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LOCALIZATION FOR AN ANDERSON-BERNOULLI MODEL WITH GENERIC INTERACTION POTENTIAL

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Abstract. We present a result of localization for a matrix-valued Anderson-Bernoulli operator acting on the space of $\boldsymbol{C}^N$-valued square-integrable functions, for an arbitrary N larger than 1, whose interaction potential is generic in the real symmetric matrices. For such a generic real symmetric matrix, we construct an explicit interval of energies on which we prove localization, in both spectral and dynamical senses, away from a finite set of critical energies. This construction is based upon the formalism of the Fürstenberg group to which we apply a general criterion of density in semisimple Lie groups. The algebraic nature of the objects we are considering allows us to prove a generic result on the interaction potential and the finiteness of the set of critical energies.

1. Introduction. In this article, we will discuss a generic result on localization properties for the random family of matrix-valued one-dimensional Anderson-Bernoulli operators

$$(1)\quad H_l(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N + V + \sum_{n\in \boldsymbol{Z}} \begin{pmatrix} c_1\omega_1^{(n)} \mathbf{1}_{[0,l]}(x-ln) & & 0 \\ & \ddots & \\ 0 & & c_N\omega_N^{(n)} \mathbf{1}_{[0,l]}(x-ln) \end{pmatrix}$$

acting on $L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$, where $N \geq 1$ is an integer, I_N is the identity matrix of order N and $l > 0$ is a real number. The matrix V is a real $N \times N$ symmetric matrix, the space of these matrices being denoted by $\mathrm{S}_N(\boldsymbol{R})$. The constants $c_1, \dots, c_N$ are non-zero real numbers.

For every $i \in \{1, \dots, N\}$, $(\omega_i^{(n)})_{n\in\boldsymbol{Z}}$ is a sequence of independent and identically distributed (i.i.d., for short) random variables on a complete probability space $(\widetilde{\Omega}_i, \widetilde{\mathcal{A}}_i, \widetilde{\mathsf{P}}_i)$, of common law ν_i such that $\{0, 1\} \subset \operatorname{supp} \nu_i$ and $\operatorname{supp} \nu_i$ is bounded. In particular, the $\omega_i^{(n)}$'s can be Bernoulli random variables. The family $\{H_l(\omega)\}_{\omega\in\Omega}$ is a family of random operators indexed by the product space

$$(\Omega, \mathcal{A}, \mathsf{P}) = \left(\bigotimes_{n\in\boldsymbol{Z}} (\widetilde{\Omega}_1 \otimes \cdots \otimes \widetilde{\Omega}_N), \bigotimes_{n\in\boldsymbol{Z}} (\widetilde{\mathcal{A}}_1 \otimes \cdots \otimes \widetilde{\mathcal{A}}_N), \bigotimes_{n\in\boldsymbol{Z}} (\widetilde{\mathsf{P}}_1 \otimes \cdots \otimes \widetilde{\mathsf{P}}_N) \right).$$

We also set, for every $n \in \boldsymbol{Z}$, $\omega^{(n)} = (\omega_1^{(n)}, \dots, \omega_N^{(n)})$, which is a random variable on $(\widetilde{\Omega}_1 \otimes \cdots \otimes \widetilde{\Omega}_N, \widetilde{\mathcal{A}}_1 \otimes \cdots \otimes \widetilde{\mathcal{A}}_N, \widetilde{\mathsf{P}}_1 \otimes \cdots \otimes \widetilde{\mathsf{P}}_N)$ of law $\nu_1 \otimes \cdots \otimes \nu_N$. The expectation value with respect to P will be denoted by $\boldsymbol{E}(\cdot)$.

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As a bounded perturbation of $-\frac{d^2}{dx^2} \otimes I_N$, the operator $H_l(\omega)$ is self-adjoint on the Sobolev space $H^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$ and thus, for every $\omega \in \Omega$, the spectrum of $H_l(\omega)$, denoted by $\sigma(H_l(\omega))$, is included in $\boldsymbol{R}$. Moreover, because of the periodicity in law of the random potential of $H_l(\omega)$, the family $\{H_l(\omega)\}_{\omega \in \Omega}$ is $l\boldsymbol{Z}$-ergodic. Thus, there exists $\Sigma \subset \boldsymbol{R}$ such that, for P-almost every $\omega \in \Omega$, $\Sigma = \sigma(H_l(\omega))$. There also exist Σ_{pp}, Σ_{ac} and Σ_{sc}, subsets of $\boldsymbol{R}$, such that, for P-almost every $\omega \in \Omega$, $\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(H_l(\omega))$, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(H_l(\omega))$ and $\Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(H_l(\omega))$, respectively the pure point, absolutely continuous and singular continuous spectrum of $H_l(\omega)$.

We can give an explicit description of the almost-sure spectrum Σ of $\{H_l(\omega)\}_{\omega \in \Omega}$. For $\omega^{(0)} = (\omega_1^{(0)}, \dots, \omega_N^{(0)}) \in \operatorname{supp}(\nu_1 \otimes \cdots \otimes \nu_N)$, we denote by $E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}$ the real eigenvalues of the real symmetric matrix $V + \operatorname{diag}(c_1\omega_1^{(0)}, \dots, c_N\omega_N^{(0)})$. Then, we have

$$\Sigma = [0, +\infty) + \bigcup_{\omega^{(0)} \in \operatorname{supp}(\nu_1 \otimes \cdots \otimes \nu_N)} \{E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}\}. \tag{2}$$

In particular, Σ does not depend on the parameter l. We will prove this explicit description in the Appendix A. In the proof, we will use the specific form of the potential, in particular the fact that V is constant and, in the random part, the fact that the single site potential is of the form $\mathbf{1}_{[0,l]}$ instead of a generic single site potential $v \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ supported on $[0, l]$.

If we want to consider the case of a generic single site potential v supported on $[0, l]$, we will face two problems. The first one is that, from our proof of the structure of Σ, we can not recover that Σ is independent of l, which may lead to an empty statement in Theorem 1.2. The second problem is that in this case, or if instead of V we choose a matrix-valued function $x \mapsto V(x)$ from $\boldsymbol{R}$ to $\mathrm{S}_N(\boldsymbol{R})$ which is not constant, the particular form of the transfer matrices introduced in Section 2 will not be simple anymore. Indeed, our analysis rests on the fact that the transfer matrices are exponentials of matrices. If neither V nor the single site potential $\mathbf{1}_{[0,l]}$ are constant, the transfer matrices become time-ordered exponentials instead of exponentials of matrices. In this case, we can not compute anymore the logarithms of these time-ordered exponentials and all the algebraic approach fails. It could eventually be possible to treat this problem using a perturbative approach based upon Lie-Trotter formula instead of using transfer matrices but, despite our attempts, we could not manage to find a rigorous proof of localization by this method.

Our main result will be about localization properties of $H_l(\omega)$. Before stating it, we give the definitions of both exponential localization and dynamical localization for $H_l(\omega)$. We denote by $E_\omega(\cdot)$ the spectral projection of the self-adjoint operator $H_l(\omega)$ and the L^2-norm is written as $\| \ \|$.

DEFINITION 1.1. Let $I \subset \boldsymbol{R}$ be an open interval. We say that :

(i) $H_l(\omega)$ exhibits *exponential localization* (EL) in I, if it has pure point spectrum in I (i.e., $\Sigma \cap I = \Sigma_{\mathrm{pp}} \cap I$ and $\Sigma_{\mathrm{ac}} \cap I = \Sigma_{\mathrm{sc}} \cap I = \emptyset$) and, for P-almost every $\omega \in \Omega$, the eigenfunctions of $H_l(\omega)$ with eigenvalues in I decay exponentially in the

L^2-sense (i.e., there exist C and $m > 0$ such that $\|\mathbf{1}_{[x-l,x+l]}\psi\| \leq C\mathrm{e}^{-m|x|}$ for an eigenfunction ψ of $H_l(\omega)$) ;

(ii) $H_l(\omega)$ exhibits *strong dynamical localization* (SDL) in I, if $\Sigma \cap I \neq \emptyset$ and, for each compact interval $\tilde{I} \subset I$ and $\psi \in L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$ with compact support, we have

$$\boldsymbol{E}\left(\sup_{t\in\boldsymbol{R}} \left\| \left(\sqrt{1+|x|^2}\right)^{n/2} E_\omega(\tilde{I})\mathrm{e}^{-\mathrm{i}tH_l(\omega)}\psi \right\|^2\right) < \infty \quad \text{for all } n \geq 0\,.$$

Before stating our main results, we need to introduce some more notations. Let $\mathrm{Sp_N}(\boldsymbol{R})$ denote the group of $2N \times 2N$ real symplectic matrices. It is the subgroup of $\mathrm{GL_{2N}}(\boldsymbol{R})$ of the matrices M satisfying

$${}^tMJM = J\,,$$

where J is the matrix of order $2N$ defined by $J = \left(\begin{smallmatrix} 0 & -I_N \\ I_N & 0 \end{smallmatrix}\right)$. Let $\mathcal{O}$ be the neighborhood of I_{2N} in $\mathrm{Sp_N}(\boldsymbol{R})$ given by Theorem 4.1 applied to $G = \mathrm{Sp_N}(\boldsymbol{R})$.

