P\big(\overline{\mathcal{E}_5^{4.7}}\big) \leq e^{-\kappa h_t/8} \leq e^{-\kappa\varepsilon h_t/2}$ by (2.32) since $\varepsilon < 1/4$, and $\mathbb{P}\big(\overline{\mathcal{E}_3^{4.7}}\big) \leq 4e^{-h_t/2}$ by (4.8). This, together with (4.24) proves (4.23).

Step 3: study of $\mathcal{J}_0$. We prove that for t large enough,

$$\mathbb{P}\big[\mathcal{J}_0 \geq 40\kappa^{-1} h_t^+ e^{-(1-4\varepsilon)h_t/2}\big] \leq C_+ e^{-\kappa\varepsilon h_t/2}. \tag{4.25}$$

Similarly as in Step 2, we introduce

$$\begin{aligned}
\mathcal{E}_6^{4.7} &:= \big\{\inf_{\tilde{L}_2^- \leq u \leq \tilde{\tau}_2^-(h_t/2)} \tilde{V}^{(2)}(u) \geq (1/2-\varepsilon)h_t\big\}, \\
\mathcal{E}_7^{4.7} &:= \big\{\sup_{s\leq 0} \mathcal{L}_{B'}(\tau^{B'}(1), s) \leq e^{\varepsilon h_t}\big\}.
\end{aligned}$$

Lemma 2.3, equation (2.27) with $\gamma = 1/2 - \varepsilon$, $\alpha = 1/2$ and $\omega = 1$, and (2.34) give

$$\begin{aligned}
P\big(\overline{\mathcal{E}_6^{4.7}}\big) \quad \leq \quad & P\big(\inf_{\tilde{L}_2^- \leq u \leq \tilde{\tau}_2^-(h_t)} \tilde{V}^{(2)}(u) < (1/2-\varepsilon)h_t\big) \\
& + P\big(\inf_{\tilde{\tau}_2^-(h_t) \leq u \leq \tilde{\tau}_2^-(h_t/2)} \tilde{V}^{(2)}(u) < (1/2-\varepsilon)h_t, \mathcal{V}_t\big) + P\big(\overline{\mathcal{V}_t}\big) \;\leq\; 3e^{-\kappa\varepsilon h_t}.
\end{aligned}$$

Moreover, by (4.9), $\mathbb{P}(\overline{\mathcal{E}_7^{4.7}}) \leq 4e^{-\varepsilon h_t}$.

Therefore, on $\mathcal{E}_5^{4.7}\cap\mathcal{E}_6^{4.7}\cap\mathcal{E}_7^{4.7}$, i.e with a probability larger than $1-C_+e^{-\kappa\varepsilon h_t/2}$, we obtain

$$\mathcal{J}_0 \leq \sup_{s\leq 0}\mathcal{L}_{B'}[\tau^{B'}(1),s]\int_{\tilde{L}_2^-}^{\tilde{\tau}_2^-(h_t/2)} e^{-\tilde{V}^{(2)}(u)}\mathrm{d}u \leq 40\kappa^{-1}h_t^+e^{-(1/2-2\varepsilon)h_t}, \tag{4.26}$$

which yields (4.25).

Step 4: study of $\mathcal{J}_1$. We prove that for t large enough,

$$\mathbb{P}\big[\mathcal{J}_1 \leq \mathbf{e}_1 e^{-\varepsilon h_t}/2\big] \leq C_+e^{-(c_-)\varepsilon h_t}. \tag{4.27}$$

First, recall that $\mathbf{e_1}=\mathcal{L}_{B'}\big(\tau^{B'}(1),0\big)$ and let

$$\mathcal{E}_8^{4.7} := \Big\{\sup_{s\in[-\delta_t,\delta_t]}\Big|\mathcal{L}_{B'}\big(\tau^{B'}(1),s\big)-\mathcal{L}_{B'}\big(\tau^{B'}(1),0\big)\Big| \leq \delta_t^{1/3}\mathcal{L}_{B'}\big(\tau^{B'}(1),0\big)\Big\}.$$

We know that $\mathbb{P}\big(\overline{\mathcal{E}_8^{4.7}}\big)\leq C_+\delta_t^{1/30}$ by (4.7). Since on $\mathcal{E}_1^{4.7}\cap\mathcal{E}_2^{4.7}$, $\tilde{A}_2(u)/\tilde{A}_2(\tilde{L}_2)\in[-\delta_t,\delta_t]$ for all $u\in[\tilde{\tau}_2^-(h_t/2),\tilde{\tau}_2(h_t/2)]$ as explained in Step 1, we get on $\mathcal{E}_1^{4.7}\cap\mathcal{E}_2^{4.7}\cap\mathcal{E}_8^{4.7}$,

$$\big(1-\delta_t^{1/3}\big)\mathcal{I}^-\mathbf{e_1} \leq \mathcal{J}_1 \leq \big(1+\delta_t^{1/3}\big)\mathcal{I}^-\mathbf{e_1}. \tag{4.28}$$

We finally prove that $\mathcal{I}^-$ is not too small, with a similar argument as before. First, we have

$$\mathcal{I}^- \geq \int_{\tilde{m}_2}^{\tilde{\tau}_2(\varepsilon h_t)} e^{-\tilde{V}^{(2)}(u)}\mathrm{d}u \geq [\tilde{\tau}_2(\varepsilon h_t)-\tilde{m}_2]e^{-\varepsilon h_t} \geq e^{-\varepsilon h_t} \tag{4.29}$$

on $\mathcal{E}_9^{4.7}$, where $\mathcal{E}_9^{4.7}:=\{\tilde{\tau}_2(\varepsilon h_t)-\tilde{m}_2\geq 1\}$. Moreover for large t,

$$P\big(\mathcal{E}_9^{4.7}\big) \geq P\big(\tau_2(\varepsilon h_t)-\tau_2(\varepsilon h_t/2)\geq 1, \mathcal{V}_t\big) \geq 1-e^{-\kappa h_t/4}$$

by Fact 2.1, (2.28) and Lemma 2.3. Let $\mathcal{E}_{10}^{4.7}:=\{\mathbf{e_1}\geq e^{-\varepsilon h_t/2}\}$, and observe that $\mathbb{P}(\mathcal{E}_{10}^{4.7})\geq 1-e^{-\varepsilon h_t/2}$ since $\mathbf{e_1}$ is exponentially distributed with mean 2. Since on $\mathcal{E}_1^{4.7}\cap\mathcal{E}_2^{4.7}\cap\mathcal{E}_8^{4.7}\cap\mathcal{E}_9^{4.7}\cap\mathcal{E}_{10}^{4.7}$,

$$\mathcal{J}_1 \geq \big(1-\delta_t^{1/3}\big)\mathcal{I}^-\mathbf{e_1} > (1/2)e^{-\varepsilon h_t}\mathbf{e_1} \geq e^{-2\varepsilon h_t} \tag{4.30}$$

for large t by (4.28) and (4.29), this gives (4.27).

Step 5: end of the proof. We have on $\cap_{i=1}^{10}\mathcal{E}_i^{4.7}$, for t large enough, by (4.18), (4.28) and (4.19),

$$\mathbf{U}/\tilde{A}_2\big(\tilde{L}_2\big) \geq \mathcal{J}_1 \geq (1-e^{-h_t(1-3\varepsilon)/6})\mathcal{I}^-\mathbf{e_1}. \tag{4.31}$$

Moreover on $\cap_{i=0}^{10}\mathcal{E}_i^{4.7}$, for t large enough,

$$\mathcal{J}_0+\mathcal{J}_2 \leq e^{(-1/2+3\varepsilon)h_t} \leq e^{(-1/2+5\varepsilon)h_t}\mathcal{J}_1 \tag{4.32}$$

by (4.26) and (4.24) for the first inequality, and (4.30) for the second one. As a consequence on $\cap_{i=0}^{10}\mathcal{E}_i^{4.7}$,

$$\mathbf{U}/\tilde{A}_2\big(\tilde{L}_2\big) \leq \big[1+e^{(-1/2+5\varepsilon)h_t}\big]\mathcal{J}_1 \leq \big[1+2e^{-h_t(1-3\varepsilon)/6}\big]\mathcal{I}^-\mathbf{e_1} \tag{4.33}$$

for large t, where we used (4.18) and (4.32) in the first inequality, and (4.28), (4.19) and $\varepsilon<2/27$ in the second one. Finally, by (4.31) and (4.33), LHS of (4.17) $\geq 1-\sum_{i=1}^{10}P\big(\overline{\mathcal{E}_i^{4.7}}\big)$, where LHS means left hand side, which proves the lemma. □

We are now ready prove Proposition 4.4, for which we use the notation of Lemma 4.5.

Proof of Proposition 4.4. The idea of the proof is that thanks to Lemma 4.7, we can already approximate $\mathbf{U}$ by $\tilde{A}_2(\tilde{L}_2)\mathcal{I}^-\mathbf{e_1}$, which is equal to $(\mathcal{I}_0^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathbf{e}_1$ on $\mathcal{V}_t$. However, $\mathcal{I}_0^+$ is not independent of $\mathcal{I}_1^-$, so we would like to replace it by a r.v $\mathcal{I}_1^+$ with the same law and independent of $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_2^+$ and $\mathbf{e_1}$. We do this by replacing in $\mathcal{I}_0^+$ the small quantity $\int_{m_2}^{\tau_2(h_t/2)} e^{V^{(2)}(s)}\mathrm{d}s$ by an independent copy of it.

More precisely, we define

$$\mathcal{I}_3^+ := \int_{\tau_2(h_t/2)}^{\tau_2(h_t)} e^{V^{(2)}(s)}\mathrm{d}s \leq \mathcal{I}_0^+. \tag{4.34}$$

By Fact 2.1 **(ii)**, $(V^{(2)}(m_2+s),\ 0\leq s\leq \tau_2(h_t)-m_2)$ is a Markov process, so $\mathcal{I}_3^+$ and $\mathcal{I}_1^-$ are independent by the strong Markov property. Moreover by Fact 2.1 **(i)**, $\mathcal{I}_2^-$ is independent of this Markov process and then is independent of $\mathcal{I}_3^+$ and $\mathcal{I}_1^-$. Also, by Fact 2.1 **(iii)**, $\mathcal{I}_2^+$ is independent of $(W_\kappa(s),\ s\leq\tau_2(h_t))$ and then of $\mathcal{I}_2^-$, $\mathcal{I}_3^+$ and $\mathcal{I}_1^-$. Finally by Lemma 4.7, $\mathbf{e_1}$ is independent of W_κ and then $\mathcal{I}_2^+$, $\mathcal{I}_3^+$, $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, and $\mathbf{e_1}$ are independent.

Furthermore, by Lemma 4.5, $\mathcal{I}_1^- \overset{\mathcal{L}}{=} F^-(h_t/2)$, $\mathcal{I}_2^- \overset{\mathcal{L}}{=} F^-(h_t/2)$ and $\mathcal{I}_2^+ \overset{\mathcal{L}}{=} G^+(h_t/2, h_t)$, $\mathbf{e}_1$ is by Lemma 4.7 independent of W_κ and exponentially distributed with mean 2. Moreover as before, by Fact 2.1, Lemma 4.5 and the strong Markov property, these r.v. $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_2^+$ and $\mathbf{e}_1$ are independent of $(V^{(2)}(s+\tau_2(h_t/2)),\ 0\leq s\leq\tau_2(h_t)-\tau_2(h_t/2))$ which has the same law as a BES$(3,\kappa/2)$ starting from $h_t/2$ and stopped when it first hits h_t.

We now consider, possibly on an enlarged probability space, a process $\big(R^{(1)}(s),\ 0\leq s\leq\tau^{R^{(1)}}(h_t/2)\big)$, independent of W_κ and $\mathbf{e_1}$ and then independent of $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_2^+$, $\mathbf{e_1}$ and $\big(V^{(2)}(s+\tau_2(h_t/2)),\ 0\leq s\leq\tau_2(h_t)-\tau_2(h_t/2)\big)$, and distributed as $\big(R(s),\ 0\leq s\leq\tau^R(h_t/2)\big)$, R being a BES$(3,\kappa/2)$ process. We now extend this process by setting $R^{(1)}(u) := V^{(2)}\big[u-\tau^{R^{(1)}}(h_t/2)+\tau_2(h_t/2)\big]$ for $\tau^{R^{(1)}}(h_t/2)\leq u\leq\tau^{R^{(1)}}(h_t/2)+\tau_2(h_t)-\tau_2(h_t/2)$. By the Strong Markov property, $\big(R^{(1)}(s),\ 0\leq s\leq\tau^{R^{(1)}}(h_t)\big)$ has the same law as $\big(R(s),\ 0\leq s\leq\tau^R(h_t)\big)$, and then

$$\mathcal{I}_1^+ := \int_0^{\tau^{R^{(1)}}(h_t)} e^{R^{(1)}(s)}\mathrm{d}s \overset{\mathcal{L}}{=} F^+(h_t), \qquad \mathcal{I}_3^+ = \int_{\tau^{R^{(1)}}(h_t/2)}^{\tau^{R^{(1)}}(h_t)} e^{R^{(1)}(s)}\mathrm{d}s. \tag{4.35}$$

Furthermore, since $\big(R^{(1)}(s),\ 0\leq s\leq\tau^{R^{(1)}}(h_t)\big)$ is obtained by gluing two processes independent of $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_2^+$ and $\mathbf{e_1}$, it is also independent of these r.v., and so $\mathcal{I}_1^+$ is also independent of these r.v. $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_2^+$ and $\mathbf{e_1}$. This already proves affirmations **(i)** and **(ii)** of Proposition 4.4.

Moreover, with the same notation as in Lemma 4.7, we have on $\mathcal{V}_t$, $\mathcal{I}^-=\mathcal{I}_1^-+\mathcal{I}_2^-$ and $\tilde{A}_2(\tilde{L}_2)=\int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)}\mathrm{d}x=\mathcal{I}_0^++\mathcal{I}_2^+$, where $\mathcal{I}_0^+=\int_{m_2}^{\tau_2(h_t)} e^{V^{(2)}(x)}\mathrm{d}x$ as defined in (4.13).

We now approximate $\mathcal{I}_0^+$ by $\mathcal{I}_1^+$. Since $\mathcal{I}_0^+-\mathcal{I}_3^+=\int_{m_2}^{\tau_2(h_t/2)} e^{V^{(2)}(s)}\mathrm{d}s \overset{\mathcal{L}}{=} F^+(h_t/2)$ by (4.34) and Fact 2.1, and since $\mathcal{I}_1^+-\mathcal{I}_3^+=\int_0^{\tau^{R^{(1)}}(h_t/2)} e^{R^{(1)}(s)}\mathrm{d}s \overset{\mathcal{L}}{=} F^+(h_t/2)$, we get

$$\begin{aligned} P\left(\mathcal{I}_1^+-\mathcal{I}_3^+ > e^{(1+\delta)h_t/2}\right) &= P\left(\mathcal{I}_0^+-\mathcal{I}_3^+ > e^{(1+\delta)h_t/2}\right) \\ = \quad P\left(\frac{F^+(h_t/2)}{e^{h_t/2}} > e^{\delta h_t/2}\right) &\leq \frac{c_1}{e^{\delta h_t/2}}, \end{aligned}$$

by Markov inequality since $E\left[F^+(y)/e^y\right]\leq c_1$ for all $y>0$ by Lemma 4.2. Moreover, $\mathcal{I}_3^+\leq\mathcal{I}_1^+$, and by (2.29), with a probability larger than $1-3e^{-\kappa\delta h_t/2}$ for large t, $\mathcal{I}_1^+\geq e^{(1-\delta)h_t}$. Therefore, with a probability greater than $1-4e^{-\kappa\delta h_t/2}$ for large t,

$$\begin{aligned} \mathcal{I}_0^+ &= \mathcal{I}_3^+ + (\mathcal{I}_0^+-\mathcal{I}_3^+) \leq \mathcal{I}_1^+ + e^{(1+\delta)h_t/2} \leq \big(1+e^{-(1-3\delta)h_t/2}\big)\mathcal{I}_1^+, \\ \mathcal{I}_0^+ &\geq \mathcal{I}_3^+ = \mathcal{I}_1^+ - (\mathcal{I}_1^+-\mathcal{I}_3^+) \geq \mathcal{I}_1^+ - e^{(1+\delta)h_t/2} \geq \big(1-e^{-(1-3\delta)h_t/2}\big)\mathcal{I}_1^+, \end{aligned}$$

and then

$$\big(1-e^{-(1-3\delta)h_t/2}\big)\mathcal{I}_1^+ \leq \mathcal{I}_0^+ \leq \big(1+e^{-(1-3\delta)h_t/2}\big)\mathcal{I}_1^+.$$

This and Lemma 2.3 give with probability at least $1-5e^{-\kappa\delta h_t/2}$,

$$\big(1-e^{-(1-3\delta)h_t/2}\big) \leq \frac{\tilde{A}_2(\tilde{L}_2)\mathcal{I}^-}{(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)} = \frac{\mathcal{I}_0^+ + \mathcal{I}_2^+}{\mathcal{I}_1^+ + \mathcal{I}_2^+} \leq \big(1+e^{-(1-3\delta)h_t/2}\big).$$

since on $\mathcal{V}_t$, $\mathcal{I}^- = \mathcal{I}_1^- + \mathcal{I}_2^-$ and $\tilde{A}_2(\tilde{L}_2) = \mathcal{I}_0^+ + \mathcal{I}_2^+$. Finally, this and Lemma 4.7 applied with $\varepsilon = \delta$ prove affirmation **(iii)** of the proposition. □

4.3 Second order of the Laplace transform of a standard exit time

We are now ready to prove Proposition 4.1. This proof is quite technical and can be skipped at first reading.

Idea of the proof of Proposition 4.1: In this proof we use Proposition 4.4 to approximate the Laplace transform of $\mathbf{U}$ by that of $(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathbf{e_1}$, and take advantage of the fact that the r.v. that appear in this product are independent. This allows us to condition first by $\mathbf{e_1}$, and then by $\sigma(\mathcal{I}_1^+, \mathcal{I}_2^+)$. We then cut this into 2 parts : one (studied in Lemma 4.8) for which $\mathcal{I}_1^+ + \mathcal{I}_2^+$ is "small", which allows us to use the approximations of the Laplace transforms provided by Lemma 4.2, and one (studied in Lemma 4.9) for which $\mathcal{I}_1^+ + \mathcal{I}_2^+$ is "big", which allows us to approximate $\mathcal{I}_1^+ + \mathcal{I}_2^+$ by a r.v. having the same law as A_∞, for which we know the density.

Proof of Proposition 4.1. We fix $\kappa \in (0,1)$, $0 < \delta < \inf(2/27, \kappa^2/2)$ and $\lambda > 0$. We have for every $t > 0$, $\mathcal{A}_t$ being defined in (4.12), and with $\varepsilon_t = 3e^{-(1-3\delta)h_t/6}$ as defined in Proposition 4.4,

$$\mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\right) = \mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\mathbb{1}_{\mathcal{A}_t}\right) + \mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\mathbb{1}_{\overline{\mathcal{A}_t}}\right) \leq \mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\mathbb{1}_{\mathcal{A}_t}\right) + \mathbb{P}\left(\overline{\mathcal{A}_t}\right).$$

Hence by the definition (4.12) of $\mathcal{A}_t$ and Proposition 4.4, we get with $\lambda_t^{\pm} := 2\lambda\left(1 \pm \varepsilon_t\right)/t$,

$$S_{0,t}^+ - C_+e^{-(c_-)\delta h_t} \leq \mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\right) \leq S_{0,t}^- + C_+e^{-(c_-)\delta h_t}, \tag{4.36}$$

where $S_{0,t}^{\pm} := \mathbb{E}\left(e^{-(\lambda_t^{\pm}/2)(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathbf{e_1}}\right)$. Let $\theta \in (3\kappa/4, \kappa)$,

$$\mathcal{B}_1 := \left\{\mathcal{I}_1^+ + \mathcal{I}_2^+ > te^{-\theta\phi(t)}\right\}, \qquad \mathcal{B}_2 := \left\{\mathcal{I}_1^+ + \mathcal{I}_2^+ \leq te^{-\theta\phi(t)}\right\} = \overline{\mathcal{B}_1}. \tag{4.37}$$

Since $\mathbf{e_1}/2$ is an exponential r.v. with mean 1 and is independent of the r.v. $\mathcal{I}_i^{\pm}$, $i \in \{1,2\}$ by Proposition 4.4, we have $S_{0,t}^{\pm} = S_{1,t}^{\pm} + S_{2,t}^{\pm}$, where for $i \in \{1,2\}$,

$$\begin{aligned} S_{i,t}^{\pm} &:= \mathbb{E}\left(e^{-(\lambda_t^{\pm}/2)(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathbf{e_1}}\mathbb{1}_{\mathcal{B}_i}\right) \\ &= \int_0^\infty E\left(\mathbb{1}_{\mathcal{B}_i}e^{-z\lambda_t^{\pm}(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)}\right)e^{-z}\mathrm{d}z \qquad (4.38) \\ &= \int_0^\infty E\Big(\mathbb{1}_{\mathcal{B}_i}\left[E\left(e^{-z\lambda_t^{\pm}(\mathcal{I}_1^+ + \mathcal{I}_2^+)\mathcal{I}_1^-}\Big|\mathcal{I}_1^+, \mathcal{I}_2^+\right)\right]^2\Big)e^{-z}\mathrm{d}z, \qquad (4.39) \end{aligned}$$

since $\mathcal{I}_1^-$ and $\mathcal{I}_2^-$ are independent and independent of $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$ and have the same law, once more by Proposition 4.4. We also define,

$$Z_t(x) := \left[1 + \frac{2\lambda_t^{\pm}x}{\kappa+1}(\mathcal{I}_1^+ + \mathcal{I}_2^+)\right]^{-1}, \quad x > 0, \qquad S_{3,t}^{\pm} := \int_0^\infty E\left(\mathbb{1}_{\mathcal{B}_1}Z_t^2(z)\right)e^{-z}\mathrm{d}z. \tag{4.40}$$

