

LOCALIZATION FOR QUANTUM GROUPS AT A ROOT OF UNITY

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1. INTRODUCTION

Let \mathbb{C} be the field of complex numbers and fix $q \in \mathbb{C}^*$. Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} and let G be the corresponding simply connected algebraic group. Let U_q be the quantized enveloping algebra of \mathfrak{g} . Let \mathcal{O}_q be the algebra of quantized functions on G . Let $\mathcal{O}_q(B)$ be the quotient Hopf algebra of \mathcal{O}_q corresponding to a Borel subgroup B of G .

In the paper [BK] we defined categories of equivariant quantum \mathcal{O}_q -modules and \mathcal{D}_q -modules on the quantum flag variety of G . We proved that the Beilinson-Bernstein localization theorem holds at a generic q . Namely, the global section functor gives an equivalence between categories of U_q -modules and \mathcal{D}_q -modules on the quantum flag variety. Thus one can translate questions about the representation theory of quantum groups to the study of the (noncommutative) geometry of the quantum flag variety. In this paper we prove that a derived version of this theorem holds at the root of unity case. Using this equivalence, we get that one can understand the representation theory of quantum groups at roots of unity through the (now-commutative) geometry of the Springer fibers.

We now recall the main results in [BK]. The constructions given there are crucial for the present paper and a fairly detailed survey of the material there is given in the next section. We defined an equivariant sheaf of quasi-coherent modules over the quantum flag variety to be a left \mathcal{O}_q -module equipped with a right $\mathcal{O}_q(B)$ -comodule structure satisfying certain compatibility conditions. Such objects form a category denoted $\mathcal{M}_{B_q}(G_q)$. It contains certain line bundles $\mathcal{O}_q(\lambda)$ for λ in the weight lattice. We proved that $\mathcal{O}_q(\lambda)$ is ample for $\lambda \gg 0$ holds for every q . This implies that the category $\mathcal{M}_{B_q}(G_q)$ is a Proj-category in the sense of Serre.

We defined the category $\mathcal{D}_{B_q}^\lambda(G_q)$ of λ -twisted quantum \mathcal{D} -modules on the quantized flag variety (see Definition 3.11). An object $M \in \mathcal{D}_{B_q}^\lambda(G_q)$ is an object in $\mathcal{M}_{B_q}(G_q)$ equipped with an additional left U_q -action satisfying certain compatibility axioms (in particular, the $U_q(\mathfrak{b})$ -action on M , that is, the restriction of the U_q -action, and the $\mathcal{O}_q(B)$ -coaction “differ by λ ”).

Received by the editors November 1, 2006.

2000 *Mathematics Subject Classification.* Primary 14A22, 17B37, 58B32; Secondary 20G42.

Key words and phrases. Quantum groups, roots of unity, localization, derived equivalence, Calabi-Yau categories, noncommutative algebraic geometry.

The second author was supported in part by NSF grant DMS-0602007.

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The global section functor Γ on $\mathcal{M}_{B_q}(G_q)$ (and on $\mathcal{D}_{B_q}^\lambda(G_q)$) is given by taking $\mathcal{O}_q(B)$ -coinvariants. Let $M \in \mathcal{M}_{B_q}(G_q)$. Its sections over an open set (i.e., a localization \mathcal{O}_{qf} of \mathcal{O}_q) are given by $\Gamma(\mathcal{O}_{qf}, M) = \Gamma(\mathcal{O}_{qf} \otimes_{\mathcal{O}_q} M)$.

The category $\mathcal{D}_{B_q}^\lambda(G_q)$ has a distinguished object \mathcal{D}_q^λ . As a vector space, $\mathcal{D}_q^\lambda = \mathcal{O}_q \otimes M_\lambda$, where M_λ is a Verma module. Here \mathcal{D}_q^λ is the (quantum) equivariant counterpart of the usual sheaf of rings of λ -twisted differential operators on the flag variety of G . (The “set” \mathcal{D}_q^λ is not a ring, but its “sections over open sets” are naturally rings.) We proved that for each q except a finite set of roots of unity (depending on \mathfrak{g}), the global sections $\Gamma(\mathcal{D}_q^\lambda)$ is isomorphic to $U_q^\lambda := U_q^{fin}/J_\lambda$, where J_λ is the annihilator of M_λ and U_q^{fin} is the maximal subalgebra of U_q on which the adjoint action of U_q is locally finite.

The main result in [BK] stated that the global section functor gives an equivalence of categories between $\mathcal{D}_{B_q}^\lambda(G_q)$ and modules over U_q^λ in the case when q is not a root of unity and λ is regular dominant. This is a quantum version of Beilinson-Bernstein’s, [BB], localization theorem.

In the present paper we study the root of unity case; q is a primitive l ’th root of unity. The situation becomes different and very interesting: It turns out that \mathcal{D}_q^λ naturally forms a sheaf of algebras over the classical (nonquantum) complex variety $T^*X^\lambda := (G \times N^{\lambda^{2l}})/B$, where $N^{\lambda^{2l}}$ is the B -invariant subvariety $\{\lambda^{2l}\} \times N$ of B (see section 3.1). If λ is integral, $N^{\lambda^{2l}} = N$ and so $T^*X^\lambda = T^*X$ is the cotangent bundle of the flag variety X of G in this case. Hence, we refer to T^*X^λ as a twisted cotangent bundle.

A key observation is that \mathcal{D}_q^λ is an Azumaya algebra over a dense subset of T^*X^λ (Proposition 3.6) and that this Azumaya algebra splits over formal neighborhoods of generalized Springer fibers (Proposition 3.23). We then show that \mathcal{D}_q^λ has no higher self-extensions, i.e., that $R\Gamma^{>0}(\mathcal{D}_q^\lambda) = 0$ (Proposition 3.25). This fact, together with the result $\Gamma(\mathcal{D}_q^\lambda) = U_q^\lambda$ and the Azumaya property implies a derived version of the localization theorem: The functor Γ induces an equivalence between bounded derived categories $D^b(\mathcal{D}_{B_q}^\lambda(G_q))$ and $D^b(U_q^\lambda\text{-mod})$ (restricted to the dense subsets).

Using the Azumaya splitting, we prove that the subcategory of $\mathcal{D}_{B_q}^\lambda(G_q)$ whose objects are supported on (a formal neighbourhood of a generalized) Springer fiber is isomorphic to the category of \mathcal{O} -modules over the twisted cotangent bundles T^*X^λ supported on the same fiber (Corollary 3.24).

Combining these results, we get an equivalence between $D^b(U_q^\lambda\text{-mod})$ and the derived category of \mathcal{O} -modules on T^*X^λ whose cohomologies are supported on certain Springer fibers.

An application of our theory is for instance the computation of the number of simple \mathfrak{u}_q -modules, because this number can be interpreted as the rank of the K -group of the category of \mathcal{O} -modules on T^*X^λ supported on the trivial Springer fiber. Of course, such a formula follows also from the fact that this is a highest weight category. Our method gives the possibility to extend this to nontrivial central characters as well. In a future paper we will use this to prove a conjecture of De Concini, Kac and Procesi regarding dimensions of irreducible modules.

In [BMR] analogous results were established for a Lie algebra \mathfrak{g}_p in characteristic p . In fact most of our methods of proofs are borrowed from that paper. They showed analogous results for $U(\mathfrak{g}_p)$ -modules, (crystalline) \mathcal{D} -modules

on the flag variety over the finite field $X_{\mathbb{F}_p}$ and certain twisted cotangent bundles of X over $\overline{\mathbb{F}_p}$.

Combining their results with ours, we see that the representation theory of U_q (when q is a p 'th root of unity) is related to the representation theory of $U(\mathfrak{g}_p)$ via cotangent bundles of X in *char* 0 and *char* p , respectively. We know furthermore that baby Verma modules go to skyscraper sheaves in both cases and [BMR] showed that the K -group of the category of quasi-coherent sheaves on the cotangent bundle is torsion-free and its rank does not depend on the characteristic of the field. Right now we are investigating what can be deduced about the representation theory of \mathfrak{g}_p , e.g., character formulas, from the representation theory of U_q with these methods.

2. GENERALITIES

2.1. **Quantum groups.** See Chari and Pressley [CP] for details about the topics in this section.

2.1.1. *Conventions.* Let \mathbb{C} be the field of complex numbers and fix $q \in \mathbb{C}^*$.

We always assume that if q is a root of unity, it is primitive of odd order and in case G has a component of type G_2 , the order is also prime to 3. Let \mathcal{A} be the local ring $\mathbb{Z}[\nu]_{\mathfrak{m}}$, where \mathfrak{m} is the maximal ideal in $\mathbb{Z}[\nu]$ generated by $\nu - 1$ and a fixed odd prime p .