We set

$$\mathrm{d}_{\log\mathcal{O}} = \max\{R > 0\,;\ B(0,R) \subset \log\mathcal{O}\}\,, \tag{3}$$

where $B(0,R)$ is the open ball, centered on 0 and of radius $R > 0$, for the metric induced on the Lie algebra $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$ of $\mathrm{Sp_N}(\boldsymbol{R})$ by the matrix norm induced by the euclidean norm on $\boldsymbol{R}^{2N}$.

For $\omega^{(0)} = (\omega_1^{(0)}, \ldots, \omega_N^{(0)}) \in \{0,1\}^N$, let

$$M_{\omega^{(0)}}(0,V) = V + \mathrm{diag}(c_1\omega_1^{(0)}, \ldots, c_N\omega_N^{(0)})\,.$$

As $M_{\omega^{(0)}}(0,V) \in \mathrm{S}_N(\boldsymbol{R})$, it has $\lambda_1^{\omega^{(0)}}, \ldots, \lambda_N^{\omega^{(0)}}$ as real eigenvalues. We set,

$$\lambda_{\min} = \min_{\omega^{(0)}\in\{0,1\}^N}\ \min_{1\leq i\leq N} \lambda_i^{\omega^{(0)}}\,, \qquad \lambda_{\max} = \max_{\omega^{(0)}\in\{0,1\}^N}\ \max_{1\leq i\leq N} \lambda_i^{\omega^{(0)}} \tag{4}$$

and $\delta = (\lambda_{\max} - \lambda_{\min})/2$. We also set

$$l_C := l_C(N,V) = \min\left(1, \frac{\mathrm{d}_{\log\mathcal{O}}}{\delta}\right) \tag{5}$$

and, for every $l \in (0, l_C)$,

$$I(N,V,l) = \left[\lambda_{\max} - \frac{\mathrm{d}_{\log\mathcal{O}}}{l}, \lambda_{\min} + \frac{\mathrm{d}_{\log\mathcal{O}}}{l}\right]. \tag{6}$$

We remark that, as l tends to 0^+, $I(N,V,l)$ tends to the whole real line. We can now state our main result.

THEOREM 1.2. *For almost every $V \in \mathrm{S}_N(\boldsymbol{R})$, there exists a finite set $\mathcal{S}_\mathrm{V} \subset \boldsymbol{R}$ such that, for every $l \in (0, l_C)$, if $I \subset I(N,V,l) \setminus \mathcal{S}_\mathrm{V}$ is an open interval with $\Sigma \cap I \neq \emptyset$, then $H_l(\omega)$ exhibits* (EL) *and* (SDL) *on I.*

Here, "almost every" is considered according to the Lebesgue measure on $\mathrm{S}_N(\boldsymbol{R})$ identified to $\boldsymbol{R}^{N(N+1)/2}$. We also remark that, as $I(N,V,l)$ tends to $\boldsymbol{R}$ when l tends to 0^+ and

Σ does not depend on l, taking $l \in (0, l_C)$ small enough ensures that we can always find a non-trivial open interval $I \subset I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}}$ such that $\Sigma \cap I \neq \emptyset$.

This theorem will follow from the next proposition. For $E \in \boldsymbol{R}$, let $G(E)$ be the Fürstenberg group associated to $H_l(\omega)$ (see Definition 2.2).

PROPOSITION 1.3. *For almost every $V \in \mathrm{S}_N(\boldsymbol{R})$, there exists a finite set $\mathcal{S}_{\mathrm{V}} \subset \boldsymbol{R}$ such that, for every $l \in (0, l_C)$,*

$$G(E) = \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R}) \quad \text{for all} \;\; E \in I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}} .$$

In particular, Proposition 1.3 will imply the separability of the Lyapunov exponents of $H_l(\omega)$ (see Definition 2.1) and the absence of absolutely continuous spectrum in $I(N, V, l)$, for $l \in (0, l_C)$.

COROLLARY 1.4. *For almost every $V \in \mathrm{S}_N(\boldsymbol{R})$, there exists a finite set $\mathcal{S}_{\mathrm{V}} \subset \boldsymbol{R}$ such that, for every $l \in (0, l_C)$, the positive Lyapunov exponents $\gamma_1(E), \dots, \gamma_N(E)$ of $H_l(\omega)$ verify*

$$\gamma_1(E) > \cdots > \gamma_N(E) > 0 \quad \text{for all} \;\; E \in I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}} . \tag{7}$$

Therefore, $H_l(\omega)$ has no absolutely continuous spectrum in $I(N, V, l)$, i.e., for every $l \in (0, l_C)$, $\Sigma_{\mathrm{ac}} \cap I(N, V, l) = \emptyset$.

In Proposition 1.3, V can not be arbitrary. For example, if V is a diagonal matrix, one can show that, for every E in $\boldsymbol{R}$, $G(E)$ is not equal to $\mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$. But, even if V is diagonal, the conclusion of Theorem 1.2 still holds. Indeed, if V is diagonal, the operator $H_l(\omega)$ splits into a direct sum of scalar-valued operators, each of them exhibiting (EL) and (SDL) on every compact interval away from a discrete subset of $\boldsymbol{R}$, as shown in [9]. It implies that $H_l(\omega)$ itself exhibits (EL) and (SDL) on every compact interval away from a discrete subset of $\boldsymbol{R}$. Up to our knowledge, there is no counter-example of V for which $H_l(\omega)$ does not exhibit (EL) or (SDL), at least on one interval away from a discrete set. We expect that Theorem 1.2 actually holds for every V in $\mathrm{S}_N(\boldsymbol{R})$ but, despite our efforts, we can not prove this result.

In dimension d higher than 2, the question of the localization remains mostly open for Anderson-Bernoulli models. Such an Anderson-Bernoulli model is given by a family of random operators of the form

$$H(\omega) = -\Delta_d + \sum_{n \in \boldsymbol{Z}^d} \omega_n V(x - n) \tag{8}$$

acting on $L^2(\boldsymbol{R}^d) \otimes \boldsymbol{C}$, where V is supported in $[0, 1]^d$ and the ω_n are i.i.d. Bernoulli random variables. By [6], it is known that there is exponential localization at the bottom of the almost sure spectrum of $H(\omega)$. In dimension $d \geq 3$, it is commonly conjectured that for high energies, there exist extended states, as for dimension $d = 2$ it is conjectured that there is localization at every energies, except maybe those in a discrete set.

To tackle the question of localization for $d = 2$, we can start by looking at a slightly simpler model, a continuous strip $\boldsymbol{R} \times [0, 1]$ in $\boldsymbol{R}^2$. This model is given by the restriction $H_{\mathrm{cs}}(\omega)$ of $H(\omega)$ to $L^2(\boldsymbol{R} \times [0, 1])$, with Dirichlet boundary conditions on $\boldsymbol{R} \times \{0\}$ and $\boldsymbol{R} \times \{1\}$. This model can be used to study transport properties of nanoconductors and so it is also of physical interest. The question of the localization at all energies for $H_{\text{cs}}(\omega)$ present difficulties of the same level as for $H(\omega)$, mostly due to the PDE's nature of the problem in both cases. But, for $H_{\text{cs}}(\omega)$, we have a possible approach by operating a discretization in the bounded direction of the strip. This can be performed by first applying discrete Fourier transform in the second variable corresponding to the bounded direction, which leads to a matrix-valued one-dimensional model with an infinite size matrix for potential. Then, by applying a cut-off in the space of Fourier frequencies, we obtain a matrix-valued one-dimensional model with a matrix of order N for potential, acting on $L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$, for an integer $N \geq 1$. It turns the nature of the initial PDE's problem to an ODE's one, which allows us to use formalism such as transfer matrices and Lyapunov exponents. The model 1 we are looking at here is not exactly the one obtained by this discretization procedure, but the understanding of localization properties for (1) should lead us to the same understanding for the discretize operator obtained from $H_{\text{cs}}(\omega)$. More precisely, the discretization procedure leads to a matrix-valued operator with a symmetric matrix as potential which has deterministic terms only on the diagonal and random terms in every coefficients of the matrix. The algebraic part of our analysis of the model 1 would have to be changed in a substantial way to be adapted to the model obtained after discretization of $H_{\text{cs}}(\omega)$.