We start with the study in the case $\mathcal{I}_1^+ + \mathcal{I}_2^+$ is "small", that is, $\mathcal{I}_1^+ + \mathcal{I}_2^+ \le te^{-\theta\phi(t)}$ as is the case on $\mathcal{B}_2$. More precisely, we prove

Lemma 4.8. *As $t \to +\infty$, with $C_0 = \Gamma(1-\kappa)\Gamma(\kappa+2)/(1+\kappa)^\kappa$ as defined in Proposition 4.1,*

$$S_{2,t}^{\pm} + S_{3,t}^{\pm} = 1 - C_0 8^\kappa \lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)}). \tag{4.41}$$

Proof of Lemma 4.8. Let $a(t) := e^{-(3/4)\kappa\phi(t)}$. Now, consider $0 \le z \le \eta_1 a(t) e^{\theta\phi(t)}/(t\lambda_t^\pm) = \eta_1 a(t) e^{\theta\phi(t)}/[2\lambda(1\pm\varepsilon_t)]$, where $\eta_1 \in (0,1)$ is defined in Lemma 4.2. We have on $\mathcal{B}_2$ for such z,

$$0 \le z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+) \le \eta_1 a(t). \tag{4.42}$$

This gives by (4.3) applied to $\mathcal{I}_1^- \overset{\mathcal{L}}{=} F^-(h_t/2)$ for t so large that $h_t/2 > M$ and $a(t) \le 1$,

$$\begin{aligned} E\left(e^{-z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)\mathcal{I}_1^-} \,\middle|\, \mathcal{I}_1^+, \mathcal{I}_2^+\right) &= E\left(e^{-\gamma F^-(h_t/2)}\right)\Big|_{\gamma = z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)} \\ &\le Z_t(z) + C_4 \max\left(e^{-\kappa h_t/2}, \left[z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)\right]^{\frac{3}{2}}\right), \end{aligned}$$

on $\mathcal{B}_2$ for such z, thanks to the independence of $\mathcal{I}_1^-$ and $(\mathcal{I}_1^+, \mathcal{I}_2^+)$ for the equality. Therefore for large t, by (4.39), and since $0 \le Z_t(z) \le 1$,

$$\begin{aligned} S_{2,t}^\pm &\le \int_0^{\frac{\eta_1 a(t) e^{\theta\phi(t)}}{t\lambda_t^\pm}} E\left(\mathbb{1}_{\mathcal{B}_2} Z_t^2(z) + \mathbb{1}_{\mathcal{B}_2}(2C_4 + C_4^2)\left([z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)]^{\frac{3}{2}} + e^{-\kappa h_t/2}\right)\right) e^{-z}\mathrm{d}z \\ &\quad + \int_{\eta_1 a(t) e^{\theta\phi(t)}/(t\lambda_t^\pm)}^\infty e^{-z}\mathrm{d}z. \end{aligned}$$

We notice that $\eta_1 a(t) e^{\theta\phi(t)}/(t\lambda_t^\pm) \ge \phi(t)$ for large t since $\theta > 3\kappa/4$, and as a consequence, $\int_{\eta_1 a(t) e^{\theta\phi(t)}/(t\lambda_t^\pm)}^\infty e^{-z}\mathrm{d}z \le e^{-\phi(t)}$. Moreover $e^{-\kappa h_t/2} = o(e^{-\kappa\phi(t)})$ since $\phi(t) = o(\log t)$, and by (4.42), $\int_0^{\eta_1 a(t) e^{\theta\phi(t)}/(t\lambda_t^\pm)} E\left(\mathbb{1}_{\mathcal{B}_2}[z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)]^{3/2}\right) e^{-z}\mathrm{d}z \le [\eta_1 a(t)]^{3/2} = o(e^{-\kappa\phi(t)})$. So for large t since $0 < \kappa < 1$,

$$S_{2,t}^\pm \le \int_0^\infty E\left[\mathbb{1}_{\mathcal{B}_2} Z_t^2(z)\right] e^{-z}\mathrm{d}z + o(e^{-\kappa\phi(t)}). \tag{4.43}$$

Recall that for any random variable $Y \ge 0$, we have by Fubini,

$$E\left((1+Y)^{-2}\right) = \int_0^\infty \mathrm{d}u \int_u^\infty E\left(e^{-x(1+Y)}\right)\mathrm{d}x.$$

So, by independence of $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$, we have for every $z > 0$,

$$E\left(Z_t^2(z)\right) = \int_0^\infty \mathrm{d}u \int_u^\infty e^{-x} E\left[\exp\left(-\frac{\rho_\kappa^\pm z x\lambda \mathcal{I}_1^+}{t}\right)\right] E\left[\exp\left(-\frac{\rho_\kappa^\pm z x\lambda \mathcal{I}_2^+}{t}\right)\right]\mathrm{d}x, \tag{4.44}$$

where

$$\rho_\kappa^\pm := \frac{4\,(1\pm\varepsilon_t)}{\kappa+1}.$$

Recall that $t = e^{h_t}e^{\phi(t)}$, $\mathcal{I}_1^+ \overset{\mathcal{L}}{=} F^+(h_t)$ and $\mathcal{I}_2^+ \overset{\mathcal{L}}{=} G^+(h_t/2, h_t)$ by Proposition 4.4. So applying (4.4) and (4.5), we have whenever $\rho_\kappa^\pm zx\lambda/e^{\phi(t)} \le \eta_1$ and $h_t > M$,

$$\begin{aligned} E\left(e^{-\frac{\rho_\kappa^\pm zx\lambda\mathcal{I}_1^+}{t}}\right) &\le 1 - \frac{2\rho_\kappa^\pm zx\lambda}{(\kappa+1)e^{\phi(t)}} + C_4\max\left(e^{-\kappa h_t}, \left(\rho_\kappa^\pm zx\lambda/e^{\phi(t)}\right)^{3/2}\right), \\ E\left(e^{-\frac{\rho_\kappa^\pm zx\lambda\mathcal{I}_2^+}{t}}\right) &\le 1 - \frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}\frac{(2\rho_\kappa^\pm zx\lambda)^\kappa}{e^{\kappa\phi(t)}} + C_4\max\left(\left(\rho_\kappa^\pm zx\lambda/e^{\phi(t)}\right)^\kappa e^{-\kappa h_t/2}, \frac{\rho_\kappa^\pm zx\lambda}{e^{\phi(t)}}\right). \end{aligned}$$

So, (4.44) gives for large t,

$$\begin{aligned}
& E\left(Z_t^2(z)\right) \\
\leq\ & \int_0^{+\infty} \mathrm{d}u \int_u^{+\infty} e^{-x}\left(1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}\frac{(2\rho_\kappa^{\pm} zx\lambda)^\kappa}{e^{\kappa\phi(t)}}\right)\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}\leq\eta_1}\mathrm{d}x \\
& +C_+\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\left(\left(\frac{\rho_\kappa^{\pm} zx\lambda}{e^{\phi(t)}}\right)^\kappa e^{-\kappa h_t/2}+\frac{\rho_\kappa^{\pm} zx\lambda}{e^{\phi(t)}}+\frac{1}{e^{\kappa h_t/2}}\right)\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}\leq\eta_1}\mathrm{d}x \\
& +\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}>\eta_1}\mathrm{d}x. \qquad (4.45)
\end{aligned}$$

Now, notice that $\int_0^{+\infty} e^{-z}\mathrm{d}z\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\rho_\kappa^{\pm} zx\lambda\mathrm{d}x/e^{\phi(t)} = 2\rho_\kappa^{\pm}\lambda/e^{\phi(t)}$. Moreover, we have $\int_0^{+\infty} e^{-z}\mathrm{d}z\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\left[\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}\right]^\kappa\mathrm{d}x = O(e^{-\kappa\phi(t)})$, and furthermore $\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}>\eta_1} \leq \rho_\kappa^{\pm} zx\lambda/\left[\eta_1 e^{\phi(t)}\right]$, so $\int_0^{+\infty}\mathrm{d}ze^{-z}\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}>\eta_1}\mathrm{d}x = O(e^{-\phi(t)})$, whereas $\int_0^{+\infty}\mathrm{d}ze^{-z}\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\mathrm{d}x = 1$. This together with (4.45) and $\phi(t)=o(h_t)$ gives since $\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}x^\kappa\mathrm{d}x = \int_0^\infty x^{\kappa+1}e^{-x}\mathrm{d}x = \Gamma(\kappa+2)$ by Fubini,

$$\begin{aligned}
& \int_0^{+\infty} E\left(Z_t^2(z)\right)e^{-z}\mathrm{d}z \\
& \leq \int_0^{+\infty}\mathrm{d}ze^{-z}\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}\left(1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}\frac{(2\rho_\kappa^{\pm} zx\lambda)^\kappa}{e^{\kappa\phi(t)}}\right)\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}\leq\eta_1}\mathrm{d}x + O(\frac{1}{e^{\phi(t)}}) \\
& = 1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)}\frac{8^\kappa\lambda^\kappa}{(1+\kappa)^\kappa e^{\kappa\phi(t)}}(1\pm\varepsilon_t)^\kappa\int_0^{+\infty} e^{-z}z^\kappa\mathrm{d}z\int_0^{+\infty} x^{\kappa+1}e^{-x}\mathrm{d}x \\
& \quad +\frac{\Gamma(1-\kappa)(2\lambda\rho_\kappa^{\pm})^\kappa}{\Gamma(1+\kappa)e^{\kappa\phi(t)}}\int_0^{+\infty}\mathrm{d}ze^{-z}\int_0^{+\infty}\mathrm{d}u\int_u^{+\infty} e^{-x}(zx)^\kappa\mathbb{1}_{\rho_\kappa^{\pm} zx\lambda/e^{\phi(t)}>\eta_1}\mathrm{d}x + O(e^{-\phi(t)}) \qquad (4.46) \\
& = 1-\frac{\Gamma(1-\kappa)\Gamma(\kappa+2)8^\kappa\lambda^\kappa}{(1+\kappa)^\kappa e^{\kappa\phi(t)}}(1\pm\varepsilon_t)^\kappa + o(e^{-\kappa\phi(t)}), \qquad (4.47)
\end{aligned}$$

since, by the dominated convergence theorem, the integral in Line (4.46) goes to 0 as $t\to+\infty$ and then Line (4.46) $= o(e^{-\kappa\phi(t)})$. Combining equations (4.43), (4.40), then (4.47) and $\lim_{t\to+\infty}\varepsilon_t = 0$, we get

$$S_{2,t}^{\pm}+S_{3,t}^{\pm} \leq \int_0^{+\infty} E\left(Z_t^2(z)\right)e^{-z}\mathrm{d}z + o(e^{-\kappa\phi(t)}) \leq 1-C_0 8^\kappa\lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)}), \quad (4.48)$$

where $C_0 = \Gamma(1-\kappa)\Gamma(\kappa+2)/(1+\kappa)^\kappa$ as defined in Proposition 4.1. We prove similarly that $S_{2,t}^{\pm}+S_{3,t}^{\pm} \geq 1-C_0 8^\kappa\lambda^\kappa e^{-\kappa\phi(t)}+o(e^{-\kappa\phi(t)})$. This proves (4.41) and then the lemma. □

We now turn to the case $\mathcal{I}_1^+ + \mathcal{I}_2^+$ is "big", as is the case on $\mathcal{B}_1$. More precisely, we prove

Lemma 4.9. *We have, with* Υ_0 *as defined in Proposition 4.1,*

$$(S_{1,t}^{\pm} - S_{3,t}^{\pm}) \sim_{t\to+\infty} (8\lambda)^\kappa\Upsilon_0 e^{-\kappa\phi(t)}. \qquad (4.49)$$

Proof of Lemma 4.9. We introduce $0<\varepsilon<1/2$.

Step 1: Approximation of $\mathcal{I}_1^+ + \mathcal{I}_2^+$. Since $\theta<\kappa$, $\varepsilon e^{(1-\theta)\phi(t)} \geq \varepsilon(\log t)^2 \geq 8h_t/\kappa$ for t large enough so that $\phi(t)\geq 2(\log\log t)/(1-\kappa)$. This gives as $t\to+\infty$,

$$\begin{aligned}
P\big(\mathcal{I}_1^+ \geq \varepsilon te^{-\theta\phi(t)}\big) &\leq P\big(\tau^R(h_t)\geq\varepsilon e^{(1-\theta)\phi(t)}\big) \\
&\leq P\big(\tau^R(h_t)\geq 8h_t/\kappa\big) \leq C_+ e^{-\kappa h_t/2\sqrt{2}} = o\big(e^{-\phi(t)}\big) \qquad (4.50)
\end{aligned}$$

since $\mathcal{I}_1^+ \overset{\mathcal{L}}{=} F^+(h_t) \le e^{h_t}\tau^R(h_t)$ by Proposition 4.4, $\phi(t) = o(\log t)$, and thanks to (2.30). Moreover, $\mathcal{I}_2^+ \overset{\mathcal{L}}{=} G^+(h_t/2, h_t)$ and $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$ are independent by Proposition 4.4. So, possibly on an enlarged probability space, there exists a random variable $\tilde{A}_\infty$ equal in law to A_∞ (A_∞ being defined in (3.1) and G^+ in (4.2)) and such that $\tilde{A}_\infty$, $\mathcal{I}_1^-$, $\mathcal{I}_2^-$, $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$ are independent under P. We now introduce $\widehat{A}_\infty := e^{-h_t}\mathcal{I}_2^+ + e^{-h_t/2}\tilde{A}_\infty$. By Markov property,

$$\mathcal{I}_2^+ = e^{h_t}\widehat{A}_\infty - e^{h_t/2}\tilde{A}_\infty, \qquad \widehat{A}_\infty \overset{\mathcal{L}}{=} \tilde{A}_\infty \overset{\mathcal{L}}{=} A_\infty, \tag{4.51}$$

and $\widehat{A}_\infty$, $\mathcal{I}_1^-$, $\mathcal{I}_2^-$ and $\mathcal{I}_1^+$ are independent. We have

$$P\big(e^{h_t/2}\tilde{A}_\infty \ge \varepsilon t e^{-\theta\phi(t)}\big) \le P\big(A_\infty \ge t^{1/3}\big) \le Ct^{-\kappa/3} = o\big(e^{-\phi(t)}\big) \tag{4.52}$$

as $t \to +\infty$, since $\phi(t) = o(\log t)$ and $P(A_\infty \ge y) \le Cy^{-\kappa}$ as explained before (3.14). Now, we have on $\mathcal{B}_1 \cap \{\mathcal{I}_1^+ < \varepsilon t e^{-\theta\phi(t)}\} \cap \{e^{h_t/2}\tilde{A}_\infty < \varepsilon t e^{-\theta\phi(t)}\}$,

$$te^{-\theta\phi(t)} < \mathcal{I}_1^+ + \mathcal{I}_2^+ = \mathcal{I}_1^+ + e^{h_t}\widehat{A}_\infty - e^{h_t/2}\tilde{A}_\infty < e^{h_t}\widehat{A}_\infty + \varepsilon t e^{-\theta\phi(t)}.$$

This yields $e^{h_t}\widehat{A}_\infty \ge (1-\varepsilon)te^{-\theta\phi(t)}$, and then $\mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\frac{\varepsilon}{1-\varepsilon})e^{h_t}\widehat{A}_\infty \le (1+2\varepsilon)e^{h_t}\widehat{A}_\infty$. Similarly, $\mathcal{I}_1^+ + \mathcal{I}_2^+ \ge e^{h_t}\widehat{A}_\infty - e^{h_t/2}\tilde{A}_\infty \ge e^{h_t}\widehat{A}_\infty - \varepsilon t e^{-\theta\phi(t)} \ge (1-\frac{\varepsilon}{1-\varepsilon})e^{h_t}\widehat{A}_\infty \ge (1-2\varepsilon)e^{h_t}\widehat{A}_\infty$ on the same event. Consequently, replacing ε by $\varepsilon/2$, we get for $0<\varepsilon<1$ for large t,

$$\begin{aligned} & P\big(\mathcal{B}_1 \cap \{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}^c\big) \\ \le\ & P\big(\mathcal{I}_1^+ \ge (\varepsilon/2)te^{-\theta\phi(t)}\big) + P\big(e^{h_t/2}\tilde{A}_\infty \ge (\varepsilon/2)te^{-\theta\phi(t)}\big) \le e^{-\phi(t)}, \end{aligned} \tag{4.53}$$

by (4.50) and (4.52). Similarly, on $\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \ge te^{-\theta\phi(t)}\} \cap \{\mathcal{I}_1^+ + \mathcal{I}_2^+ > (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}$, we have by (4.51),

$$\varepsilon te^{-\theta\phi(t)}/(1+\varepsilon) \le \varepsilon e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ - e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+.$$

Moreover, we have on $\big\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \ge te^{-\theta\phi(t)}\big\} \cap \big\{\mathcal{I}_1^+ + \mathcal{I}_2^+ < (1-\varepsilon)e^{h_t}\widehat{A}_\infty\big\}$, again by (4.51),

$$\varepsilon te^{-\theta\phi(t)}/(1+\varepsilon) \le \varepsilon e^{h_t}\widehat{A}_\infty \le e^{h_t}\widehat{A}_\infty - \mathcal{I}_1^+ - \mathcal{I}_2^+ \le e^{h_t/2}\tilde{A}_\infty.$$

Consequently for large t, the last inequality being obtained as in (4.50) and (4.52),

$$\begin{aligned} & P\big(\big\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \ge te^{-\theta\phi(t)}\big\} \cap \big\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\big\}^c\big) \\ \le\ & P\big(\mathcal{I}_1^+ \ge [\varepsilon/(1+\varepsilon)]te^{-\theta\phi(t)}\big) + P\big(e^{h_t/2}\tilde{A}_\infty \ge [\varepsilon/(1+\varepsilon)]te^{-\theta\phi(t)}\big) \le e^{-\phi(t)}. \end{aligned} \tag{4.54}$$

Step 2: Simplification. Thanks to (4.38), (4.40) first and then the definition (4.37) of $\mathcal{B}_1$, (4.53) and $|e^{-(\dots)} - Z_t^2(u))| \le 1$, we get for large t,

$$\begin{aligned} S_{1,t}^\pm - S_{3,t}^\pm &= \int_0^\infty E\left[\left(e^{-\lambda_t^\pm u(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_t^2(u)\right)\mathbb{1}_{\mathcal{B}_1}\right]e^{-u}\mathrm{d}u \\ &\le \int_0^\infty E\left[\left(e^{-\lambda_t^\pm u(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_t^2(u)\right)\mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ > te^{-\theta\phi(t)}\}}\right. \\ &\qquad \left.\times \mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]e^{-u}\mathrm{d}u + e^{-\phi(t)}. \end{aligned}$$

Consequently, using $0 \le Z_{t,\varepsilon,\infty}(u) \le Z_t(u)$ in the expectation for $Z_{t,\varepsilon,\infty}$ defined below,

$$\begin{aligned} S_{1,t}^\pm - S_{3,t}^\pm &\le \int_0^\infty E\left[\left(e^{-\lambda_t^\pm u(1-\varepsilon)e^{h_t}\widehat{A}_\infty(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_{t,\varepsilon,\infty}^2(u)\right)\mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ > te^{-\theta\phi(t)}\}}\right. \\ &\qquad \left.\times \mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]e^{-u}\mathrm{d}u + e^{-\phi(t)} \\ &= S_{4,t,\varepsilon}^\pm + S_{5,t,\varepsilon}^\pm + e^{-\phi(t)}, \end{aligned} \tag{4.55}$$

where

$$Z_{t,\varepsilon,\infty}(u) := \left[1 + 2\lambda_t^{\pm} u(1+\varepsilon)e^{h_t}\widehat{A}_\infty/(\kappa+1)\right]^{-1},$$

and

$$S_{4,t,\varepsilon}^{\pm} := \int_0^\infty e^{-u} E\left[\left(e^{-\lambda_t^{\pm} u(1-\varepsilon)e^{h_t}\widehat{A}_\infty(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_{t,\varepsilon,\infty}^2(u)\right)\mathbb{1}_{\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \geq te^{-\theta\phi(t)}\}}\right.$$
$$\left.\mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]\mathrm{d}u,$$

$$S_{5,t,\varepsilon}^{\pm} := \int_0^\infty e^{-u} E\left[\left(e^{-\lambda_t^{\pm} u(1-\varepsilon)e^{h_t}\widehat{A}_\infty(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_{t,\varepsilon\infty}^2(u)\right)\right.$$
$$\left.[\mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ > te^{-\theta\phi(t)}\}} - \mathbb{1}_{\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \geq te^{-\theta\phi(t)}\}}]\mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]\mathrm{d}u.$$

We will prove in our Step 5 that $S_{5,t,\varepsilon}^{\pm}$ is negligible. So, we start with $S_{4,t,\varepsilon}^{\pm}$. Using (4.54) and $|e^{-(\dots)} - Z_{t,\varepsilon,\infty}(u)^2)| \leq 1$ leads to

$$\left|S_{4,t,\varepsilon}^{\pm} - \int_0^\infty E\left[\left(e^{-\lambda_t^{\pm} u(1-\varepsilon)e^{h_t}\widehat{A}_\infty(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_{t,\varepsilon,\infty}^2(u)\right)\mathbb{1}_{\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \geq te^{-\theta\phi(t)}\}}\right]\frac{\mathrm{d}u}{e^u}\right| \leq \frac{1}{e^{\phi(t)}}. \tag{4.56}$$