2.1.2. *Root data.* Let \mathfrak{g} be a semi-simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra contained in a Borel subalgebra of \mathfrak{g} . Let R be the root system, $\Delta \subset R_+ \subset R$ a basis and the positive roots. Let $P \subset \mathfrak{h}^*$ be the weight lattice and P_+ the positive weights; the i 'th fundamental weight is denoted by ω_i and ρ denotes the half sum of the positive roots. Let $Q \subset P$ be the root lattice and $Q_+ \subset Q$ those elements which have nonnegative coefficients with respect to the basis of simple roots. Let \mathcal{W} be the Weyl group of \mathfrak{g} . We let $\langle \cdot, \cdot \rangle$ denote a \mathcal{W} -invariant bilinear form on \mathfrak{h}^* normalized by $\langle \gamma, \gamma \rangle = 2$ for each short root γ .

Let $T_P = \text{Hom}_{groups}(P, \mathbb{C}^*)$ be the character group of P with values in \mathbb{C} (we use additive notation for this group). If $\mu \in P$, then $\langle \mu, P \rangle \subset \mathbb{Z}$ and hence we can define $q^\mu \in T_P$ by the formula $q^\mu(\gamma) = q^{\langle \mu, \gamma \rangle}$, for $\gamma \in P$. If $\mu \in P, \lambda \in T_P$, we write $\mu + \lambda = q^\mu + \lambda$. Note that the Weyl group naturally acts on T_P .

2.1.3. *Quantized enveloping algebra U_q and quantized algebra of functions \mathcal{O}_q .* Let U_q be the simply connected quantized enveloping algebra of \mathfrak{g} over \mathbb{C} . Recall that U_q has algebra generators $E_\alpha, F_\alpha, K_\mu$, α, β are simple roots, $\mu \in P$ subject to the relations

$$(2.1) \quad K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1,$$

$$(2.2) \quad K_\mu E_\alpha K_{-\mu} = q^{\langle \mu, \alpha \rangle} E_\alpha, \quad K_\mu F_\alpha K_{-\mu} = q^{-\langle \mu, \alpha \rangle} F_\alpha,$$

$$(2.3) \quad [E_\alpha, F_\beta] = \delta_{\alpha, \beta} \frac{K_\alpha - K_{-\alpha}}{q_\alpha - q_\alpha^{-1}}$$

and certain Serre relations that we do not recall here. Here

$$q_\alpha = q^{d_\alpha}, \quad d_\alpha = \langle \alpha, \alpha \rangle / 2.$$

(We have assumed that $q_\alpha^2 \neq 1$.)

Let G be the simply connected algebraic group with Lie algebra \mathfrak{g} , B a Borel subgroup of G and $N \subset B$ its unipotent radical. Let $\mathfrak{b} = \text{Lie } B$ and $\mathfrak{n} = \text{Lie } N$ and denote by $U_q(\mathfrak{b})$ and $U_q(\mathfrak{n})$ the corresponding subalgebras of U_q . Then $U_q(\mathfrak{b})$ is a Hopf algebra, while $U_q(\mathfrak{n})$ is only an algebra. Let $\mathcal{O}_q = \mathcal{O}_q(G)$ be the algebra of matrix coefficients of finite dimensional type-1 representations of U_q . There is a natural pairing $(\ , \) : U_q \otimes \mathcal{O}_q \rightarrow \mathbb{C}$. This gives a U_q -bimodule structure on \mathcal{O}_q as follows:

$$(2.4) \quad ua = a_1(u, a_2), \quad au = (u, a_1)a_2, \quad u \in U_q, \quad a \in \mathcal{O}_q.$$

Then \mathcal{O}_q is the (restricted) dual of U_q with respect to this pairing. We let $\mathcal{O}_q(B)$ and $\mathcal{O}_q(N)$ be the quotient algebras of \mathcal{O}_q corresponding to the subalgebras $U_q(\mathfrak{b})$ and $U_q(\mathfrak{n})$ of U_q , respectively, by means of this duality. Then $\mathcal{O}_q(B)$ is a Hopf algebra and $\mathcal{O}_q(N)$ is only an algebra.

There is a braid group action on U_q . For each $w \in \mathcal{W}$, we get an automorphism T_w of U_q .

2.1.4. *Integral versions of U_q .* Let $U_{\mathcal{A}}^{res}$ be the Lusztig integral form of U_q , the \mathcal{A} -algebra in U_q generated by divided powers $E_{\alpha}^{(n)} = E_{\alpha}^n/[n]_{d_{\alpha}}!$, $F_{\alpha}^{(n)} = F_{\alpha}^n/[n]_{d_{\alpha}}!$, α a simple root, $n \geq 1$ (where $[m]_d = \frac{\prod_{s=1}^m (q^{d \cdot s} - q^{-d \cdot s})}{q^d - q^{-d}}$) and the K_{μ} 's, $\mu \in P$. There is also the De Concini-Kac integral form $U_{\mathcal{A}}$, which is generated over \mathcal{A} by the E_{α}, F_{α} and K_{μ} 's. The subalgebra $U_{\mathcal{A}}$ is preserved by the adjoint action of $U_{\mathcal{A}}^{res}$: $\text{ad}_{U_{\mathcal{A}}^{res}}(U_{\mathcal{A}}) \subset U_{\mathcal{A}}$. The operators T_w from section 2.1.2 preserve the integral versions.

$\mathcal{O}_{\mathcal{A}}$ is defined to be the dual of $U_{\mathcal{A}}^{res}$. This is an \mathcal{A} -sub-Hopf algebra of \mathcal{O}_q .

2.1.5. *Finite part of U_q .* The algebra U_q acts on itself by the adjoint action $\text{ad} : U_q \rightarrow U_q$ where $\text{ad}(u)(v) = u_1 v S(u_2)$. Let U_q^{fin} be the finite part of U_q with respect to this action:

$$U_q^{fin} = \{v \in U_q; \dim \text{ad}(U_q)(v) < \infty\}.$$

This is a subalgebra. (See [JL].)

We can also give an integral version of the finite part as the finite part of the action of $U_{\mathcal{A}}^{res}$ on $U_{\mathcal{A}}$. Thus by specializing we get a subalgebra of U_q for every q . Of course, when specialized to generic q , this coincides with the previous definition.

2.1.6. *Specializations and Frobenius maps.* For any ring map $\phi : \mathcal{A} \rightarrow R$ we put $U_R = U_{\mathcal{A}} \otimes_{\mathcal{A}} R$ and $U_R^{res} = U_{\mathcal{A}}^{res} \otimes_{\mathcal{A}} R$. If $R = \mathbb{C}$ and $\phi(\nu) = q$, there are three different cases: q is a root of unity, $q = 1$ and q is generic. Then $U_R = U_q$.

There is also a ring map $\mathcal{A} \rightarrow \mathbb{F}_p$, sending $\nu \rightarrow 1$. Then $U_{\mathbb{F}_p}/(K - 1) = U(\mathfrak{g}_p)$, the enveloping algebra of the Lie algebra \mathfrak{g}_p in characteristic p .

For any $U_{\mathcal{A}}$ -module (resp. $U_{\mathcal{A}}^{res}$ -module) $M_{\mathcal{A}}$ we put $M_R = M_{\mathcal{A}} \otimes_{\mathcal{A}} R$. This is an U_R -module (resp. U_R^{res} -module). When $R = \mathbb{C}$, we simply write $M = M_{\mathbb{C}}$.

When q is a root of unity, we have the Frobenius map: $U_q^{res} \rightarrow U(\mathfrak{g})$. Its algebra kernel is denoted by u_q . We also have the Frobenius map $U_q^{res}(\mathfrak{b}) \rightarrow U(\mathfrak{b})$, with algebra kernel b_q . These maps induce dual maps $\mathcal{O} = \mathcal{O}(G) \hookrightarrow \mathcal{O}_q$ and $\mathcal{O}(B) \hookrightarrow \mathcal{O}_q(B)$. It is important to note that the Frobenius map $U_q^{res}(\mathfrak{b}) \rightarrow U(\mathfrak{b})$ has a splitting. From this splitting we can deduce that the functor of b_q -invariants on the category of $U_q^{res}(\mathfrak{b})$ is exact as follows: the splitting implies that $U_q^{res}(\mathfrak{b})$ is a semi-direct product of b_q and $U(\mathfrak{b})$. Hence b_q is exact in $U_q^{res}(\mathfrak{b})$ (the induction

functor is exact). Note that we are working with algebraic induction so we should write everything in terms of the dual Hopf algebras. So we get

$$(2.5) \quad \begin{aligned} H^*(b_q, Res_{b_q}^{B_q}(M)) &= H^*(B_q, Ind_{b_q}^{B_q} Res_{b_q}^{B_q}(M)) \\ &= H^*(B_q, \mathcal{O}(B) \otimes M^{triv}) = H^*(B_q, Ind_{b_q}^{B_q}(\mathbb{C})) \otimes M = H^*(b_q, \mathbb{C}) \otimes M. \end{aligned}$$

Here $H^*(B_q, ?)$ is cohomology w.r.t. the big quantum Borel. Note that the cohomology of b_q is zero in odd degrees. Hence when we apply the functor of invariants to an exact sequence of integrable $U_q^{res}(\mathfrak{b})$ -modules, we get an exact sequence. Note that we have proven a stronger statement, we only need to assume that the first object in the sequence is a B_q -module.