The model 1 is a matrix-valued model with a continuous Laplacian acting on $L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$. In previous works of Goldsheid and Margulis (see [11, 10]), the analysis of Lyapunov exponents has been already done for discrete models acting on $\ell^2(\boldsymbol{Z}) \otimes \boldsymbol{C}^N$. The model 1 is a continuous analog of the discrete model studied in [11]. The main difference between these discrete and continuous models is that, in the discrete case, one can have energies where some of the Lyapunov exponents may vanish or may not be distinct (see [8] for $N = 1$ and [5, 2] for $N = 2$). From a technical point of view, the algebraic part of the analysis, where we reconstruct $\mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$ from the transfer matrices, is more difficult in the continuous case than in the discrete one. It is mostly due to the form of the transfer matrices. In the discrete case, their form is simple enough to compute directly the Zariski closure of $G(E)$ (in particular, the transfer matrices multiply well together) and to prove that it is equal to $\mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$ by showing that the Lie algebra of the Zariski closure of $G(E)$ is equal to the Lie algebra $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$ of $\mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$. In the continuous case, the transfer matrices are exponentials of matrices and difficulties arise when we try to multiply elements of $G(E)$. Indeed, the form of a matrix obtained after multiplication of two exponentials of matrices is not simply related to the form of the multiplied exponentials. To avoid this problem, the idea is to work only in the Lie algebra of $G(E)$. This approach is different from what was done in the discrete case where, to prove equality of Lie algebras, it was allowed to make computations both in the Zariski closure of $G(E)$ and in its Lie algebra. Here, by using a general result on Lie groups due to Breuillard and Gelander, one is brought to make computations only in the Lie algebra generated by the logarithms of the transfer matrices. To avoid problems of determination of logarithms, as encountered in [2], we have to introduce the parameter l in the model 1. All those difficulties do not appear in the case $N = 1$ (see [9]). When $N = 1$, we have only to prove the positivity of one Lyapunov exponent and apply Fürstenberg's theorem directly. In this case, no Lie algebra argument is needed and there is no problem due to the determination of logarithms of matrices. To prove localization for general N, one can follow most of the steps of the proof done in [9] for $N = 1$, and in particular the use of the multiscale analysis (see [4] for a presentation in the continuous matrix-valued case). What completely changes for general N is the proof of the separability of Lyapunov exponents which, for all the reasons we have just given, is much more difficult than for $N = 1$.

In [4], we had already obtained a result of localization for a particular case of the model 1, where V was fixed to the tridiagonal matrix with coefficients on the diagonal equal to zero and coefficients on the lower and upper diagonals equal to one. We had also proved a result of separability of the Lyapunov exponents for all energies in some explicitely constructed interval of $\boldsymbol{R}$. The present article generalize the results of [4] to the case of a generic V in $\mathrm{S}_N(\boldsymbol{R})$. The algebraic setting in which our result takes place gives rise naturally to a genericity argument as we will show in Section 3. This fact was not understood yet at the time of the publication of [4], and one of the goal of the present article is also to illustrate how one can get a generic result from a particular one, just by using the algebraic nature of the considered objects.

We finish this introduction by giving the outline of the article. In Section 2, we present the formalism of transfer matrices and compute them for $H_l(\omega)$. We also define the Lyapunov exponents and the Fürstenberg group associated to $H_l(\omega)$. In Section 3, we study the Lie algebra generated by the matrices $X_{\omega^{(0)}}(E, V)$ defined at (15). In this section we also prove the genericity argument and we construct the finite set $\mathcal{S}_{\mathrm{V}}$ in Theorem 1.2, Proposition 1.3 and Corollary 1.4. This genericity argument is mostly based upon algebraic geometry considerations and the Lebesgue measure on affine algebraic manifolds. In Section 4, we prove Proposition 1.3 and Corollary 1.4 and we explicitely construct l_C and $I(N, V, l)$ for $l \in (0, l_C)$. The proofs of this section are based upon a general result on Lie groups due to Breuillard and Gelander (see Theorem 4.1). In Section 5, we recall localization results of [4] and we deduce from them the proof of Theorem 1.2. Finally, in Section 6, we state a result of existence and regularity of the integrated density of states associated to $H_l(\omega)$.

The general idea of the proof of Theorem 1.2 can be briefly sketched. First we change the initial spectral and dynamical problem of the localization into a topological problem to prove that a Lie group with a finite number of generators is dense in the real symplectic group $\mathrm{Sp_N}(\boldsymbol{R})$, which is the statement of Proposition 1.3. Then, we use the general criterion on Lie groups of Breuillard and Gelander to transform this topological problem into a purely algebraic problem to generate the Lie algebra $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$. The algebraic nature of the objects we are considering at this last step allows us to prove a generic result on V and the finiteness of the set $\mathcal{S}_{\mathrm{V}}$ of critical energies.

2. Transfer matrices and the Fürstenberg group. Let $E \in \boldsymbol{R}$. We want to understand the exponential asymptotic behaviour of a solution $u : \boldsymbol{R} \to \boldsymbol{C}^N$ of the second order differential system

$$H_l(\omega)u = Eu \,. \tag{9}$$

For this, we transform (9) into an Hamiltonian differential system of order 1 and we introduce the transfer matrix $T_{\omega^{(n)}}(E)$ of $H_l(\omega)$ from ln to $l(n+1)$ which maps a solution (u, u') of the order 1 system at time ln to the solution at time $l(n+1)$. The transfer matrix $T_{\omega^{(n)}}(E)$ is therefore defined by the relation

$$\begin{pmatrix} u(l(n+1)) \\ u'(l(n+1)) \end{pmatrix} = T_{\omega^{(n)}}(E) \begin{pmatrix} u(ln) \\ u'(ln) \end{pmatrix} \tag{10}$$

for all $n \in \boldsymbol{Z}$. Since $T_{\omega^{(n)}}(E)$ is the solution of an Hamiltonian differential system of order 1 at time 1, the transfer matrix $T_{\omega^{(n)}}(E)$ lies in the symplectic group $\mathrm{Sp_N}(\boldsymbol{R})$. The sequence $(T_{\omega^{(n)}}(E))_{n\in\boldsymbol{Z}}$ is also a sequence of i.i.d. symplectic matrices because of the i.i.d. character of the $\omega_i^{(n)}$'s for every i in $\{1, \ldots, N\}$ and the non-overlapping of these random variables. By iterating the relation (10) we get the asymptotic behaviour of (u, u'). To get the *exponential* asymptotic behaviour of (u, u'), we can define the exponential growth (or decay) exponents of the product of random matrices $T_{\omega^{(n-1)}}(E) \cdots T_{\omega^{(0)}}(E)$.

DEFINITION 2.1. Let $E \in \boldsymbol{R}$. The Lyapunov exponents $\gamma_1(E), \ldots, \gamma_{2N}(E)$ associated to the sequence $(T_{\omega^{(n)}}(E))_{n\in\boldsymbol{Z}}$ are defined inductively by

$$\sum_{i=1}^{p} \gamma_i(E) = \lim_{n\to\infty} \frac{1}{n} \boldsymbol{E}(\log \| \textstyle\bigwedge^p (T_{\omega^{(n-1)}}(E) \cdots T_{\omega^{(0)}}(E)) \|) \tag{11}$$

for every $p \in \{1, \ldots, 2N\}$. Here, $\bigwedge^p M$ denotes the p-th exterior power of the matrix M, acting on the p-th exterior power of $\boldsymbol{R}^{2N}$.

One has $\gamma_1(E) \geq \cdots \geq \gamma_{2N}(E)$. Moreover, due to the symplecticity of the random matrices $T_{\omega^{(n)}}(E)$, we have the symmetry property $\gamma_{2N-i+1} = -\gamma_i$ for every $i \in \{1, \ldots, N\}$. Thus, we will only have to study the first N Lyapunov exponents to obtain Corollary 1.4. To prove the separability of the Lyapunov exponents, we introduce the group which contains all the different products of transfer matrices, the so-called Fürstenberg group.