Recall that by (4.51) and Proposition 4.4, $\mathcal{I}_1^-$, $\mathcal{I}_2^-$ and $\widehat{A}_\infty$ are independent, and $\mathcal{I}_1^- \overset{\mathcal{L}}{=} \mathcal{I}_2^-$. So, the integral in (4.56) can be written as

$$\int_0^\infty E\left[\left(E\left(e^{-\lambda_t^{\pm} u(1-\varepsilon)e^{h_t}\widehat{A}_\infty\mathcal{I}_1^-}|\widehat{A}_\infty\right)^2 - Z_{t,\varepsilon\infty}^2(u)\right)\mathbb{1}_{\widehat{A}_\infty \geq \frac{e^{(1-\theta)\phi(t)}}{1+\varepsilon}}\right]e^{-u}\mathrm{d}u$$
$$\leq \int_0^\infty E\left[\left(E\left(e^{-4\lambda(1-\varepsilon)^2 ue^{-\phi(t)}\gamma_\kappa^{-1}\mathcal{I}_1^-}|\gamma_\kappa\right)^2 - \left[1 + \frac{8\lambda(1+\varepsilon)^2 u}{(\kappa+1)e^{\phi(t)}\gamma_\kappa}\right]^{-2}\right)\mathbb{1}_{\gamma_\kappa \leq \frac{2(1+\varepsilon)}{e^{(1-\theta)\phi(t)}}}\right]\frac{\mathrm{d}u}{e^u}, \tag{4.57}$$

where $\gamma_\kappa := 2/\widehat{A}_\infty$, for t large enough so that $2\lambda(1-\varepsilon)/t \leq \lambda_t^{\pm} \leq 2\lambda(1+\varepsilon)/t$. Since γ_κ has density $x^{\kappa-1}e^{-x}\mathbb{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$ by Fact 3.1 (Dufresne), and is, as $\widehat{A}_\infty$, independent of $\mathcal{I}_1^-$, the RHS of (4.57) is equal to

$$\int_0^\infty e^{-u}\mathrm{d}u\int_0^{\frac{2(1+\varepsilon)}{e^{(1-\theta)\phi(t)}}}\left[E\left(e^{-4\lambda(1-\varepsilon)^2 ue^{-\phi(t)}x^{-1}\mathcal{I}_1^-}\right)^2 - \left(1 + \frac{8\lambda(1+\varepsilon)^2 u}{(\kappa+1)e^{\phi(t)}x}\right)^{-2}\right]\frac{x^{\kappa-1}e^{-x}}{\Gamma(\kappa)}\mathrm{d}x.$$

With the change of variables $y = 8u\lambda e^{-\phi(t)}x^{-1}$, this is equal to $(8\lambda)^\kappa e^{-\kappa\phi(t)}\Upsilon_{t,\epsilon}$, with $\Upsilon_{t,\epsilon} := \int_0^\infty\int_0^\infty f_{t,\varepsilon}(u,y)\mathrm{d}y\mathrm{d}u$ and for every $u > 0$ and $y > 0$,

$$f_{t,\varepsilon}(u,y)$$
$$:= \mathbb{1}_{\{y \geq \frac{4\lambda ue^{-\theta\phi(t)}}{1+\varepsilon}\}}\frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left[E\left(e^{-(1-\varepsilon)^2 y\mathcal{I}_1^-/2}\right)^2 - \left(1 + \frac{(1+\varepsilon)^2 y}{\kappa+1}\right)^{-2}\right]\frac{e^{-8\lambda uy^{-1}e^{-\phi(t)}}}{y^{\kappa+1}}.$$

Consequently by (4.56),

$$S_{4,t,\varepsilon}^{\pm} \leq (8\lambda)^\kappa e^{-\kappa\phi(t)}\Upsilon_{t,\epsilon} + e^{-\phi(t)}. \tag{4.58}$$

Step 3: pointwise convergence.

Notice that thanks to (4.6), $\lim_{x\to+\infty} E\big(e^{-\gamma F^-(x)}\big) = \frac{(2\gamma)^{\kappa/2}}{\kappa\Gamma(\kappa)I_\kappa(2\sqrt{2\gamma})}$ for $\gamma > 0$, and recall that $\mathcal{I}_1^- \overset{\mathcal{L}}{=} F^-(h_t/2)$ by Proposition 4.4. Hence, for every $0 < \varepsilon < 1$, $u > 0$ and $y > 0$,

$$f_{t,\varepsilon}(u,y) \underset{t\to+\infty}{\longrightarrow} f_\varepsilon(u,y) := \frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left[\frac{(1-\varepsilon)^{2\kappa}y^\kappa}{[\Gamma(\kappa+1)I_\kappa(2(1-\varepsilon)\sqrt{y})]^2} - \left(1 + \frac{(1+\varepsilon)^2 y}{\kappa+1}\right)^{-2}\right]\frac{1}{y^{\kappa+1}}. \tag{4.59}$$

Step 4: dominated convergence. We notice that $f_{t,\varepsilon}(u,y) = a_{t,\varepsilon}(u,y) + b_{t,\varepsilon}(u,y)$ and $f_\varepsilon(u,y) = a_\varepsilon(u,y) + b_\varepsilon(u,y)$ for every $0 < \varepsilon < 1$, $u > 0$ and $y > 0$, where

$$\begin{aligned} & a_{t,\varepsilon}(u,y) \\ := \ & \mathbb{1}_{\{y \geq \frac{4\lambda u e^{-\theta\phi(t)}}{1+\varepsilon}\}} \frac{u^\kappa e^{-u}}{\Gamma(\kappa)} \left[E\left(e^{-(1-\varepsilon)^2 y \mathcal{I}_1^-/2} \right)^2 - \left(1 + \frac{(1-\varepsilon)^2 y}{\kappa+1} \right)^{-2} \right] \frac{e^{-8\lambda u y^{-1} e^{-\phi(t)}}}{y^{\kappa+1}}, \\ & b_{t,\varepsilon}(u,y) \\ := \ & \mathbb{1}_{\{y \geq \frac{4\lambda u e^{-\theta\phi(t)}}{1+\varepsilon}\}} \frac{u^\kappa e^{-u}}{\Gamma(\kappa)} \left[\left(1 + \frac{(1-\varepsilon)^2 y}{\kappa+1} \right)^{-2} - \left(1 + \frac{(1+\varepsilon)^2 y}{\kappa+1} \right)^{-2} \right] \frac{e^{-8\lambda u y^{-1} e^{-\phi(t)}}}{y^{\kappa+1}}, \end{aligned}$$

and their pointwise limits on $(\mathbb{R}_+^*)^2$ as $t \to +\infty$ are respectively

$$\begin{aligned} a_\varepsilon(u,y) &:= \frac{u^\kappa e^{-u}}{\Gamma(\kappa)} \left[\frac{(1-\varepsilon)^{2\kappa} y^\kappa}{[\Gamma(\kappa+1) I_\kappa(2(1-\varepsilon)\sqrt{y})]^2} - \left(1 + \frac{(1-\varepsilon)^2 y}{\kappa+1} \right)^{-2} \right] y^{-\kappa-1}, \\ b_\varepsilon(u,y) &:= \frac{u^\kappa e^{-u}}{\Gamma(\kappa)} \left[\left(1 + \frac{(1-\varepsilon)^2 y}{\kappa+1} \right)^{-2} - \left(1 + \frac{(1+\varepsilon)^2 y}{\kappa+1} \right)^{-2} \right] y^{-\kappa-1}. \end{aligned} \tag{4.60}$$

Since $\forall x > 0,\ I_\kappa(x) > \frac{(x/2)^\kappa}{\Gamma(\kappa+1)} + \frac{(x/2)^{\kappa+2}}{\Gamma(\kappa+2)}$ due to the series expansion of I_κ (see e.g. [7] p. 638), we get $\frac{\gamma^\kappa}{[\Gamma(\kappa+1) I_\kappa(2\sqrt{\gamma})]^2} - \left(1 + \frac{\gamma}{\kappa+1} \right)^{-2} < 0$ for every $\gamma > 0$. One consequence of this inequality is that $\Upsilon_0 < 0$, Υ_0 being defined in (4.1), and another one is that $\forall (u,y) \in (\mathbb{R}_+^*)^2,\ a_\varepsilon(u,y) < 0$. This, and the fact that $x \mapsto E\left(e^{-\gamma F^-(x)} \right)$ is nonincreasing for $\gamma \geq 0$, lead to

$$\forall (u,y) \in (\mathbb{R}_+^*)^2, \quad a_\varepsilon(u,y) \leq a_{t,\varepsilon}(u,y) \leq \frac{u^\kappa e^{-u}}{\Gamma(\kappa)} \left[1 - \left(1 + \frac{(1-\varepsilon)^2 y}{\kappa+1} \right)^{-2} \right] y^{-\kappa-1} =: h_\varepsilon(u,y). \tag{4.61}$$

Hence, $|a_{t,\varepsilon}(u,y)| \leq |a_\varepsilon(u,y)| + |h_\varepsilon(u,y)|$ and $|b_{t,\varepsilon}(u,y)| \leq |b_\varepsilon(u,y)|$. for every $(u,y) \in \mathbb{R}_+^2$ and $0 < \varepsilon < 1$. Moreover, since $0 < \kappa < 1$, $|h_\varepsilon|$, $|b_\varepsilon|$ and $|a_\varepsilon|$ have finite integrals over $\mathbb{R}_+^2$ (notice for example that $e^u u^{-\kappa} a(u,y) = O(y^{-\kappa})$ as $y \to 0$, since $I_\kappa(x) = (x/2)^\kappa/\Gamma(\kappa+1) + (x/2)^{\kappa+2}/\Gamma(\kappa+2) + o(x^{\kappa+2})$ as $x \to 0$ by (6.1) below, and $-y^{-\kappa-1} u^\kappa e^{-u}/\Gamma(\kappa) \leq a_\varepsilon(u,y) < 0$ as $y \to +\infty$). Thus, by the dominated convergence theorem,

$$\lim_{t \to +\infty} \Upsilon_{t,\epsilon} = \int_0^\infty \int_0^\infty f_\varepsilon(u,y) \mathrm{d}y \mathrm{d}u =: \Upsilon_\epsilon, \qquad 0 < \varepsilon < 1. \tag{4.62}$$

Hence applying (4.58),

$$\limsup_{t \to +\infty} \left(S_{4,t,\varepsilon}^\pm e^{\kappa\phi(t)} \lambda^{-\kappa} \right) \leq 8^\kappa \Upsilon_\epsilon, \qquad 0 < \varepsilon < 1. \tag{4.63}$$

Step 5: Let $0 < \varepsilon < 1$; we now prove that $S_{5,t,\varepsilon}^\pm$ is negligible. First, we have

$$\begin{aligned} S_{5,t,\varepsilon}^\pm = & \int_0^\infty e^{-u} E\Big[\Big(Z_{t,\varepsilon,\infty}^2(u) - e^{-\lambda_t^\pm u(1-\varepsilon) e^{h_t} \widehat{A}_\infty (\mathcal{I}_1^- + \mathcal{I}_2^-)} \Big) \\ & \mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ \leq t e^{-\theta\phi(t)}\}} \mathbb{1}_{\{(1+\varepsilon) e^{h_t} \widehat{A}_\infty \geq t e^{-\theta\phi(t)}\}} \mathbb{1}_{\{(1-\varepsilon) e^{h_t} \widehat{A}_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon) e^{h_t} \widehat{A}_\infty\}} \Big] \mathrm{d}u, \end{aligned}$$

where we used $(\mathbb{1}_{E_1} - \mathbb{1}_{E_2}) \mathbb{1}_{E_3} = -\mathbb{1}_{\overline{E_1}} \mathbb{1}_{E_2} \mathbb{1}_{E_3} + \mathbb{1}_{E_1} \mathbb{1}_{\overline{E_2}} \mathbb{1}_{E_3}$, for events E_i, $1 \leq i \leq 3$ with $\mathbb{1}_{E_1} \mathbb{1}_{\overline{E_2}} \mathbb{1}_{E_3} = 0$ in our case. Conditioning by $\sigma\big(\widehat{A}_\infty, \mathcal{I}_1^+, \mathcal{I}_2^+ \big)$, which is independent of $\mathcal{I}_1^-$

and $\mathcal{I}_2^-$ gives since $\mathcal{I}_1^-$ and $\mathcal{I}_2^-$ are independent,

$$\begin{aligned} & S_{5,t,\varepsilon}^{\pm} \\ = & \int_0^\infty e^{-u} E\left[\left(Z_{t,\varepsilon,\infty}^2(u) - E\left(e^{-\lambda_t^\pm u(1-\varepsilon)e^{h_t}\widehat{A}_\infty \mathcal{I}_1^-}\Big|\widehat{A}_\infty\right)^2\right)\right. \\ & \left.\mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ \le te^{-\theta\phi(t)}\}}\mathbb{1}_{\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \ge te^{-\theta\phi(t)}\}}\mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]\mathrm{d}u \\ \le & \int_0^\infty e^{-u} E\left[\left(\left[1 + \frac{4\lambda u(1-\varepsilon^2)e^{-\phi(t)}\widehat{A}_\infty}{\kappa+1}\right]^{-2} - E\left(e^{-2\lambda u(1-\varepsilon^2)e^{-\phi(t)}\widehat{A}_\infty \mathcal{I}_1^-}\Big|\widehat{A}_\infty\right)^2\right)\right. \\ & \left.\mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ \le te^{-\theta\phi(t)}\}}\mathbb{1}_{\{(1+\varepsilon)e^{h_t}\widehat{A}_\infty \ge te^{-\theta\phi(t)}\}}\mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le \mathcal{I}_1^+ + \mathcal{I}_2^+ \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]\mathrm{d}u \quad (4.64) \end{aligned}$$

for large t where we used $2\lambda(1-\varepsilon)/t \le \lambda_t^\pm \le 2\lambda(1+\varepsilon)/t$. Consequently,

$$\begin{aligned} S_{5,t,\varepsilon}^{\pm} \le & \int_0^\infty e^{-u} E\left[\left|\left[1 + \frac{4\lambda u(1-\varepsilon^2)e^{-\phi(t)}\widehat{A}_\infty}{\kappa+1}\right]^{-2} - E\left(e^{-2\lambda u(1-\varepsilon^2)e^{-\phi(t)}\widehat{A}_\infty \mathcal{I}_1^-}\Big|\widehat{A}_\infty\right)^2\right|\right. \\ & \left.\mathbb{1}_{\{(1-\varepsilon)e^{h_t}\widehat{A}_\infty \le te^{-\theta\phi(t)} \le (1+\varepsilon)e^{h_t}\widehat{A}_\infty\}}\right]\mathrm{d}u. \end{aligned} \quad (4.65)$$

As in Step 2 after (4.57), since $\gamma_\kappa = 2/\widehat{A}_\infty$ has density $x^{\kappa-1}e^{-x}\mathbb{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$, and $\widehat{A}_\infty$ and then γ_κ are independent of $\mathcal{I}_1^-$, the right hand side of (4.65) is equal to

$$\int_0^\infty e^{-u}\int_{\frac{2(1-\varepsilon)}{e^{(1-\theta)\phi(t)}}}^{\frac{2(1+\varepsilon)}{e^{(1-\theta)\phi(t)}}}\left|\left[1 + \frac{8\lambda u(1-\varepsilon^2)}{(\kappa+1)e^{\phi(t)}x}\right]^{-2} - E\left(e^{-4\lambda u(1-\varepsilon^2)e^{-\phi(t)}x^{-1}\mathcal{I}_1^-}\right)^2\right|\frac{x^{\kappa-1}e^{-x}}{\Gamma(\kappa)}\mathrm{d}x\mathrm{d}u. \quad (4.66)$$

With the change of variables $y = 8u\lambda e^{-\phi(t)}x^{-1}$, this is equal to

$$\frac{(8\lambda)^\kappa}{e^{\kappa\phi(t)}}\int_0^\infty \frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\int_{\frac{4u\lambda}{(1+\varepsilon)e^{\theta\phi(t)}}}^{\frac{4u\lambda}{(1-\varepsilon)e^{\theta\phi(t)}}}\left|E\left(e^{-y(1-\varepsilon^2)\mathcal{I}_1^-/2}\right)^2 - \left[1 + \frac{(1-\varepsilon^2)y}{\kappa+1}\right]^{-2}\right|\frac{\mathrm{d}y\mathrm{d}u}{y^{\kappa+1}e^{8u\lambda e^{-\phi(t)}y^{-1}}}.$$

So by (4.65),

$$S_{5,t,\varepsilon}^{\pm} \le \frac{(8\lambda)^\kappa}{e^{\kappa\phi(t)}}\int_0^\infty\int_0^\infty \left|F_{t,(1-\varepsilon^2),(1-\varepsilon^2),\varepsilon}(u,y)\right|\mathrm{d}y\mathrm{d}u \quad (4.67)$$

where for $z_1 \ge 0$ and $z_2 \ge 0$,

$$F_{t,z_1,z_2,\varepsilon}(u,y) := \mathbb{1}_{\{\frac{4\lambda ue^{-\theta\phi(t)}}{1+\varepsilon} \le y \le \frac{4\lambda ue^{-\theta\phi(t)}}{1-\varepsilon}\}}\frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left[E\left(e^{-z_1 y\mathcal{I}_1^-/2}\right)^2 - \left(1 + \frac{z_2 y}{\kappa+1}\right)^{-2}\right]\frac{1}{y^{\kappa+1}}.$$

Using the same method, we get for large t,

$$S_{5,t,\varepsilon}^{\pm} \ge -\frac{(8\lambda)^\kappa}{e^{\kappa\phi(t)}}\int_0^\infty\int_0^\infty \left|F_{t,(1-\varepsilon)^2,(1+\varepsilon)^2,\varepsilon}(u,y)\right|\mathrm{d}y\mathrm{d}u.$$

This together with (4.67) gives

$$|S_{5,t,\varepsilon}^{\pm}|\frac{e^{\kappa\phi(t)}}{(8\lambda)^\kappa} \le \int_0^\infty\int_0^\infty \left(\left|F_{t,(1-\varepsilon^2),(1-\varepsilon^2),\varepsilon}(u,y)\right| + \left|F_{t,(1-\varepsilon)^2,(1+\varepsilon)^2,\varepsilon}(u,y)\right|\right)\mathrm{d}y\mathrm{d}u. \quad (4.68)$$

We have for every $z_1 \ge 0$, $z_2 \ge 0$,

$$F_{t,z_1,z_2,\varepsilon}(u,y) \le \frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left[1 - \left(1 + \frac{z_2 y}{\kappa+1}\right)^{-2}\right]\frac{1}{y^{\kappa+1}}, \qquad u \ge 0,\ y > 0, \quad (4.69)$$

which has a finite integral over $\mathbb{R}_+^2$ since $0<\kappa<1$. We recall that $\mathcal{I}_1^- \overset{\mathcal{L}}{=} F^-(h_t/2)$ by Proposition 4.4, and that $\lim_{x\to+\infty} E\left(e^{-\gamma F^-(x)}\right) = \frac{(2\gamma)^{\kappa/2}}{\kappa\Gamma(\kappa)I_\kappa(2\sqrt{2\gamma})} \in [0,1]$ for $\gamma>0$ by (4.6). Also, $x\mapsto E\left(e^{-\gamma F^-(x)}\right)$ is nonincreasing for $\gamma>0$. So,

$$\begin{aligned} & F_{t,z_1,z_2,\varepsilon}(u,y) \\ \geq\ & \mathbb{1}_{\{\frac{4\lambda u e^{-\theta\phi(t)}}{1+\varepsilon}\leq y\leq \frac{4\lambda u e^{-\theta\phi(t)}}{1-\varepsilon}\}} \frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left[\frac{(z_1y)^\kappa}{[\kappa\Gamma(\kappa)I_\kappa(2\sqrt{z_1y})]^2} - \left(1+\frac{z_2y}{\kappa+1}\right)^{-2}\right]\frac{1}{y^{\kappa+1}} \\ \geq\ & -\frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left|\frac{(z_1y)^\kappa}{[\kappa\Gamma(\kappa)I_\kappa(2\sqrt{z_1y})]^2} - \left(1+\frac{z_2y}{\kappa+1}\right)^{-2}\right|\frac{1}{y^{\kappa+1}}. \end{aligned} \tag{4.70}$$

Notice that RHS of (4.70) has a finite integral over $\mathbb{R}_+^2$, since it is, for fixed u, $O(y^{-\kappa})$ as $y\to 0$ and $O(y^{-\kappa-1})$ as $y\to+\infty$, where we used $I_\kappa(x)=(x/2)^\kappa/\Gamma(\kappa+1)+(x/2)^{\kappa+2}/\Gamma(\kappa+2)+o(x^{\kappa+2})$ as $x\to 0$ by (6.1) below, and RHS of (4.70) $\geq -2u^\kappa e^{-u}y^{-\kappa-1}/\Gamma(\kappa)$. This and (4.69) prove that $|F_{t,z_1,z_2,\varepsilon}|$ is dominated by some function which does not depend on t and has a finite integral over $\mathbb{R}_+^2$. Moreover, we notice that $\left|F_{t,(1-\varepsilon^2),(1-\varepsilon^2),\varepsilon}(u,y)\right| + \left|F_{t,(1-\varepsilon)^2,(1+\varepsilon)^2,\varepsilon}(u,y)\right| \to_{t\to+\infty} 0$ for every $(u,y)\in(\mathbb{R}_+^*)^2$ and $0<\varepsilon<1$. So, by the dominated convergence theorem, RHS of (4.68) $\to_{t\to+\infty} 0$. So by (4.68), we get for every $0<\varepsilon<1$,

$$S^\pm_{5,t,\varepsilon} = o(e^{-\kappa\phi(t)}), \qquad t\to+\infty. \tag{4.71}$$

Step 6: We now prove that $\Upsilon_\epsilon \to_{\varepsilon\to 0} \Upsilon_0$, and end the proof of Lemma 4.9. We recall that $\Upsilon_\epsilon = \int_0^\infty\int_0^\infty f_\varepsilon(u,y)\mathrm{d}y\mathrm{d}u$ as introduced in (4.62), where f_ε is defined in (4.59). First, notice that for every $(u,y)\in(0,\infty)^2$,

$$f_\varepsilon(u,y) \to_{\varepsilon\to 0} \frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\left[\frac{y^\kappa}{[\Gamma(\kappa+1)I_\kappa(2\sqrt{y})]^2} - \left(1+\frac{y}{\kappa+1}\right)^{-2}\right]y^{-\kappa-1} =: f_0(u,y).$$

Using $|(1+c(1-x)^2)^{-2} - (1+c(1+x)^2)^{-2}| \leq 16cx$ every for $x\in[0,1]$ and $c>0$, coming from Mean value theorem, we have for b_ε defined in (4.60), for every $0<\varepsilon<1$,

$$|b_\varepsilon(u,y)| \leq u^\kappa e^{-u}y^{-\kappa-1}\Gamma(\kappa)^{-1}[16y\mathbb{1}_{\{y<\kappa+1\}}/(\kappa+1) + 2\mathbb{1}_{\{y\geq\kappa+1\}}], \qquad u>0,\ y>0,$$

which has a finite integral over $(\mathbb{R}_+^*)^2$. Moreover, $\left|\frac{\gamma^\kappa}{[\Gamma(\kappa+1)I_\kappa(2\sqrt{\gamma})]^2} - \left(1+\frac{\gamma}{\kappa+1}\right)^{-2}\right|$ is bounded by 2 on $\mathbb{R}_+$ as in (4.70), and is $o(\gamma)$ as $\gamma\to 0$. and so is less than $|\gamma|$ for every γ in some nonempty interval $[0,c)$. So, $|a_\varepsilon(u,y)| \leq u^\kappa e^{-u}y^{-\kappa-1}\Gamma(\kappa)^{-1}[|y|\mathbb{1}_{\{y\leq c\}} + 2\mathbb{1}_{\{y>c\}}]$ which also has a finite integral over $(\mathbb{R}_+^*)^2$. Since $|f_\varepsilon(u,y)| \leq |a_\varepsilon(u,y)| + |b_\varepsilon(u,y)|$ for every $(u,y)\in(\mathbb{R}_+^*)^2$ (see the start of Step 4), the dominated convergence theorem gives

$$\Upsilon_\epsilon \to_{\varepsilon\to 0} \int_0^\infty \frac{u^\kappa e^{-u}}{\Gamma(\kappa)}\mathrm{d}u\int_0^\infty\left[\frac{y^\kappa}{[\Gamma(\kappa+1)I_\kappa(2\sqrt{y})]^2} - \left(1+\frac{y}{\kappa+1}\right)^{-2}\right]y^{-\kappa-1}\mathrm{d}y = \Upsilon_0, \tag{4.72}$$

as defined in (4.1).

