Tohoku Math. J.
61 (2009), 483–497

LOCALIZATION FOR BRANCHING BROWNIAN MOTIONS IN RANDOM ENVIRONMENT

YUICHI SHIOZAWA

(Received January 23, 2009, revised June 15, 2009)

Abstract. We consider a model of branching Brownian motions in random environment associated with the Poisson random measure. We find a relation between the slow population growth and the localization property in terms of the replica overlap. Applying this result, we prove that, if the randomness of the environment is strong enough, this model possesses the strong localization property, that is, particles gather together at small sets.

1. Introduction. We studied in [13] the population growth rate and the diffusivity of the population density for branching Brownian motions in random environment associated with the Poisson random measure. We proved there that our model possesses the phase transition in terms of the population growth rate. Our purpose in this paper is to show the existence of the phase transition in terms of the diffusive behavior of the population density.

Smith and Wilkinson [14] introduced a model of branching processes in random environment as a generalization of the classical Galton-Watson process. This model is then generalized to the continuous time model ([10]) and the model of branching random walks in random environment ([2], [8] and [15]). In particular, Yoshida [15] proved that, if the randomness of the environment is moderated by that of the random walk, the population growth rate is the same as its expectation with positive probability and the population density satisfies the central limit theorem in terms of convergence in probability, that is, particles spread over the whole space diffusively. These two results are then refined to hold almost surely by Hu and Yoshida [8] and Nakashima [11], respectively. Yoshida [15] also showed that, if the randomness of the environment is strong enough, the population growth rate is less than its expectation almost surely. Furthermore, Hu and Yoshida [8] proved that this model possesses the localization property, that is, particles gather together at small sets. The latter two properties contrast with those of the non-random environment model, and arise from the fluctuations in the randomness of the environment.

Here we consider a model of branching Brownian motions in time-space random environment associated with the Poisson random measure; the places occupied by Poisson points are suitable for particles to live, and the branching rate of each particle is proportional to the number of Poisson points which influence the particle. This model is introduced in [13] as

2000 *Mathematics Subject Classification.* Primary 60K37; Secondary 60G44, 60G57, 60J80.
Key words and phrases. Branching Brownian motion, random environment, localization, Poisson random measure, Ito's formula.
Partly supported by the Global COE program at Department of Mathematics and Research Institute for Mathematical Sciences, Kyoto University.

a continuous counterpart of branching random walks in random environment, and is closely related to Brownian directed polymers in random environment introduced by Comets and Yoshida ([6]). We are now concerned with the diffusive behavior of the population density for this model. To be precise, let $\boldsymbol{P}$ be the law of the branching Brownian motion in random environment (see Section 2 below for the definition of our model). We denote by $N_t(A)$ the number of particles on $A \subset \boldsymbol{R}^d$ at time t and set $\bar{N}_t := N_t(\boldsymbol{R}^d)$. We can then recognize $N_t(\cdot)$ as a configuration measure of particles at time t on $\boldsymbol{R}^d$. We associate a population density $\rho_t(\mathrm{d}x)$ at time t defined by

$$\rho_t(\mathrm{d}x) = \frac{N_t(\mathrm{d}x)}{\bar{N}_t} .$$

We obtained in [13] the continuous counterparts of the results by Yoshida [15] which we mentioned above. In this paper, we show the following localization property (Corollary 2.3): if the randomness of the environment is strong enough, then there exists a non-random positive constant $c > 0$ such that

$$\limsup_{t \to \infty} \bar{\rho}_t \geq c \quad \boldsymbol{P}\text{-a.s.} \tag{1.1}$$

Here $U(x)$ is a closed ball centered at $x \in \boldsymbol{R}^d$ with unit volume and $\bar{\rho}_t := \sup_{x \in \boldsymbol{R}^d} \rho_t(U(x))$ is the density at the most populated site. We should note that this kind of the localization property is studied for many models, such as directed polymers in random environment ([3], [4], [5] and [6]), branching random walks in random environment ([8]), linear stochastic evolutions ([16]) and linear systems ([17]).

Our result is a continuous counterpart of that established by Hu and Yoshida [8], and we take an approach similar to theirs. However, unlike the discrete time model, the splitting time of each particle is randomly determined. To overcome this difficulty, we use Ito's formula for semimartingales and the asymptotic equivalence between the quadratic variation and the predictable quadratic variation of a martingale. We now explain how to obtain the localization property of our model. Let $\bar{M}_t$ be the normalization of $\bar{N}_t$ with respect to its expectation (see (2.1) below for the definition of $\bar{M}_t$), which is a martingale as we proved in [13]. We can then calculate the predictable quadratic variation of $\bar{M}_t$ explicitly (Proposition 3.3). Moreover, by applying Ito's formula to $-\log \bar{M}_t$ and by the asymptotic equivalence as we mentioned above (see also (4.6) below), $-\log \bar{M}_t$ is comparable to the replica overlap defined by

$$R_t = \int_{\boldsymbol{R}^d} \rho_t(U(x))^2 \,\mathrm{d}x ,$$

which measures the degree to which pairs of particles meet together. In fact, we obtain the following relation: if the randomness of the environment is strong enough, then there exists a non-random positive constant c such that

$$\int_0^t R_s \,\mathrm{d}s \geq -c \log \bar{M}_t \quad \text{for large } t .$$

We finally get (1.1) by combining this relation with the inequality $\bar{\rho}_t \geq R_t$ for any $t \geq 0$ and by the fact that the population growth rate is strictly less than its expectation almost surely as we proved in [13] (see also Theorem 2.1 below).

2. Model and results.

2.1. Model. We begin by recalling the model of branching Brownian motions in random environment introduced in [13]. Define $\boldsymbol{R}_+ = [0, \infty)$ and let η denote the Poisson random measure on $\boldsymbol{R}_+ \times \boldsymbol{R}^d$ with unit intensity on a probability space $(\mathcal{M}, \mathcal{G}, Q)$. Namely, η is a non-negative integer valued random measure such that $\eta(A_1), \dots, \eta(A_n)$ are mutually independent for disjoint and bounded sets $A_1, \dots, A_n \in \mathcal{B}(\boldsymbol{R}_+ \times \boldsymbol{R}^d)$ and

$$Q(\eta(A) = k) = e^{-|A|}\frac{|A|^k}{k!} \quad \text{for } A \in \mathcal{B}(\boldsymbol{R}_+ \times \boldsymbol{R}^d),$$

where $\mathcal{B}(\boldsymbol{R}_+ \times \boldsymbol{R}^d)$ is the family of all Borel measurable sets on $\boldsymbol{R}_+ \times \boldsymbol{R}^d$ and $|\cdot|$ is the Lebesgue measure on $\boldsymbol{R}^{1+d}$. Let $\{\theta_t\}_{t\geq 0}$ be the time shift operator of the Poisson random measure, that is, for $\eta \in \mathcal{M}$, $\theta_t\eta = \theta_t\eta(\mathrm{d}s, \mathrm{d}x) = \eta(\{t\} + \mathrm{d}s, \mathrm{d}x)$ identically for any $s, t \geq 0$. The notation $\theta_t\eta$ is often abbreviated to η_t. We denote by $\{\mathcal{G}_t\}_{t\geq 0}$ the family of the sub-σ-field of $\mathcal{G}$ defined by

$$\mathcal{G}_t = \sigma\big(\eta(A \cap ((0, t] \times \boldsymbol{R}^d)), A \in \mathcal{B}(\boldsymbol{R}_+ \times \boldsymbol{R}^d)\big).$$