DEFINITION 2.2. For every $E \in \boldsymbol{R}$, the *Fürstenberg group* of $H_l(\omega)$ is defined by

$$G(E) = \overline{\langle \mathrm{supp}\ \mu_E \rangle},$$

where μ_E is the common distribution of the $T_{\omega^{(n)}}(E)$ and the closure is taken for the usual topology in $\mathrm{Sp_N}(\boldsymbol{R})$.

As the $T_{\omega^{(n)}}(E)$ are i.i.d., $\mu_E = (T_{\omega^{(0)}}(E))_* (\nu_1 \otimes \cdots \otimes \nu_N)$ and we have the internal description of $G(E)$:

$$G(E) = \overline{\langle T_{\omega^{(0)}}(E)\ ;\ \omega^{(0)} \in \mathrm{supp}\ (\nu_1 \otimes \cdots \otimes \nu_N) \rangle} \quad \text{for all } E \in \boldsymbol{R}. \tag{12}$$

As, for every $i \in \{1, \ldots, N\}$, $\{0, 1\} \subset \mathrm{supp}\ \nu_i$, we also have

$$\overline{\langle T_{\omega^{(0)}}(E)\ ;\ \omega^{(0)} \in \{0, 1\}^N \rangle} \subset G(E). \tag{13}$$

We will denote by $G_{\{0,1\}}(E)$ the subgroup $\langle T_{\omega^{(0)}}(E)\ ;\ \omega^{(0)} \in \{0, 1\}^N \rangle$ of $G(E)$ with 2^N generators.

In Section 4, we will prove that, for almost every $V \in \mathrm{S}_N(\boldsymbol{R})$ and for all $E \in \boldsymbol{R}$ except those in a finite set, $G_{\{0,1\}}(E)$ is dense in $\mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$.

We finish this section by giving the explicit form of the transfer matrices $T_{\omega^{(n)}}(E)$. Let $V \in \mathrm{S}_N(\boldsymbol{R})$, $E \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$ and $\omega^{(n)} \in \mathrm{supp}\,(\nu_1 \otimes \cdots \otimes \nu_N)$. We set

$$M_{\omega^{(n)}}(E, V) = V + \mathrm{diag}(c_1\omega_1^{(n)}, \ldots, c_N\omega_N^{(n)}) - EI_{\mathrm{N}}\,. \tag{14}$$

Then, we set the matrix in the Lie algebra $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}) \subset \mathcal{M}_{2\mathrm{N}}(\boldsymbol{R})$

$$X_{\omega^{(n)}}(E, V) = \begin{pmatrix} 0 & I_{\mathrm{N}} \\ M_{\omega^{(n)}}(E, V) & 0 \end{pmatrix}. \tag{15}$$

By solving the constant coefficients system (9) on $[ln, l(n+1)]$, we have

$$T_{\omega^{(n)}}(E) = \exp\big(lX_{\omega^{(n)}}(E, V)\big) \tag{16}$$

for every $l > 0$, every $n \in \boldsymbol{Z}$, every $V \in \mathrm{S}_N(\boldsymbol{R})$ and every $E \in \boldsymbol{R}$.

It is important here to notice that $T_{\omega^{(n)}}(E)$ is the exponential of a matrix, as it will be crucial to be able to apply Theorem 4.1 to the subgroup $G_{\{0,1\}}(E)$.

3. The genericity argument. In this section we will present in details the proof of the genericity argument needed to prove Proposition 1.3 and Theorem 1.2. We start by looking at the geometry of the set of k-tuples in $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$ which does not generate $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$ in the sense of Lie algebras.

LEMMA 3.1. *Let $k \in \boldsymbol{N}^*$ and*

$$\mathcal{V}_k = \big\{(X_1, \ldots, X_k) \in (\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k\ ;\ (X_1, \ldots, X_k) \text{ does not generate } \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})\big\}. \tag{17}$$

Then, there exist $Q_{r_1}, \ldots, Q_{r_k} \in \boldsymbol{R}[(\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k]$ such that

$$\mathcal{V}_k = \big\{(X_1, \ldots, X_k) \in (\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k\ ;\ Q_{r_1}(X_1, \ldots, X_k) = 0, \ldots, Q_{r_k}(X_1, \ldots, X_k) = 0\big\}. \tag{18}$$

Thus, $\mathcal{V}_k$ is the affine algebraic manifold of $\{Q_{r_1}, \ldots, Q_{r_k}\}$ which will be denoted by $V(\{Q_{r_1}, \ldots, Q_{r_k}\})$. We will also identify the ring of the polynomials over $(\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k$, $\boldsymbol{R}[(\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k]$, with the ring of polynomials in $k(2N^2+N)$ variables, $\boldsymbol{R}[T_1, \ldots, T_{k(2N^2+N)}]$.

PROOF. Let $(X_1, \ldots, X_k) \in (\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k$ and let $\mathrm{Lie}\{X_1, \ldots, X_k\}$ be the Lie algebra generated by $X_1, \ldots, X_k$. If we denote by $\{Y_1, \ldots, Y_l, \ldots\}$ the countable set of all the successives brackets constructed from $\{X_1, \ldots, X_k\}$, we have

$$\mathrm{Lie}\{X_1, \ldots, X_k\} = \mathrm{span}(\{Y_1, \ldots, Y_l, \ldots\})\,, \tag{19}$$

the vector space spanned by $\{Y_1, \ldots, Y_l, \ldots\}$. Then we have

$$\mathrm{Lie}\{X_1, \ldots, X_k\} \neq \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}) \text{ if and only if } \mathrm{rk}(\{Y_1, \ldots, Y_l, \ldots\}) < 2N^2 + N\,, \tag{20}$$

since $\dim \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}) = 2N^2 + N$. At each $Y_l \in \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$, we associate $\tilde{Y}_l \in \boldsymbol{R}^{2N^2+N}$ whose coefficients are those which define the matrix Y_l. The coefficients of $\tilde{Y}_l$ are polynomial in the $k(2N^2+N)$ coefficients which define the matrices $X_1, \ldots, X_k$. For $m \in (\boldsymbol{N}^*)^{2N^2+N}$, we set

$$Q_m(X_1, \ldots, X_k) = \det(\tilde{Y}_{m_1}, \ldots, \tilde{Y}_{m_{2N^2+N}}) \in \boldsymbol{R}[(\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k]\,. \tag{21}$$

Then, $\mathrm{rk}(\{Y_1,\dots,Y_l,\dots\}) < 2N^2+N$ if and only if $Q_m(X_1,\dots,X_k)=0$ for all $m \in (\boldsymbol{N}^*)^{2N^2+N}$. Thus,

$$\mathcal{V}_k = \bigcap_{m\in(\boldsymbol{N}^*)^{2N^2+N}} \left\{(X_1,\dots,X_k)\in(\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^k\ ;\ Q_m(X_1,\dots,X_k)=0\right\}. \tag{22}$$

With the definition of the affine algebraic manifold, we can rewrite (22) as

$$\mathcal{V}_k = V(\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\}). \tag{23}$$

But, if $I(\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\})$ denote the ideal generated by the family $\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\}$, we have

$$V(\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\}) = V(I(\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\})). \tag{24}$$

Since the ring $\boldsymbol{R}[T_1,\dots,T_{k(2N^2+N)}]$ is Noetherian, $I(\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\})$ is of finite type, i.e., there exist $r_1,\dots,r_k\in(\boldsymbol{N}^*)^{2N^2+N}$ such that,

$$I(\{Q_m\ ;\ m\in(\boldsymbol{N}^*)^{2N^2+N}\}) = I(\{Q_{r_1},\dots,Q_{r_k}\}). \tag{25}$$

Finally,

$$\mathcal{V}_k = V(I(\{Q_{r_1},\dots,Q_{r_k}\})) = V(\{Q_{r_1},\dots,Q_{r_k}\}). \tag{26}$$

□

For $E\in\boldsymbol{R}$ and $V\in\mathrm{S}_N(\boldsymbol{R})$, we will reindex the family $\{X_{\omega^{(0)}}(E,V)\}_{\omega^{(0)}\in\{0,1\}^N}$ as $(X_1(E,V),\dots,X_{2^N}(E,V))$. Let $E\in\boldsymbol{R}$ be fixed and let

$$\mathcal{V}_{(E)} = \left\{V\in\mathrm{S}_N(\boldsymbol{R})\ ;\ (X_1(E,V),\dots,X_{2^N}(E,V)) \text{ does not generate } \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})\right\}. \tag{27}$$

LEMMA 3.2. *We have*, $\mathrm{Leb}_{N(N+1)/2}(\mathcal{V}_{(E)})=0$.