We proved that $S^\pm_{5,t,\varepsilon} = o(e^{-\kappa\phi(t)})$ (in (4.71) in Step 5) and also (in (4.63) in Step 4) that $\limsup_{t\to+\infty}[S^\pm_{4,t,\varepsilon}e^{\kappa\phi(t)}\lambda^{-\kappa}] \leq 8^\kappa\Upsilon_\epsilon$. So, $\limsup_{t\to+\infty}[(S^\pm_{4,t,\varepsilon}+S^\pm_{5,t,\varepsilon})e^{\kappa\phi(t)}\lambda^{-\kappa}] \leq 8^\kappa\Upsilon_\epsilon$. Then by (4.55), (4.72) and since $\kappa<1$,

$$\limsup_{t\to+\infty}\left[(S^\pm_{1,t} - S^\pm_{3,t})e^{\kappa\phi(t)}\lambda^{-\kappa}\right] \leq 8^\kappa\Upsilon_\epsilon \to_{\varepsilon\to 0} 8^\kappa\Upsilon_0, \tag{4.73}$$

so LHS of (4.73) $\leq 8^\kappa\Upsilon_0$. We prove similarly that $\liminf_{t\to+\infty}\left[(S^\pm_{1,t} - S^\pm_{3,t})e^{\kappa\phi(t)}\lambda^{-\kappa}\right] \geq 8^\kappa\Upsilon_0$. This proves (4.49) and then Lemma 4.9. □

Conclusion: This and (4.41) gives $S_{0,t}^{\pm} = S_{1,t}^{\pm} + S_{2,t}^{\pm} = 1 + (\Upsilon_0 - C_0)8^\kappa \lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)})$. This and (4.36) give, since $\phi(t) = o(\log t)$,

$$1 - \mathbb{E}\left(e^{-\frac{\lambda}{t}\mathbf{U}}\right) \sim_{t\to+\infty} 8^\kappa (C_0 - \Upsilon_0)\lambda^\kappa e^{-\kappa\phi(t)},$$

which proves Proposition 4.1. □

5 Proof of the main results

5.1 The renewal results :

In this subsection we prove Propositions 1.4, 1.6 and Corollary 1.5. We start with some important intermediate result on the exit time $\mathbf{U}$.
In what follows, we mainly use the same ideas as in Enriquez et al. ([24] pages 441 to 443), inspired by the book of Feller ([27] pages 470 to 472). We start with the following lemma, for which we recall that for all $i \geq 1$, $U_i = H(\tilde{L}_i) - H(\tilde{m}_i)$, and C_κ is defined in Proposition 4.1.

Lemma 5.1. *Assume $0 < \kappa < 1$ and $0 < \delta < \inf\{2/27, \kappa^2/2\}$. For $t > 0$, let μ_t be the positive measure on $\mathbb{R}_+$ such that*

$$\forall x \geq 0, \qquad \mu_t([0,x]) := e^{-\kappa\phi(t)} \sum_{j=1}^{n_t-2} \mathbb{P}\left(\sum_{i=1}^{j} \frac{U_i}{t} \leq x\right).$$

Then, $(\mu_t)_t$ converges vaguely as $t \to +\infty$ to μ defined by

$$\mathrm{d}\mu(x) := (C_\kappa \Gamma(\kappa))^{-1} x^{\kappa-1} \mathbb{1}_{(0,+\infty)}(x)\mathrm{d}x.$$

Moreover when $t \to +\infty$, $x \mapsto e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t \geq x)$ converges uniformly on every compact subset of $(0,+\infty)$ to $x \mapsto C_\kappa x^{-\kappa}/\Gamma(1-\kappa)$.

Proof of Lemma 5.1. First, let us prove that for all $\lambda > 0$, we have as $t \to +\infty$,

$$\int_0^{+\infty} e^{-\lambda x}\mathrm{d}\mu_t(x) = \int_0^{+\infty} \frac{e^{-\lambda x}x^{\kappa-1}}{C_\kappa\Gamma(\kappa)}\mathrm{d}x + o(1), \tag{5.1}$$

$$\int_0^{+\infty} e^{-\lambda x}e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t \geq x)\mathrm{d}x = \int_0^{+\infty} e^{-\lambda x}\frac{C_\kappa}{\Gamma(1-\kappa)x^\kappa}\mathrm{d}x + o(1). \tag{5.2}$$

Let $\lambda > 0$. First, we have, by Proposition 3.4,

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda x}\mathrm{d}\mu_t(x) &= \frac{1}{e^{\kappa\phi(t)}} \sum_{j=1}^{n_t-2} \mathbb{E}\Big(e^{-\lambda\sum_{i=1}^{j}\frac{U_i}{t}}\Big) \\ &= \sum_{j=1}^{n_t-2} \frac{1}{e^{\kappa\phi(t)}}\left[\mathbb{E}\left(e^{-\lambda\frac{\mathbf{U}}{t}}\right)\right]^j + O\left(\frac{(n_t)^2 e^{-\delta\kappa h_t}}{e^{\kappa\phi(t)}}\right).\end{aligned}$$

We notice that, by Proposition 4.1, $[\mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\right)]^{n_t-1} = o(1)$, since $n_t e^{-\kappa\phi(t)} \to_{t\to+\infty} +\infty$ and $C_\kappa > 0$. Hence, we get as $t \to +\infty$, again by Proposition 4.1 and since $(n_t)^2 e^{-\delta\kappa h_t}e^{-\kappa\phi(t)} = o(1)$ because $\phi(t) = o(\log t)$,

$$\int_0^{+\infty} e^{-\lambda x}\mathrm{d}\mu_t(x) = \frac{e^{-\kappa\phi(t)}(1+o(1))}{1-\mathbb{E}\left(e^{-\lambda\mathbf{U}/t}\right)} + o(1) = \frac{1}{C_\kappa\lambda^\kappa} + o(1) = \int_0^{+\infty} \frac{e^{-\lambda x}x^{\kappa-1}}{C_\kappa\Gamma(\kappa)}\mathrm{d}x + o(1),$$

which gives (5.1). This implies the pointwise convergence of the Laplace transform of $(\mu_t)_t$ to that of μ, and then the vague convergence of $(\mu_t)_t$ to μ (e.g. by Feller [27], XIII.1 Th. 2c). Now, we have as $t \to +\infty$ by Fubini and then Proposition 4.1,

$$\begin{aligned}&\lambda \int_0^{+\infty} e^{-\lambda x} \mathbb{P}\left(\mathbf{U}/t \geq x\right) \mathrm{d}x \\ = \quad & \int_0^{\infty} \int_0^{u} \lambda e^{-\lambda x} \mathrm{d}x \mathbb{P}(\mathbf{U}/t \in \mathrm{d}u) = \mathbb{E}\left(1 - e^{-\frac{\lambda}{t}\mathbf{U}}\right) = \frac{C_\kappa \lambda^\kappa + o(1)}{e^{\kappa\phi(t)}}.\end{aligned}$$

Since $\lambda^\kappa = \lambda \int_0^{+\infty} e^{-\lambda x} x^{-\kappa} \mathrm{d}x / \Gamma(1-\kappa)$, we get (5.2). Again, this pointwise convergence of Laplace transforms gives the vague convergence of $e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{U}/t \geq x\right) \mathrm{d}x$ to $C_\kappa x^{-\kappa} \mathrm{d}x / \Gamma(1-\kappa)$ as $t \to +\infty$. Since $x \mapsto \mathbb{P}(\mathbf{U}/t \geq x)$ is nonincreasing, we have for all $0 < \varepsilon < x$,

$$\frac{1}{\varepsilon} \int_x^{x+\varepsilon} e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq u\right) \mathrm{d}u \leq e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq x\right) \leq \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x} e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq u\right) \mathrm{d}u.$$

Using the previous vague convergence gives

$$\limsup_{t\to+\infty} e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq x\right) \leq \frac{1}{\varepsilon} \lim_{t\to+\infty} \int_{x-\varepsilon}^{x} e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq u\right) \mathrm{d}u = \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x} \frac{C_\kappa u^{-\kappa}}{\Gamma(1-\kappa)} \mathrm{d}u.$$

Taking the limit as $\varepsilon \downarrow 0$ gives $\limsup_{t\to+\infty} e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq x\right) \leq C_\kappa x^{-\kappa}/\Gamma(1-\kappa)$. We prove similarly that $\liminf_{t\to+\infty} e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq x\right) \geq C_\kappa x^{-\kappa}/\Gamma(1-\kappa)$. This gives the pointwise convergence of $x \mapsto e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t \geq x)$ to $x \mapsto C_\kappa x^{-\kappa}/\Gamma(1-\kappa)$ on $(0,+\infty)$ as $t \to +\infty$. Finally, since $x \mapsto e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t \geq x)$ is monotone and its pointwise limit is continuous on $(0,\infty)$, Dini's second theorem proves that this convergence is uniform on every compact of $(0,\infty)$. □

We now introduce for $t > 0$ the unique integer $\tilde{N}_t$ such that $H(\tilde{m}_{\tilde{N}_t}) \leq t < H(\tilde{m}_{\tilde{N}_t+1})$. We notice that $\tilde{N}_t = N_t$ on $\mathcal{V}_t$ if $N_t < n_t$ or $\tilde{N}_t < n_t$ due to Remark 2.4. We prove the following lemma:

Lemma 5.2. *Assume* $0 < \kappa < 1$ *and* $0 < \delta < \inf\{2/27, \kappa^2/2\}$. *We have, as* $t \to +\infty$,

$$\mathbb{P}\left(N_t \geq n_t\right) = o(1), \qquad \mathbb{P}\big(N_t = 0\big) = o(1), \qquad \mathbb{P}\big(\tilde{N}_t \geq n_t\big) = o(1), \qquad \mathbb{P}\big(\tilde{N}_t = 0\big) = o(1).$$

Proof of Lemma 5.2. First, by equation (3.40) and the exponential Markov inequality,

$$\mathbb{P}\big(\tilde{N}_t \geq n_t\big) \leq \mathbb{P}\left(\sum_{j=1}^{n_t-1} U_j \leq H(\tilde{m}_{n_t}) \leq t\right) \leq e\mathbb{E}\Big(e^{-\sum_{i=1}^{n_t-1} U_i/t}\Big).$$

This, together with Propositions 3.4 and 4.1, gives since $\phi(t) = o(\log t)$,

$$\begin{aligned}\mathbb{P}\big(\tilde{N}_t \geq n_t\big) \quad &\leq \quad e\left(\mathbb{E}\big(e^{-\mathbf{U}/t}\big)\right)^{n_t} + eC_2 n_t e^{-\delta\kappa h_t} \\ &\leq \quad C_+ \exp\big(-c_- n_t e^{-\kappa\phi(t)}\big) + eC_2 n_t e^{-\delta\kappa h_t} = o(1). \end{aligned} \tag{5.3}$$

This proves the third inequality of the lemma. Moreover, $\mathbb{P}(\tilde{N}_t = 0) = o(1)$ by Lemma 3.7. Finally, the first two inequalities follow from Lemma 2.3, since $\tilde{N}_t = N_t$ on $\mathcal{V}_t$ if $N_t < n_t$ or $\tilde{N}_t < n_t$. □

Proof of Proposition 1.4. First, let $0 < \kappa < 1$, $0 < \delta < \inf\{2/27, \kappa^2/2\}$ such that $(1+2\delta)\kappa < 1$, $0 < r < s < 1$, and $a > 0$. Remark 2.4 and then Lemmas 2.3, 3.7 and 5.2 give as $t \to +\infty$,

$$\begin{aligned}
&\mathbb{P}\big[1-s \leq H(m_{N_t})/t \leq 1-r,\ H(m_{N_t+1})/t \geq 1+a\big] \qquad (5.4)\\
&\leq \mathbb{P}\bigg(\bigcup_{j=1}^{n_t-1}\bigg\{1-s \leq \frac{H(\tilde{m}_{\tilde{N}_t})}{t} \leq 1-r,\ \frac{H(\tilde{m}_{\tilde{N}_t+1})}{t} \geq 1+a,\ \tilde{N}_t = j,\ \mathcal{V}_t\bigg\}\bigg)\\
&\quad + \mathbb{P}\big(\tilde{N}_t \geq n_t\big) + \mathbb{P}\big(\tilde{N}_t = 0\big) + P(\overline{\mathcal{V}}_t)\\
&\leq \sum_{j=2}^{n_t-1}\mathbb{P}\bigg(1-s-\frac{2}{\log h_t} \leq \sum_{i=1}^{j-1}\frac{U_i}{t} \leq 1-r,\ \sum_{i=1}^{j}\frac{U_i}{t} \geq 1+a-\frac{2}{\log h_t}\bigg) + o(1).
\end{aligned}$$

We now use (3.20) of Proposition 3.4 and get for small $\varepsilon > 0$, for large t,

$$(5.4) \leq \int_{1-s-\varepsilon}^{1-r} e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t > 1+a-\varepsilon-x)\mathrm{d}\mu_t(x) + o(1). \qquad (5.5)$$

Using first the uniform convergence of $u \mapsto e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t > u)$ on the compact $[a+r-\varepsilon, a+s] \subset (0,\infty)$ and then the vague convergence of μ_t (see Lemma 5.1), we get

$$\lim_{t\to+\infty}\int_{1-s-\varepsilon}^{1-r} e^{\kappa\phi(t)}\mathbb{P}(\mathbf{U}/t > 1+a-\varepsilon-x)\mathrm{d}\mu_t(x) = \int_{1-s-\varepsilon}^{1-r}\frac{x^{\kappa-1}(1+a-\varepsilon-x)^{-\kappa}}{\Gamma(\kappa)\Gamma(1-\kappa)}\mathrm{d}x.$$

Consequently, by letting $\varepsilon \to 0$, we obtain the first inequality of the following line:

$$\limsup_{t\to+\infty}(5.4) \leq \int_{1-s}^{1-r}\frac{x^{\kappa-1}}{\Gamma(1-\kappa)\Gamma(\kappa)}\int_a^{\infty}\kappa(1+y-x)^{-\kappa-1}\mathrm{d}y\mathrm{d}x \leq \liminf_{t\to+\infty}(5.4).$$

We prove similarly the second inequality. Consequently, $\mathbb{P}[(H(m_{N_t})/t, H(m_{N_t+1})/t) \in \Delta]$ converges to $\int_\Delta \kappa x^{\kappa-1}(y-x)^{-\kappa-1}\mathbb{1}_{(0,1)}(x)\mathbb{1}_{(1,\infty)}(y)/(\Gamma(1-\kappa)\Gamma(\kappa))\mathrm{d}x\mathrm{d}y$ as $t \to +\infty$ for $\Delta = [1-s, 1-r] \times [1+a, +\infty)$ and so for every $\Delta = [a,b] \times [c,d] \subset (0,1) \times (1,\infty)$. This gives the vague convergence on $(0,1) \times (1,+\infty)$, and then the convergence in law of $(H(m_{N_t})/t, H(m_{N_t+1})/t)$ to $\kappa x^{\kappa-1}(y-x)^{-\kappa-1}\mathbb{1}_{(0,1)}(x)\mathbb{1}_{(1,\infty)}(y)/(\Gamma(1-\kappa)\Gamma(\kappa))\mathrm{d}x\mathrm{d}y$, which proves Proposition 1.4 since $\Gamma(1-\kappa)\Gamma(\kappa) = \pi/\sin(\pi\kappa)$. □

Proof of Corollary 1.5. First by Proposition 1.4, for $v \geq 0$ as $t \to +\infty$,

$$\lim_{t\to+\infty}\mathbb{P}\bigg(\frac{H(m_{N_t+1})}{t} \geq 1+v\bigg) = \frac{\sin(\pi\kappa)}{\pi}\int_v^{\infty}\mathrm{d}u\int_0^1 \kappa x^{\kappa-1}(1+u-x)^{-\kappa-1}\mathrm{d}x.$$

Using the change of variables $z = x/(1+u-x)$ in the second integral leads to (1.3). (1.2) follows from Proposition 1.4 by straightforward computations. Finally, for every continuous bounded function φ, Proposition 1.4 and the change of variables $u = y - x$ give

$$\begin{aligned}
&\mathbb{E}\big[\varphi\big((H(m_{N_t+1}) - H(m_{N_t}))/t\big)\big]\\
\to_{t\to+\infty}\quad &\int\varphi(u)\frac{\sin(\pi\kappa)}{\pi}u^{-\kappa-1}\int\kappa x^{\kappa-1}\mathbb{1}_{[0,1]}(x)\mathbb{1}_{\{x\geq 1-u\}}\mathrm{d}x\mathrm{d}u,
\end{aligned}$$

which gives the convergence in law of $(H(m_{N_t+1}) - H(m_{N_t}))/t$ under $\mathbb{P}$ as $t \to +\infty$. □

Proof of Proposition 1.6. Let $0 < \kappa < 1$, $\lambda > 0$, and $0 < \delta < \inf\{2/27, \kappa^2/2\}$ such that $(1+2\delta)\kappa < 1$. For $t > 0$, we denote by ν_t the measure on $\mathbb{R}_+$ such that

$$\nu_t([0,x]) = e^{-\kappa\phi(t)} \sum_{j=1}^{n_t-1} \exp\Big(- \frac{C_\kappa \lambda^\kappa j}{e^{\kappa\phi(t)}}\Big) \mathbb{P}\Big(\sum_{i=1}^{j-1} \frac{U_i}{t} \leq x \Big), \qquad x \geq 0.$$

In particular for $j = 1$, $\mathbb{P}(\sum_{i=1}^{j-1} U_i/t \in .) = \mathbb{P}(0 \in .)$ denotes the Dirac measure at 0. We first show that the Laplace transform of the measure ν_t converges when t goes to infinity. We consider α such that $0 < \lambda < \alpha$. We get as in the proof of Lemma 5.1,

$$\begin{aligned} &\int_0^{+\infty} e^{-\alpha u} \mathrm{d}\nu_t(u) \\ = \quad & e^{-\kappa\phi(t)} \sum_{j=1}^{n_t-1} \exp\Big(- \frac{C_\kappa \lambda^\kappa j}{e^{\kappa\phi(t)}}\Big) \Big[\mathbb{E}\left(e^{-\alpha \mathbf{U}/t}\right)\Big]^{j-1} + O(e^{-\kappa\phi(t)} n_t^2 e^{-\delta\kappa h_t}) \\ = \quad & \frac{1 - \Big[\exp\Big(- \frac{C_\kappa\lambda^\kappa}{e^{\kappa\phi(t)}}\Big) \mathbb{E}\left(e^{-\alpha\mathbf{U}/t}\right)\Big]^{n_t-1}}{\exp\Big(\kappa\phi(t) + \frac{C_\kappa\lambda^\kappa}{e^{\kappa\phi(t)}}\Big)\Big[1 - \exp\Big(- \frac{C_\kappa\lambda^\kappa}{e^{\kappa\phi(t)}}\Big)\mathbb{E}\left(e^{-\alpha\mathbf{U}/t}\right)\Big]} + o(1) = \frac{1}{C_\kappa(\alpha^\kappa + \lambda^\kappa)} + o(1), \end{aligned}$$

by Propositions 3.4 and then 4.1. We also notice that

$$\frac{1}{\alpha^\kappa + \lambda^\kappa} = \sum_{j=0}^{+\infty} \frac{(-\lambda^\kappa)^j}{\alpha^{\kappa(1+j)}} = \sum_{j=0}^{+\infty} \frac{(-\lambda^\kappa)^j}{\Gamma[\kappa(1+j)]} \int_0^{+\infty} e^{-\alpha u} u^{\kappa(1+j)-1} \mathrm{d}u.$$

Moreover, $\int_0^\infty \sum_{j=0}^\infty \big| \frac{(-\lambda^\kappa)^j}{\Gamma[\kappa(1+j)]} \frac{e^{-\alpha u} u^{\kappa(1+j)-1}}{C_\kappa} \big| \mathrm{d}u < \infty$, since $\sum_{j=0}^{+\infty} \frac{(\lambda u)^{\kappa j}}{\Gamma[\kappa(1+j)]} = O(e^{(\lambda+\varepsilon)u})$ as $u \to +\infty$ for any $\varepsilon > 0$, and $\kappa > 0$. So Fubini gives,

$$\forall \alpha > \lambda, \qquad \lim_{t\to+\infty} \int_0^{+\infty} e^{-\alpha u} \mathrm{d}\nu_t(u) = \int_0^{+\infty} e^{-\alpha u} \mathrm{d}\nu(u),$$

where ν is the measure defined by $\mathrm{d}\nu(u) = \frac{1}{C_\kappa} \sum_{j=0}^{+\infty} \frac{(-\lambda^\kappa)^j u^{\kappa(1+j)-1}}{\Gamma[\kappa(1+j)]} \mathbb{1}_{(0,+\infty)}(u) \mathrm{d}u$. This pointwise convergence of the Laplace transform of ν_t on $(\lambda, +\infty)$ leads to the vague convergence of ν_t to ν on $[0, +\infty)$ as $t \to +\infty$ (e.g. by Feller [27], XIII.1 Th. 2c).