Let $\mathbf{M} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{B_t\}_{t\geq 0}, \{P_x\}_{x\in\boldsymbol{R}^d}, \{\theta_t\}_{t\geq 0})$ be the Brownian motion on $\boldsymbol{R}^d$, where $\{\theta_t\}_{t\geq 0}$ is the time shift operator of paths, that is, for $\omega \in \Omega$, $B_t(\theta_s\omega) = B_{t+s}(\omega)$ identically for any $s, t \geq 0$. Note that we use the common notation $\{\theta_t\}_{t\geq 0}$ as the time shift operators of paths and of the Poisson random measure, respectively. Denote by V_t the tube around the graph $\{(s, B_s)\}_{0<s\leq t}$ defined by

$$V_t = V_t(\omega) = \{(s, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^d\,;\ s \in (0, t],\ x \in U(B_s(\omega))\} \quad \text{for } \omega \in \Omega,$$

where $U(x)$ is a closed ball in $\boldsymbol{R}^d$ centered at $x \in \boldsymbol{R}^d$ with unit volume.

Let τ be a non-negative random variable on $(\Omega, \mathcal{F}, P_x)$, independently of the Brownian motion, of exponential distribution with the mean 1; $P_x(\tau > a) = e^{-a}$ for any $a \geq 0$. Fix a parameter $\alpha > 0$ and set

$$S = S(\omega, \eta) = \inf\{t > 0\,;\ \alpha\eta(V_t(\omega)) > \tau(\omega)\} \quad \text{for } (\omega, \eta) \in \Omega \times \mathcal{M}.$$

Then we have

$$P_x(S(\cdot, \eta) > t) = E_x[e^{-\alpha\eta(V_t)}].$$

Here we note that, if a path $\omega \in \Omega$ is fixed, $\{\eta(V_t(\omega))\}_{t\geq 0}$ is a standard Poisson process on the half line. In particular, the jump size of this process is equal to one Q-a.s. (for instance, see [12, p. 472, Proposition 1.4]). Let $\{p_n\}_{n=0}^{\infty}$ be a probability function, that is, $p_n \geq 0$ for any $n \geq 0$ and $\sum_{n=0}^{\infty} p_n = 1$. In what follows, we assume $p_0 + p_1 < 1$ to avoid the case where the numbers of particles do not increase for branching Brownian motions which are introduced below. We define

$$m^{(q)} = \sum_{n=0}^{\infty} n^q p_n \quad \text{for } q \geq 0.$$

We also let I be an $\boldsymbol{N} \cup \{0\}$-valued random variable on $(\Omega, \mathcal{F}, P_x)$, independently of the Brownian motion and τ, such that $P_x(I = n) = p_n$.

We now introduce the index sets. For each k in $\boldsymbol{N}$, we define

$$K_k^0 = \{()\}, \quad K_k^1 = \{(k)\}, \quad K_k^n = \{(k, k_2, \dots, k_n)\,;\, k_2, \dots, k_n \in \boldsymbol{N}\} \quad \text{for } n \geq 2$$

and

$$\mathbf{K}_k = \coprod_{n=0}^{\infty} K_k^n ,$$

where () is the empty sequence. In addition, it is useful to define

$$\overline{K_k^n} = K_k^{n+1} \quad \text{for } n \geq 0 \quad \text{and} \quad \overline{\mathbf{K}_k} = \coprod_{n=0}^{\infty} \overline{K_k^n} .$$

If $\mathbf{k} = (k, k_2, \dots, k_n) \in K_k^n$ for some $n \geq 1$ and $k_{n+1} \in \boldsymbol{N}$, then we define $\mathbf{k} \cdot k_{n+1} = (k, k_2, \dots, k_n, k_{n+1}) \in \overline{K_k^n}$. By the same way, we identify $(k) \in \overline{K_k^0}$ with $() \cdot k$.

Let $\{B_t^{\mathbf{k}}\}_{t \geq 0}$ and $\tau^{\mathbf{k}}$, $\mathbf{k} \in \mathbf{K}_k$, be independent copies of $\{B_t\}_{t \geq 0}$ and τ, respectively. Denote by $V_t^{\mathbf{k}}$ the tube V_t associated with the Brownian motion $\{B_t^{\mathbf{k}}\}_{t \geq 0}$, and by $S^{\mathbf{k}}$ the random variable S with τ and V_t replaced by $\tau^{\mathbf{k}}$ and $V_t^{\mathbf{k}}$, respectively. In addition, we set $I^{()} = 1$ and let $I^{\mathbf{k}}$, $\mathbf{k} \in \mathbf{K}_k \setminus K_k^0$, be independent copies of I, respectively.

Let us denote by $K^0, \dots, \overline{K^0}, \dots, \mathbf{K}, \bar{\mathbf{K}}$ the quantities $K_k^0, \dots, \overline{K_k^0}, \dots, \mathbf{K}_k, \overline{\mathbf{K}_k}$ with $k = 1$, respectively. We consider the family of random variables $T^{\mathbf{k}}$ and $\{\mathbf{B}_t^{\mathbf{k}}\}_{t \geq 0}$ indexed by $\mathbf{k} \in \mathbf{K}$ on the measurable space $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G})$ as follows; for each fixed $(\omega, \eta) \in \Omega \times \mathcal{M}$, let $T^{()}(\omega, \eta) = 0$ and $\mathbf{B}_t^{()}(\omega, \eta) = B_t^{()}(\omega)$ identically for any $t \geq 0$. We then define inductively for $\mathbf{k} \cdot k \in \bar{\mathbf{K}}$,

$$\begin{aligned} T^{\mathbf{k}\cdot k} &= T^{\mathbf{k}\cdot k}(\omega, \eta) \\ &= \begin{cases} T^{\mathbf{k}}(\omega, \eta) + S^{\mathbf{k}\cdot k}(\theta_{T^{\mathbf{k}}(\omega,\eta)}\omega, \theta_{T^{\mathbf{k}}(\omega,\eta)}\eta) & \text{if } k \leq I^{\mathbf{k}}(\omega) \\ \infty & \text{if } k \geq I^{\mathbf{k}}(\omega) + 1 , \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbf{B}_t^{\mathbf{k}\cdot k} &= \mathbf{B}_t^{\mathbf{k}\cdot k}(\omega, \eta) \\ &= \begin{cases} \mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}(\omega,\eta)}(\omega, \eta) + B_t^{\mathbf{k}\cdot k}(\omega) - B^{\mathbf{k}\cdot k}_{T^{\mathbf{k}}(\omega,\eta)}(\omega) & \\ \qquad\qquad \text{for } T^{\mathbf{k}}(\omega, \eta) \leq t < T^{\mathbf{k}\cdot k}(\omega, \eta) \quad \text{if } k \leq I^{\mathbf{k}}(\omega) & \\ \Delta \qquad\qquad \text{otherwise} , & \end{cases} \end{aligned}$$

where Δ is a cemetery point. We use the notations $\mathbf{B}_t^{\mathbf{k}}$ and $T^{\mathbf{k}}$ to denote, respectively, the position and the splitting time of the particle with index $\mathbf{k}$ of a branching Brownian motion. More precisely, we can describe our branching Brownian motion as follows:

- At time 0, the Brownian particle with index 1 starts from $\mathbf{B}_0^{()}$.
- The Brownian particle with index $\mathbf{k} \in \mathbf{K} \setminus K^0$ splits into n Brownian particles with probability p_n at site $\mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}}$ at time $T^{\mathbf{k}}$.