For each q there exists a map $U_q \rightarrow U_q^{res}$ whose image is \mathfrak{u}_q and whose algebra kernel is $\mathcal{Z}^{(l)}$ (see section 2.1.7 below for the definition of $\mathcal{Z}^{(l)}$).

2.1.7. *Verma modules.* For each $\lambda \in T_P$ there is the one dimensional $U_q(\mathfrak{b})$ -module \mathbb{C}_λ which is given by extending λ to act by zero on the E_α 's. The Verma module M_λ is the U_q -module induced from \mathbb{C}_λ . If $\mu \in P$, we write $M_\mu = M_{q^\mu}$. An important point for us is that M_λ carries an $U_q^{res}(\mathfrak{b})$ -module structure defined as follows: $U_q^{res}(\mathfrak{b})$ acts on U_q by restricting the adjoint action of U_q^{res} on U_q . This induces a $U_q(\mathfrak{b})$ -action on the quotient M_λ of U_q . Since this action is locally finite, it corresponds to an $\mathcal{O}_q(B)$ -comodule action on M_λ . **Note!!** As a $U_q^{res}(\mathfrak{b})$ -module M_λ has *trivial highest weight*. (In case q is generic, $U_q^{res}(\mathfrak{b}) = U_q(\mathfrak{b})$ and then the $U_q^{res}(\mathfrak{b})$ -action on M_λ described above is the same as the $U(\mathfrak{b})$ -action on $M_\lambda \otimes \mathbb{C}_{-\lambda}$.)

The Verma module M_λ has an integral version $M_{\lambda, \mathcal{A}}$.

2.1.8. *Centers of U_q and definition of \tilde{U}_q .* Let \mathcal{Z} denote the center of U_q . When q is a p 'th root of unity, \mathcal{Z} contains the Harish-Chandra center \mathcal{Z}^{HC} and the l 'th center $\mathcal{Z}^{(l)}$ which is generated by the $E_\alpha^l, F_\alpha^l, K_\mu^l$ and K_μ^{-l} 's. In fact, $\mathcal{Z} = \mathcal{Z}^{(l)} \otimes_{\mathcal{Z}^{(l)} \cap \mathcal{Z}^{HC}} \mathcal{Z}^{HC}$. There is the Harish-Chandra homomorphism $\mathcal{Z}^{HC} \rightarrow \mathcal{O}(T_P)$ that maps isomorphically to the W -invariant even part of $\mathcal{O}(T_P)$. We define $\tilde{U}_q = U_q \otimes_{\mathcal{Z}^{HC}} \mathcal{O}(T_P)$.

2.1.9. *Some conventions.* We shall frequently refer to a right (resp. left) \mathcal{O}_q -comodule as a left (resp. right) G_q -module, etc. If we have two right \mathcal{O}_q -comodules V and W , then $V \otimes W$ carries the structure of a right \mathcal{O}_q -comodule via the formula

$$\delta(v \otimes w) = v_1 \otimes w_1 \otimes v_2 w_2.$$

We shall refer to this action as the *tensor* or *diagonal* action. A similar formula exist for left comodules.

2.2. **Quantum flag variety.** Here we recall the definition and basic properties of the quantum flag variety from [BK].

2.2.1. *Category $\mathcal{M}_{B_q}(G_q)$.* The composition

$$(2.6) \quad \mathcal{O}_q \rightarrow \mathcal{O}_q \otimes \mathcal{O}_q \rightarrow \mathcal{O}_q \otimes \mathcal{O}_q(B)$$

defines a right $\mathcal{O}_q(B)$ -comodule structure on \mathcal{O}_q . A B_q -equivariant sheaf on G_q is a triple (F, α, β) where F is a vector space, $\alpha : \mathcal{O}_q \otimes F \rightarrow F$ a left \mathcal{O}_q -module action and $\beta : F \rightarrow F \otimes \mathcal{O}_q(B)$ a right $\mathcal{O}_q(B)$ -comodule action such that α is a right comodule map, where we consider the tensor comodule structure on $\mathcal{O}_q(G) \otimes F$.

Definition 2.1. We denote $\mathcal{M}_{B_q}(G_q)$ to be the category of B_q -equivariant sheaves on G_q . Morphisms in $\mathcal{M}_{B_q}(G_q)$ are those compatible with all structures.

If $q = 1$, the category $\mathcal{M}_B(G)$ is equivalent to the category $\mathcal{M}(G/B)$ of quasi-coherent sheaves on G/B .

Definition 2.2. We define the induction functor $\text{Ind} : B_q\text{-mod} \rightarrow \mathcal{M}_{B_q}(G_q)$, $\text{Ind } V = \mathcal{O}_q \otimes V$ with the tensor B_q -action and the \mathcal{O}_q -action on the first factor. For $\lambda \in P$ we define a line bundle $\mathcal{O}_q(\lambda) = \text{Ind } \mathbb{C}_{-\lambda}$.

Definition 2.3. The global section functor $\Gamma : \mathcal{M}_{B_q}(G_q) \rightarrow \mathbb{C}\text{-mod}$ is defined by

$$\Gamma(M) = \text{Hom}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, M) = \{m \in M; \Delta_B(m) = m \otimes 1\}.$$

This is the set of B_q -invariants in M .

The category $\mathcal{M}_{B_q}(G_q)$ has enough injectives, so derived functors $R\Gamma$ are well-defined. We showed that $R^i\Gamma(\text{Ind } V) = H^i(G_q/B_q, V)$, where $H^i(G_q/B_q, \)$ is the i 'th derived functor of the functor $V \rightarrow \Gamma(\text{Ind } V)$ from $B_q\text{-mod}$ to $\mathbb{C}\text{-mod}$.

We proved a quantum version of Serre's basic theorem on projective schemes: Each $M \in \mathcal{M}_{B_q}(G_q)$ is a quotient of a direct sum of $\mathcal{O}_q(\lambda)$'s and each surjection $M \rightarrow M'$ of noetherian objects in $\mathcal{M}_{B_q}(G_q)$ induces a surjection $\Gamma(M(\lambda)) \rightarrow \Gamma(M'(\lambda))$ for $\lambda \gg 0$.

Here the notation $\lambda \gg 0$ means that $\langle \lambda, \alpha^\wedge \rangle$ is a sufficiently large integer for each simple root α and $M(\lambda) = M \otimes \mathbb{C}_{-\lambda}$ is the λ -twist of M .

Let $V \in G_q\text{-mod}$. Denote by $V|_{B_q}$ the restriction of V to B_q and by V^{triv} the trivial B_q -module whose underlying space is V . We showed that $\text{Ind } V|_{B_q}$ and $\text{Ind } V^{triv}$ are isomorphic in $\mathcal{M}_{B_q}(G_q)$. In particular

$$(2.7) \quad \Gamma(\text{Ind } V|_{B_q}) = V|_{B_q} \otimes \Gamma(\mathcal{O}_q) = V|_{B_q}, \quad \text{for } V \in G_q\text{-mod}.$$

2.2.2. $\mathcal{M}_{B_q}(G_q)$ at a root of unity. In case q is a root of unity, we have the following Frobenius morphism:

$$(2.8) \quad Fr_* : \mathcal{M}_{B_q}(G_q) \rightarrow \mathcal{M}(G/B),$$

defined as

$$(2.9) \quad N \mapsto N^{b_q}.$$

Using the description of $\mathcal{M}_{B_q}(G_q)$ as $Proj(A_q)$ where $A_q = \bigoplus V_{q,\lambda}$ [BK] and similarly $\mathcal{M}(G/B) = Proj(A)$ where $A = \bigoplus V_\lambda$, we see that Fr_* is induced from the quantum Frobenius map $A \hookrightarrow A_q$. It follows that

Proposition 2.4. *Fr_* is exact and faithful.*

This functor has a left adjoint.

3. \mathcal{D}_q^λ -MODULES AT A ROOT OF UNITY

3.1. **First construction.** From now on q is an l 'th root of unity (recall the restrictions of 2.1.1).

In this section we shall give a representation theoretic construction of the sheaf of quantum differential operators. This will turn out to be a sheaf of algebras over the Springer resolution—the sheaf of endomorphisms of the (nonexistent) universal baby Verma module.

Recall the Frobenius map $O \hookrightarrow O_q$ and the fact that $O_q^{u_q} = O$. This allows us to define the functor of (finite) induction

$$(3.1) \quad \text{Ind} : \mathfrak{u}_q\text{-mod} \rightarrow \mathcal{M}_{G_q}(G),$$

$$(3.2) \quad N \mapsto (O_q \otimes N)^{\mathfrak{u}_q}.$$

Here $\mathcal{M}_{G_q}(G)$ is the category of G_q -equivariant O -modules, that is, an O -module which is also an O_q -comodule and such that the O -module structure map is a map of O_q -comodules.