We have, with the arguments already used between (5.4) and (5.5), for any $a > 0$ and $\varepsilon \in (0, a)$,

$$\begin{aligned} & \mathbb{E}\left[\exp\Big(- \frac{C_\kappa\lambda^\kappa N_t}{e^{\kappa\phi(t)}}\Big), \ \frac{H(m_{N_t+1})}{t} \geq 1 + a\right] \qquad (5.6) \\ \leq \quad & \mathbb{E}\left[\sum_{j=1}^{n_t-1} \exp\Big(- \frac{C_\kappa\lambda^\kappa j}{e^{\kappa\phi(t)}}\Big) \mathbb{1}_{\{H(\tilde m_j)/t \leq 1, H(\tilde m_{j+1})/t \geq 1+a, \tilde N_t = j\}} \mathbb{1}_{\mathcal{V}_t}\right] + o(1) \\ \leq \quad & \sum_{j=1}^{n_t-1} \exp\Big(- \frac{C_\kappa\lambda^\kappa j}{e^{\kappa\phi(t)}}\Big) \mathbb{P}\left[\sum_{i=1}^{j-1} \frac{U_i}{t} \leq 1, \ 1 + a - \frac{2}{\log h_t} \leq \sum_{i=1}^{j} \frac{U_i}{t}\right] + o(1) \\ \leq \quad & \int_0^1 e^{\kappa\phi(t)} \mathbb{P}\left(\frac{\mathbf{U}}{t} \geq 1 + a - \varepsilon - x\right) \mathrm{d}\nu_t(x) + o(1), \end{aligned}$$

since the probability for $j = 1$ in the third line is for large t less than $\mathbb{P}[U_1/t \geq 1 + a - \varepsilon] \leq \mathbb{P}(\mathbf{U}/t \geq 1 + a - \varepsilon) + o(1) = o(1)$ by the case $n = 1$ just after (3.20).

Using the uniform convergence of $x \mapsto e^{\kappa\phi(t)} \mathbb{P}\left(\mathbf{U}/t \geq 1 + a - \varepsilon - x\right)$ on $[0, 1]$ provided by Lemma 5.1 followed by the vague convergence of ν_t to ν, we get the first inequality of

$$\limsup_{t\to+\infty} (5.6) \leq \int_0^1 \frac{C_\kappa}{\Gamma(1-\kappa)} (1 + a - x)^{-\kappa} \mathrm{d}\nu(x) \leq \liminf_{t\to+\infty} (5.6). \qquad (5.7)$$

We obtain the second one similarly. Since $\lim_{a\downarrow 0}\lim_{t\to+\infty}\mathbb{P}(H(m_{N_t+1})/t<1+a)=0$ by equation (1.3) of Corrolary 1.5, letting $a\downarrow 0$ in (5.7) gives

$$\begin{aligned}\lim_{t\to+\infty}\mathbb{E}\left[\exp\left(-\frac{C_\kappa\lambda^\kappa N_t}{e^{\kappa\phi(t)}}\right)\right] &= \int_0^1 \frac{C_\kappa(1-x)^{-\kappa}}{\Gamma(1-\kappa)}\mathrm{d}\nu(x)\\ &= \sum_{j=0}^{+\infty}(-\lambda^\kappa)^j\int_0^1\frac{(1-x)^{-\kappa}x^{\kappa(1+j)-1}}{\Gamma[\kappa(1+j)]\Gamma(1-\kappa)}\mathrm{d}x.\end{aligned}$$

Since $\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ for every $a>0$ and $b>0$, changing $C_\kappa\lambda^\kappa$ into u gives the pointwise convergence of $\mathbb{E}[\exp(-uN_t/e^{\kappa\phi(t)})]$ to $\sum_{j=0}^{+\infty}\frac{1}{\Gamma(\kappa j+1)}\left(\frac{-u}{C_\kappa}\right)^j$, which ends the proof of Proposition 1.6. □

5.2 The localization : proof of Theorem 1.3

We recall the notation $H_{x\to y}=\inf\{s>H(x),\ X(s)=y\}-H(x)$ for $(x,y)\in\mathbb{R}_+^2$, which is equal to $H(y)-H(x)$ if $x<y$. Let $\phi^*(t):=\phi(t)/\zeta$, where $0<\zeta<1$ will be chosen later. We define $t^*:=t-e^{(1+2\delta)\phi^*(t)}$,

$$\mathcal{A}_0:=\{1\leq\tilde{N}_t<n_t\},\qquad \mathcal{A}_1:=\cap_{j=1}^{n_t-1}\{H_{\tilde{L}_j\to\tilde{m}_{j+1}}\leq 2t/\log h_t\},$$
$$\mathcal{A}_2:=\cap_{j=1}^{n_t-1}\{H_{\tilde{m}_j\to\tilde{m}_{j+1}}<H_{\tilde{m}_j\to\tilde{L}_j^-}\},\qquad \mathcal{A}_3:=\{H(\tilde{m}_{\tilde{N}_t})\leq t^*\}.$$

We also introduce $\mathbf{I}_j:=[\tilde{m}_j-\phi^*(t)/\zeta,\tilde{m}_j+\phi^*(t)/\zeta]$, $j\in\mathbb{N}^*$ (see Figure 2 page 46). Let $\varepsilon>0$. We have:

$$\begin{aligned}\mathbb{P}\left(X(t)\notin\mathbf{I}_{\tilde{N}_t}\right) \leq\ & \mathbb{P}\left(X(t)\notin\mathbf{I}_{\tilde{N}_t},\tilde{N}_t=\tilde{N}_{t(1+\varepsilon)},\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\right)\\ &+\mathbb{P}\left(X(t)\notin\mathbf{I}_{\tilde{N}_t},\tilde{N}_t=\tilde{N}_{t(1+\varepsilon)},\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2,\overline{\mathcal{A}_3}\right)\\ &+\mathbb{P}\big(\tilde{N}_t\neq\tilde{N}_{t(1+\varepsilon)}\big)+\mathbb{P}\left(\overline{\mathcal{A}_0}\right)+\mathbb{P}\left(\overline{\mathcal{A}_1}\right)+\mathbb{P}\left(\overline{\mathcal{A}_2}\right).\end{aligned}\tag{5.8}$$

We split the proof into three parts, in which we estimate these different probabilities. We start with:

Part 1: We prove that for large t,

$$\mathbb{P}\left(X(t)\notin\mathbf{I}_{\tilde{N}_t},\tilde{N}_t=\tilde{N}_{t(1+\varepsilon)},\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\right)\leq n_t\left(C_+h_te^{-\kappa(1-\delta)\phi^*(t)/16}+e^{-(c_-)\delta\phi^*(t)}\right).\tag{5.9}$$

Let $\mathcal{B}_j:=\{\tilde{N}_t=\tilde{N}_{t(1+\varepsilon)}=j\}\cap\mathcal{A}_1\cap\mathcal{A}_2$ for $j\in\mathbb{N}^*$. We have

$$\mathbb{P}\left(X(t)\notin\mathbf{I}_{\tilde{N}_t},\ \tilde{N}_t=\tilde{N}_{t(1+\varepsilon)},\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\right)=\sum_{j=1}^{n_t-1}\mathbb{P}\left(X(t)\notin\mathbf{I}_j,\mathcal{B}_j\cap\mathcal{A}_3\right).\tag{5.10}$$

Let $1\leq j<n_t$. Notice that on $\mathcal{B}_j$, $\tilde{N}_{t(1+\varepsilon)}=j$, so $H(\tilde{L}_j)+H_{\tilde{L}_j\to\tilde{m}_{j+1}}=H(\tilde{m}_{j+1})>t(1+\varepsilon)$; moreover for large t, $H_{\tilde{L}_j\to\tilde{m}_{j+1}}\leq\varepsilon t/2$, then $H(\tilde{L}_j)>(1+\varepsilon/2)t$, and so for all $u\in[H(\tilde{m}_j),(1+\varepsilon/2)t]$, $u<H(\tilde{L}_j)<H(\tilde{m}_{j+1})=H(\tilde{m}_j)+H_{\tilde{m}_j\to\tilde{m}_{j+1}}<H(\tilde{m}_j)+H_{\tilde{m}_j\to\tilde{L}_j^-}$, and then $X(u)\in(\tilde{L}_j^-,\tilde{L}_j)$. That is, if t is large enough, on $\mathcal{B}_j$, after first hitting $\tilde{m}_j$, X stays in $(\tilde{L}_j^-,\tilde{L}_j)$ at least until time $(1+\varepsilon/2)t$. Therefore, conditioning on $H(\tilde{m}_j)$ and

using the strong Markov property,

$$\begin{aligned}
&\mathbb{P}\left(X(t)\notin \mathbf{I}_j, \mathcal{B}_j\cap\mathcal{A}_3\right)\\
&\quad\le E\Big(\mathbb{P}^{W_\kappa}\left[X(t)\notin \mathbf{I}_j,\ H(\tilde m_j)\le t^*,\ \forall u\in[H(\tilde m_j),(1+\varepsilon/2)t], X(u)\in\big(\tilde L_j^-,\tilde L_j)\right]\Big)\\
&\quad= E\Big(\int_0^{t^*}\mathbb{P}^{W_\kappa}(H(\tilde m_j)\in ds)\\
&\qquad\qquad \mathbb{P}^{W_\kappa}_{\tilde m_j}\Big[X(t-s)\notin \mathbf{I}_j,\ \forall u\in\big[0,(1+\varepsilon/2)t-s\big], X(u)\in\big(\tilde L_j^-,\tilde L_j)\Big]\Big)\\
&\quad\le E\Big(\sup_{0\le s\le t^*}\mathbb{P}^{W_\kappa}_{\tilde m_j}\Big[X(t-s)\notin \mathbf{I}_j,\forall u\in[0,(1+\varepsilon/2)t-s], X(u)\in\big(\tilde L_j^-,\tilde L_j)\Big]\Big). \qquad (5.11)
\end{aligned}$$

Now, as in Brox ([10], proof of Prop. 4.1) we introduce a coupling between X (under $\mathbb{P}^{W_\kappa}_{\tilde m_j}$) and a reflected diffusion Y_j defined below. To this aim, let $\big(Y_j^{(x)}(u),\ u\ge 0\big)$ be (in words) a diffusion in the potential W_κ, starting from $x\in\big[\tilde L_j^-,\tilde L_j\big]$ and reflected at $\tilde L_j^-$ and $\tilde L_j$. We denote its law by $\widehat P^{W_\kappa}_{j,x}$. More precisely, this process $Y_j^{(x)}$ is defined as in Brox ([10], p. 1216) by

$$Y_j^{(x)}(u):=A^{-1}\Big(\widehat B_j^{(x)}\Big(\widehat T_{j,x}^{-1}(u)\Big)\Big),\qquad u\ge 0,\ x\in[\tilde L_j^-,\tilde L_j],$$

where $(\widehat B_j^{(x)}(s),\ s\ge 0)$ is a one-dimensional Brownian motion independent from W_κ, starting from $A(x)$ and reflected at $A\big(\tilde L_j^-\big)$ and $A\big(\tilde L_j\big)$, and $\widehat T_{j,x}$ is defined as T (see (3.3)) replacing B by $\hat B_j^{(x)}$. This enables us to define $(Y_j(s),\ s\ge 0)$ by $\widehat P_j^{W_\kappa}(Y_j\in .):=\int_{\tilde L_j^-}^{\tilde L_j}\widehat P^{W_\kappa}_{j,x}(.)\mathrm{d}\tilde\mu_j(x)$, where

$$\mathrm{d}\tilde\mu_j(x):=\exp(-\tilde V^{(j)}(x))\mathbb{1}_{[\tilde L_j^-,\tilde L_j]}(x)\mathrm{d}x\Big(\int_{\tilde L_j^-}^{\tilde L_j}\exp(-\tilde V^{(j)}(y))dy\Big)^{-1}. \qquad (5.12)$$

As is proved in ([10], proof of Prop. 4.1), $\tilde\mu_j$ is invariant for the semi-group of Y_j; in particular $\widehat P_j^{W_\kappa}(Y_j(s)\in\Delta)=\tilde\mu_j(\Delta)$ for every $s\ge 0$ and $\Delta\subset[\tilde L_j^-,\tilde L_j]$.

We can now, as in [10], build a coupling $Q^{W_\kappa}_{\tilde m_j}$ of X and Y_j, such that $Q^{W_\kappa}_{\tilde m_j}(Y_j\in .)=\widehat P_j^{W_\kappa}(Y_j\in .)$, and $Q^{W_\kappa}_{\tilde m_j}(X\in .)=\mathbb{P}^{W_\kappa}_{\tilde m_j}(X\in .)$, these two Markov processes Y_j and X move independently until

$$\widehat H_{\{X=Y_j\}}:=\inf\{u\ge 0,\ X(u)=Y_j(u)\},$$

which is the first collision, then $X(u)=Y_j(u)$ until the next exit time

$$\widehat H_j^{\text{exit}}:=\inf\big\{u>\widehat H_{\{X=Y_j\}},\ X(u)\notin(\tilde L_j^-,\tilde L_j)\big\},$$

and then X and Y_j move independently again.

We introduce (see Figure 2)

$$t_1:=t-t^*=e^{(1+2\delta)\phi^*(t)},\qquad \widehat L_j^+:=\tilde\tau_j(\phi^*(t)),\qquad \widehat L_j^-:=\tilde\tau_j^-(\phi^*(t)).$$

We now prove that, with a large probability, under $Q^{W_\kappa}_{\tilde m_j}$, X and Y_j first collide before time t_1, that is, $\widehat H_{\{X=Y_j\}}\le t_1$. To this aim, we first provide a result concerning only the hitting times H of X:

Lemma 5.3. *For large t, with a probability larger than $1-e^{-(c_-)\delta\phi^*(t)}$,*

$$Q^{W_\kappa}_{\tilde m_j}\Big(H\big(\widehat L_j^+\big)\vee H\big(\widehat L_j^-\big)>e^{(1+\delta)\phi^*(t)}\Big)\le C_+\phi^*(t)e^{-\delta\phi^*(t)/8}, \qquad (5.13)$$

uniformly for $j\in[1,n_t)$ and with $x\vee y:=\max(x,y)$,

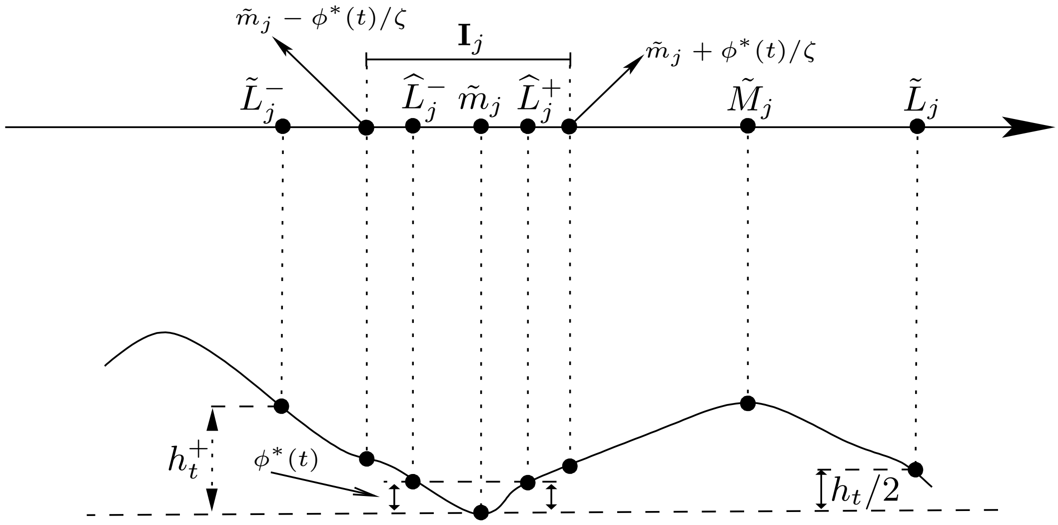

Figure 2: Schema of valley number j and some notation around its minimum for small ζ

The proof of this lemma is deferred to Subsection 5.4. We deduce from (5.13) and the definition of $x \vee y = \max(x, y)$ that with probability larger than $1 - e^{-(c_-)\delta\phi^*(t)}$,

$$\begin{aligned}
& Q^{W_\kappa}_{\tilde m_j}\big(\widehat H_{\{X=Y_j\}} > t_1\big) \\
\leq\ & Q^{W_\kappa}_{\tilde m_j}\big[\widehat H_{\{X=Y_j\}} > H\big(\widehat L_j^+\big) \vee H\big(\widehat L_j^-\big)\big] + Q^{W_\kappa}_{\tilde m_j}\big[H\big(\widehat L_j^+\big) \vee H\big(\widehat L_j^-\big) > t_1\big] \\
\leq\ & Q^{W_\kappa}_{\tilde m_j}\Big(\widehat H_{\{X=Y_j\}} > H\big(\widehat L_j^-\big), Y_j(0) < \tilde m_j\Big) \\
& + Q^{W_\kappa}_{\tilde m_j}\Big(\widehat H_{\{X=Y_j\}} > H\big(\widehat L_j^+\big), Y_j(0) \geq \tilde m_j\Big) + \frac{C_+\phi^*(t)}{e^{\delta\phi^*(t)/8}}.
\end{aligned}$$

On $\big\{\widehat H_{\{X=Y_j\}} > H(\widehat L_j^-), Y_j(0) < \tilde m_j\big\}$, $Y_j(0) - X(0) = Y_j(0) - \tilde m_j < 0$ under $Q^{W_\kappa}_{\tilde m_j}$, and by continuity $Y_j - X < 0$ up to time $\widehat H_{\{X=Y_j\}}$ and in particular at time $H(\widehat L_j^-)$, so

$$\begin{aligned}
Q^{W_\kappa}_{\tilde m_j}\big(\widehat H_{\{X=Y_j\}} > t_1\big) \leq\ & Q^{W_\kappa}_{\tilde m_j}\Big(\ Y_j[H(\widehat L_j^-)] \in [\tilde L_j^-, \widehat L_j^-],\ \widehat H_{\{X=Y_j\}} > H(\widehat L_j^-)\Big) \\
& + Q^{W_\kappa}_{\tilde m_j}\Big(\ Y_j[H(\widehat L_j^+)] \in [\widehat L_j^+, \tilde L_j],\ \widehat H_{\{X=Y_j\}} > H(\widehat L_j^+)\Big) + \frac{C_+\phi^*(t)}{e^{\delta\phi^*(t)/8}} \\
\leq\ & \tilde\mu_j\big([\tilde L_j^-, \widehat L_j^-]\big) + \tilde\mu_j\big([\widehat L_j^+, \tilde L_j]\big) + C_+\phi^*(t)e^{-\delta\phi^*(t)/8}, \qquad (5.14)
\end{aligned}$$

where the last line comes from the independence of X and Y_j until $\widehat H_{\{X=Y_j\}}$ and since $Q^{W_\kappa}_{\tilde m_j}(Y_j(s) \in \Delta) = \widehat P^{W_\kappa}_j(Y_j(s) \in \Delta) = \tilde\mu_j(\Delta)$, for every $s \geq 0$ and $\Delta \subset [\tilde L_j^-, \tilde L_j]$ as explained after (5.12).