- These Brownian particles, indexed by $\mathbf{k}\cdot 1, \mathbf{k}\cdot 2, \dots, \mathbf{k}\cdot n$, respectively, start from $\mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}}$ independently.

Here we note that particles indexed respectively by $\mathbf{k}\cdot 1, \mathbf{k}\cdot 2, \dots, \mathbf{k}\cdot n$ at time $T^{\mathbf{k}}$ can be identified with n particles starting from $\mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}}$ at time t in environment $\eta_{T^{\mathbf{k}}}$.

Let us introduce the notion of branching Brownian motions in random environment. We define the probability measures $\{\boldsymbol{P}^{\eta}_x\}_{x\in\boldsymbol{R}^d}$ and $\{\boldsymbol{P}_x\}_{x\in\boldsymbol{R}^d}$ on $(\Omega\times\mathcal{M}, \mathcal{F}\otimes\mathcal{G})$, respectively, by

$$\boldsymbol{P}^{\eta}_x = P_x\otimes\delta_\eta \quad \text{and} \quad \boldsymbol{P}_x = \int_{\mathcal{M}} Q(\mathrm{d}\eta)\boldsymbol{P}^{\eta}_x\,,$$

where δ_η is the Dirac measure at $\eta\in\mathcal{M}$. We call $(\Omega\times\mathcal{M}, \mathcal{F}\otimes\mathcal{G}, \{\{\mathbf{B}^{\mathbf{k}}_t\}_{t\ge 0}\}_{\mathbf{k}\in\mathbf{K}}, \{T^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{K}}, \{\boldsymbol{P}^{\eta}_x\}_{x\in\boldsymbol{R}^d})$ the branching Brownian motion in environment η with offspring distribution $\{p_n\}_{n=0}^{\infty}$, and $(\Omega\times\mathcal{M}, \mathcal{F}\otimes\mathcal{G}, \{\{\mathbf{B}^{\mathbf{k}}_t\}_{t\ge 0}\}_{\mathbf{k}\in\mathbf{K}}, \{T^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{K}}, \{\boldsymbol{P}_x\}_{x\in\boldsymbol{R}^d})$ the branching Brownian motion in random environment with offspring distribution $\{p_n\}_{n=0}^{\infty}$.

Let $N_t(A)$ be the number of particles on the set $A\in\mathcal{B}(\boldsymbol{R}^d)$ at time t, that is,

$$N_t(A) = \sum_{\mathbf{k}\cdot k\in\bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k},\ \mathbf{B}^{\mathbf{k}\cdot k}_t\in A\}}\,.$$

We denote by $\bar{N}_t$ the total number of particles at time t, that is, $\bar{N}_t = N_t(\boldsymbol{R}^d)$. We also use the notation

$$N_t(f) = \sum_{\mathbf{k}\cdot k\in\bar{\mathbf{K}}} f(\mathbf{B}^{\mathbf{k}\cdot k}_t)\mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}, \mathbf{B}^{\mathbf{k}\cdot k}_t\in\boldsymbol{R}^d\}} \quad \text{for } f\in\mathcal{B}_b(\boldsymbol{R}^d)\,,$$

where $\mathcal{B}_b(\boldsymbol{R}^d)$ stands for the set of all bounded Borel measurable functions on $\boldsymbol{R}^d$.

2.2. Results. In this subsection, we state the results in this paper. From now on, we denote by P, $\boldsymbol{P}^{\eta}$, $\boldsymbol{P}$, etc. the quantities P_x, $\boldsymbol{P}^{\eta}_x$, $\boldsymbol{P}_x$, etc. for $x=0$, respectively. Let us define

$$\bar{M}_t = e^{-\lambda t}\bar{N}_t \tag{2.1}$$

for

$$\beta := \log\{m^{(1)} - e^{-\alpha}(m^{(1)}-1)\} \quad \text{and} \quad \lambda = \lambda(\beta) := e^{\beta}-1\,.$$

Since $\bar{M}_t$ is a positive $\boldsymbol{P}$-martingale as we proved in [13], there exists a limit $\lim_{t\to\infty}\bar{M}_t =: \bar{M}_\infty$ $\boldsymbol{P}$-a.s. Let $\rho_t(\mathrm{d}x)$ be the population density at time t defined by

$$\rho_t(\mathrm{d}x) = \frac{N_t(\mathrm{d}x)}{\bar{N}_t}\,.$$

As we mentioned in Introduction, we proved in [13] that, if the randomness of the environment is dominated by that of the Brownian motion, the properties of the population growth rate and the population density are similar to the non-random environment model. In contrast with these properties, if the effect from the randomness of the environment is strong enough, then the situation is different from the non-environmental case. In fact, our model possesses the phase transition in terms of the population growth rate as follows:

THEOREM 2.1 ([13]). *For $d = 1$ or 2, $\boldsymbol{P}(\bar{M}_\infty = 0) = 1$ holds for any $\beta > 0$. On the other hand, for $d \geq 3$, there exists a positive constant $\beta_0(d) > 0$ such that $\boldsymbol{P}(\bar{M}_\infty = 0) = 1$ holds for any $\beta > \beta_0(d)$. Moreover, for any dimension d, there exists a non-negative constant $\beta_1(d) \geq 0$ such that, for each $\beta > \beta_1(d)$,*

$$\limsup_{t\to\infty} \frac{\log \bar{M}_t}{t} < -c(\beta) \quad \boldsymbol{P}\text{-a.s.}$$

holds with some non-random constant $c(\beta) > 0$. In particular, we have $\beta_1(1) = \beta_1(2) = 0$ and $\beta_1(d) > 0$ for $d \geq 3$.

Theorem 2.1 means that the exponential growth rate of the population size is strictly less than its expectation almost surely. Here we are concerned with the diffusive behavior of the population density in the situation of Theorem 2.1. To confirm this property, we define

$$R_t = \int_{\boldsymbol{R}^d} \rho_t(U(x))^2 \, \mathrm{d}x \, .$$

We can then recognize R_t as so-called the replica overlap by analogy with the spin glass theory. We first prove the following relations between the slow population growth and the localization property in terms of the replica overlap:

THEOREM 2.2. (i) *Assume*

$$p_0 = 0\,, \quad m^{(1)} > 1 \quad \textit{and} \quad m^{(2)} < \infty\,. \tag{2.2}$$

Then we have the relation

$$\{\bar{M}_\infty = 0\} \subset \left\{ \int_0^\infty R_t \, \mathrm{d}t = \infty \right\} \quad \boldsymbol{P}\text{-a.s.}$$

Furthermore, if $\boldsymbol{P}(\bar{M}_\infty = 0) = 1$ holds, then there exists a non-random positive constant $c > 0$ such that

$$\int_0^t R_s \, \mathrm{d}s \geq -c \log \bar{M}_t \quad \textit{for any} \ \ t \geq T$$

for some random positive constant $T > 0$.