We have the following important proposition [AG]:

Proposition 3.1. *Ind : $\mathfrak{u}_q\text{-mod} \rightarrow \mathcal{M}_{G_q}(G)$ is an equivalence of categories.*

Notice that both categories are tensor categories (in $\mathcal{M}_{G_q}(G)$ it is tensoring over O) and that Ind is a tensor functor. This will be used later. Note also that $\mathcal{M}_{G_q}(G)$ has an obvious action by G , that is, for any $g \in G$ we have a functor $F_g : \mathcal{M}_{G_q}(G) \rightarrow \mathcal{M}_{G_q}(G)$ and natural transformations $\alpha_{g,h} : F_g \circ F_h \Rightarrow F_{gh}$ satisfying a certain cocycle condition. Moreover starting from any \mathfrak{u}_q -module, we can form a family of such modules indexed by G , more precisely an $O(G)$ -module in the category of \mathfrak{u}_q -modules. We will use this to construct the universal family of endomorphisms of baby Verma modules.

Remark 3.2. The article [CKP] defined an action of an infinite dimensional group \mathfrak{G} on U_q preserving the l -center (and the augmentation ideal of the l -center) and thus acting also on \mathfrak{u}_q . This is defined by observing that the derivation defined by commuting with the divided powers $E^{(l)}, F^{(l)}$ actually preserves the algebra generated by the nondivided powers. These derivations are then exponentiated to get automorphisms of U_q at a root of unity. The group they generate is infinite dimensional as the action is not locally finite. This action also induces an action on \mathfrak{u}_q , but here the group is finite dimensional \mathfrak{G}_0 . We thus have another group action on the category. Note that the action of G on the category gives us an infinitesimal action of the Lie algebra \mathfrak{g} on the category, which as weak action is the same as a map of \mathfrak{g} to outer derivations. But for \mathfrak{G}_0 the Lie algebra action is given by the derivation defined by commuting with a divided power and for G it is given by the adjoint action and both have the same image inside $\text{Ext}^1(\mathfrak{u}_q, \mathfrak{u}_q)$. Both actions are also faithful, so as weak actions they are the same.

Recall that $M_\lambda = U_q/J_\lambda$; put

$$(3.3) \quad I_\lambda = \mathcal{Z}^{(l)} \cap J_\lambda,$$

$$(3.4) \quad \mathcal{Z}_{-, \lambda}^{(l)} = \mathcal{Z}^{(l)} / I_\lambda,$$

so that $\mathcal{Z}_{-, \lambda}^{(l)} \subset M_\lambda$.

Then $\mathcal{Z}_{-, \lambda}^{(l)}$ is a $U_q^{res}(\mathfrak{b})$ -module algebra and M_λ is a $U_q^{res}(\mathfrak{b})$ -module for this algebra (see section 2.1.7).

In fact, $\mathcal{O} \otimes \mathcal{Z}_{-, \lambda}^{(l)}$ is a B -equivariant algebra: Recall the Frobenius map $Fr : U_q^{res}(\mathfrak{b}) \rightarrow U(\mathfrak{b})$ and denote the algebra kernel of Fr by b_q .

Definition 3.3. *i)* Consider $\mathcal{O}(B)$ as a B -module under the adjoint action. For each $t \in T$ we have the B -subvariety $N^t = \{t\} \times N$ of B . $\mathcal{O}(N^t)$ is isomorphic to $\mathcal{O}(N)$ as an algebra, but not as a B -module, unless $t = 1$.

ii) Think of T as characters on $2l \cdot P$ and of T_P as characters on P . Lattice inclusion induces a natural map $()^{2l} : T_P \rightarrow T$.

Note that for λ integral, $\lambda^{2l} = 1$.

We now have

Lemma 3.4. *b_q acts trivially on $\mathcal{Z}_{-\lambda}^{(l)}$; this module is a pullback by the Frobenius of the B -module $\mathcal{O}(N^{\lambda^{2l}})$.*

Proof of Lemma 3.4. This will follow from Lemma 3.18. □

b_q also acts trivially on \mathcal{O} , so $\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)}$ is a B -module. Hence, we have the category $\mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)})$ of B -equivariant $\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)}$ -modules; from Lemma 3.4 we conclude that

$$(3.5) \quad \mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)}) \cong qcoh(\mathcal{O}_{(G \times N^{\lambda^{2l}})/B}).$$

Here the B -action on $G \times N^\lambda$ is given by $b \cdot (g, x) = (bg, b \cdot x)$. Similarly we can define the category ${}_{G_q}\mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)})$ of G_q -equivariant objects in $\mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)})$.

We shall denote

$$(3.6) \quad G \times N^{\lambda^{2l}}/B = T^*X^\lambda.$$

Note that for integral λ this is the cotangent bundle to the flag variety, also known as the Springer resolution. For nonintegral λ this is a twisted cotangent bundle.

For any $\tau \in \text{maxspec}(\mathcal{Z}_{-\lambda}^{(l)})$ we have the central reduction $M_{\lambda,\tau}$ (a baby Verma module). Only for trivial τ (corresponding to the augmentation ideal) we get a \mathfrak{u}_q -module. But for any τ , $\text{End}(M_{\lambda,\tau}) = M_{\lambda,\tau} \otimes M_{\lambda,\tau}^*$ is a \mathfrak{u}_q -module since its l -central character is trivial, and likewise $\text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_\lambda)$ is a \mathfrak{u}_q -module. Hence we can define

Definition 3.5. $D = (\text{Ind}(\text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_{\lambda+2\rho})))$.

(The shift by 2ρ will become clear later.) Since $\text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_\lambda)$ is a $\mathcal{Z}_{-\lambda}^{(l)}$ -module and also a B_q -module ($O_q(B)$ -comodule) in a compatible way we get that D actually lives in ${}_{G_q}\mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)})$. Since Ind is a tensor functor and $\text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_\lambda)$ is a \mathfrak{u}_q algebra, we get that D is a sheaf of algebras over the (twisted) cotangent bundle.

By construction we know that the algebras sitting over the fiber over B are matrix algebras (endomorphisms of baby Verma modules); hence we get

Proposition 3.6. *Over a dense subset of T^*X^λ , D is an Azumaya algebra.*

Note that by the construction of the sheaf of algebras D , its fibers over the zero section of T^*X^λ are endomorphisms of baby Verma modules. Hence we see that over the zero section D is the endomorphism sheaf of the vector bundle whose fibers are baby Verma modules. This implies that the Azumaya locus contains the zero section and in fact the algebra is split over the zero section. It will also follow from Proposition 3.23 that the Azumaya locus contains the preimage of the big cell under the Springer map.

Remark 3.7. Note that all our constructions can also be defined over a formal neighbourhood of a prime p that is over a p -adic field and that when specialized to \mathbf{F}_p , they give the usual characteristic p crystalline differential operators which are

Azumaya and thus we would get that over the p -adic field our algebra is Azumaya as well. This will not be used in this paper since we will look at complex representations, but in a subsequent paper we will use this to construct t -structures in zero characteristic relating to the ones constructed by Bezrukavnikov, Mirkovic and Rumynin [BMR].

Remark 3.8. For any rigid braided tensor category one can define the notion of an Azumaya algebra. In the category of \mathfrak{u}_q -modules, $\text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_\lambda)$ is an Azumaya algebra. Hence, using the equivalence Proposition 3.1, we get that D is an Azumaya algebra over T^*X^λ , not with respect to the usual braiding (flip), but with respect to the braiding induced from U_q^{res} .

3.2. Second construction—the ring \mathcal{D}_{G_q} and the category of \mathcal{D}^λ -modules.

We need the following important

Remark 3.9. All objects described in the preceding sections are defined over \mathcal{A} . For any specialization $\mathcal{A} \rightarrow R$ and any object Obj we denote by Obj_R its R -form. For the functors, we do not use any subscripts; so, for instance, there is the functor $\text{Ind} : B_{q,R} - \text{mod} \rightarrow \mathcal{M}_{B_q}(G_q)_R$.

Recall the U_q -bimodule structure on \mathcal{O}_q given by (2.4). Now, as we have two versions of the quantum group, we pick the following definition of the ring of differential operators on the group (the *crystalline* version).