We would like to bound (5.11). Let $s \in [0, t^*]$. Using first $t_1 = t - t^* \leq t - s \leq (1+\varepsilon/2)t - s$, and second $X(u) = Y_j(u)$ for every $\widehat H_{\{X=Y_j\}} \leq u \leq \widehat H_j^{\text{exit}}$ and $Q^{W_\kappa}_{\tilde m_j}(Y_j(t-s) \in .) = \tilde\mu_j(.)$,

$$\begin{aligned}
& Q^{W_\kappa}_{\tilde m_j}\Big(X(t-s) \notin \mathbf{I}_j,\ \forall u \in [0, (1+\varepsilon/2)t - s],\ X(u) \in (\tilde L_j^-, \tilde L_j)\Big) \qquad (5.15) \\
\leq\ & Q^{W_\kappa}_{\tilde m_j}\big(\widehat H_{\{X=Y_j\}} > t_1\big) \\
& + Q^{W_\kappa}_{\tilde m_j}\Big(\widehat H_{\{X=Y_j\}} \leq t_1 \leq t - s \leq (1+\varepsilon/2)t - s \leq \widehat H_j^{\text{exit}}, X(t-s) \notin \mathbf{I}_j\Big) \\
\leq\ & Q^{W_\kappa}_{\tilde m_j}\big(\widehat H_{\{X=Y_j\}} > t_1\big) + Q^{W_\kappa}_{\tilde m_j}\big(Y_j(t-s) \notin \mathbf{I}_j\big) \\
\leq\ & Q^{W_\kappa}_{\tilde m_j}\big(\widehat H_{\{X=Y_j\}} > t_1\big) + \tilde\mu_j\big([\tilde L_j^-, \tilde L_j] \smallsetminus \mathbf{I}_j\big). \qquad (5.16)
\end{aligned}$$

In the following lemma, we show that with high probability, the invariant probability measure $\tilde{\mu}_j$ is highly concentrated on the small neighborhood $\mathbf{I}_j$ of $\tilde{m}_j$. The proof of this lemma is deferred to Subsection 5.4. More precisely:

Lemma 5.4. *Assume $\zeta < \kappa/8$. For large t, for all $1 \leq j < n_t$, with a probability greater than $1 - e^{-(c_-)\delta\phi^*(t)}$,*

$$\tilde{\mu}_j\big(\big[\tilde{L}_j^-, \tilde{L}_j\big] \smallsetminus \mathbf{I}_j\big) \leq \tilde{\mu}_j\big(\big[\tilde{L}_j^-, \widehat{L}_j^-\big]\big) + \tilde{\mu}_j\big(\big[\widehat{L}_j^+, \tilde{L}_j\big]\big) \leq C_+ h_t e^{-(1-\delta)\kappa\phi^*(t)/16}. \tag{5.17}$$

Notice that for any $\zeta > 0$, the right hand side of (5.17) go to 0 as $t \to +\infty$ since $h_t \leq \log t$ and $\log\log t = o(\phi(t)) = o(\phi^*(t))$, and is $o(1/n_t)$ if ζ is chosen small enough.

Moreover, notice that we can replace $Q_{\tilde{m}_j}^{W_\kappa}$ by $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ in the first line (5.15) since $Q_{\tilde{m}_j}^{W_\kappa}(X \in .) = \mathbb{P}_{\tilde{m}_j}^{W_\kappa}(X \in .)$. So, (5.16), (5.14) and then Lemma 5.4 give

$$\begin{aligned}
&\sup_{0\leq s\leq t^*} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(X(t-s) \notin \mathbf{I}_j,\ \forall u \in [0, (1+\varepsilon/2)t - s],\ X(u) \in (\tilde{L}_j^-, \tilde{L}_j)\right) \\
\leq\ & \tilde{\mu}_j\big(\big[\tilde{L}_j^-, \widehat{L}_j^-\big]\big) + \tilde{\mu}_j\big(\big[\widehat{L}_j^+, \tilde{L}_j\big]\big) + \frac{C_+\phi^*(t)}{e^{\delta\phi^*(t)/8}} + \tilde{\mu}_j\big(\big[\tilde{L}_j^-, \tilde{L}_j\big] \smallsetminus \mathbf{I}_j\big) \\
\leq\ & \frac{C_+ h_t}{e^{(1-\delta)\kappa\phi^*(t)/16}} + \frac{C_+\phi^*(t)}{e^{\delta\phi^*(t)/8}}
\end{aligned}$$

with probability at least $1 - e^{-(c_-)\delta\phi^*(t)}$. Finally, integrating this and applying successively (5.10) and (5.11) leads to (5.9), which ends the proof of this Part 1.

Part 2 : We prove that there exists $c_3 > 0$ such that if $\zeta \leq \kappa/48$,

$$\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_t},\ \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \overline{\mathcal{A}_3}\right) \leq n_t e^{-c_3\phi^*(t)/\zeta}. \tag{5.18}$$

First, we prove similarly as in Part 1 from (5.10) to (5.11) that, using $H(\tilde{m}_j) \leq t$ on $\mathcal{B}_j$,

$$\begin{aligned}
&\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_t},\ \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \overline{\mathcal{A}_3}\right) = \sum_{j=1}^{n_t-1} \mathbb{P}\left(X(t) \notin \mathbf{I}_j, \mathcal{B}_j \cap \overline{\mathcal{A}_3}\right) \qquad (5.19) \\
&\leq \sum_{j=1}^{n_t-1} E\left(\sup_{t^*\leq s\leq t} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(X(t-s) \notin \mathbf{I}_j,\ \forall u \in [0, (1+\varepsilon/2)t - s],\ X(u) \in \left(\tilde{L}_j^-, \tilde{L}_j\right)\right)\right) \\
&\leq \sum_{j=1}^{n_t-1} E\left(\sup_{t^*\leq s\leq t} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left[H(\tilde{m}_j - \phi^*(t)/\zeta) \wedge H(\tilde{m}_j + \phi^*(t)/\zeta) < t - s\right]\right),
\end{aligned}$$

by definition of $\mathbf{I}_j = [\tilde{m}_j - \phi^*(t)/\zeta, \tilde{m}_j + \phi^*(t)/\zeta]$, and with $x \wedge y := \inf(x, y)$. Since $t - t^* = e^{(1+2\delta)\phi^*(t)}$, this gives

$$(5.19) \leq n_t \sup_{1\leq j\leq n_t-1} E\left(\mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left[H(\tilde{m}_j - \phi^*(t)/\zeta) \wedge H(\tilde{m}_j + \phi^*(t)/\zeta) < e^{(1+2\delta)\phi^*(t)}\right]\right). \tag{5.20}$$

We estimate this with the following lemma, the proof of which is deferred to Subsection 5.4:

Lemma 5.5. *Assume $\zeta \leq \kappa/48$. For large t, for each $1 \leq j < n_t$, with probability at least $1 - e^{-(c_-)\phi^*(t)/\zeta}$,*

$$\mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(H(\tilde{m}_j \pm \phi^*(t)/\zeta) \leq e^{(1+2\delta)\phi^*(t)}\right) \leq e^{-(c_-)\phi^*(t)/\zeta}.$$

This together with (5.20) leads to (5.18), which ends this Part 2.

Part 3: We prove that if $0 < \delta < 1/16$, $\delta < \kappa^2/2$ and $(1+2\delta)\kappa < 1$, as $t \to +\infty$,

$$\mathbb{P}\big(\tilde{N}_t \neq \tilde{N}_{t(1+\varepsilon)}\big) + \mathbb{P}\left(\overline{\mathcal{A}_0}\right) + \mathbb{P}\left(\overline{\mathcal{A}_1}\right) + \mathbb{P}\left(\overline{\mathcal{A}_2}\right) \leq \varepsilon^{1-\kappa}/(1-\kappa) + o(1). \tag{5.21}$$

First, $\mathbb{P}\big(\overline{\mathcal{A}_0}\big) = o(1)$ by Lemma 5.2. The fact that $\mathbb{P}\big(\overline{\mathcal{A}_1}\big) = o(1)$ for such δ follows from (3.44) since $(\log h_t) n_t e^{h_t}/t = o(1)$ as stated in Lemma 3.7. Furthermore, by Lemma 2.3 and then (1.3),

$$\mathbb{P}\big(\tilde{N}_t \neq \tilde{N}_{t(1+\varepsilon)}\big) \leq \mathbb{P}[H(m_{N_t+1}) \leq t(1+\varepsilon)] + P\big(\overline{\mathcal{V}}_t\big) + \mathbb{P}\big(\overline{\mathcal{A}_0}\big) \leq \varepsilon^{1-\kappa}/(1-\kappa) + o(1).$$

Moreover by the strong Markov property, recalling that $\tilde{L}_i^- < \tilde{m}_j < \tilde{L}_i^* < \tilde{L}_i < \tilde{m}_{j+1}$ by (3.11),

$$\begin{aligned}
\mathbb{P}_{\tilde{m}_j}^{W_\kappa}\Big(H(\tilde{m}_{j+1}) < H(\tilde{L}_j^-)\Big) &= \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\Big(H(\tilde{L}_j) < H(\tilde{L}_j^-)\Big) \times \mathbb{P}_{\tilde{L}_j}^{W_\kappa}\Big(H(\tilde{m}_{j+1}) < H(\tilde{L}_j^-)\Big) \\
&\geq \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\Big(H(\tilde{L}_j) < H(\tilde{L}_j^-)\Big) \times \mathbb{P}_{\tilde{L}_j}^{W_\kappa}\Big(H(\tilde{m}_{j+1}) < H(\tilde{L}_j^*)\Big) \\
&\geq \big(1 - e^{-\kappa h_t/2}\big) \times \big(1 - 2n_t e^{-h_t/8}\big) = 1 - o(1/n_t),
\end{aligned}$$

for all $1 \leq j \leq n_t - 1$ with probability $\geq 1 - C_+ n_t e^{-\kappa\delta h_t}$ by Lemmas 3.2 and 3.3 since $\delta < 1/16$. This proves that $\mathbb{P}\left(\overline{\mathcal{A}_2}\right) = o(1)$. This leads to (5.21), which ends this Part 3.

Conclusion: We now choose $\zeta = \delta^2$. We recall that $\phi(t) \leq h_t \leq \log t = \exp(o(\phi(t)))$ since $\log\log t = o(\phi(t))$. Combining (5.8), (5.9), (5.18) with (5.21), and choosing δ small enough gives $\limsup_{t\to+\infty} \mathbb{P}\big(X(t) \notin \mathbf{I}_{\tilde{N}_t}\big) \leq \varepsilon^{1-\kappa}/(1-\kappa)$, for every $\varepsilon > 0$, and so is 0. So,

$$\begin{aligned}
\lim_{t\to+\infty} \mathbb{P}\big(|X(t) - m_{N_t}| > \phi(t)/\zeta^2\big) &= \lim_{t\to+\infty} \mathbb{P}\big(|X(t) - \tilde{m}_{\tilde{N}_t}| > \phi(t)/\zeta^2\big) \\
&= \lim_{t\to+\infty} \mathbb{P}\big(X(t) \notin \mathbf{I}_{\tilde{N}_t}\big) = 0,
\end{aligned}$$

since $\tilde{N}_t = N_t$ and $\tilde{m}_j = m_j$ for $1 \leq j \leq n_t$ on $\mathcal{V}_t \cap \mathcal{A}_0$ by Remark 2.4, $P\big(\overline{\mathcal{V}}_t\big) = o(1)$ by Lemma 2.3, $\mathbb{P}\big(\overline{\mathcal{A}_0}\big) = o(1) = \mathbb{P}(\tilde{N}_t \notin [1, n_t]) = o(1)$ by Lemma 5.2. This proves Theorem 1.3. ☐

5.3 The aging : proof of Proposition 1.2

Proof of Proposition 1.2. We fix $\alpha > 1$ and $\delta > 0$. We recall that the r.v. $(m_i)_i$ depend on h_t. In what follows, we apply Theorem 1.3 first at time t with function ϕ, and second at time αt with a function ϕ_α defined by $\log(\alpha t) - \phi_\alpha(\alpha t) = \log t - \phi(t)$, so that the r.v. m_i are the same in both cases. By Theorem 1.3 and Lemma 5.2, we get

$$\begin{aligned}
&\mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t),\ N_t < N_{\alpha t}) \\
&\leq \mathbb{P}\big[\,|m_{N_{\alpha t}} - m_{N_t}| \leq \mathcal{C}_1[4\phi(t) + \phi_\alpha(\alpha t)], N_t < N_{\alpha t} \leq n_{\alpha t}, N_t \in [1, n_t)\big] + \mathbb{P}\big[N_t \notin [1, n_t)\big] \\
&+ \mathbb{P}\big[|X(t) - m_{N_t}| > \mathcal{C}_1\phi(t)\big] + \mathbb{P}\big[|X(\alpha t) - m_{N_{\alpha t}}| > \mathcal{C}_1\phi_\alpha(\alpha t)\big] + \mathbb{P}\big[N_{\alpha t} \notin [1, n_{\alpha t})\big] \\
&\leq \sum_{i=1}^{n_t}\sum_{j=i+1}^{n_{\alpha t}} \mathbb{P}\Big(m_j - m_i \leq 4\mathcal{C}_1\phi(t) + \mathcal{C}_1\phi_\alpha(\alpha t)\Big) + o(1),
\end{aligned}$$

as $t \to +\infty$. So, (2.21) and Lemma 2.3 leads to $\mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t), N_t < N_{\alpha t}) = o(1)$ since $n_t = e^{o(h_t)}$ and $n_{\alpha t} = e^{o(h_t)}$. Consequently,

$$\begin{aligned}
\mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t)) &= \mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t),\ N_t = N_{\alpha t}) + o(1) \\
&\leq \mathbb{P}(N_t = N_{\alpha t}) + o(1).
\end{aligned} \tag{5.22}$$

Moreover, by Theorem 1.3 applied at time t with function ϕ, we have for large t,

$$\begin{aligned}
& \mathbb{P}(|X(\alpha t) - X(t)| > 3\mathcal{C}_1\phi(t),\ N_t = N_{\alpha t}) \\
\leq\ & \mathbb{P}(|X(\alpha t) - m_{N_t}| > 3\mathcal{C}_1\phi(t) - \mathcal{C}_1\phi(t),\ N_t = N_{\alpha t}) + \mathbb{P}[|X(t) - m_{N_t}| > \mathcal{C}_1\phi(t)] \\
\leq\ & \mathbb{P}[|X(\alpha t) - m_{N_{\alpha t}}| > 2\mathcal{C}_1\phi(t)] + o(1) \\
\leq\ & \mathbb{P}(|X(\alpha t) - m_{N_{\alpha t}}| > \mathcal{C}_1\phi_\alpha(\alpha t)) + o(1).
\end{aligned}$$

This is $o(1)$ as $t \to +\infty$, by Theorem 1.3 applied at time αt with function ϕ_α. Therefore,

$$\begin{aligned}
\mathbb{P}(N_t = N_{\alpha t}) &= \mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t), N_t = N_{\alpha t}) + o(1) \\
&\leq \mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t)) + o(1).
\end{aligned}$$

This together with (5.22) gives

$$\mathbb{P}(|X(\alpha t) - X(t)| \leq 3\mathcal{C}_1\phi(t)) = \mathbb{P}(N_t = N_{\alpha t}) + o(1) = \mathbb{P}[H(m_{N_t+1}) > \alpha t] + o(1).$$

This, combined with (1.3) and the change of variables $u = 1/(1+x)$ proves Proposition 1.2, since ϕ is choosen up to a multiplicative constant. $\square$

5.4 Proof of Lemmas 5.3, 5.4 and 5.5

Proof of Lemma 5.3. Let $j \in [1, n_t)$. First by (3.2),

$$\mathbb{P}^{W_\kappa}_{\tilde{m}_j}\left(H(\widehat{L}_j^+) > H\left[\tilde{\tau}_j^-((1+\delta/2)\phi^*(t))\right]\right) \leq \widehat{Q}_j/\widehat{D}_j,$$

where $\widehat{Q}_j := \int_{\tilde{m}_j}^{\widehat{L}_j^+} e^{\tilde{V}^{(j)}(x)}\mathrm{d}x$ and $\widehat{D}_j := \int_{\tilde{\tau}_j^-((1+\delta/2)\phi^*(t))}^{\widehat{m}_j} e^{\tilde{V}^{(j)}(x)}\mathrm{d}x$. Recall that $\widehat{L}_j^+ = \tilde{\tau}_j(\phi^*(t))$. With a method similar as for (3.10), we get with a probability larger than $1 - e^{-(c_-)\delta\phi^*(t)}$,

$$\widehat{Q}_j \leq (\widehat{L}_j^+ - \tilde{m}_j)\exp\left[\max_{[\tilde{m}_j, \widehat{L}_j^+]} \tilde{V}^{(j)}\right] \leq 8\kappa^{-1}\phi^*(t)\exp(\phi^*(t)), \qquad \widehat{D}_j \geq e^{(1+\delta/8)\phi^*(t)}$$

by (2.22) for the first inequality and by Fact 2.1 **(ii)** and (2.29) with $h = (1+\delta/2)\phi^*(t)$ and $\alpha = 1 - \frac{1+\delta/8}{1+\delta/2}$ for the second one, together with Lemma 2.3 since $\phi(t) = o(\log t)$. Hence, with such a probability,

$$\mathbb{P}^{W_\kappa}_{\tilde{m}_j}\left[H(\widehat{L}_j^+) > H[\tilde{\tau}_j^-((1+\delta/2)\phi^*(t))]\right] \leq C_+\phi^*(t)e^{-\delta\phi^*(t)/8}. \tag{5.23}$$

Moreover, as in (3.23) and by scaling, we have under $\mathbb{P}^{W_\kappa}_{\tilde{m}_j}$ on $\left\{H(\widehat{L}_j^+) < H[\tilde{\tau}_j^-((1+\delta/2)\phi^*(t))]\right\}$,

$$H(\widehat{L}_j^+) \stackrel{\mathcal{L}}{=} \tilde{A}_j(\widehat{L}_j^+)\int_{\tilde{\tau}_j^-[(1+\delta/2)\phi^*(t)]}^{\widehat{L}_j^+} e^{-\tilde{V}^{(j)}(u)}\mathcal{L}_B\left[\tau_B(1), \tilde{A}_j(u)/\tilde{A}_j(\widehat{L}_j^+)\right]\mathrm{d}u, \tag{5.24}$$

where $\tilde{A}_j(u) = \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)}\mathrm{d}x$ for $u \in \mathbb{R}$ as before, and $(B(s),\ s \geq 0)$ is a Brownian motion independent of W_κ. Since $\tilde{A}_j(\widehat{L}_j^+) \leq [\tilde{\tau}_j(\phi^*(t)) - \tilde{m}_j]e^{\phi^*(t)}$, we get

$$\text{RHS of (5.24)} \leq e^{\phi^*(t)}\left[\tilde{\tau}_j(\phi^*(t)) - \tilde{\tau}_j^-[(1+\delta/2)\phi^*(t)]\right]^2 \sup_{x\in\mathbb{R}} \mathcal{L}_B\left[\tau_B(1), x\right].$$

We know that $P\left[\tilde{\tau}_j(\phi^*(t)) - \tilde{m}_j \geq e^{\delta\phi^*(t)/3}\right] \leq C_+e^{-\kappa\phi^*(t)/(2\sqrt{2})}$ and $P\left[\tilde{m}_j - \tilde{\tau}_j^-[(1+\delta/2)\phi^*(t)] \geq e^{\delta\phi^*(t)/3}\right] \leq C_+e^{-\kappa\phi^*(t)/(2\sqrt{2})}$ by (2.22) and (2.23). So, we get with probability at least $1 - C_+e^{-\kappa\phi^*(t)/(2\sqrt{2})}$, $\tilde{\tau}_j(\phi^*(t)) - \tilde{\tau}_j^-[(1+\delta/2)\phi^*(t)] \leq 2e^{\delta\phi^*(t)/3}$ and then

$$\begin{aligned}
P^{W_\kappa}_{\tilde{m}_j}\left(\text{RHS of (5.24)} \geq e^{(1+\delta)\phi^*(t)}\right) &\leq \mathbb{P}^{W_\kappa}_{\tilde{m}_j}\left(\sup_{x\in\mathbb{R}} \mathcal{L}_B(\tau^B(1), x) \geq e^{\delta\phi^*(t)/3}/4\right) \\
&\leq C_+e^{-\delta\phi^*(t)/3},
\end{aligned}$$

where the last inequality comes from (4.8), (4.9) and the independence of B and W_κ. This, (5.23), (5.24) and $Q^{W_\kappa}_{\tilde m_j}(X\in .)=\mathbb{P}^{W_\kappa}_{\tilde m_j}(X\in .)$ give with a probability larger than $1-e^{-c_-\delta\phi^*(t)}$,

$$Q^{W_\kappa}_{\tilde m_j}\left(H(\widehat L_j^+)>e^{(1+\delta)\phi^*(t)}\right)=P^{W_\kappa}_{\tilde m_j}\left(H(\widehat L_j^+)>e^{(1+\delta)\phi^*(t)}\right)\le C_+\phi^*(t)e^{-\delta\phi^*(t)/8}. \quad (5.25)$$

We get the same result for $H(\widehat L_j^-)$, since the law of $V^{(j)}$ restricted to $[\tau_j^-(h_t),\tau_j(h_t)]$ is symmetric with respect to m_j for $j\ge 2$ by Fact 2.1 and $P(\overline{\mathcal V}_t)=o(e^{-\delta\phi^*(t)/8})$ by Lemma 2.3; the result from $j=1$ follows from the fact that the valleys for $j=1$ and $j=2$ have the same law by Lemma 2.3. This together with (5.25) gives (5.13). □

Proof of Lemma 5.4. Let $1\le j\le n_t-1$ and assume $0<\zeta<\kappa/8$. Recall that $\widehat L_j^-=\tilde\tau_j^-(\phi^*(t))$ and $\widehat L_j^+=\tilde\tau_j(\phi^*(t))$.

First by (2.22) since $\phi^*(t)=o(\log t)$ and $\zeta<\kappa/8$, we have for uniformly large t,

$$P\big[\widehat L_j^+\ge\tilde m_j+\phi^*(t)/\zeta\big]\le P\big[\tilde\tau_j(\phi^*(t))-\tilde m_j>8\phi^*(t)/\kappa\big]\le C_+e^{-\kappa\phi^*(t)/(2\sqrt2)}.$$

Similarly by (2.23), $P\big[\widehat L_j^-\le\tilde m_j-\phi^*(t)/\zeta\big]\le C_+e^{-\kappa\phi^*(t)/(2\sqrt2)}$. So with probability at least $1-C_+e^{-\kappa\phi^*(t)/(2\sqrt2)}$, $\big[\widehat L_j^-,\widehat L_j^+\big]\subset[\tilde m_j-\phi^*(t)/\zeta,\tilde m_j+\phi^*(t)/\zeta]=\mathbf I_j$ as in Figure 2, and then $\tilde\mu_j\big([\tilde L_j^-,\tilde L_j]\smallsetminus\mathbf I_j\big)\le\tilde\mu_j\big([\tilde L_j^-,\widehat L_j^-]\big)+\tilde\mu_j\big([\widehat L_j^+,\tilde L_j]\big)$. This gives the first inequality of (5.17) with such probability.