(ii) *Assume*

$$p_0 = 0 \quad \textit{and there exists} \ \ L \geq 2 \ \ \textit{such that} \ \ p_n = 0 \ \ \textit{for any} \ \ n \geq L+1\,. \tag{2.3}$$

Then we also have the relation

$$\{\bar{M}_\infty = 0\} = \left\{ \int_0^\infty R_t \, \mathrm{d}t = \infty \right\} \quad \boldsymbol{P}\text{-a.s.}$$

If $\boldsymbol{P}(\bar{M}_\infty = 0) = 1$, then there exist non-random positive constants $c_1, c_2 > 0$ such that

$$-c_1 \log \bar{M}_t \leq \int_0^t R_s \, \mathrm{d}s \leq -c_2 \log \bar{M}_t \quad \textit{for any} \ \ t \geq T$$

for some random positive constant $T > 0$.

Let $\bar{\rho}_t$ be the density at the most populated site at time t defined by

$$\bar{\rho}_t = \sup_{x \in \boldsymbol{R}^d} \rho_t(U(x)) .$$

As we mentioned in [13], there exists a constant $c = c(d) \in (0, 1)$ such that the inequality $c\bar{\rho}_t^2 \leq R_t \leq \bar{\rho}_t$ holds for any $t \geq 0$. Hence, combining Theorems 2.1 and 2.2 (i), we finally find that the branching Brownian motion in random environment possesses the following strong localization property.

COROLLARY 2.3 (Localization). *Assume the condition* (2.2). *Then*, *for any* $\beta > \beta_1(d)$, *we have*

$$\limsup_{t\to\infty} \bar{\rho}_t \geq \limsup_{t\to\infty} R_t \geq c'(\beta) \quad \boldsymbol{P}\text{-a.s.}$$

with some non-random positive constant $c'(\beta) \in (0, 1)$.

3. Moments. To prove Theorem 2.2, we first calculate the conditional second moment of $\bar{N}_t$. Here, we assume that $m^{(2)}$ is finite. Define

$$c = m^{(2)} - m^{(1)} \quad \text{and} \quad \mu = 1 - e^{-\alpha} .$$

Then we have the following lemma.

LEMMA 3.1. *For any* $s, t \geq 0$ *and* $f, g \in \mathcal{B}_b(\boldsymbol{R}^d)$, *we have*

$$\begin{aligned}
\boldsymbol{E}_x^\eta[N_{t+s}(f)N_{t+s}(g)|\mathcal{F}_t \otimes \mathcal{G}_t] &= \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}\}} \biggl(E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}[e^{\beta\eta_t(V_s)} f(B_s) g(B_s)] \\
&\quad + c\mu E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\biggl[\int_{(0,s]} e^{\beta\eta_t(V_{u-})} E_{B_u}[e^{\beta\eta_{t+u}(V_{s-u})} f(B_{s-u})] E_{B_u}[e^{\beta\eta_{t+u}(V_{s-u})} g(B_{s-u})]\, \mathrm{d}\eta_t(V_u)\biggr]\biggr) \\
&\quad + \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \leq t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}[e^{\beta\eta_t(V_s)} f(B_s)] E_{\mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}}[e^{\beta\eta_t(V_s)} g(B_s)] \quad Q\text{-a.s.}
\end{aligned}$$

PROOF. We prove this lemma only for $f \equiv 1$ and $g \equiv 1$ because the proof for the general case is done by a modification of the notation. Since we have

$$\bar{N}_{t+s} = \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}\}} \biggl(\sum_{\mathbf{k}'\cdot k' \in \overline{\mathbf{K}_k}} \mathbf{1}_{\{T^{\mathbf{k}\cdot\mathbf{k}'} \leq t+s < T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'}\}} \biggr),$$

it follows that

$$\begin{aligned}
\bar{N}_{t+s}^2 &= \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}\}} \biggl(\sum_{\mathbf{k}'\cdot k', \tilde{\mathbf{k}}'\cdot\tilde{k}' \in \overline{\mathbf{K}_k}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}\cdot\mathbf{k}'} \leq t+s < T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'} \\ T^{\mathbf{k}\cdot\tilde{\mathbf{k}}'} \leq t+s < T^{\mathbf{k}\cdot\tilde{\mathbf{k}}'\cdot\tilde{k}'}}\right\}} \biggr) \\
&\quad + \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \leq t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} \sum_{\mathbf{k}'\cdot k' \in \overline{\mathbf{K}_k}} \mathbf{1}_{\{T^{\mathbf{k}\cdot\mathbf{k}'} \leq t+s < T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'}\}} \sum_{\tilde{\mathbf{k}}'\cdot\tilde{k}' \in \overline{\mathbf{K}_k}} \mathbf{1}_{\{T^{\tilde{\mathbf{k}}\cdot\tilde{\mathbf{k}}'} \leq t+s < T^{\tilde{\mathbf{k}}\cdot\tilde{\mathbf{k}}'\cdot\tilde{k}'}\}} .
\end{aligned}$$

Hence the Markov property yields

$$\boldsymbol{E}_x^{\eta}[\bar{N}_{t+s}^2 | \mathcal{F}_t \otimes \mathcal{G}_t] = \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} \boldsymbol{E}^{\eta_t}_{\mathbf{B}_t^{\mathbf{k}\cdot k}}[\bar{N}_s^2] + \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} \boldsymbol{E}^{\eta_t}_{\mathbf{B}_t^{\mathbf{k}\cdot k}}[\bar{N}_s] \boldsymbol{E}^{\eta_t}_{\mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}}[\bar{N}_s] .$$

We thus complete the proof by applying [13, Lemmas 3.1 and 3.3] to this equality. □

For two independent Brownian motions $(\{B_t^1\}_{t\ge 0}, \{P_x^1\}_{x\in \boldsymbol{R}^d})$ and $(\{B_t^2\}_{t\ge 0}, \{P_x^2\}_{x\in \boldsymbol{R}^d})$ on $\boldsymbol{R}^d$, we let $P_{x,y} = P_x^1 \otimes P_y^2$ and abbreviate $P_{x,x}$ to P_x. By combining Lemma 3.1 with [13, Lemma 3.4], we obtain the following lemma.