Definition 3.10. We define the ring of quantum differential operators on G_q to be the smash product algebra $\mathcal{D}_{G_q} := \mathcal{O}_q \star U_q$. So $\mathcal{D}_{G_q} = \mathcal{O}_q \otimes U_q$ as a vector space and multiplication is given by

$$(3.7) \quad a \otimes u \cdot b \otimes v = au_1(b) \otimes u_2v.$$

We consider now the ring \mathcal{D}_{G_q} as a left U_q^{res} -module, via the left U_q^{res} -action on \mathcal{O}_q in (2.4) and the left adjoint action of U_q^{res} on U_q ; this way \mathcal{D}_{G_q} becomes a module algebra for U_q^{res} : In the following we will use the restriction of this action to $U_q^{res}(\mathfrak{b}) \subset U_q^{res}$. As U_q is not locally finite with respect to the adjoint action, this $U_q^{res}(\mathfrak{b})$ -action does not integrate to a B_q -action. Thus \mathcal{D}_{G_q} is not an object of $\mathcal{M}_{B_q}(G_q)$; however, \mathcal{D}_{G_q} has a subalgebra $\mathcal{D}_{G_q}^{fin} = \mathcal{O}_q \star U_q^{fin}$ which belongs to $\mathcal{M}_{B_q}(G_q)$. This fact will be used below.

Definition 3.11. Let $\lambda \in T_P$. A (B_q, λ) -equivariant \mathcal{D}_{G_q} -module is a triple (M, α, β) , where M is a \mathbb{C} -module, $\alpha : \mathcal{D}_{G_q} \otimes M \rightarrow M$ a left \mathcal{D}_{G_q} -action and $\beta^{res} : M \rightarrow M \otimes \mathcal{O}_q(B)$ a right $\mathcal{O}_q(B)$ -coaction. The latter action induces an $U_q^{res}(\mathfrak{b})$ -action on M , again denoted by β^{res} . We have the natural map $U_q(\mathfrak{b}) \rightarrow U_q^{res}(\mathfrak{b})$ which together with β^{res} gives an action β of $U_q(\mathfrak{b})$ on M . We require

- i) The $U_q(\mathfrak{b})$ -actions on $M \otimes \mathbb{C}_\lambda$ given by $\beta \otimes \lambda$ and by $(\alpha|_{U_q(\mathfrak{b})}) \otimes \text{Id}$ coincide.
- ii) The map α is $U_q^{res}(\mathfrak{b})$ -linear with respect to the β -action on M and the action on \mathcal{D}_{G_q} .

These objects form a category denoted by $\mathcal{D}_{B_q}^\lambda(G_q)$. There is the forgetful functor $\mathcal{D}_{B_q}^\lambda(G_q) \rightarrow \mathcal{M}_{B_q}(G_q)$. Morphisms in $\mathcal{D}_{B_q}^\lambda(G_q)$ are morphisms in $\mathcal{M}_{B_q}(G_q)$ that are \mathcal{D}_{G_q} -linear.

We defined ([BK]) \mathcal{D}_q^λ as the maximal quotient of \mathcal{D}_{G_q} which is an object of $\mathcal{D}_{B_q}^\lambda(G_q)$ and showed that

$$(3.8) \quad \mathcal{D}_q^\lambda = \text{Ind } M_\lambda$$

as an object in $\mathcal{D}_{B_q}^\lambda(G_q)$. (See section 2.1.7 for the B_q -action = $U_q^{res}(\mathfrak{b})$ -action on M_λ .) The global section functor $\Gamma : \mathcal{D}_{B_q}^\lambda(G_q) \rightarrow \mathcal{M}$ is the functor of taking B_q -invariants (with respect to the action β); we have $\Gamma = \text{Hom}_{\mathcal{D}_{B_q}^\lambda(G_q)}(\mathcal{D}_q^\lambda, \cdot)$.

Hence, in particular $\Gamma(\mathcal{D}_q^\lambda) = \text{End}_{\mathcal{D}_{B_q}^\lambda(G_q)}(\mathcal{D}_q^\lambda)$ (which explains the ring structure on $\Gamma(\mathcal{D}_q^\lambda)$).

If we view \mathcal{D}_q^λ as an object in ${}_{G_q}\mathcal{M}_{B_q}(G_q)$, we have that $Fr_*(\mathcal{D}_q^\lambda)$ is an object of ${}_{G_q}\mathcal{M}(G/B)$. But it actually lies in ${}_{G_q}\mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)})$ since M_λ is a $\mathcal{Z}_{-\lambda}^{(l)}$ -module. In the next section we shall show that $\mathcal{D}_q^\lambda = D$ and that the category of modules over this sheaf of algebras is $\mathcal{D}_{B_q}^\lambda(G_q)$.

3.3. \mathcal{D}^λ as a sheaf of algebras. We have a natural functor:

$$(3.9) \quad F : \mathcal{D}_{B_q}^\lambda(G_q) \rightarrow \mathcal{M}_B(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)}),$$

$$(3.10) \quad F(N) = N^{\mathfrak{b}_q}.$$

Since taking \mathfrak{b}_q -invariants is exact on B_q -modules ($O_q(B)$ -comodules) (see (2.5)) and is faithful on $\mathcal{M}_{B_q}(G_q)$ (see (2.4)), we get

Proposition 3.12. *This functor is exact and faithful.*

The functor F has a left adjoint. Let us denote it G .

Proposition 3.13. $F(G(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)})) = (\mathcal{D}_q^\lambda)^{\mathfrak{b}_q}$ is a sheaf of algebras over T^*X^λ . The category of modules over it is equivalent to $\mathcal{D}_{B_q}^\lambda(G_q)$.

Proof of Proposition 3.13. This theorem follows from a Barr-Beck type argument. F is an exact and faithful functor with left adjoint G . $F \circ G$ is a monad in $qCoh(T^*X^\lambda)$ and modules over this monad are equivalent to the category $\mathcal{D}_{B_q}^\lambda(G_q)$. Note also that $F \circ G$ is a right exact functor from the category $qCoh(T^*X^\lambda)$ to itself. Note that any right exact functor L from $qCoh(X)$ to $qCoh(Y)$ where X and Y are varieties is given by tensoring with the bimodule $L(O_X)$. In our case this is $F(G(\mathcal{O} \otimes \mathcal{Z}_{-\lambda}^{(l)}))$. The monad structure induces the algebra structure on this sheaf. □

We want to prove the following:

Proposition 3.14. $(\mathcal{D}_q^\lambda)^{\mathfrak{b}_q} = D$ as sheaves of algebras.

Proof of Proposition 3.14. We have that

$$(3.11) \quad (\mathcal{D}_q^\lambda)^{\mathfrak{b}_q} = (O_q \otimes M_\lambda)^{\mathfrak{b}_q} = \text{Ind}_{\mathfrak{b}_q}^{G_q}(M_\lambda) = \text{Ind}_{\mathfrak{u}_q}^{G_q} \circ \text{Ind}_{\mathfrak{b}_q}^{\mathfrak{u}_q}(M_\lambda).$$

Hence it is enough to prove that

$$(3.12) \quad \text{Ind}_{\mathfrak{b}_q}^{\mathfrak{u}_q}(M_\lambda) = \text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_{\lambda+2\rho})$$

and that this map is compatible with all relevant structures. Here we use two conventions about Verma modules, where M_λ gets its action from the adjoint action of the restricted quantum group on the nonrestricted and in $\text{End}_{\mathcal{Z}_{-\lambda}^{(l)}}(M_{\lambda+2\rho})$ we think of the Verma module as induced from the Borel. It is enough to prove that

$$(3.13) \quad \text{Ind}_{\mathfrak{b}_q}^{\mathfrak{u}_q}(M_{\lambda,\chi}) = \text{End}(M_{\lambda+2\rho,\chi}).$$

First note that if we denote by $o_q(G)$ the functions on the quantum Frobenius kernel (the fiber over the identity of sheaf of algebras $O_q(G)$ over $O(G)$), we have that $o_q(G) = o_q(N_-) \otimes o_q(B)$ and so $\text{Ind}_{\mathfrak{b}_q}^{u_q}(M_{\lambda,\chi}) = o_q(N_-) \otimes M_{\lambda,\chi}$ as vector spaces (actually as N_q -modules). Now by definition $M_{\lambda,\chi}$ is an algebra (it is a quotient of $U_q(\mathfrak{b})$) and using duality, one can define an action of $o_q(N_-)$ on $M_{\lambda,\chi}$ as in [J] (which gives explicit formulas on page 91 and proposition on page 92), getting the desired isomorphism by specializing the map from [J] (lemma on page 93) to the χ central character. Notice that [J] works with generic q but that all the formulas and proofs make sense and are compatible with the action of the center for q a root of unity if we work with the De Concini-Kac integral form. \square

3.3.1. *Category $\mathcal{D}_{B_q}^\lambda(G_q)$ as a gluing of module categories over quantum Weyl algebras.* In this section we will give another way of viewing \mathcal{D}_q^λ as a sheaf of algebras over the cotangent bundle.