PROOF. Let

$$\begin{array}{rccl} f_E & : \mathrm{S}_N(\boldsymbol{R}) & \to & (\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^{2^N} \\ & V & \mapsto & (X_1(E,V),\dots,X_{2^N}(E,V)). \end{array} \tag{28}$$

Then f_E is a polynomial in the $N(N+1)/2$ coefficients which define V. Indeed, we can identify $\mathrm{S}_N(\boldsymbol{R})\simeq\boldsymbol{R}^{N(N+1)/2}$ and $(\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^{2^N}\simeq\boldsymbol{R}^{2^N(2N^2+N)}$ and, after this identification, each of the $2^N(2N^2+N)$ components of f_E is a polynomial in $N(N+1)/2$ variables. We have $\mathcal{V}_{(E)}=f_E^{-1}(\mathcal{V}_{2^N})$. Then, by Lemma 3.1, $V\in\mathcal{V}_{(E)}$ if and only if

$$Q_{r_1}(X_1(E,V),\dots,X_{2^N}(E,V))=0,\dots,Q_{r_{2^N}}(X_1(E,V),\dots,X_{2^N}(E,V))=0,$$

which can be rewritten

$$V\in\mathcal{V}_{(E)} \text{ if and only if } (Q_{r_1}\circ f_E)(V)=0,\dots,(Q_{r_{2^N}}\circ f_E)(V)=0. \tag{29}$$

But, we can prove that, if V_0 is the tridiagonal matrix with zeros on the diagonal and all coefficients on its upper and lower diagonals equal to 1, then, for any $E\in\boldsymbol{R}$, V_0 is not in $\mathcal{V}_{(E)}$ (see [4, Lemma 1]). Thus, there exists $i_0\in\{r_1,\dots,r_{2^N}\}$ such that $(Q_{i_0}\circ f_E)(V_0)\neq 0$ and, since the function $Q_{i_0}\circ f_E$ is polynomial and does not vanish identically,

$$\mathrm{Leb}_{N(N+1)/2}\left(\{V\in\mathrm{S}_N(\boldsymbol{R})\ ;\ (Q_{i_0}\circ f_E)(V)=0)\}\right)=0 \tag{30}$$

and, by inclusion,

$$\mathrm{Leb}_{N(N+1)/2}(\mathcal{V}_{(E)}) = 0\,. \tag{31}$$ □

Finally, we introduce the set

$$\begin{aligned}\mathcal{V} &= \bigcap_{E\in\boldsymbol{R}} \mathcal{V}_{(E)} \\ &= \big\{V \in \mathrm{S}_N(\boldsymbol{R})\ ;\ (X_1(E,V),\dots,X_{2^N}(E,V)) \text{ does not generate } \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})\,,\ \text{for all } E \in \boldsymbol{R}\big\}\,.\end{aligned} \tag{32}$$

Then, by Lemma 3.2 and by inclusion, we have

$$\mathrm{Leb}_{N(N+1)/2}(\mathcal{V}) = 0\,. \tag{33}$$

Now we can prove the last result of this section.

LEMMA 3.3. *For any $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$, there exists a finite set $\mathcal{S}_{\mathrm{V}} \subset \boldsymbol{R}$ such that*

$$(X_1(E,V),\dots,X_{2^N}(E,V)) \textit{ generates } \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}) \textit{ for all } E \in \boldsymbol{R}\setminus\mathcal{S}_{\mathrm{V}}\,.$$

PROOF. Let $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$. Then, there exists $E_0 \in \boldsymbol{R}$ such that the family $(X_1(E_0,V),\dots,X_{2^N}(E_0,V))$ generates $\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$. Thus, there exists $i_0 \in \{r_1,\dots,r_{2^N}\}$ such that $(Q_{i_0}\circ f)(E_0,V) \neq 0$, where

$$\begin{array}{llll} f & : & \boldsymbol{R}\times \mathrm{S}_N(\boldsymbol{R}) & \to (\mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}))^{2^N} \\ & & (E,V) & \mapsto (X_1(E,V),\dots,X_{2^N}(E,V))\,. \end{array} \tag{34}$$

But, for V fixed, $E \mapsto (Q_{i_0}\circ f)(E,V)$ is polynomial and, as it is not identically vanishing, it has only a finite set $\mathcal{S}_{\mathrm{V}}$ of roots. Thus, we have

$$(Q_{i_0}\circ f)(E,V) \neq 0 \quad \text{for all}\ \ E \in \boldsymbol{R}\setminus\mathcal{S}_{\mathrm{V}}\,, \tag{35}$$

which is equivalent to the condition that

$$(X_1(E,V),\dots,X_{2^N}(E,V)) \notin \mathcal{V}_{2^N} \quad \text{for all}\ \ E \in \boldsymbol{R}\setminus\mathcal{S}_{\mathrm{V}}\,. \tag{36}$$ □

We will prove Proposition 1.3 by using Lemma 3.3 in the next section.

REMARK 3.4. For the special case $N = 2$, it is possible to compute the explicit representation of $\mathcal{V}_{(E)}$ for any real number E. Let $V \in \mathrm{S}_2(\boldsymbol{R})$ and we denote its coefficients by

$$V = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad a,b,c \in \boldsymbol{R}\,. \tag{37}$$

We fix $E \in \boldsymbol{R}$. Then, by algebraic computations similar to those done in [4, Lemma 1], one can prove that, if $c \neq 0$,

$$\mathrm{Lie}\{X_1(E,V),\dots,X_4(E,V)\} = \mathfrak{sp}_2(\boldsymbol{R}) \quad \text{for any}\ \ a,b \in \boldsymbol{R}\,. \tag{38}$$

If $c = 0$, for any $a, b \in \boldsymbol{R}$,

$$\begin{aligned}&\mathrm{Lie}\{X_1(E,V),\dots,X_4(E,V)\} \\ &\qquad = \left\{\begin{pmatrix} d_1 & d_2 \\ d_3 & -d_1 \end{pmatrix}\ ;\ d_1,d_2,d_3 \in \mathcal{M}_2(\boldsymbol{R}) \text{ are diagonal matrices}\right\}\end{aligned} \tag{39}$$

and

$$\text{Lie}\{X_1(E,V),\dots,X_4(E,V)\} \subsetneq \mathfrak{sp}_2(\boldsymbol{R})\,. \tag{40}$$

We deduce from (38) and (39) that

$$\mathcal{V}_{(E)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \; ; a, b \in \boldsymbol{R} \right\} \quad \text{for any} \;\; E \in \boldsymbol{R}\,, \tag{41}$$

and thus

$$\mathcal{V} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \; ; a, b \in \boldsymbol{R} \right\}\,, \tag{42}$$

which is indeed of Lebesgue measure zero in $\mathrm{S}_2(\boldsymbol{R})$. Moreover, if $V \notin \mathcal{V}$, because (38) is true for any $E \in \boldsymbol{R}$, we get that $\mathcal{S}_{\mathrm{V}}$ is empty.

For $N \geq 3$, the structure of $\mathcal{V}_{(E)}$ seems to be more complicated and we were not able to determine it rigorously. One may conjecture that for a general N, $\mathcal{V}_{(E)}$ is the set of matrices $V \in \mathrm{S}_N(\boldsymbol{R})$ which are irreducible and thus does not depend on E. If so, we would have $\mathcal{S}_{\mathrm{V}} = \emptyset$ for any $V \notin \mathcal{V}$ and Proposition 1.3 would be true for $\mathcal{S}_{\mathrm{V}}$ empty. It would also mean that for l small enough and for $V \notin \mathcal{V}$, $\mathcal{S}_{\mathrm{V}}$ would be empty in the statement of Theorem 1.2. But, for $l = 1$ and $N = 1$, it was shown in [8] that the localization result in Theorem 1.2 only holds on compact intervals away from a discrete set. The same result was obtained for $l = 1$ and $N = 2$ in [2] for the particular V corresponding to $a = b = 0$ and $c = 1$. Thus, if we do not suppose that l is small, one cannot expect that $\mathcal{S}_{\mathrm{V}}$ is empty in general. In this case $\mathcal{S}_{\mathrm{V}}$ would not appear as a set of energies to exclude in order to apply our genericity argument, but it would appear as energies for which the logarithms of the transfer matrices are not defined. One may see that by looking at [2, Section 4.2].