We now prove the second inequality of (5.17). First, we observe that for large t,

$$\tilde Z:=\int_{\tilde L_j^-}^{\tilde L_j}\frac{\mathrm dy}{e^{\tilde V^{(j)}(y)}}\ge\int_{\tau_j[\alpha\phi^*(t)/2]}^{\tau_j[\alpha\phi^*(t)]}\frac{\mathrm dy}{e^{V^{(j)}(y)}}\ge\big[\tau_j[\alpha\phi^*(t)]-\tau_j[\alpha\phi^*(t)/2]\big]e^{-\alpha\phi^*(t)}\ge e^{-\alpha\phi^*(t)}$$

on $\mathcal E_1^{5.4}\cap\mathcal V_t$, where $\mathcal E_1^{5.4}:=\{\tau_j[\alpha\phi^*(t)]-\tau_j[\alpha\phi^*(t)/2]\ge1\}$ and $\alpha:=(1-\delta)\kappa/16$. By (2.28) and Fact 2.1 **(ii)**, $P(\overline{\mathcal E_1^{5.4}})\le4e^{-\alpha^2\phi^*(t)^2/12}$. Moreover, due to the definition (5.12) of $\tilde\mu_j$, we have

$$\tilde\mu_j\big([\tilde L_j^-,\widehat L_j^-]\big)+\tilde\mu_j\big([\widehat L_j^+,\tilde L_j]\big)=[\mathcal J_3+\mathcal J_4+\mathcal J_5+\mathcal J_6]/\tilde Z, \quad (5.26)$$

where

$$\mathcal J_3:=\int_{\tilde L_j^-}^{\tilde\tau_j^-(h_t)}\frac{\mathrm dy}{e^{\tilde V^{(j)}(y)}},\qquad \mathcal J_4:=\int_{\tilde\tau_j^-(h_t)}^{\tilde\tau_j^-[\phi^*(t)]}\frac{\mathrm dy}{e^{\tilde V^{(j)}(y)}},$$
$$\mathcal J_5:=\int_{\tilde\tau_j[\phi^*(t)]}^{\tilde\tau_j(h_t)}\frac{\mathrm dy}{e^{\tilde V^{(j)}(y)}},\qquad \mathcal J_6:=\int_{\tilde\tau_j(h_t)}^{\tilde L_j}\frac{\mathrm dy}{e^{\tilde V^{(j)}(y)}}.$$

Recalling that $\tilde V^{(j)}=V^{(j)}$ on $\mathcal V_t$ by Remark 2.4, we introduce $\gamma=(1-\delta)\kappa/8$,

$$\mathcal E_2^{5.4}:=\big\{\inf\{V^{(j)}(s),\ \tau_j(\phi^*(t))\le s\le\tau_j(h_t)\}>\gamma\phi^*(t)\big\},$$
$$\mathcal E_3^{5.4}:=\big\{\tilde\tau_j(h_t)-\tilde m_j\le 8h_t/\kappa\big\}.\qquad \mathcal E_4^{5.4}:=\big\{\tilde L_j-\tilde\tau_j(h_t)\le 2h_t/\kappa\big\}.$$

Equation (2.35) with $h=h_t$, $\alpha=1/2$ and $\omega=1$ gives for large t since $\tilde\tau_j(h_t)$ is a stopping time,

$$P\big(\overline{\mathcal E_4^{5.4}}\big)=P\big(\tau^{W_\kappa}(-h_t/2)>2h_t/\kappa\big)\le e^{-\kappa h_t/16}.$$

We have $P\big(\overline{\mathcal E_3^{5.4}}\big)\le C_+e^{-\kappa h_t/(2\sqrt2)}$ by (2.22). Moreover using Fact 2.1 **(ii)** if $j\ge2$ and taking the limit as $t\to+\infty$ in (2.31) applied with $h=\phi^*(t)$, $\alpha=1$, $\gamma=(1-\delta)\kappa/8$ and $\omega=h_t/\phi^*(t)$ gives $P(\overline{\mathcal E_2^{5.4}})\le2e^{\kappa(\gamma-1)\phi^*(t)}\le2e^{-\delta\kappa^2\phi^*(t)/8}$. We have on $\mathcal E_2^{5.4}\cap\mathcal E_3^{5.4}\cap\mathcal E_4^{5.4}\cap\mathcal V_t$,

$$\mathcal J_5\le\big[\tilde\tau_j(h_t)-\tilde m_j\big]e^{-\gamma\phi^*(t)}\le8\kappa^{-1}h_te^{-\gamma\phi^*(t)},\qquad \mathcal J_6\le\big[\tilde L_j-\tilde\tau_j(h_t)\big]e^{-h_t/2}\le2\kappa^{-1}h_te^{-h_t/2}.$$

We prove similarly that there exists an event $\mathcal{E}_5^{5.4}$ such that $P\big(\overline{\mathcal{E}_5^{5.4}} \cap \mathcal{V}_t\big) \leq 2e^{\kappa(\gamma-1)\phi^*(t)} + C_+ e^{-\kappa h_t/(2\sqrt{2})}$ and $\mathcal{J}_4 \leq 8\kappa^{-1} h_t e^{-\gamma\phi^*(t)}$ on $\mathcal{E}_5^{5.4} \cap \mathcal{V}_t$. Furthermore, by Lemma 2.7 equations (2.32) and (2.34), on some event $\mathcal{E}_6^{5.4}$ which has probability at least $1 - e^{-\kappa h_t/8} \geq 1 - C_+ e^{-(c_-)\delta\phi^*(t)}$,

$$\mathcal{J}_3 \leq (\tilde{\tau}_j^-(h_t) - \tilde{L}_j^-) e^{-h_t/2} \leq 40\kappa^{-1} h_t^+ e^{-h_t/2} \leq 8\kappa^{-1} h_t e^{-\gamma\phi^*(t)}$$

for large t. These inequalities combined with (5.26) and Lemma 2.3 give on $\cap_{i=1}^6 \mathcal{E}_i^{5.4} \cap \mathcal{V}_t$,

$$\tilde{\mu}_j\big([\tilde{L}_j^-, \widehat{L}_j^-]\big) + \tilde{\mu}_j\big([\widehat{L}_j^+, \tilde{L}_j]\big) \leq 8\kappa^{-1} h_t [3e^{-\gamma\phi^*(t)} + e^{-h_t/2}] e^{\alpha\phi^*(t)} \leq C_+ h_t e^{-(1-\delta)\kappa\phi^*(t)/16},$$

since $\phi^*(t) = o(\log t)$. Since $P(\cap_{i=1}^6 \mathcal{E}_i^{5.4} \cap \mathcal{V}_t) \geq 1 - C_+ e^{-(c_-)\delta\phi^*(t)}$, due to the previous inequalities and to Lemma 2.3, this proves the second inequality of (5.17) for $2 \leq j \leq n_t - 1$. This is also true if $j = 1$ since the first valley has the same law by Lemma 2.3. □

Proof of Lemma 5.5. Let $j \in [1, n_t)$. By (3.22) applied with $i = j$ and $r = \tilde{\tau}_j[\kappa\phi^*(t)/(8\zeta)] - \tilde{m}_j$, there exists a Brownian motion $(\tilde{B}(s),\ s \geq 0)$, independent of $\tilde{V}^{(j)}$, such that under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$,

$$H\Big(\tilde{\tau}_j[\kappa\phi^*(t)/(8\zeta)]\Big) \geq \int_{\tilde{m}_j}^{\tilde{\tau}_j[\kappa\phi^*(t)/(8\zeta)]} e^{-V^{(j)}(u)} \mathcal{L}_{\tilde{B}}\Big[\tau^{\tilde{B}}\Big(\tilde{A}_j\Big(\tilde{\tau}_j\Big[\frac{\kappa\phi^*(t)}{8\zeta}\Big]\Big)\Big), \tilde{A}_j(u)\Big] \mathrm{d}u =: H_j^+,$$

where for all $z \in \mathbb{R}$, $\tilde{A}_j(z) = \int_{\tilde{m}_j}^z e^{\tilde{V}^{(j)}(x)} \mathrm{d}x$. Now, let $\mathcal{E}_7^{5.4} := \{\tilde{\tau}_j[\kappa\phi^*(t)/(8\zeta)] - \tilde{m}_j \leq \phi^*(t)/\zeta\}$. By (2.22), $P\big(\mathcal{E}_7^{5.4}\big) \geq 1 - C_+ e^{-\kappa^2\phi^*(t)/(16\zeta\sqrt{2})}$. We have on $\mathcal{E}_7^{5.4}$ under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$,

$$H\big(\tilde{m}_j + \phi^*(t)/\zeta\big) \geq H\big(\tilde{\tau}_j[\kappa\phi^*(t)/(8\zeta)]\big) \geq H_j^+.$$

Assume $\zeta \leq \kappa/48$. In order to estimate H_j^+, we introduce

$$\delta_t^* := e^{-\kappa\phi^*(t)/(48\zeta)} \in (0,1), \qquad \mathcal{J}_7 := \tilde{A}_j\Big(\tilde{\tau}_j\Big[\frac{\kappa\phi^*(t)}{8\zeta}\Big]\Big), \qquad \mathcal{J}_8 := \int_{\tilde{m}_j}^{\tilde{\tau}_j[\frac{\kappa\phi^*(t)}{16\zeta}]} e^{-\tilde{V}^{(j)}(u)} \mathrm{d}u.$$

By scaling, there exists some Brownian motion B' independent of $\tilde{V}^{(j)}$ such that

$$H_j^+ \geq \int_{\tilde{m}_j}^{\tilde{\tau}_j[\frac{\kappa\phi^*(t)}{16\zeta}]} e^{-\tilde{V}^{(j)}(u)} \mathcal{J}_7 \mathcal{L}_{B'}\big(\tau^{B'}(1), \tilde{A}_j(u)/\mathcal{J}_7\big) \mathrm{d}u \geq \mathcal{J}_7 \mathcal{J}_8 \mathcal{J}_9$$

on $\mathcal{E}_8^{5.4} := \{\tilde{A}_j\big(\tilde{\tau}_j\big[\frac{\kappa\phi^*(t)}{16\zeta}\big]\big) \leq \delta_t^* \mathcal{J}_7\}$ with $\mathcal{J}_9 := \inf_{x \in [0, \delta_t^*]} \mathcal{L}_{B'}\big(\tau^{B'}(1), x\big)$. We have

$$\mathcal{J}_8 \geq \big[\tilde{\tau}_j[\kappa\phi^*(t)/(48\zeta)] - \tilde{\tau}_j[\kappa\phi^*(t)/(96\zeta)]\big] \exp[-\kappa\phi^*(t)/(48\zeta)] \geq \exp[-\kappa\phi^*(t)/(48\zeta)]$$

with probability $\geq 1 - e^{-(c_-)(\phi^*(t))^2/\zeta^2} - e^{-h_t/4}$ for large t by (2.28), Fact 2.1 **(ii)** and Lemma 2.3. Moreover,

$$\mathcal{J}_9 \geq \big[1 - (\delta_t^*)^{1/3}\big] \mathcal{L}_{B'}\big(\tau^{B'}(1), 0\big) \geq (1/2) \mathcal{L}_{B'}\big(\tau^{B'}(1), 0\big) \geq e^{-\kappa\phi^*(t)/(48\zeta)}$$

for large t with probability $\geq 1 - c_2(\delta_t^*)^{1/30}$ by (4.7) for the first inequality, and with probability $\geq 1 - e^{-\kappa\phi^*(t)/(48\zeta)}$ for the last one since $\mathcal{L}_{B'}\big(\tau^{B'}(1), 0\big)$ is exponentially distributed with mean 2 as before by the first Ray-Knight theorem. Furthermore, by Lemma 2.3, Fact 2.1 and (2.29),

$$P\Big(\mathcal{J}_7 \geq e^{\frac{5\kappa\phi^*(t)}{48\zeta}}\Big) \geq P\Big(\int_0^{\tau^R(\kappa\phi^*(t)/(8\zeta))} e^{R(u)} \mathrm{d}u \geq e^{(1-1/6)\frac{\kappa\phi^*(t)}{8\zeta}}\Big) - P\big(\overline{\mathcal{V}_t}\big) \geq 1 - 4e^{-\frac{(c_-)\phi^*(t)}{\zeta}}.$$

Finally by (2.22), with probability at least $1-C_+\exp(-\frac{\kappa^2\phi^*(t)}{32\zeta\sqrt{2}})$,

$$\tilde{A}_j\Big(\tilde{\tau}_j\Big[\frac{\kappa\phi^*(t)}{16\zeta}\Big]\Big)\leq\Big[\tilde{\tau}_j\Big[\frac{\kappa\phi^*(t)}{16\zeta}\Big]-\tilde{m}_j\Big]\exp\Big(\frac{\kappa\phi^*(t)}{16\zeta}\Big)\leq\frac{\phi^*(t)}{2\zeta}\exp\Big(\frac{\kappa\phi^*(t)}{16\zeta}\Big).$$

The last two inequalities give $P[\mathcal{E}_8^{5.4}]\geq 1-e^{-(c_-)\phi^*(t)/\zeta}$. As a consequence, we have

$$\mathbb{P}_{\tilde{m}_j}^{W_\kappa}\Big[H\big(\tilde{m}_j+\phi^*(t)/\zeta\big)\geq H_j^+\geq\mathcal{J}_7\mathcal{J}_8\mathcal{J}_9\geq e^{\kappa\phi^*(t)/(16\zeta)}\geq e^{(1+2\delta)\phi^*(t)}\Big]\geq 1-e^{-(c_-)\phi^*(t)/\zeta}$$

with probability at least $1-e^{-(c_-)\phi^*(t)/\zeta}$ since $\zeta\leq\kappa/48$ and $\delta<1$. We obtain the same result for $H(\tilde{m}_j-\phi^*(t)/\zeta)$ by symmetry of the law of $V^{(j)}$ for $j\geq 2$ by Fact 2.1 **(ii)**, and then for $j=1$ by Lemma 2.3 as before. □

6 Proofs of some technical estimates related to the environment

6.1 Proof of Lemma 4.2

We denote by I_κ and K_κ the modified Bessel functions, respectively of the first and second kind. We remind that as $x\downarrow 0$ (see e.g. [7] p. 638), since $0<\kappa<1$,

$$I_\kappa(x)=\frac{1}{\Gamma(\kappa+1)}\Big(\frac{x}{2}\Big)^\kappa+\frac{1}{\Gamma(\kappa+2)}\Big(\frac{x}{2}\Big)^{\kappa+2}+O\big(x^{\kappa+4}\big),\tag{6.1}$$

$$K_\kappa(x)=\frac{\pi\big[I_{-\kappa}(x)-I_\kappa(x)\big]}{2\sin(\pi\kappa)}=\frac{\pi}{2\sin(\pi\kappa)}\Big[\frac{(x/2)^{-\kappa}}{\Gamma(1-\kappa)}-\frac{(x/2)^\kappa}{\Gamma(\kappa+1)}+\frac{(x/2)^{2-\kappa}}{\Gamma(2-\kappa)}\Big]+O(x^{\kappa+2}).\tag{6.2}$$

Moreover, we remind that (see e.g. [7] p. 638),

$$I_\kappa'(u)K_\kappa(u)-I_\kappa(u)K_\kappa'(u)=1/u,\qquad u>0.\tag{6.3}$$

Let $y>0$. First, ([7], 2.10.3 page 302) with $\alpha=0$, $x=y$, $z=y/2<x$, $\beta=1/2$, and $\mu=-\kappa/2$ gives for G^+, which is defined in (4.2),

$$E\left(e^{-\gamma G^+(y/2,y)}\right)=E\bigg[\exp\bigg(-\gamma\int_0^{\tau^{W_\kappa^y}(y/2)}e^{W_\kappa^y(s)}\mathrm{d}s\bigg)\bigg]=e^{\kappa y/4}\frac{K_\kappa(2\sqrt{2\gamma}e^{y/2})}{K_\kappa(2\sqrt{2\gamma}e^{y/4})},\qquad\gamma>0.$$

So, $E\left(e^{-\gamma G^+(y/2,y)/e^y}\right)=e^{\kappa y/4}\frac{K_\kappa(2\sqrt{2\gamma})}{K_\kappa(2\sqrt{2\gamma}e^{-y/4})}=g_+\big(2\sqrt{2\gamma},[2\sqrt{2\gamma}e^{-y/4}]^\kappa\big)$. where

$$g_+(u,v):=\frac{u^\kappa K_\kappa(u)}{vK_\kappa(v^{1/\kappa})},\qquad u>0,\ v>0.$$

We have, as $\max(u,v)\downarrow 0$, by (6.1) and (6.2) since $0<\kappa<1$,

$$\begin{aligned}g_+(u,v)&=\frac{u^\kappa\frac{\pi}{2\sin(\pi\kappa)}\big[\frac{1}{\Gamma(1-\kappa)}(u/2)^{-\kappa}-\frac{1}{\Gamma(\kappa+1)}(u/2)^\kappa+\frac{1}{\Gamma(2-\kappa)}(u/2)^{2-\kappa}+o(u^{2-\kappa})\big]}{v\frac{\pi}{2\sin(\pi\kappa)}\big[\frac{1}{\Gamma(1-\kappa)}\frac{v^{-1}}{2^{-\kappa}}-\frac{1}{\Gamma(\kappa+1)}\frac{v}{2^\kappa}+o(v)\big]}\\&=1-\frac{\Gamma(1-\kappa)}{\Gamma(\kappa+1)}\frac{u^{2\kappa}}{4^\kappa}+O\big([\max(u,v)]^2\big).\end{aligned}$$

This gives, with $u=2\sqrt{2\gamma}$ and $v=[2\sqrt{2\gamma}e^{-y/4}]^\kappa$ as $\gamma\downarrow 0$ and $y\to+\infty$,

$$E\left(e^{-\gamma G^+(y/2,y)/e^y}\right)=1-\frac{\Gamma(1-\kappa)}{\Gamma(\kappa+1)}(2\gamma)^\kappa+O\big(\max(\gamma,\gamma^\kappa e^{-\kappa y/2})\big).\tag{6.4}$$

We now turn to $F^{\pm}(y)$, defined in (4.2). We have for $\gamma > 0$, by (2.4),

$$E\left(e^{-\gamma F^{\pm}(y)}\right) = \lim_{x\downarrow 0} E\Bigg[\exp\left(-\gamma\int_0^{\tau^{W^x_{-\kappa}}(y)} e^{\pm W^x_{-\kappa}(s)}\mathrm{d}s\right)\frac{\mathbb{1}_{\{\tau^{W^x_{-\kappa}}(0)=\infty\}}}{P\big[\tau^{W^x_{-\kappa}}(0)=\infty\big]}\Bigg].$$

The expectation in the right hand side of this equality is equal to, first by the strong Markov property, and second by ([7], 3.10.7(b) page 317) with $\alpha = 0$, $a = 0$, $b = y$, $\beta = \pm 1/2$, $\mu = \kappa/2$ and $x > 0$, and since $P\big(\tau^{W^x_{-\kappa}}(0) = \infty\big) = 1 - e^{-\kappa x}$ due to the scale function (2.26),

$$\begin{aligned}
&E\Bigg[\exp\left(-\gamma\int_0^{\tau^{W^x_{-\kappa}}(y)} e^{\pm W^x_{-\kappa}(s)}\mathrm{d}s\right)\mathbb{1}_{\{\tau^{W^x_{-\kappa}}(y)<\tau^{W^x_{-\kappa}}(0)\}}\Bigg]\frac{P\big[\tau^{W^y_{-\kappa}}(0)=\infty\big]}{P\big[\tau^{W^x_{-\kappa}}(0)=\infty\big]}\\
= \; & \frac{e^{(\mu-|\mu|(\pm 1))y}S_\kappa\big(2\sqrt{2\gamma}e^{\pm x/2}, 2\sqrt{2\gamma}\big)}{e^{(\mu-|\mu|(\pm 1))x}S_\kappa\big(2\sqrt{2\gamma}e^{\pm y/2}, 2\sqrt{2\gamma}\big)}\,\frac{1-e^{-\kappa y}}{1-e^{-\kappa x}},
\end{aligned}$$

where $S_\kappa(u,v) := (uv)^{-\kappa}[I_\kappa(u)K_\kappa(v) - K_\kappa(u)I_\kappa(v)]$ as defined in ([7] p. 645). Consequently, $E\big(e^{-\gamma F^{\pm}(y)}\big)$ is the limit, as $x\downarrow 0$, of

$$\frac{\big[I_\kappa(2\sqrt{2\gamma}e^{\pm x/2})K_\kappa(2\sqrt{2\gamma}) - K_\kappa(2\sqrt{2\gamma}e^{\pm x/2})I_\kappa(2\sqrt{2\gamma})\big]}{\big[I_\kappa(2\sqrt{2\gamma}e^{\pm y/2})K_\kappa(2\sqrt{2\gamma}) - K_\kappa(2\sqrt{2\gamma}e^{\pm y/2})I_\kappa(2\sqrt{2\gamma})\big]}\,\frac{\sinh(\kappa y/2)}{\sinh(\kappa x/2)} =: \frac{N_\pm(x,y)}{D_\pm(x,y)}.$$

Now, notice that for $\gamma > 0$, as $x\downarrow 0$,

$$\begin{aligned}
I_\kappa\big(2\sqrt{2\gamma}e^{\pm x/2}\big) &= I_\kappa\big[2\sqrt{2\gamma}(1\pm x/2+o(x))\big] = I_\kappa\big[2\sqrt{2\gamma}\big] \pm \sqrt{2\gamma}I'_\kappa[2\sqrt{2\gamma}]x + o(x),\\
K_\kappa\big(2\sqrt{2\gamma}e^{\pm x/2}\big) &= K_\kappa\big[2\sqrt{2\gamma}(1\pm x/2+o(x))\big] = K_\kappa\big[2\sqrt{2\gamma}\big] \pm \sqrt{2\gamma}K'_\kappa[2\sqrt{2\gamma}]x + o(x).
\end{aligned}$$