For $w \in \mathcal{W}$, De Concini and Lyubashenko [DL] introduced localizations $\mathcal{O}_{q,w}$ of \mathcal{O}_q . These induce localizations \mathcal{O}_w of \mathcal{O} corresponding to covering G by translates of the big cell B_-B . Joseph has introduced localizations of the representation ring $\mathcal{O}_q^{N_q}$ ([J]), and it is easy to see that they are exactly $\mathcal{O}_{q,w}^{N_q}$. We thus have a covering of the category of \mathcal{O}_q -modules by the categories $\mathcal{O}_{q,w}$. By this we mean that we have a pair of adjoint functors

$$(3.14) \quad f^* : \mathcal{O}_q\text{-mod} \rightarrow \prod \mathcal{O}_{q,w}\text{-mod} : f_*$$

such that f^* is exact and faithful, thus inducing an equivalence between \mathcal{O}_q -modules and modules over the monad $f^* \circ f_*$ (Barr-Beck). We will denote such an equivalence by

$$(3.15) \quad \mathcal{O}_q\text{-mod} = \lim_{\leftarrow w \in \mathcal{W}} \mathcal{O}_{q,w}\text{-mod}.$$

We have corresponding localizations $\mathcal{M}_{B_q}(G_q)_w$ (i.e., B_q -equivariant $\mathcal{O}_{q,w}$ -modules), of the category $\mathcal{M}_{B_q}(G_q)$. Then

$$(3.16) \quad \mathcal{M}_{B_q}(G_q) = \lim_{\leftarrow w \in \mathcal{W}} \mathcal{M}_{B_q}(G_q)_w.$$

Using the description of $\mathcal{M}_{B_q}(G_q)$ as a Proj-category from [BK], it is clear that $\mathcal{M}_{B_q}(G_q)_w$ is affine, i.e., $\mathcal{O}_{q,w}$ is a projective generator of $\mathcal{M}_{B_q}(G_q)_w$ and hence $\mathcal{M}_{B_q}(G_q)_w \cong \text{mod} - \text{End}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_{q,w})$. The functors that induce this equivalence are the adjoint pair $(? \otimes_{\text{End}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_{q,w})} \mathcal{O}_{q,w}, ?^{B_q})$. Now $\text{End}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_{q,w}) = \text{Hom}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, \mathcal{O}_{q,w}) = \Gamma(\mathcal{O}_{q,w})$. Joseph [J] showed that

$$(3.17) \quad \text{End}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_{q,w}) = (M_0^*)^{T_w}$$

as U_q -modules, where T_w are the algebra automorphisms of U_q from section 2.1.3. Note that these algebras are nonisomorphic for different w in general. When $w = e$, we have $\text{End}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_e) \cong U_q(\mathfrak{n}_-)$ as an algebra.

Similarly to $\mathcal{M}_{B_q}(G_q)$, $\mathcal{M}_{B_q}(G_q)_w$ also has an induction functor from B_q -modules, $V \mapsto \mathcal{O}_{q,w} \otimes V$. It follows from the fact that $\mathcal{M}_{B_q}(G_q)_w$ is affine that

$$(3.18) \quad (\mathcal{O}_{q,w} \otimes V)^{B_q} \cong \mathcal{O}_{q,w}^{B_q} \otimes V.$$

Next we have the forgetful functor $f_* : \mathcal{D}_{B_q}^\lambda(G_q) \rightarrow \mathcal{M}_{B_q}(G_q)$, which is exact and faithful. It is easy to see that it has left adjoint $f^* : \mathcal{M}_{B_q}(G_q) \rightarrow \mathcal{D}_{B_q}^\lambda(G_q)$ (just tensor with \mathcal{D}_{G_q} over \mathcal{O}_q and factor out the necessary relations).

Define $\mathcal{D}_{B_q}^\lambda(G_q)_w$ to be the localization of $\mathcal{D}_{B_q}^\lambda(G_q)$ lying over $\mathcal{M}_{B_q}(G_q)_w$ (i.e., replace \mathcal{O}_q by $\mathcal{O}_{q,w}$ in the definition of $\mathcal{D}_{B_q}^\lambda(G_q)$). We get the adjoint pair of functors $(f_w^*, f_{w,*})$ between these categories with the same properties as above. Hence, abstract nonsense shows that

Proposition 3.15. $f_w^* \mathcal{O}_{q,w} = \mathcal{O}_{q,w} \otimes_{\mathcal{O}_q} \mathcal{D}_q^\lambda$ is a projective generator of $\mathcal{D}_{B_q}^\lambda(G_q)_w$ and therefore

$$(3.19) \quad \mathcal{D}_{B_q}^\lambda(G_q)_w \cong \text{mod} - \text{End}_{\mathcal{D}_{B_q}^\lambda(G_q)}(\mathcal{O}_{q,w} \otimes_{\mathcal{O}_q} \mathcal{D}_q^\lambda).$$

Put

$$(3.20) \quad A_{q,w}^\lambda = \Gamma(\mathcal{O}_{q,w} \otimes_{\mathcal{O}_q} \mathcal{D}_q^\lambda) = \text{End}_{\mathcal{D}_{B_q}^\lambda(G_q)}(\mathcal{O}_{q,w} \otimes_{\mathcal{O}_q} \mathcal{D}_q^\lambda).$$

We can calculate these $A_{q,w}^\lambda$ explicitly. We have

$$(3.21) \quad A_{q,w}^\lambda = \Gamma(\mathcal{O}_{q,w} \otimes_{\mathcal{O}_q} \mathcal{D}_q^\lambda) = (\mathcal{O}_{q,w} \otimes M_\lambda)^{B_q} = \mathcal{O}_{q,w}^{B_q} \otimes M_\lambda$$

where the ring structure is induced from the one on \mathcal{D}_{G_q} .

It follows that the $A_{q,w}^\lambda$ coincide with the rings introduced by Joseph [J]. In the generic case these can be described as the $B_q^{T_w}$ -finite part of the full endomorphism ring $\text{End}((M_\lambda^*)^{T_w})$.

When $w = e$, Joseph describes this ring explicitly, a quantum Weyl algebra

$$(3.22) \quad A_{q,e}^\lambda \cong U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+).$$

Here, the algebra structures on $U_q(\mathfrak{n}_-)$ and on $U_q(\mathfrak{n}_+)$ are the usual ones and we have the commutation relations

$$(3.23) \quad q^{-(\alpha,\beta)} E_\alpha \otimes F_\beta - q^{(\alpha,\beta)} F_\beta \otimes E_\alpha = \delta_{\alpha,\beta}$$

for $\alpha, \beta \in \Delta$. (So $A_{q,e}^\lambda$'s algebra structure is independent of λ .)

The category $\mathcal{D}_{B_q}^\lambda(G_q)$ can now be described as the gluing of module categories

$$(3.24) \quad \mathcal{D}_{B_q}^\lambda(G_q) = \lim_{\leftarrow w \in \mathcal{W}} A_{q,w}^\lambda - \text{mod}.$$

From the description of $A_{q,w}^\lambda$ as $\mathcal{O}_{q,w}^{B_q} \otimes M_\lambda$ with the algebra structure induced from \mathcal{D}_{G_q} , we get

Proposition 3.16. *i) $Z(A_{q,w}^\lambda) = (\mathcal{O}_w \otimes \mathcal{Z}_{-\lambda}^{(l)})^B = \mathcal{O}_w^B \otimes \mathcal{Z}_{-\lambda}^{(l)}$ for each $w \in \mathcal{W}$.
ii) Consequently,*

$$\text{qcoh}(T^* X^\lambda) = \lim_{\leftarrow w \in \mathcal{W}} Z(A_{q,w}^\lambda) - \text{mod}$$

where $Z(A_{q,w}^\lambda)$ denotes the center of $A_{q,w}^\lambda$.

Let us remark that for generic λ the variety $(G \times N^{\lambda^{2l}})/B$ is an affine variety.

3.3.2. *Torsors.* We first define a “torsor” as a category $\tilde{\mathcal{D}}_{B_q}(G_q)$ of \mathcal{D}_{G_q} -modules that contains all \mathcal{D}_q^λ , $\lambda \in T_P$.

Definition 3.17. An object of $\tilde{\mathcal{D}}_{B_q}(G_q)$ is a triple (M, α, β) , where $\alpha : \mathcal{D}_{G_q} \otimes M \rightarrow M$ is a left \mathcal{D}_{G_q} -action and $\beta^{res} : M \rightarrow M \otimes \mathcal{O}_q(B)$ is a right $\mathcal{O}_q(B)$ -coaction.

- i) The $U_q(\mathfrak{n}_+)$ -actions on M given by $\beta|_{U_q(\mathfrak{n}_+)}$ and by $(\alpha|_{U_q(\mathfrak{n}_+)})$ coincide.
- ii) The map α is $U_q(\mathfrak{b})$ -linear with respect to the β -action on M and the action on \mathcal{D}_{G_q} .

Put $\tilde{\mathcal{M}} = U_q(\mathfrak{g}) / \sum_{\alpha \in R_+} U_q(\mathfrak{g}) \cdot E_\alpha$ (a “universal” Verma module) and define

$$(3.25) \quad \tilde{\mathcal{D}}_q = \mathcal{O}_q \otimes \tilde{\mathcal{M}}.$$

$\tilde{\mathcal{D}}_q$ inherits an $U_q^{res}(\mathfrak{b})$ -module structure from \mathcal{D}_{G_q} , so $\tilde{\mathcal{D}}_q$ is an object in $\tilde{\mathcal{D}}_{B_q}(G_q)$.