LEMMA 3.2. *For any $s, t \ge 0$ and $f, g \in \mathcal{B}_b(\boldsymbol{R}^d)$, we have*

$$\begin{aligned} \boldsymbol{E}_x[N_{t+s}(f)N_{t+s}(g) | \mathcal{F}_t \otimes \mathcal{G}_t] &= \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} \Bigg(e^{\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}[f(B_s)g(B_s)] + c\mu e^{2\lambda s} \\ &\times E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[\int_0^s e^{-\lambda u} E_{B_u}\left[\exp\left(\lambda^2 \int_0^{s-u} |U(B_v^1) \cap U(B_v^2)|\,\mathrm{d}v\right) f(B_{s-u}^1) g(B_{s-u}^2)\right] \mathrm{d}u\right]\Bigg) \\ &+ \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} e^{2\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\left[\exp\left(\lambda^2 \int_0^s |U(B_u^1) \cap U(B_u^2)|\,\mathrm{d}u\right) f(B_s^1) g(B_s^2)\right]. \end{aligned}$$

In particular, we have

$$\begin{aligned} &\boldsymbol{E}_x[\bar{N}_{t+s}^2 | \mathcal{F}_t \otimes \mathcal{G}_t] \\ &\quad = \bar{N}_t \left(e^{\lambda s} + c\mu e^{2\lambda s} \int_0^s e^{-\lambda u} E\left[\exp\left(\frac{\lambda^2}{2} \int_0^{2(s-u)} |U(0) \cap U(B_v)|\,\mathrm{d}v\right)\right] \mathrm{d}u\right) \\ &\quad\quad + \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} e^{2\lambda s} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\left[\exp\left(\lambda^2 \int_0^s |U(B_u^1) \cap U(B_u^2)|\,\mathrm{d}u\right)\right]. \end{aligned}$$

Let $\langle \bar{M} \rangle_t$ be a predictable quadratic variation of the martingale $\bar{M}_t$, that is, $\langle \bar{M} \rangle_t$ is a unique predictable and locally integrable increasing process such that $\bar{M}_t^2 - \langle \bar{M} \rangle_t$ is a locally square integrable martingale (see [7, p. 199, 7.28 Lemma]). Let us define

$$M_t(\mathrm{d}x) = e^{-\lambda t} N_t(\mathrm{d}x) .$$

PROPOSITION 3.3. *In above notation, we get the following equality.*

(3.1)
$$\langle \bar{M} \rangle_t = \left\{ \sum_{n=1}^{\infty} (n-1)^2 p_n \right\} \mu \int_0^t e^{-\lambda s} \bar{M}_s \,\mathrm{d}s + \lambda^2 \int_0^t \left(\int_{\boldsymbol{R}^d} M_s(U(x))^2 \,\mathrm{d}x - e^{-\lambda s} \bar{M}_s \right) \mathrm{d}s , \quad t \ge 0 .$$

Before we prove this proposition, we should note the equality

$$\int_{\boldsymbol{R}^d} M_s(U(x))^2\,\mathrm{d}x - e^{-\lambda s}\bar{M}_s = e^{-2\lambda s}\int_{\boldsymbol{R}^d} \sum_{\substack{\mathbf{k}\cdot k,\tilde{\mathbf{k}}\cdot\tilde{k}\in\bar{\mathbf{K}}\\ \mathbf{k}\cdot k\neq\tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}}\leq s<T^{\mathbf{k}\cdot k}\\ \mathbf{B}_s^{\mathbf{k}\cdot k}\in U(x)}\right\}} \mathbf{1}_{\left\{\substack{T^{\tilde{\mathbf{k}}}\leq s<T^{\tilde{\mathbf{k}}\cdot\tilde{k}}\\ \mathbf{B}_s^{\tilde{\mathbf{k}}\cdot\tilde{k}}\in U(x)}\right\}}\,\mathrm{d}x\,. \tag{3.2}$$

In particular, this implies that the second term of the right hand side of (3.1) is closely related to the correlation among particles because the magnitude of the correlation is proportional to the degree to which pairs of particles meet together as we mentioned in [13].

PROOF OF PROPOSITION 3.3. We prove this proposition only for $p_2 = 1$ because the proof for the general case is done by a modification of the notation. Then we have $\mu = \lambda$. To establish this proposition, it is enough to show the equality

(3.3)

$$\begin{aligned}&\boldsymbol{E}_x\left[\bar{M}_{t+s}^2 - \int_0^{t+s}\left\{\lambda e^{-\lambda u}\bar{M}_u + \lambda^2\left(\int_{\boldsymbol{R}^d} M_u(U(x))^2\,\mathrm{d}x - e^{-\lambda u}\bar{M}_u\right)\right\}\mathrm{d}u \,\middle|\, \mathcal{F}_t\otimes\mathcal{G}_t\right]\\ &\quad = \bar{M}_t^2 - \int_0^t\left\{\lambda e^{-\lambda u}\bar{M}_u + \lambda^2\left(\int_{\boldsymbol{R}^d} M_u(U(x))^2\,\mathrm{d}x - e^{-\lambda u}\bar{M}_u\right)\right\}\mathrm{d}u\,.\end{aligned}$$

From Lemma 3.2, we get

$$\begin{aligned}&\boldsymbol{E}[\bar{M}_{t+s}^2|\mathcal{F}_t\otimes\mathcal{G}_t]\\ &\quad = e^{-2\lambda t}\bar{N}_t\left(e^{-\lambda s} + 2\lambda\int_0^s e^{-\lambda u}E\left[\exp\left(\frac{\lambda^2}{2}\int_0^{2(s-u)}|U(0)\cap U(B_v)|\,\mathrm{d}v\right)\right]\mathrm{d}u\right)\\ &\qquad + e^{-2\lambda t}\sum_{\substack{\mathbf{k}\cdot k,\tilde{\mathbf{k}}\cdot\tilde{k}\in\bar{\mathbf{K}}\\ \mathbf{k}\cdot k\neq\tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}}\leq t<T^{\mathbf{k}\cdot k}\\ T^{\tilde{\mathbf{k}}}\leq t<T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k},\mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\left[\exp\left(\lambda^2\int_0^s|U(B_u^1)\cap U(B_u^2)|\,\mathrm{d}u\right)\right].\end{aligned} \tag{3.4}$$

On the other hand, (3.2) yields the equality

$$\boldsymbol{E}\left[\int_t^{t+s}\left\{\lambda e^{-\lambda u}\bar{M}_u + \lambda^2\left(\int_{\boldsymbol{R}^d} M_u(U(x))^2\,\mathrm{d}x - e^{-\lambda u}\bar{M}_u\right)\right\}\mathrm{d}u\,\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right] = (\mathrm{I}) + (\mathrm{II})$$

for

$$(\mathrm{I}) := \boldsymbol{E}\left[\int_t^{t+s}\lambda e^{-\lambda u}\bar{M}_u\,\mathrm{d}u\,\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right]$$

and

$$(\mathrm{II}) := \lambda^2\int_t^{t+s} e^{-2\lambda u}\int_{\boldsymbol{R}^d}\boldsymbol{E}\left[\sum_{\substack{\mathbf{k}\cdot k,\tilde{\mathbf{k}}\cdot\tilde{k}\in\bar{\mathbf{K}}\\ \mathbf{k}\cdot k\neq\tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}}\leq u<T^{\mathbf{k}\cdot k}\\ \mathbf{B}_u^{\mathbf{k}\cdot k}\in U(x)}\right\}} \mathbf{1}_{\left\{\substack{T^{\tilde{\mathbf{k}}}\leq u<T^{\tilde{\mathbf{k}}\cdot\tilde{k}}\\ \mathbf{B}_u^{\tilde{\mathbf{k}}\cdot\tilde{k}}\in U(x)}\right\}}\,\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right]\mathrm{d}x\mathrm{d}u.$$