4. Proof of Proposition 1.3 and Corollary 1.4. The proof of Proposition 1.3 is based upon a general criterion of density in semisimple Lie groups due to Breuillard and Gelander.

THEOREM 4.1 ([7, Theorem 2.1]). *Let G be a real, connected, semisimple Lie group, whose Lie algebra is $\mathfrak{g}$. Then, there is a neighborhood $\mathcal{O}$ of* 1 *in G, on which* $\log = \exp^{-1}$ *is a well-defined diffeomorphism, such that $g_1, \dots, g_m \in \mathcal{O}$ generate a dense subgroup whenever* $\log g_1, \dots, \log g_m$ *generate* $\mathfrak{g}$.

This criterion, applied to $G = \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$, gives us the proof of Proposition 1.3, which is outlined as follows.

1. We construct l_C and $I(N,V,l)$ such that, for $l \in (0, l_C)$ and $E \in I(N,V,l)$, $T_{\omega^{(0)}}(E) \in \mathcal{O}$ for every $\omega^{(0)} \in \{0,1\}^N$.
2. We compute $\log T_{\omega^{(0)}}(E)$.
3. We justify that $\text{Lie}\{\log T_{\omega^{(0)}}(E)\;;\; \omega^{(0)} \in \{0,1\}^N\} = \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$ for $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$ and $E \in I(N,V,l) \setminus \mathcal{S}_{\mathrm{V}}$.
4. We deduce that $G_{\{0,1\}}(E)$ is dense with respect to the usual topology of $\mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$ for $E \in I(N,V,l) \setminus \mathcal{S}_{\mathrm{V}}$.

PROOF. We fix $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$. We start by constructing l_C and, for $l \in (0, l_C)$, the interval $I(N, V, l)$ as given in (5) and (6). Now, let $\lambda_1^{\omega^{(0)}}, \dots, \lambda_N^{\omega^{(0)}}$ be the real eigenvalues of $M_{\omega^{(0)}}(0, V)$ (see (14)). Then, the eigenvalues of $X_{\omega^{(0)}}(E, V)^t X_{\omega^{(0)}}(E, V)$ are 1, $(\lambda_1^{\omega^{(0)}} - E)^2$, $\dots$, $(\lambda_N^{\omega^{(0)}} - E)^2$. Thus we have

$$\|X_{\omega^{(0)}}(E, V)\| = \max\Big(1, \max_{1 \le i \le N} |\lambda_i^{\omega^{(0)}} - E|\Big), \tag{43}$$

where $\| \ \|$ is the matrix norm associated to the euclidian norm on $\boldsymbol{R}^{2N}$.

Let $\mathcal{O}$ be the neighborhood of the identity given by Theorem 4.1 applied to the group $G = \mathrm{Sp_N}(\boldsymbol{R})$. Then, for $\mathrm{d}_{\log \mathcal{O}}$ as defined in (3), we take $l \le \mathrm{d}_{\log \mathcal{O}}$ and we set $r_l = (1/l)\,\mathrm{d}_{\log \mathcal{O}} \ge 1$. If we set

$$I(N, V, l) = \Big\{E \in \boldsymbol{R}\ ;\ \max\Big(1, \max_{\omega^{(0)} \in \{0,1\}^N} \max_{1 \le i \le N} |\lambda_i^{\omega^{(0)}} - E|\Big) \le r_l\Big\}, \tag{44}$$

then, since $r_l \ge 1$, we have

$$I(N, V, l) = \bigcap_{\omega^{(0)} \in \{0,1\}^N} \bigcap_{1 \le i \le N} [\lambda_i^{\omega^{(0)}} - r_l, \lambda_i^{\omega^{(0)}} + r_l]\,. \tag{45}$$

Let $\lambda_{\min}$, $\lambda_{\max}$ and δ be as in (4). If $\delta < r_l$ then $I(N, V, l) \neq \emptyset$ and we have

$$I(N, V, l) = [\lambda_{\max} - r_l, \lambda_{\min} + r_l]\,, \tag{46}$$

which is the definition we took in (6). This interval is centered in $(\lambda_{\min} + \lambda_{\max})/2$ and is of length $2r_l - 2\delta > 0$, which tends to $+\infty$ when l tends to 0^+. We also note that $\lambda_{\min}$, $\lambda_{\max}$ and $\mathrm{d}_{\log \mathcal{O}}$ depend only on N and V, and thus $I(N, V, l)$ depends only on N, V and l. Finally, the condition $\delta < r_l$, which ensures that $I(N, V, l) \neq \emptyset$, is equivalent to

$$0 < l < \frac{\mathrm{d}_{\log \mathcal{O}}}{\delta} = l_C(N, V)\,.$$

So, we have just proved that

$$0 < l\|X_{\omega^{(0)}}(E, V)\| \le \mathrm{d}_{\log \mathcal{O}} \tag{47}$$

for every $l \in (0, l_C)$, for every $E \in I(N, V, l)$ and for every $\omega^{(0)} \in \{0, 1\}^N$. Thus, for every $l \in (0, l_C)$ and every $E \in I(N, V, l)$,

$$lX_{\omega^{(0)}}(E, V) \in \log \mathcal{O} \tag{48}$$

for every $\omega^{(0)} \in \{0, 1\}^N$. From this, we deduce that

$$T_{\omega^{(0)}}(E) \in \mathcal{O} \tag{49}$$

for every $l \in (0, l_C)$, for every $E \in I(N, V, l)$ and for every $\omega^{(0)} \in \{0, 1\}^N$. We actually get more from (48). As the exponential is a diffeomorphism from $\log \mathcal{O}$ into $\mathcal{O}$, we also have

$$\log\, T_{\omega^{(0)}}(E) = lX_{\omega^{(0)}}(E, V)\,, \tag{50}$$

for every $l \in (0, l_C)$, for every $E \in I(N, V, l)$ and for every $\omega^{(0)} \in \{0, 1\}^N$. But, from the beginning, we chose $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$ and, by Lemma 3.3, there exists a finite set $\mathcal{S}_{\mathrm{V}} \subset \boldsymbol{R}$ such that

$$\text{(51)} \qquad \mathrm{Lie}\{X_{\omega^{(0)}}(E, V)\ ;\ \omega^{(0)} \in \{0, 1\}^N\} = \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R}) \quad \text{for all } E \in \boldsymbol{R} \setminus \mathcal{S}_{\mathrm{V}} .$$

Now, by (50) and (51), as $l \in (0, l_C)$ is different from 0,

$$\text{(52)} \qquad \mathrm{Lie}\{\log\, T_{\omega^{(0)}}(E)\ ;\ \omega^{(0)} \in \{0, 1\}^N\} = \mathfrak{sp}_{\mathrm{N}}(\boldsymbol{R})$$

for every $l \in (0, l_C)$ and every $E \in I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}}$. By applying Theorem 4.1, we obtain that

$$\text{(53)} \qquad G_{\{0,1\}}(E) \text{ is dense in } \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R}) \text{ for all } l \in (0, l_C) \text{ and all } E \in I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}}.$$

Now, as the Fürstenberg group $G(E)$ is the closure of $G_{\{0,1\}}(E)$, we get

$$\text{(54)} \qquad G(E) = \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R}) \quad \text{for all } l \in (0, l_C) \text{ and all } E \in I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}} .$$

We have proved Proposition 1.3 because $\mathcal{V}$ is of Lebesgue measure 0 (see (33)) and $\mathcal{S}_{\mathrm{V}}$ is finite. □

We deduce Corollary 1.4 by using the fact that, for l and E such that $G(E) = \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$, $G(E)$ is p-contracting and L_p-strongly irreducible for every $p \in \{1, \dots, N\}$ (see [1, Definitions A.IV.3.3 and A.IV.1.1] for the definitions of these notions). Thus, by [1, Proposition IV.3.4], we get the separability and the positivity of the Lyapunov exponents $\gamma_1(E), \dots, \gamma_N(E)$ (see (7)). Because $\mathcal{S}_{\mathrm{V}}$ is finite, it is of Lebesgue measure zero in $\boldsymbol{R}$ and we can apply Kotani's theory (see [13]) to prove the absence of absolutely continuous spectrum in $I(N, V, l)$ for $l \in (0, l_C)$ and $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$, which finish the proof of Corollary 1.4.

REMARK 4.2. We also note that, by applying [3, Theorem 2], we get that the functions $E \mapsto \gamma_p(E)$ for $p \in \{1, \dots, N\}$ are Hölder continuous on every compact interval $I \subset I(N, V, l) \setminus \mathcal{S}_{\mathrm{V}}$ for $l \in (0, l_C)$ and $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$.