$\tilde{\mathcal{M}}$ has the $U_q^{res}(\mathfrak{b})$ -submodule $\mathcal{Z}_{\mathfrak{b}_-}^{(l)} = \mathcal{Z}^{(l)} / (\mathcal{Z}^{(l)} \cap U_q \cdot U_q(\mathfrak{n}_+)_+)$, which is an algebra generated by $K_\mu^{(l)}, E_\mu^{(l)}, \mu \in R_+$. b_q acts trivially on $\mathcal{Z}_{\mathfrak{b}_-}^{(l)}$, so the $U_q^{res}(\mathfrak{b})$ -action factors to a $U(\mathfrak{b})$ -action on $\mathcal{Z}_{\mathfrak{b}_-}^{(l)}$. We have

Lemma 3.18. *As a B -module, $\mathcal{Z}_{\mathfrak{b}_-}^{(l)} \cong \mathcal{O}(B)$, where the module structure comes as follows: consider the map $B \rightarrow B$ defined as $tn \rightarrow t^2n$; this is an unramified covering. Use this covering to pull back the adjoint action of B on itself.*

Proof of Lemma 3.18. To calculate this, one should notice that in the generic case the action induced from the adjoint action of $U_q(\mathfrak{n}_+)$ on the universal Verma module is the same as the one induced from the commutator action. Hence one can use the calculations of [CKP] which give the required result. As for the torus part, the adjoint action is by the grading and it is easy to see that we get the required action. \square

As in the previous sections we see that $\mathcal{O}(G \times B)$ imbeds to $\tilde{\mathcal{D}}_q$, B_q -equivariantly. Note that this embedding corresponds to the surjection $(\)^l : T_P \rightarrow T$. Define B^{twist} to be the cover of B induced from the map $B = TN \ni tn \rightarrow t^{2l}n \in B$ with the natural B -structure. We have

Definition-Proposition 3.19. We define $\tilde{T}^*X^{twist} = (G \times B^{twist})/B$ (where B acts on $G \times B^{twist}$ by $b \cdot (g, b') = (bg, b \cdot b')$). \tilde{T}^*X^{twist} is a T_P -torsor over T^*X . $\tilde{\mathcal{D}}_q$ is a sheaf of Azumaya algebras over \tilde{T}^*X^{twist} .

3.4. **Azumaya splitting over fibers.** Recall the $U_{\mathcal{A}}^{res}$ -action on U_q . The center of U_q is exactly the submodule of invariants with respect to the subalgebra u_q . Hence the center of U_q is a \mathfrak{g} -module. Recall also that U^{fin} is the finite part of U_q with respect to the $U_{\mathcal{A}}^{res}$ -action. We have

Lemma 3.20. *Let \mathcal{Z}^{fin} be the center of U^{fin} . We have $\mathcal{Z}^{fin} = \mathcal{Z} \cap U^{fin}$. \mathcal{Z}^{fin} is an integrable \mathfrak{g} -module and $\text{Spec}(\mathcal{Z}^{fin})$ is isomorphic to an unramified cover of $G \times_{T/W} T$ where the map $T \rightarrow T/W$ is induced from the l 'th power map and the action is the adjoint action.*

Proof of Lemma 3.20. The first claim follows from Joseph’s [JL] description of U_q^{fin} inside U_q . The second statement follows from the calculations of [CKP]. \square

Consider the diagram

$$(3.26) \quad \begin{array}{ccc} T_P & \longleftarrow & \tilde{T}^*X^{twist} \\ \downarrow 2l & & \downarrow \pi \\ T/(\bullet, W) & \longleftarrow & G \end{array}$$

Definition 3.21. $t \in T_P$ is unramified if for every $\alpha \in \tilde{\Delta} = \Delta \cup \{-\alpha_0\}$ (where α_0 is the longest root), $(t(K_\alpha))^{2l} = 1$ implies $(t(K_\alpha))^2 = q^{-(2\rho, \alpha)}$. T_P^{unram} denotes the set of unramified t 's.

Note that any $t \in T_P$ can be made unramified by adding an integral weight to it. We put $\mathcal{Z}^{unram} = \mathcal{O}(T_P^{unram} \times_{T/W} B_-B)$ and $U_q^{unram} = U_q \otimes_{\mathcal{Z}^{unram}} \mathcal{O}(T_P^{unram})$. Brown and Gordon [BG] proved that

Lemma 3.22. *i) U_q^{unram} is Azumaya over \mathcal{Z}^{unram} and
 ii) for each $\chi \in \text{maxspec}(\mathcal{Z}^{unram})$, we have $\text{End}(M_\chi^{baby}) = U_q/\chi U_q$ (where M_χ^{baby} is the baby Verma module).*

Let $\sigma : \tilde{T}^*X^{twist} \rightarrow T_P \times_{T/W} G$ be the natural map. Note that this is a G_q -equivariant map. Now since both sheaves are also G_q -equivariant, from the description of the \mathcal{D}_{G_q} as induced from endomorphism of baby Vermas and from Lemma 3.22 that describes the sheaf over a dense subset (the big cell), we have

Proposition 3.23. *The action map $U_q^{fin} \otimes_{\mathcal{Z}} \mathcal{O}_{\tilde{T}^*X^{twist}} \rightarrow \tilde{\mathcal{D}}_q$ induces an isomorphism $U_q^{fin} \otimes_{\mathcal{Z}} \mathcal{O}_{\tilde{T}^*X^{twist, unram}} \cong \tilde{\mathcal{D}}_q|_{\tilde{T}^*X^{twist, unram}}$.*

Thus we see that over the preimage of the big cell \mathcal{D}_q^λ is Azumaya. (Note again that in a formal neighbourhood of p it is Azumaya everywhere.)

Hence it follows as in [BMR] (vanishing of the Brauer group of a local ring with separably closed residue field) that $\tilde{\mathcal{D}}_q$ Azumaya splits over the formal neighborhood of any fiber of σ lying over the big cell. Hence, we get

Corollary 3.24. *The category of \mathcal{D}_q^λ -modules supported on the fiber of σ over the big cell is equivalent to \mathcal{O} -modules supported on the same fiber.*

3.5. Derived \mathcal{D} -affinity.

3.5.1. *Global sections and vanishing cohomology of \mathcal{D}_q^λ .*

Proposition 3.25. *We have i) $U_q^\lambda \cong R\Gamma(\mathcal{D}_q^\lambda)$ and ii) $\tilde{U}_q \cong R\Gamma(\tilde{\mathcal{D}}_q)$ (if l is a prime $p >$ Coxeter number of G).*

We recall from [BK] that the natural map $U_q^\lambda \rightarrow \Gamma(\mathcal{D}_q^\lambda)$ was given as follows: There is the natural surjection $U_q^\lambda \rightarrow M_\lambda$. It induces a surjective map

$$(3.27) \quad \text{Ind } U_q^\lambda \rightarrow \text{Ind } M_\lambda = \mathcal{D}_q^\lambda.$$

Since U_q^λ is a G_q -module, (2.7) shows that $\Gamma(\text{Ind } U_q^\lambda) = U_q^\lambda$, which gives the desired map, by applying Γ to (3.27).

Remark 3.26. Recall that $U_q^\lambda = U^{fin}/J_\lambda$. In order to get global differential operators equal to U_q/J_λ , one can enlarge \mathcal{D}_q^λ by adding some grading operators; the vanishing of higher self-extensions will remain true.

Proof of Proposition 3.25. We have $\mathcal{D}_q^\lambda = \text{Ind } M_\lambda$. We have the integral version $\text{Ind } M_{\lambda, \mathcal{A}} \in \mathcal{M}_{B_q}(G_q)_{\mathcal{A}}$.

Consider the specialization $\mathcal{A} \rightarrow \mathbb{F}_p, \nu \rightarrow 1$. In [BK] we showed that the statement about global sections in *i*) holds for a generic q . The argument given there transforms to the case of p 'th roots of unity if we can show that

$$(3.28) \quad \dim_{\mathbb{C}} \Gamma(\text{Ind gr}_i M_\lambda) = \dim_{\mathbb{C}} \Gamma(\text{Ind gr}_i M_{\lambda, q=1}).$$

Now, by [AJ], $\dim_{\mathbb{C}} \Gamma(\text{Ind gr}_i M_{\lambda, q=1}) = \dim_{\mathbb{F}_p} \Gamma(\text{Ind gr}_i M_{\lambda, \mathbb{F}_p})$. On the other hand, it follows from [APW] that $\dim_{\mathbb{C}} \Gamma(\text{Ind gr}_i M_\lambda) = \dim_{\mathbb{F}_p} \Gamma(\text{Ind gr}_i M_{\lambda, \mathbb{F}_p})$.