Since $\bar{M}_t$ is a martingale, a direct calculation implies

$$(\mathrm{I}) = \bar{M}_t\int_t^{t+s}\lambda e^{-\lambda u}\,\mathrm{d}u = (1-e^{-\lambda s})e^{-\lambda t}\bar{M}_t\,.$$

Let us define

$$U_{x,y} = U(x) \cap U(y) \quad \text{for } x, y \in \boldsymbol{R}^d .$$

Then, by the same way as that in the proof of Lemmas 3.1 and 3.2, we have

$$\text{(II)} = 2\lambda e^{-2\lambda t}\text{(III)} + e^{-2\lambda t}\text{(IV)}$$

for

$$\begin{aligned}\text{(III)} := \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} \int_0^s E_{\mathbf{B}_t^{\mathbf{k}\cdot k}} \bigg[\int_0^u e^{-\lambda v} \\ \times E_{B_v}\bigg[\exp\bigg(\lambda^2 \int_0^{u-v} |U_{B_w^1, B_w^2}| \, \mathrm{d}w\bigg) \lambda^2 |U_{B_{u-v}^1, B_{u-v}^2}| \bigg] \mathrm{d}v \bigg] \mathrm{d}u\end{aligned}$$

and

$$\text{(IV)} := \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} \int_0^s E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}} \bigg[\exp\bigg(\lambda^2 \int_0^u |U_{B_v^1, B_v^2}| \, \mathrm{d}v\bigg) \lambda^2 |U_{B_u^1, B_u^2}| \bigg] \mathrm{d}u \, .$$

By Fubini's theorem, we have

$$\begin{aligned}\text{(III)} &= \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}} \bigg[\int_0^s e^{-\lambda v} \\ &\qquad \times E_{B_v}\bigg[\int_v^s \exp\bigg(\lambda^2 \int_0^{u-v} |U_{B_w^1, B_w^2}| \, \mathrm{d}w\bigg) \lambda^2 |U_{B_{u-v}^1, B_{u-v}^2}| \, \mathrm{d}u \bigg] \mathrm{d}v \bigg] \\ &= \sum_{\mathbf{k}\cdot k \in \bar{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}} \bigg[\int_0^s e^{-\lambda v} \bigg(E_{B_v}\bigg[\exp\bigg(\lambda^2 \int_0^{s-v} |U_{B_w^1, B_w^2}| \, \mathrm{d}w\bigg) \bigg] - 1 \bigg) \mathrm{d}v \bigg] \\ &= \bar{N}_t \int_0^s e^{-\lambda v} E\bigg[\exp\bigg(\frac{\lambda^2}{2} \int_0^{2(s-v)} |U_{0, B_w}| \, \mathrm{d}w \bigg) \bigg] \mathrm{d}v - \frac{1}{\lambda}(1 - e^{-\lambda s}) \bar{N}_t\end{aligned}$$

and

$$\begin{aligned}\text{(IV)} &= \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} \bigg(E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}} \bigg[\exp\bigg(\lambda^2 \int_0^s |U_{B_v^1, B_v^2}| \, \mathrm{d}v\bigg) \bigg] - 1 \bigg) \\ &= \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \bar{\mathbf{K}} \\ \mathbf{k}\cdot k \neq \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\left\{\substack{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k} \\ T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}} \bigg[\exp\bigg(\lambda^2 \int_0^s |U_{B_v^1, B_v^2}| \, \mathrm{d}v\bigg) \bigg] - (\bar{N}_t^2 - \bar{N}_t) \, .\end{aligned}$$

Hence we obtain

$$(\mathrm{I})+(\mathrm{II}) = -\bar{M}_t^2 + e^{-2\lambda t}\bar{N}_t\left(e^{-\lambda s} + 2\lambda\int_0^s e^{-\lambda v}E\left[\exp\left(\frac{\lambda^2}{2}\int_0^{2(s-v)}|U_{0,B_w}|\,\mathrm{d}w\right)\right]\mathrm{d}v\right)$$
$$+e^{-2\lambda t}\sum_{\substack{\mathbf{k}\cdot k,\tilde{\mathbf{k}}\cdot\tilde{k}\in\bar{\mathbf{K}}\\ \mathbf{k}\cdot k\neq\tilde{\mathbf{k}}\cdot\tilde{k}}}\mathbf{1}_{\left\{\substack{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\\ T^{\tilde{\mathbf{k}}}\le t<T^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\right\}}E_{\mathbf{B}_t^{\mathbf{k}\cdot k},\mathbf{B}_t^{\tilde{\mathbf{k}}\cdot\tilde{k}}}\left[\exp\left(\lambda^2\int_0^s|U_{B_v^1,B_v^2}|\,\mathrm{d}v\right)\right].$$

Combining this with (3.4), we have (3.3) and the proof is completed. □

The next lemma gives us a lower bound of the exponential growth rate of $\bar{N}_t$.

LEMMA 3.4. *Assume*

$$p_0 = 0 \quad \text{and} \quad 1 < m^{(1)} < \infty. \tag{3.5}$$

Then we have the following inequality.

$$\liminf_{t\to\infty}\frac{1}{t}\log\bar{N}_t \ge -\log\left(\boldsymbol{E}\left[\frac{1}{\bar{N}_1}\right]\right) \quad \boldsymbol{P}\text{-a.s.}$$

PROOF. We prove this lemma in a similar way to that in [8, Lemma 3.1.3]. Since we obtain

$$\frac{\bar{N}_{t+1}}{\bar{N}_t} = \frac{\sum_{\mathbf{k}\cdot k\in\bar{\mathbf{K}}}\mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}}\left(\sum_{\mathbf{k}'\cdot k'\in\overline{\mathbf{K}_k}}\mathbf{1}_{\{T^{\mathbf{k}\cdot\mathbf{k}'}\le t+1<T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'}\}}\right)}{\sum_{\mathbf{k}\cdot k\in\bar{\mathbf{K}}}\mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}}},$$

Jensen's inequality implies

$$\left(\frac{\bar{N}_{t+1}}{\bar{N}_t}\right)^{-1} \le \frac{1}{\bar{N}_t}\left\{\sum_{\mathbf{k}\cdot k\in\bar{\mathbf{K}}}\mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}}\left(\sum_{\mathbf{k}'\cdot k'\in\overline{\mathbf{K}_k}}\mathbf{1}_{\{T^{\mathbf{k}\cdot\mathbf{k}'}\le t+1<T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'}\}}\right)^{-1}\right\}.$$