5. Proof of Theorem 1.2. Using Proposition 1.3, Theorem 1.2 will be a consequence of the following result.

THEOREM 5.1 ([4, Theorem 1]). *Let $I \subset \boldsymbol{R}$ be a compact interval such that $\Sigma \cap I \neq \emptyset$ and let $\tilde{I}$ be an open interval with $I \subset \tilde{I}$, such that, for every $E \in \tilde{I}$, $G(E)$ is p-contracting and L_p-strongly irreducible for every $p \in \{1, \dots, N\}$. Then, $H_l(\omega)$ exhibits* (EL) *and* (SDL) *in I.*

To prove this result we had to:

1. Obtain an integral representation of the Lyapunov exponents of $H_l(\omega)$ which, in particular, implies their positivity.
2. Deduce from this integral representation some Hölder regularity of the Lyapunov exponents (see Remark 4.2).
3. Show that the integrated density of states of $H_l(\omega)$ has the same Hölder regularity (see Proposition 6.2).

4. Prove a Wegner estimate using the Hölder regularity of the integrated density of states.

5. Obtain (EL) and (SDL) by using multiscale analysis.

PROOF OF THEOREM 1.2. Let $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{S}_{\mathrm{V}}$ and assume that $l \in (0, l_C)$. Let $\tilde{I} \subset I(N,V,l) \setminus \mathcal{S}_{\mathrm{V}}$ be an open interval such that there exists a compact interval $I \subset \tilde{I}$, with $\Sigma \cap I \neq \emptyset$. If we take l small enough, as the intervals $I(N,V,l)$ tends to $\boldsymbol{R}$ and Σ does not depend on l, we can always find such intervals $\tilde{I}$ and I. Now, as $\tilde{I} \subset I(N,V,l) \setminus \mathcal{S}_{\mathrm{V}}$, by Proposition 1.3, for every $E \in \tilde{I}$, we have $G(E) = \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$. Thus, we can apply Theorem 5.1 to obtain that $H_l(\omega)$ exhibits (EL) and (SDL) in I, which proves Theorem 1.2. □

6. Results on the integrated density of states. The integrated density of states is the distribution function of the energy levels of $H_l(\omega)$, per unit volume. To define it properly, we first need to restrict the operator $H_l(\omega)$ to finite length intervals. Let $L \geq 1$ be an integer and let $H_l^{(L)}(\omega)$ be the restriction of $H_l(\omega)$ to $L^2([-lL, lL]) \otimes \boldsymbol{C}^N$, with Dirichlet (or Neumann) boundary conditions at $\pm lL$.

DEFINITION 6.1. For P-almost every $\omega \in \Omega$, the integrated density of states associated to $H_l(\omega)$ is the function from $\boldsymbol{R}$ to $\boldsymbol{R}_+$, $E \mapsto N(E)$, where $N(E)$, for $E \in \boldsymbol{R}$, is defined by

$$N(E) = \lim_{L \to +\infty} \frac{1}{2lL} \#\{\lambda \leq E \,;\; \lambda \in \sigma(H_l^{(L)}(\omega))\}\,. \tag{55}$$

For the integrated density of states associated to $H_l(\omega)$, we have the following results.

PROPOSITION 6.2. (1) *For any $V \in \mathrm{S}_N(\boldsymbol{R})$, $l > 0$ and $E \in \boldsymbol{R}$, the limit* (55) *exists and is P-almost surely independent of $\omega \in \Omega$.*

(2) *Let $V \in \mathrm{S}_N(\boldsymbol{R}) \setminus \mathcal{V}$ and $l \in (0, l_C)$. Let $I \subset I(N,V,l) \setminus \mathcal{S}_{\mathrm{V}}$ be an open interval. Then the integrated density of states of $H_l(\omega)$, $E \mapsto N(E)$, is Hölder continuous on I.*

PROOF. For (1), we directly apply [3, Corollary 1]. For (2), we use the fact that, for $E \in I$ with $I \subset I(N,V,l) \setminus \mathcal{S}_{\mathrm{V}}$, we have $G(E) = \mathrm{Sp}_{\mathrm{N}}(\boldsymbol{R})$ and thus, $G(E)$ is p-contracting and L_p-strongly irreducible for every $p \in \{1, \dots, N\}$. As $I(N,V,l)$ is compact, we can directly apply [3, Theorem 4] to I which proves that $H_l(\omega)$ is Hölder continuous on I. □

A. Appendix : the almost-sure spectrum. We fix $l > 0$. Let $\omega \in \Omega$ and $n \in \boldsymbol{Z}$. If V is a matrix in $\mathrm{S}_N(\boldsymbol{R})$, we set

$$V_{\omega^{(n)}} = V + \mathrm{diag}(c_1 \omega_1^{(n)}, \dots, c_N \omega_N^{(n)}) \in \mathrm{S}_N(\boldsymbol{R})\,. \tag{56}$$

In this appendix, we prove that the almost-sure spectrum Σ of $\{H_l(\omega)\}_{\omega \in \Omega}$ verifies

$$\Sigma = [0, +\infty) + \bigcup_{\omega^{(0)} \in \mathrm{supp}\,(\nu_1 \otimes \cdots \otimes \nu_N)} \{E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}\}\,, \tag{57}$$

where $E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}$ are the real eigenvalues of the real symmetric matrix $V_{\omega^{(0)}}$.
Let $x \in \boldsymbol{R}$. We set

$$V_\omega(x) = \sum_{n \in \boldsymbol{Z}} \mathbf{1}_{[0,l]}(x - ln) \otimes V_{\omega^{(n)}} . \tag{58}$$

We denote by V_ω the maximal multiplication operator by the function $x \mapsto V_\omega(x)$. Since $(\omega^{(n)})_{n\in\boldsymbol{Z}}$ is a sequence of i.i.d. random variables and since for every $n \in \boldsymbol{Z}$ the function $x \mapsto \mathbf{1}_{[0,l]}(x - ln) \otimes V_{\omega^{(n)}}$ is constant on $[ln, l(n+1)]$, the almost-sure spectrum of the $l\boldsymbol{Z}$-ergodic family $\{V_\omega\}_{\omega\in\Omega}$ is

$$\Sigma(V_\omega) = \bigcup_{\omega^{(0)} \in \operatorname{supp}(\nu_1 \otimes \cdots \otimes \nu_N)} \{E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}\} . \tag{59}$$

We recall that if we consider the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L^2(\boldsymbol{R}) \otimes \boldsymbol{C}$ of domain $H^2(\boldsymbol{R}) \otimes \boldsymbol{C}$, we have

$$\sigma\Big(- \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big) = [0, +\infty) . \tag{60}$$

Now, since the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N$ is deterministic, its almost-sure spectrum is

$$\Sigma\Big(- \frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N\Big) = [0, +\infty) . \tag{61}$$

For any $\omega \in \Omega$, V_ω is a bounded self-adjoint operator on $L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$ and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}$ is self-adjoint on $H^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$. Then, using [12, Theorem 4.10, Chapter V], we get that every $\lambda \in \Sigma$ satisfies $\lambda \in \Sigma\big(- \frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N\big) + \Sigma(V_\omega)$ and thus, $\Sigma \subset \Sigma\big(- \frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N\big) + \Sigma(V_\omega)$.

Conversely, let $\alpha \in \Sigma\big(- \frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}\big)$ and let $\beta \in \Sigma(V_\omega)$. Then, $\alpha \in [0, +\infty)$ and in particular, $\alpha \in \sigma\big(- \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big)$. One can find a sequence $(g_p)_{p\in\boldsymbol{N}}$ of elements of $H^2(\boldsymbol{R}) \otimes \boldsymbol{C}$ such that

(i) $\|g_p\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}} = 1$ for every $p \in \boldsymbol{N}$,

(ii) $\| - g_p'' - \alpha g_p\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}}$ tends to 0 as p tends to infinity,

(iii) $\operatorname{supp} g_p \subset [-(p+1), (p+1)]$ for every $p \in \boldsymbol{N}$.

To construct such a sequence $(g_p)_{p\in\boldsymbol{N}}$, we can consider a solution of the ordinary differential equation $-u'' = \alpha u$, for example the function $u : x \mapsto \mathrm{e}^{\mathrm{i}\sqrt{\alpha}x}$. Then, we multiply this solution by a sequence of functions $(\chi_p)_{p\in\boldsymbol{N}}$ which are compactly supported in the interval $[-(p+1), (p+1)]$, constant on the interval $[-p, p]$ and such that $\|\chi_p\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}} = 1$ for every $p \in \boldsymbol{N}$.