So for $\gamma > 0$, by (6.3),

$$\begin{aligned}
N_\pm(x,y) \quad &\sim_{x\downarrow 0} \quad \pm\sqrt{2\gamma}\big[I'_\kappa[2\sqrt{2\gamma}]K_\kappa(2\sqrt{2\gamma}) - K'_\kappa[2\sqrt{2\gamma}]I_\kappa(2\sqrt{2\gamma})\big]\sinh(\kappa y/2)x\\
&\sim_{x\downarrow 0} \quad \pm\sinh(\kappa y/2)x/2.
\end{aligned}$$

Moreover, $\sinh(\kappa x/2) \sim_{\downarrow 0} \kappa x/2$, and then

$$E\left(e^{-\gamma F^{\pm}(y)}\right) = \lim_{x\downarrow 0}\frac{N_\pm(x,y)}{D_\pm(x,y)} = \frac{\pm\kappa^{-1}\sinh(\kappa y/2)}{I_\kappa(2\sqrt{2\gamma}e^{\pm y/2})K_\kappa(2\sqrt{2\gamma}) - K_\kappa(2\sqrt{2\gamma}e^{\pm y/2})I_\kappa(2\sqrt{2\gamma})}. \tag{6.5}$$

We now consider

$$f(u,v) := \frac{-(v^{-\kappa}-v^{\kappa})}{2\kappa[I_\kappa(uv)K_\kappa(u) - K_\kappa(uv)I_\kappa(u)]} \tag{6.6}$$

so that $E\big(e^{-\gamma F^-(y)}\big) = E\big(e^{-\gamma F^+(y)/e^y}\big) = f\big(2\sqrt{2\gamma}, e^{-y/2}\big)$ for every $\gamma > 0$, and so $F^-(y) \overset{\mathcal{L}}{=} F^+(y)/e^y$. We get successively, as $\max\big(v^{2\kappa}, u^3\big) \to 0$, by (6.1) and (6.2), using $\Gamma(1-\kappa)\Gamma(\kappa) = \pi/\sin(\pi\kappa)$,

$$\begin{aligned}
K_\kappa(uv)I_\kappa(u) &= \frac{1}{2\kappa}v^{-\kappa} + \frac{1}{8(\kappa+1)\kappa}v^{-\kappa}u^2 + v^{-\kappa}o\big(\max\big(v^{2\kappa}, u^3\big)\big),\\
I_\kappa(uv)K_\kappa(u) &= v^\kappa/(2\kappa) + v^{-\kappa}o\big(\max\big(v^{2\kappa}, u^3\big)\big),\\
2\kappa\big[I_\kappa(uv)K_\kappa(u) - K_\kappa(uv)I_\kappa(u)\big] &= v^\kappa - v^{-\kappa} - \frac{1}{4(\kappa+1)}v^{-\kappa}u^2 + v^{-\kappa}o\big(\max\big(v^{2\kappa}, u^3\big)\big).
\end{aligned}$$

This yields

$$f(u,v) = \frac{-(v^{-\kappa}-v^{\kappa})}{v^\kappa - v^{-\kappa} - \frac{1}{4(\kappa+1)}v^{-\kappa}u^2 + v^{-\kappa}o(\max(v^\kappa, u^3))} = 1 - \frac{1}{4(\kappa+1)}u^2 + o(\max(v^{2\kappa}, u^3)).$$

Consequently, as $\max(e^{-\kappa y},\gamma^{3/2})\to 0$,

$$f(2\sqrt{2\gamma},e^{-y/2})=1-\frac{2\gamma}{\kappa+1}+O(\max(e^{-\kappa y},\gamma^{3/2}))=\left(1+\frac{2\gamma}{\kappa+1}\right)^{-1}+O(\max(e^{-\kappa y},\gamma^{3/2})).$$

Since $E\big(e^{-\gamma F^+(y)/e^y}\big)=E\big(e^{-\gamma F^-(y)}\big)=f(2\sqrt{2\gamma},e^{-y/2})$, this and (6.4) proves the existence of $C_4>0$, $M>0$ and $\eta_1\in(0,1)$ such that (4.3), (4.4) and (4.5) are satisfied simultaneously for all $y>M$ and $\gamma\in(0,\eta_1]$.

We now prove $E[F^+(y)/e^y]$ exists and is bounded, by computing $\frac{\partial}{\partial\gamma}E\big(e^{-\gamma F^+(y)/e^y}\big)$ at $\gamma=0$. To this aim, we fix $y>0$ and observe that as $\gamma\downarrow 0$, once more by (6.1) and (6.2),

$$\begin{aligned}E\left(e^{-\gamma F^+(y)/e^y}\right)&=f\big(2\sqrt{2\gamma},e^{-y/2}\big)\\&=1-\frac{[e^{\kappa y/2}-e^{-\kappa y/2-y}]}{(\kappa+1)\sinh(\kappa y/2)}\gamma-\frac{(e^{-y+\kappa y/2}-e^{-\kappa y/2})}{(1-\kappa)\sinh(\kappa y/2)}\gamma+o(\gamma).\end{aligned}$$

Hence,

$$E[F^+(y)/e^y]=-\left(\frac{\partial}{\partial\gamma}\right)_{\gamma=0}E\left(e^{-\gamma F^+(y)/e^y}\right)=\frac{[e^{\kappa y/2}-e^{-\kappa y/2-y}]}{(\kappa+1)\sinh(\kappa y/2)}+\frac{(e^{-y+\kappa y/2}-e^{-\kappa y/2})}{(1-\kappa)\sinh(\kappa y/2)},$$

which is a bounded function of y on $\mathbb{R}_+$. Moreover, $F^-(y)\overset{\mathcal{L}}{=}F^+(y)/e^y$, so $E[F^-(y)]=E[F^+(y)/e^y]$.

Finally, taking the limit of $E\big(e^{-\gamma F^-(y)}\big)$ in (6.5) for fixed $\gamma>0$ as $y\to+\infty$ with the help of (6.1) and (6.2) proves (4.6). □

6.2 Proof of Lemma 2.7

We consider $\mathcal{E}_1^{6.2}:=\cap_{i=1}^{n_t}\{\tilde{L}_i^+-\tilde{\tau}_i^-(h_t^+)\le 40h_t^+/\kappa\}$. We also introduce $\mathcal{E}_2^{6.2}:=\{W_\kappa(\tilde{m}_{n_t})\ge -n_te^{5\kappa h_t/4}\}$. We have,

$$\begin{aligned}P\big[\overline{\mathcal{E}_2^{6.2}}\big]&=P\big[W_\kappa(\tilde{m}_{n_t})<-n_te^{5\kappa h_t/4}\big]\\&=P\bigg(-h_t^++\sum_{i=1}^{n_t}\big(2h_t^++W_\kappa(\tilde{L}_i^\sharp)-W_\kappa(\tilde{m}_i)\big)>n_te^{5\kappa h_t/4}\bigg).\end{aligned}$$

Recall that the r.v. $W_\kappa(\tilde{L}_i^\sharp)-W_\kappa(\tilde{m}_i)$, $i\ge 1$ are equal in law to $-\inf_{[0,\tau_1^*(h_t)]}W_\kappa$, which expectation is $\frac{2}{\kappa}\sinh(\kappa h_t/2)e^{\kappa h_t/2}$ (as recalled before (2.9)). By Markov inequality, for large t,

$$P\big[\overline{\mathcal{E}_2^{6.2}}\big]\le\frac{n_t}{n_te^{5\kappa h_t/4}+h_t^+}\Big(2h_t^++\frac{2}{\kappa}\sinh(\kappa h_t/2)e^{\kappa h_t/2}\Big)\le C_+e^{-\kappa h_t/4}.\tag{6.7}$$

On $\overline{\mathcal{E}_1^{6.2}}$ there exists $1\le i\le n_t$ such that $\tilde{L}_i^+-\tilde{\tau}_i^-(h_t^+)>40h_t^+/\kappa$. There exists an integer $j\in\mathbb{Z}$ such that $-j\ge W_\kappa[\tilde{\tau}_i^-(h_t^+)]>-j-1$. So on $\mathcal{E}_2^{6.2}$ by (2.16), $-j>W_\kappa(\tilde{m}_i)\ge W_\kappa(\tilde{m}_{n_t})\ge -n_te^{5\kappa h_t/4}$, thus $j<n_te^{5\kappa h_t/4}$. Moreover, we have by (2.16),

$$W_\kappa[\tilde{\tau}_i^-(h_t^+)]=W_\kappa[\tilde{m}_i]+h_t^+\le W_\kappa\big(\tilde{L}_i^\sharp\big)+h_t^+=W_\kappa\big(\tilde{L}_{i-1}^+\big)\le 0,\tag{6.8}$$

so $j\ge 0$. Since $0\ge -j\ge W_\kappa[\tilde{\tau}_i^-(h_t^+)]$, we have $\tau^{W_\kappa}(-j)\le\tilde{\tau}_i^-(h_t^+)$. On the other hand, $-j-1<W_\kappa[\tilde{\tau}_i^-(h_t^+)]=W_\kappa(\tilde{m}_i)+h_t^+$, so by (2.16),

$$W_\kappa(\tilde{L}_i^+)=\inf_{[0,\tilde{L}_i^+]}W_\kappa=W_\kappa(\tilde{m}_i)-h_t^+>-j-1-2h_t^+,$$

and then $\tau^{W_\kappa}[-j-1-2h_t^+]\geq \tilde L_i^+$. Hence, $\tau^{W_\kappa}[-j-1-2h_t^+]-\tau^{W_\kappa}(-j)\geq \tilde L_i^+-\tilde\tau_i^-(h_t^+)>40h_t^+/\kappa$. So for large t,

$$\begin{aligned}P\big(\overline{\mathcal{E}_1^{6.2}}\cap\mathcal{E}_2^{6.2}\big) &\leq P\Big(\cup_{j=0}^{\lfloor n_te^{5\kappa h_t/4}\rfloor}\big\{\tau^{W_\kappa}[-j-1-2h_t^+]-\tau^{W_\kappa}(-j)\geq 40h_t^+/\kappa\big\}\Big)\\ &\leq (n_te^{5\kappa h_t/4}+1)P\Big(\tau^{W_\kappa}[-1-2h_t^+]\geq 40h_t^+/\kappa\Big)\\ &\leq 2n_te^{5\kappa h_t/4}P\Big(\tau^{W_\kappa}[-3h_t^+]\geq 40h_t^+/\kappa\Big)\\ &\leq 2n_te^{5\kappa h_t/4}\exp\big[-3\kappa h_t^+\big]\leq e^{-\kappa h_t},\end{aligned}$$

by Lemma 2.8 with $h=h_t^+$, $\alpha=3$ and $\omega=20$ and since $n_t=e^{o(h_t)}$ because $\phi(t)=o(\log t)$.

We also consider

$$\mathcal{E}_3^{6.2}:=\cap_{i=1}^{n_t}\big\{\inf_{[\tilde\tau_i^-(h_t^+),\tilde\tau_i^-(h_t^+)+1]}\tilde V^{(i)}\geq h_t^+-\kappa h_t/2\big\},\qquad \mathcal{E}_4^{6.2}:=\{\tilde m_{n_t}\leq e^{2\kappa h_t}\}.$$

We notice that since $E[\tau^{W_\kappa}(-1)]<\infty$,

$$E\big[\tilde L_1^+-\tilde\tau_1(h_t)\big]=E\big[\tau^{W_\kappa}(-h_t^+-h_t)\big]\leq(\lfloor h_t\rfloor+1)E\big[\tau^{W_\kappa}(-1)\big]\leq C_+h_t.$$

Similarly, $E\big[\tilde L_1^\sharp\big]=E\big[\tau^{W_\kappa}(-h_t^+)\big]\leq C_+h_t$. Moreover, $E\big[\tilde\tau_1(h_t)-\tilde L_1^\sharp\big]=E\big[\tau_1^*(h_t)\big]\leq C_+e^{\kappa h_t}$ by (2.10). Combining these inequalities gives $E\big[\tilde L_1^+\big]\leq C_+e^{\kappa h_t}$, and as a consequence, $E[\tilde m_{n_t}]\leq E\big[\tilde L_{n_t}^+\big]=n_tE\big[\tilde L_1^+\big]\leq C_+n_te^{\kappa h_t}$ since $\big(\tilde L_{i+1}^+-\tilde L_i^+\big)$, $i\geq 0$, are i.i.d. Consequently for large t,

$$P\big[\overline{\mathcal{E}_4^{6.2}}\big]\leq E[\tilde m_{n_t}]/e^{2\kappa h_t}\leq e^{-\kappa h_t/2}.$$

On $\overline{\mathcal{E}_3^{6.2}}$, there exists $1\leq i\leq n_t$ such that $\inf_{[\tilde\tau_i^-(h_t^+),\tilde\tau_i^-(h_t^+)+1]}\tilde V^{(i)}<h_t^+-\kappa h_t/2$. Since $\tilde V^{(i)}[\tilde\tau_i^-(h_t^+)]=h_t^+$, for $k=\lfloor\tilde\tau_i^-(h_t^+)\rfloor$, we have $\sup_{[k,k+2]}W_\kappa-\inf_{[k,k+2]}W_\kappa\geq\kappa h_t/2$, and $0\leq k\leq\tilde m_{n_t}\leq e^{2\kappa h_t}$ on $\mathcal{E}_4^{6.2}$. Moreover we have for every $k\in\mathbb{N}$,

$$\begin{aligned}P\Big[\sup_{[k,k+2]}W_\kappa-\inf_{[k,k+2]}W_\kappa\geq\kappa h_t/2\Big] &\leq P\Big[\sup_{[0,2]}W\geq\frac{\kappa h_t}{4}-\kappa\Big]+P\Big[-\inf_{[0,2]}W\geq\frac{\kappa h_t}{4}-\kappa\Big]\\ &= 2P\big[|W(2)|\geq\kappa h_t/4-\kappa\big]\\ &\leq 4\exp\big[-(\kappa h_t/4-\kappa)^2/4\big]\end{aligned}$$

for large t since $\sup_{[0,2]}W\overset{\mathcal{L}}{=}|W(2)|$ and $P(W(1)\geq x)\leq e^{-x^2/2}$ for large x. Consequently,

$$\begin{aligned}P\big(\overline{\mathcal{E}_3^{6.2}}\big) &\leq P\big(\overline{\mathcal{E}_3^{6.2}}\cap\mathcal{E}_4^{6.2}\big)+P\big(\overline{\mathcal{E}_4^{6.2}}\big)\\ &\leq \sum_{k=0}^{\lfloor e^{2\kappa h_t}\rfloor}P\Big[\sup_{[k,k+2]}W_\kappa-\inf_{[k,k+2]}W_\kappa\geq\frac{\kappa h_t}{2}\Big]+P\big(\overline{\mathcal{E}_4^{6.2}}\big)\leq e^{-\kappa h_t/4}\end{aligned}$$

for large t. Notice in particular that on $\mathcal{E}_3^{6.2}$, we have for every $1\leq i\leq n_t$, $\tilde\tau_i^-(h_t^+)+1<\tilde m_i$ since $\tilde V^{(i)}(\tilde m_i)=0<h_t^+-\kappa h_t/2$, and

$$\int_{\tilde L_i^-}^{\tilde m_i}e^{\tilde V^{(i)}(u)}\mathrm{d}u\geq\int_{\tilde\tau_i^-(h_t^+)}^{\tilde\tau_i^-(h_t^+)+1}e^{\tilde V^{(i)}(u)}\mathrm{d}u\geq e^{h_t^+-\kappa h_t/2}. \tag{6.9}$$

Finally, let $\mathcal{E}_5^{6.2}:=\cap_{i=1}^{n_t}\big\{\inf_{[\tilde\tau_i^-(h_t^+),\tilde\tau_i^-(h_t)]}\tilde V^{(i)}\geq h_t/2\big\}$. On $\overline{\mathcal{E}_5^{6.2}}$, we consider $1\leq i\leq n_t$ such that $\inf_{[\tilde\tau_i^-(h_t^+),\tilde\tau_i^-(h_t)]}\tilde V^{(i)}<h_t/2$. There exists an integer $j\in\mathbb{Z}$ such that $-j\geq W_\kappa[\tilde\tau_i^-(h_t^+)]>-j-1$. As before, just before and after (6.8), this yields $0\leq j<n_te^{5\kappa h_t/4}$ on $\mathcal{E}_2^{6.2}$.

Consider now $y_j := \inf\left\{x \geq \tau^{W_\kappa}(-j-1),\ W_\kappa(x) - \inf_{[\tau^{W_\kappa}(-j-1),x]} W_\kappa \geq h_t/2\right\}$. We have $y_j > \tau^{W_\kappa}(-j-1) \geq 0$. Moreover, due to the definition of our i, and since for all $0 \leq x \leq \tau^{W_\kappa}(-j-1)$, $W_\kappa(x) \geq -j-1 \geq W_\kappa(\tilde{m}_i) + h_t^+ - 1$ and so, for large t, $\tilde{V}^{(i)}(x) \geq h_t^+ - 1 > h_t/2 > \inf_{[\tilde{\tau}_i^-(h_t^+),\tilde{\tau}_i^-(h_t)]} \tilde{V}^{(i)}$, we have

$$\tilde{V}^{(i)}[\tilde{\tau}_i^-(h_t)] - \inf_{[\tau^{W_\kappa}(-j-1),\tilde{\tau}_i^-(h_t)]} \tilde{V}^{(i)} \geq \tilde{V}^{(i)}[\tilde{\tau}_i^-(h_t)] - \inf_{[\tilde{\tau}_i^-(h_t^+),\tilde{\tau}_i^-(h_t)]} \tilde{V}^{(i)} \geq h_t/2,$$

in every case, whether $\tau^{W_\kappa}(-j-1) < \tilde{\tau}_i^-(h_t^+)$ or not. Since for large t, $\tilde{\tau}_i^-(h_t) \geq \tau^{W_\kappa}(-j-1)$ because $-j-1 > W_\kappa(\tilde{m}_i) + h_t$, this gives $y_j \leq \tilde{\tau}_i^-(h_t) \leq \tilde{\tau}_i(h_t)$.

Hence, $\inf_{[\tau^{W_\kappa}(-j-1),y_j]} W_\kappa \geq \inf_{[0,\tilde{\tau}_i(h_t)]} W_\kappa = W_\kappa(\tilde{m}_i) = \inf_{[y_j,\tilde{\tau}_i(h_t)]} W_\kappa$ by (2.16). Moreover, let $z_j := \inf\{x \geq y_j,\ W_\kappa(x) - \inf_{[y_j,x]} W_\kappa \geq h_t\} \geq y_j \geq 0$. Since $y_j \leq \tilde{\tau}_i^-(h_t) \leq \tilde{m}_i \leq \tilde{\tau}_i(h_t)$, we get $z_j \leq \tilde{\tau}_i(h_t)$. Thus for these j and i,

$$\inf_{[\tau^{W_\kappa}(-j-1),y_j]} W_\kappa - (-j-1) \geq W_\kappa(\tilde{m}_i) - (-j-1) \geq W_\kappa(\tilde{m}_i) - W_\kappa[\tilde{\tau}_i^-(h_t^+)] = -h_t^+, \quad (6.10)$$

$$\inf_{[y_j,z_j]} W_\kappa - W_\kappa(y_j) \geq W_\kappa(\tilde{m}_i) - [(-j-1) + h_t/2] \geq -h_t^+ - h_t/2. \quad (6.11)$$

So on $\overline{\mathcal{E}_5^{6.2}} \cap \mathcal{E}_2^{6.2}$ there exists $0 \leq j \leq \lfloor n_t e^{5\kappa h_t/4}\rfloor$ and some $1 \leq i \leq n_t$ such that such that (6.10) and (6.11) are satisfied. For this j, $z_j - y_j$ plays the role of $\tau_1^*(h_t)$ for the process $W_\kappa(.+y_j) - W_\kappa(y_j)$. Moreover $y_j - \tau^{W_\kappa}(-j-1)$ plays the role of $\tau_1^*(h_t/2)$ for $W_\kappa[.+\tau^{W_\kappa}(-j-1)] - (-j-1)$. Applying the strong Markov property for every $0 \leq j \leq \lfloor n_t e^{5\kappa h_t/4}\rfloor$ at stopping times $\tau^{W_\kappa}(-j-1)$ and y_j, we get for large t,

$$\begin{aligned} P\left[\overline{\mathcal{E}_5^{6.2}} \cap \mathcal{E}_2^{6.2}\right] &\leq \sum_{j=0}^{\lfloor n_t e^{5\kappa h_t/4}\rfloor} P\left[\inf_{[0,\tau_1^*(h_t/2)]} W_\kappa \geq -h_t^+\right] P\left[\inf_{[0,\tau_1^*(h_t)]} W_\kappa \geq -h_t^+ - h_t/2\right] \\ &\leq 2n_t e^{5\kappa h_t/4}\left[h_t^+ e^{-\kappa h_t/2}\right]\left[(h_t^+ + h_t/2)e^{-\kappa h_t}\right] \leq e^{-\kappa h_t/8}/10 \end{aligned}$$

where we applied (2.9) and $n_t = e^{o(h_t)}$ since $\phi(t) = o(\log t)$. This together with (6.7) gives $P\left[\overline{\mathcal{E}_5^{6.2}}\right] \leq e^{-\kappa h_t/8}/2$.

Since (2.32), (2.33) and (2.34) are true for $1 \leq i \leq n_t$ on $\mathcal{E}_1^{6.2} \cap \mathcal{E}_3^{6.2} \cap \mathcal{E}_5^{6.2}$, due to (6.9) for the second one, the lemma is proved. □

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Acknowledgments. We would like to thank Dominique Lepingle for pointing out reference [40]. We are grateful to an anonymous referee for comments that were useful to help improve the presentation of the paper.