By the Markov property and by the space uniformity of Poisson random measures, we get

$$\boldsymbol{E}\left[\left(\sum_{\mathbf{k}'\cdot k'\in\overline{\mathbf{K}_k}}\mathbf{1}_{\{T^{\mathbf{k}\cdot\mathbf{k}'}\le t+1<T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'}\}}\right)^{-1}\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right] = \boldsymbol{E}_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[\frac{1}{\bar{N}_1}\right] = \boldsymbol{E}\left[\frac{1}{\bar{N}_1}\right]$$

for $T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}$. Thus we have

$$\begin{aligned}\boldsymbol{E}\left[\frac{1}{\bar{N}_{t+1}}\,\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right] &= \frac{1}{\bar{N}_t}\boldsymbol{E}\left[\left(\frac{\bar{N}_{t+1}}{\bar{N}_t}\right)^{-1}\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right]\\ &\le \frac{1}{\bar{N}_t^2}\sum_{\mathbf{k}\cdot k\in\bar{\mathbf{K}}}\mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}}\boldsymbol{E}\left[\left(\sum_{\mathbf{k}'\cdot k'\in\overline{\mathbf{K}_k}}\mathbf{1}_{\{T^{\mathbf{k}\cdot\mathbf{k}'}\le t+1<T^{\mathbf{k}\cdot\mathbf{k}'\cdot k'}\}}\right)^{-1}\middle|\,\mathcal{F}_t\otimes\mathcal{G}_t\right]\\ &= \frac{1}{\bar{N}_t}\boldsymbol{E}\left[\frac{1}{\bar{N}_1}\right]\end{aligned}$$

for any $t \ge 0$. Inductively, we get

$$\boldsymbol{E}\left[\frac{1}{\bar{N}_n}\right] \le \boldsymbol{E}\left[\frac{1}{\bar{N}_1}\right]^n \quad \text{for any } n \in \boldsymbol{N}.$$

Since $\boldsymbol{E}[1/\bar{N}_1] < 1$ holds, Borel-Cantelli's lemma implies

$$\liminf_{n\to\infty} \frac{1}{n} \log \bar{N}_n \geq -\log\left(\boldsymbol{E}\left[\frac{1}{\bar{N}_1}\right]\right) \quad \boldsymbol{P}\text{-a.s.}$$

We complete the proof by letting n go infinity in the inequality

$$\frac{1}{t}\log \bar{N}_t \geq \frac{n}{n+1}\cdot\frac{1}{n}\log \bar{N}_n \quad \text{for any } t \in [n, n+1)\,.$$

□

As a consequence of Lemma 3.4, if the condition (3.5) holds, we have

$$\int_0^\infty \frac{1}{\bar{N}_t}\,\mathrm{d}t < \infty \quad \boldsymbol{P}\text{-a.s.} \tag{3.6}$$

4. Proofs of Theorem 2.2 and Corollary 2.3. We are now in a position to prove Theorem 2.2 and Corollary 2.3. In this section, we use the following notations: for functions f and g defined on a set $A \subset \boldsymbol{R}^d$, we write $f \asymp g$ on a set A if there exist two positive constants $c_1, c_2 > 0$ such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ holds for any $x \in A$. For functions f and g defined on $\boldsymbol{R}_+$, we write $f \sim g$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 1$ holds.

PROOF OF THEOREM 2.2. We first note that $\bar{M}_t$ is a purely discontinuous martingale because $\bar{M}_t$ is of finite variation on each finite time interval (see [9, p. 41, 4.14 Lemma (b)]). Therefore, if $[\bar{M}]_t$ denotes the quadratic variation of $\bar{M}_t$, then we have

$$[\bar{M}]_t = \bar{M}_0^2 + \sum_{\substack{0<s\leq t\\ \Delta\bar{M}_s\neq 0}} (\Delta\bar{M}_s)^2$$

for $\Delta\bar{M}_t := \bar{M}_t - \bar{M}_{t-}$, $t > 0$. Furthermore, by Ito's formula ([9, p. 57, Theorem 4.57]) applied to $-\log\bar{M}_t$, we get

$$-\log\bar{M}_t = -\int_0^t \frac{1}{\bar{M}_{s-}}\,\mathrm{d}\bar{M}_s - \sum_{\substack{0<s\leq t\\ \Delta\bar{M}_s\neq 0}} \left(\log\bar{M}_s - \log\bar{M}_{s-} - \frac{\Delta\bar{M}_s}{\bar{M}_{s-}}\right). \tag{4.1}$$

On the other hand, since Proposition 3.3 yields

$$\int_0^t \frac{1}{\bar{M}_s^2}\,\mathrm{d}\langle\bar{M}\rangle_s = \lambda^2\int_0^t R_s\,\mathrm{d}s - \left\{\lambda^2 - \mu\sum\nolimits_{n=1}^\infty (n-1)^2 p_n\right\}\int_0^t \frac{1}{\bar{N}_s}\,\mathrm{d}s,$$

we see

$$\int_0^\infty \frac{1}{\bar{M}_t^2}\,\mathrm{d}\langle\bar{M}\rangle_t < \infty \Leftrightarrow \int_0^\infty R_t\,\mathrm{d}t < \infty$$

from (3.6). Moreover, if $\int_0^\infty R_t\,\mathrm{d}t = \infty$ holds, then we have

$$\int_0^t \frac{1}{\bar{M}_s^2}\,\mathrm{d}\langle\bar{M}\rangle_s \sim \lambda^2\int_0^t R_s\,\mathrm{d}s \quad \text{as } t\to\infty. \tag{4.2}$$

(i) Since there exists a constant $C > 0$ such that $0 \le x - \log(1+x) \le Cx^2$ for any $x \ge 0$, we obtain

$$
\begin{aligned}
-\sum_{\substack{0<s\le t\\ \Delta\bar{M}_s\neq 0}} \left(\log \bar{M}_s - \log \bar{M}_{s-} - \frac{\Delta \bar{M}_s}{\bar{M}_{s-}}\right) &= \sum_{\substack{0<s\le t\\ \Delta\bar{M}_s\neq 0}} \left\{\frac{\Delta \bar{M}_s}{\bar{M}_{s-}} - \log\left(1 + \frac{\Delta \bar{M}_s}{\bar{M}_{s-}}\right)\right\} \\
&\le C \sum_{\substack{0<s\le t\\ \Delta\bar{M}_s\neq 0}} \frac{(\Delta \bar{M}_s)^2}{\bar{M}_{s-}^2} \\
&= \int_0^t \frac{1}{\bar{M}_{s-}^2}\, \mathrm{d}[\bar{M}]_s \quad \text{for any } t > 0\,,
\end{aligned}
$$

that is,

$$
-\log \bar{M}_t \le -\int_0^t \frac{1}{\bar{M}_{s-}}\, \mathrm{d}\bar{M}_s + C\int_0^t \frac{1}{\bar{M}_{s-}^2}\, \mathrm{d}[\bar{M}]_s \quad \text{for any } t > 0\,. \tag{4.3}
$$

Assume $\int_0^\infty R_t\, \mathrm{d}t < \infty$, that is, $\int_0^\infty (1/\bar{M}_t^2)\, \mathrm{d}\langle \bar{M}\rangle_t < \infty$. Since we know

$$
\left\{\int_0^\infty \frac{1}{\bar{M}_t^2}\, \mathrm{d}\langle \bar{M}\rangle_t < \infty\right\} \subset \left\{\int_0^\infty \frac{1}{\bar{M}_{t-}^2}\, \mathrm{d}[\bar{M}]_t < \infty\right\} \quad \boldsymbol{P}\text{-a.s.}
$$

from [7, p. 222, 8.30 Corollary] and

$$
\left\{\int_0^\infty \frac{1}{\bar{M}_t^2}\, \mathrm{d}\langle \bar{M}\rangle_t < \infty\right\} \subset \left\{\lim_{t\to\infty} \int_0^t \frac{1}{\bar{M}_{s-}}\, \mathrm{d}\bar{M}_s \text{ exists and is finite}\right\} \quad \boldsymbol{P}\text{-a.s.}
$$

from [7, p. 222, 8.32 Theorem], $-\log \bar{M}_\infty$ is finite by (4.3), that is, $\bar{M}_\infty$ is strictly positive.