Let $m \in \boldsymbol{N}$ and $n \in \boldsymbol{Z}$. We set

$$\Omega_n^{(m)} = \{\omega \in \Omega \ ; \ \beta \in \sigma(V_{\omega^{(n)}}) \quad \text{and} \quad \omega^{(n)} = \omega^{(n+1)} = \cdots = \omega^{(n+m)}\} , \tag{62}$$

where $\sigma(V_{\omega^{(n)}}) = \{E_1^{\omega^{(n)}}, \ldots, E_N^{\omega^{(n)}}\}$. We remark that

(63)
$$\Sigma(V_\omega) = \bigcup_{\omega^{(0)} \in \mathrm{supp}\,(\nu_1 \otimes \cdots \otimes \nu_N)} \sigma(V_{\omega^{(0)}}) = \bigcup_{\omega^{(n)} \in \mathrm{supp}\,(\nu_1 \otimes \cdots \otimes \nu_N)} \sigma(V_{\omega^{(n)}}), \quad \text{for any } n \in \boldsymbol{Z}\,.$$

We also set

(64) $\Omega^{(m)} = \{\omega \in \Omega\ ;\ \beta \in \sigma(V_{\omega^{(n)}})$ and $\omega^{(n)} = \cdots = \omega^{(n+m)}$ for infinitely many $n \in \boldsymbol{Z}\}$.

Since $(\omega^{(n)})_{n\in \boldsymbol{Z}}$ is a sequence of i.i.d. random variables, we have

$$\mathsf{P}(\Omega_i^{(m)}) = \mathsf{P}(\Omega_j^{(m)}) \quad \text{for any } (i,j) \in \boldsymbol{Z}^2 \text{ such that } i \neq j\,. \tag{65}$$

Moreover, as $\beta \in \Sigma(V_\omega)$, by (63) and the fact that the random variables $\omega^{(n)}$ are i.i.d., we have

$$\mathsf{P}(\Omega_n^{(m)}) > 0 \quad \text{for any } n \in \boldsymbol{Z}\,. \tag{66}$$

Finally, the events $(\Omega_{(m+1)n}^{(m)})_{n\in \boldsymbol{Z}}$ are independent and we can apply Borel-Cantelli's lemma to obtain

$$\mathsf{P}(\Omega^{(m)}) = 1 \quad \text{for any } m \in \boldsymbol{N}\,. \tag{67}$$

We set

$$\Omega_1 = \{\omega \in \Omega\ ; \forall m \in \boldsymbol{N},\ \beta \in \sigma(V_{\omega^{(n)}}) \text{ and } \omega^{(n)} = \cdots = \omega^{(n+m)} \text{ for infinitely many } n \in \boldsymbol{Z}\}\,.$$

Then, Ω_1 is the countable intersection of the $\Omega^{(m)}$ and, by (67), $\mathsf{P}(\Omega_1) = 1$. We set

$$\Omega_2 = \{\omega \in \Omega\ ;\ \sigma(H_l(\omega)) = \Sigma\}\,.$$

By definition of Σ, we have $\mathsf{P}(\Omega_2) = 1$. Thus, if we set $\Omega_0 = \Omega_1 \cap \Omega_2$, we have $\mathsf{P}(\Omega_0) = 1$. In particular, $\Omega_0 \neq \emptyset$. We fix $\omega \in \Omega_0$.

Let $p \in \boldsymbol{N}$ and let $m \in \boldsymbol{N}$ such that $m > 2p + 2$. Since $\omega \in \Omega_0$, there exists $n \in \boldsymbol{Z}$ such that $\beta \in \sigma(V_{\omega^{(n)}})$ and $\omega^{(n+i)} = \omega^{(n)}$ for every $i \in \{1, \ldots, m\}$. Since properties (i), (ii) and (iii) of $(g_p)_{p\in \boldsymbol{N}}$ are invariant by translation, one may assume that supp $g_p \subset [n, n+m]$. Moreover, since $\omega^{(n+i)} = \omega^{(n)}$ for every $i \in \{1, \ldots, m\}$, $V_{\omega^{(n+i)}} = V_{\omega^{(n)}}$ for every $i \in \{1, \ldots, m\}$. Since $\beta \in \sigma(V_{\omega^{(n)}})$, one can find an eigenvector $f_n \in \boldsymbol{C}^N$ of the matrix $V_{\omega^{(n)}}$ associated to β which is also an eigenvector of $V_{\omega^{(n+i)}}$ associated to β for every $i \in \{1, \ldots, m\}$. We can assume that $\|f_n\|_{\boldsymbol{C}^N} = 1$, where $\|\cdot\|_{\boldsymbol{C}^N}$ is any norm on $\boldsymbol{C}^N$.

Then, we set $f \in L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$ which is equal to f_n on $[n, n+m]$ and equal to 0 on $\boldsymbol{R} \setminus [n, n+m]$. We have

$$(V_\omega - \beta) \cdot f = \sum_{j=n}^{n+m} (V_{\omega^{(j)}} - \beta) \cdot f_n + \sum_{j \in \boldsymbol{Z} \setminus \{n, \ldots, n+m\}} (V_{\omega^{(j)}} - \beta) \cdot 0 = 0\,, \tag{68}$$

since f_n is a common eigenvector to the matrices $V_{\omega^{(j)}}$ for $j \in \{n, \ldots, n+m\}$.

Now we can define $h_p = g_p f$ for every $p \in \boldsymbol{N}$. Then $h_p \in H^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$ because $g_p \in H^2(\boldsymbol{R}) \otimes \boldsymbol{C}$, supp $g_p \subset [n, n+m]$ and $f \in L^2(\boldsymbol{R}) \otimes \boldsymbol{C}^N$ is constant on $[n, n+m]$. We have

$$
\begin{aligned}
\|h_p\|^2_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N} &= \int_{\boldsymbol{R}} \|g_p(x) f(x)\|^2_{\boldsymbol{C}^N} \mathrm{d}x \\
&= \int_{[n,n+m]} |g_p(x)|^2 \, \|f_n\|^2_{\boldsymbol{C}^N} \mathrm{d}x \\
&= \int_{[n,n+m]} |g_p(x)|^2 \mathrm{d}x = \|g_p\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N} = 1 \,,
\end{aligned} \tag{69}
$$

since supp $g_p \subset [n, n+m]$. We also have, using (68),

$$
\begin{aligned}
\|(H_l(\omega) - (\alpha+\beta)) \cdot h_p\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N} &= \left\| \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N - \alpha \right) \cdot h_p + (V_\omega - \beta) \cdot h_p \right\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N} \\
&= \left\| (-g_p'' - \alpha g_p) f + g_p (V_\omega - \beta) \cdot f \right\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N} \\
&= \left\| (-g_p'' - \alpha g_p) f \right\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N} \\
&\leq \left\| - g_p'' - \alpha g_p \right\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}} \times \left(\sup_{x\in\boldsymbol{R}} \|f(x)\|^2_{\boldsymbol{C}^N} \right) \\
&= \left\| - g_p'' - \alpha g_p \right\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}}
\end{aligned}
$$

by $\sup_{x\in\boldsymbol{R}} \|f(x)\|^2_{\boldsymbol{C}^N} = \|f_n\|^2_{\boldsymbol{C}^N} = 1$. By (ii), we obtain that $\|(H_l(\omega) - (\alpha+\beta)) \cdot h_p\|_{L^2(\boldsymbol{R})\otimes\boldsymbol{C}^N}$ tends to 0 when p tends to infinity. Combining this with (69) and applying Weyl's criterion with the sequence $(h_p)_{p\in\boldsymbol{N}}$, we obtain $\alpha+\beta \in \sigma(H_l(\omega))$. Since $\omega \in \Omega_2$, we have $\alpha+\beta \in \Sigma$. Thus, $\Sigma\big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}\big) + \Sigma(V_\omega) \subset \Sigma$ and finally

$$
\Sigma = \Sigma \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}} \right) + \Sigma(V_\omega) = [0, +\infty) + \bigcup_{\omega^{(0)} \in \mathrm{supp}\,(\nu_1 \otimes \cdots \otimes \nu_N)} \{E_1^{\omega^{(0)}}, \ldots, E_N^{\omega^{(0)}}\} \,. \tag{70}
$$

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