(ii) Assume $\int_0^\infty R_t\, \mathrm{d}t = \infty$, that is, (4.2) holds. Then we get

$$
\lim_{t\to\infty} \int_0^t \frac{1}{\bar{M}_{s-}}\, \mathrm{d}\bar{M}_s \bigg/ \left(\int_0^t \frac{1}{\bar{M}_s^2}\, \mathrm{d}\langle \bar{M}\rangle_s\right) = 0 \quad \boldsymbol{P}\text{-a.s.}
$$

by [7, p. 247, 9.38 Corollary] and by the equality

$$
\int_0^t \frac{1}{\bar{M}_s^2}\, \mathrm{d}\langle \bar{M}\rangle_s = \left\langle \int_0^\cdot \frac{1}{\bar{M}_{s-}}\, \mathrm{d}\bar{M}_s \right\rangle_t .
$$

On the other hand, since (2.3) implies

$$
0 \le \Delta N_t \le (L-1)N_{t-} \quad \text{for all } t > 0 \quad \boldsymbol{P}\text{-a.s.}\,, \tag{4.4}
$$

we have

$$
-\log \bar{M}_t \asymp -\int_0^t \frac{1}{\bar{M}_{s-}}\, \mathrm{d}\bar{M}_s + \int_0^t \frac{1}{\bar{M}_{s-}^2}\, \mathrm{d}[\bar{M}]_s \quad \text{for any } t > 0 \tag{4.5}
$$

by (4.1) and the relation $x - \log(1+x) \asymp x^2$ on $[0, L-1]$. Furthermore, we obtain

$$
\int_0^t \frac{1}{\bar{M}_{s-}^2}\, \mathrm{d}[\bar{M}]_s \sim \int_0^t \frac{1}{\bar{M}_s^2}\, \mathrm{d}\langle \bar{M}\rangle_s \quad \text{as } t \to \infty \tag{4.6}
$$

by [7, p. 291, 10.7] because (4.4) yields the inequality

$$\boldsymbol{E}\left[\left(\frac{\Delta \bar{M}_T}{\bar{M}_{T-}}\right)^2 ; T < \infty\right] = \boldsymbol{E}\left[\left(\frac{\Delta \bar{N}_T}{\bar{N}_{T-}}\right)^2 ; T < \infty\right] \leq (L-1)^2 < \infty$$

for any stopping time T. Hence we get

$$\frac{-\log \bar{M}_t}{\int_0^t R_s \, \mathrm{d}s} \asymp -\int_0^t \frac{1}{\bar{M}_{s-}} \mathrm{d}\bar{M}_s \Big/ \left(\int_0^t \frac{1}{\bar{M}_s^2} \mathrm{d}\langle \bar{M}\rangle_s\right) + \int_0^t \frac{1}{\bar{M}_{s-}^2} \mathrm{d}[\bar{M}]_s \Big/ \left(\int_0^t \frac{1}{\bar{M}_s^2} \mathrm{d}\langle \bar{M}\rangle_s\right)$$
$$\to 1 \quad \text{as } t \to \infty,$$

by (4.2) and (4.5), which completes the proof. □

PROOF OF COROLLARY 2.3. Theorems 2.1 and 2.2 imply $\boldsymbol{P}(\int_0^\infty R_t \, \mathrm{d}t = \infty) = 1$ for any $\beta > \beta_1(d)$. Therefore, in a similar way to that in Theorem 2.2 (ii), we have

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t R_s \, \mathrm{d}s \geq -c_1 \limsup_{t\to\infty} \frac{\log \bar{M}_t}{t} > c_1 \cdot c(\beta) =: c'(\beta) \quad \boldsymbol{P}\text{-a.s.},$$

whence the inequality $\limsup_{t\to\infty} R_t \geq c'(\beta)$ follows. From the inequality $\bar{\rho}_t \geq R_t$ for any $t \geq 0$, we complete the proof. □

Acknowledgments. The author would like to thank Professor Nobuo Yoshida for his valuable comment and fruitful discussion. He is also grateful to Professor Mark Yor for his useful comment and suggestion. He is thankful to the referee and the editor for their careful reading of the manuscript and for their invaluable comment.

REFERENCES

[1] K. B. ATHREYA AND P. E. NEY, Branching processes, Die Grundlehren der mathematischen Wissenschaften, Band 196, Springer-Verlag, New York-Heidelberg, 1972.

[2] M. BIRKNER, J. GEIGER AND G. KERSTING, Branching processes in random environment—a view on critical and subcritical cases, Interacting stochastic systems, 269–291, Springer, Berlin, 2005.

[3] P. CARMONA AND Y. HU, On the partition function of a directed polymer in a Gaussian random environment, Probab. Theory Related Fields 124 (2002), 431–457.

[4] P. CARMONA AND Y. HU, Strong disorder implies strong localization for directed polymers in a random environment, ALEA Lat. Am. J. Probab. Math. Stat. 2 (2006), 217–229.

[5] F. COMETS, T. SHIGA AND N. YOSHIDA, Directed polymers in a random environment: path localization and strong disorder, Bernoulli 9 (2003), 705–723.

[6] F. COMETS AND N. YOSHIDA, Brownian directed polymers in random environment, Comm. Math. Phys. 254 (2005), 257–287.

[7] S. W. HE, J. G. WANG AND J. A. YAN, Semimartingale theory and stochastic calculus, Science Press, Beijing, 1992.

[8] Y. HU AND N. YOSHIDA, Localization for branching random walks in random environment, Stochastic Process. Appl. 119 (2009), 1632–1651.

[9] J. JACOD AND A. N. SHIRYAEV, Limit theorems for stochastic processes, 2nd ed., Grundlehren Math. Wiss., 288, Springer-Verlag, Berlin, 2003.

[10] N. KAPLAN, A continuous time Markov branching model with random environments, Advances in Appl. Probability 5 (1973), 37–54.

[11] M. NAKASHIMA, Almost sure central limit theorem for branching random walks in random environment, preprint.

[12] D. REVUZ AND M. YOR, Continuous martingales and Brownian motion, 3rd ed., Grundlehren Math. Wiss. 293, Springer-Verlag, Berlin, 1999.

[13] Y. SHIOZAWA, Central limit theorem for branching Brownian motions in random environment, J. Stat. Phys. 136 (2009), 145–163.

[14] W. SMITH AND W. WILKINSON, On branching processes in random environments, Ann. Math. Statist. 40 (1969), 814–827.

[15] N. YOSHIDA, Central limit theorem for branching random walks in random environment, Ann. Appl. Probab. 18 (2008), 1619–1635.

[16] N. YOSHIDA, Localization for linear stochastic evolutions, preprint.

[17] Y. NAGAHATA AND N. YOSHIDA, Localization for a class of linear systems, preprint.

GRADUATE SCHOOL OF
NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY
OKAYAMA 700–8530
JAPAN

E-mail address: shiozawa@ems.okayama-u.ac.jp