In order to prove that higher cohomologies vanishes in *i*), it suffices by [APW] (page 26) to show that

$$(3.29) \quad R\Gamma_{\mathbb{F}_p}^{>0}(\text{Ind } M_{\lambda, \mathbb{F}_p}) = 0.$$

This holds because $R\Gamma^{>0}(\text{Ind gr } M_{\lambda, \mathbb{F}_p}) = R\Gamma^{>0}(\text{Ind } S(\mathfrak{n}_{-, p})) = 0$, by [AJ] (gr is taken with respect to the filtration on M_λ coming from the identification $M_\lambda = U_q(\mathfrak{n}_-)$ and putting each F_μ in degree 1). This proves *i*). The proof of *ii*) is similar. \square

Remark. In [BK] we explain how the restrictions on the roots of unity can be dropped by using the work of [K] or [W] in order to drop some of the assumptions of [APW].

3.5.2. *Localization functor.* Recall the definition of the localization functor $Loc_\lambda : U_q^\lambda - \text{mod} \rightarrow \mathcal{D}_{B_q}^\lambda(G_q)$ [BK].

Definition 3.27. Define the localization functor $Loc_\lambda : \Gamma(\mathcal{D}_q^\lambda) - \text{mod} \rightarrow \mathcal{D}_{B_q}^\lambda(G_q)$ by $M \rightarrow \mathcal{D}_q^\lambda \otimes_{U_q^\lambda} M$, where we have used $U_q^\lambda = \Gamma(\mathcal{D}_q^\lambda)$.

This is a left adjoint to the global sections functor. Note that $Loc_\lambda(U_q^\lambda) = \mathcal{D}_q^\lambda$. Similarly, we can define localization $\widetilde{Loc} : U_q^{fin} - \text{mod} \rightarrow \widetilde{\mathcal{D}}_{B_q}(G_q)$ and also localizations with generalized central character $\widetilde{Loc}^\lambda : \text{mod}^\lambda(U_q^{fin}) \rightarrow \widetilde{\mathcal{D}}_{B_q}(G_q)^\lambda$. Since U_q^{fin} has finite cohomological dimension, we get that \widetilde{Loc} has left derived functor

$$(3.30) \quad \widetilde{\mathcal{L}} : D^b(U_q^{fin} - \text{mod}) \rightarrow D^b(\widetilde{\mathcal{D}}_{B_q}(G_q))$$

and hence also \widetilde{Loc}^λ . Note that a priori we do not know if U_q^λ has finite cohomological dimension (it will follow later that for regular λ it has); hence Loc_λ has a left derived functor which is defined for the bounded above derived category:

$$(3.31) \quad \mathcal{L}_\lambda : D^-(U_q^\lambda - \text{mod}) \rightarrow D^-(\mathcal{D}_{B_q}^\lambda(G_q)).$$

For regular λ this functor is compatible with restriction from a generalized central character as in [BMR] and so we get that it is also defined on the bounded derived category. From now on we will use \mathcal{L}_λ to denote the functor on the bounded derived category in case λ is regular. We thus have

Lemma 3.28. (i) $\widetilde{\mathcal{L}}$ is left adjoint to $R\widetilde{\Gamma}$ and (ii) if λ is regular, \mathcal{L}_λ is left adjoint to $R\Gamma$.

If λ is not regular, the second statement in the lemma will hold if we replace D^b by D^- .

Proposition 3.29. *i) The functor $R\tilde{\Gamma} \circ \tilde{\mathcal{L}} : D^b(U_q) \rightarrow D^b(\tilde{U}_q)$ is isomorphic to the functor $M \rightarrow M \otimes_{\mathcal{Z}HC} \mathcal{O}(T_P)$.*

ii) For regular λ , the adjunction map $id \rightarrow R\Gamma \circ \mathcal{L}_\lambda$ is an isomorphism.

Proof. Part *i)* follows from part *ii)* of Lemma 3.25 for free modules and then from the same lemma again for general modules, by considering free resolutions.

For *ii)* observe that for regular λ , for any $M \in D^b(U_q^\lambda\text{-mod})$, we have canonically $M \otimes_{\mathcal{Z}HC} \mathcal{O}(T_P) = \bigoplus_{w \in \mathcal{W}} \bigoplus_{\text{Hom}(P, \{+1, -1\})} M$. Now the claim follows since $R\Gamma \circ \mathcal{L}_\lambda(M)$ is one of these direct summands. □

3.5.3. *Derived \mathcal{D} -affinity.*

Theorem 3.30. *$R\Gamma : D^b(\mathcal{D}_{B_q}^\lambda(G_q)) \rightarrow D^b(U_q^\lambda\text{-mod})$ is an equivalence of categories.*

Noting that the canonical bundle of T^*X^λ is trivial and that π from (3.26) is a projective morphism, the theorem now follows from the following lemma of [BMR]:

Lemma 3.31. *Let \mathcal{A} be a generically Azumaya algebra over a smooth variety X (i.e., Azumaya over a generic point). Suppose that X is Calabi-Yau (i.e., $\omega_X \cong \mathcal{O}_X$) and that we have a projective map $\pi : X \rightarrow \text{Spec}R$ for some commutative algebra R . Suppose also that the derived global section functor $R\Gamma : D^b(\mathcal{A}\text{-mod}) \rightarrow D^b(\Gamma(\mathcal{A})\text{-mod})$ has a right adjoint \mathcal{L} and the adjunction morphism $id \rightarrow R\Gamma \circ \mathcal{L}$ is an isomorphism. Then $R\Gamma$ is an equivalence of categories.*

The first point in the proof of this lemma is the following proposition:

Proposition 3.32. *Let $L : C \rightarrow D$ be a functor between triangulated categories with D O -triangulated Calabi-Yau for some finitely generated commutative algebra O (we have a natural functor $R\text{Hom}(X, Y)$ defined on $D \times D^{op}$ with values in $D^b(O\text{-mod})$ whose first cohomology is the Hom in the category and there is a functorial isomorphism $R\text{Hom}(X, Y) \cong \mathbb{D}_O(R\text{Hom}(Y, X[n]))$ where \mathbb{D} is the duality functor over O). Suppose L has a right adjoint R such that the adjunction $id \rightarrow R \circ L$ is an isomorphism. Suppose also that D is indecomposable. Then L is an equivalence.*

The second point is the observation that the bounded derived category of modules over an Azumaya algebra is indecomposable.

Note that, for generic λ , T^*X^λ is affine and hence we then get an equivalence $\Gamma : \mathcal{D}_{B_q}^\lambda(G_q) \cong U_q^\lambda\text{-mod}$.

4. APPLICATIONS

Assume for simplicity λ is integral and regular and that χ belongs to B_-B and is unipotent. We know that

$$(4.1) \quad D^b(\mathcal{D}_{B_q}^\lambda(G_q)) \cong D_\lambda^b(U_q\text{-mod}).$$

We get that

$$(4.2) \quad D^b(\mathcal{D}_{B_q}^\lambda(G_q)_{(\chi, \lambda)}) \cong D_{\chi, \lambda}^b(U_q\text{-mod})$$

where the left hand side denotes those (complexes of) \mathcal{D}_q^λ -modules supported on the Springer fiber of (χ, λ) and the right hand side denotes U_q^λ -modules that are locally annihilated by a power of the maximal ideal in \mathcal{Z} corresponding to χ (generalized central l -character χ).

By the Azumaya splitting we have

$$(4.3) \quad \mathcal{D}_{B_q}^\lambda(G_q)_{(\chi,\lambda)} \cong \text{qcoh}(T^*X)_\chi$$

where the latter category is the quasi-coherent \mathcal{O} -modules on T^*X supported on the Springer fiber of χ (with respect to the usual Springer resolution $T^*X \rightarrow \mathcal{N} =$ unipotent variety of G). Note that for the trivial central character this equivalence is Koszul dual to the equivalence in [ABG], thus giving a geometric proof of their equivalence.

We deduce an equivalence of K -groups

$$(4.4) \quad K_{\tilde{\chi}}(U_q^\lambda - \text{mod}) \cong K(\text{qcoh}(T^*X)_\chi).$$

Note that $K(\text{qcoh}(T^*X)_{(\chi)})$ is torsion-free. From (4.4) we see for instance that the number of irreducible $U_{q,\chi}$ -modules equals the rank of $K(\text{qcoh}(T^*X)_{(\chi)})$.

In [Kr] we use the methods here to prove a conjecture made by De Concini, Kac and Procesi stating that the dimensions of irreducible $U_{q,\chi}$ -modules is divisible by $l^{\text{codim}(O)/2}$ where l is the order of the root of unity and O is the orbit through χ .

Notice also that [BMR] showed that $K(\text{qcoh}(T^*X)_\chi) \cong K(\text{qcoh}(T^*X)_\chi(\mathbb{F}_p))$ for $p >$ Coxeter number of G . This relates our work to the representation theory of \mathfrak{g}_p via the results in [BMR].

Actually, it is possible to relate our work to [BMR] in order to get more transparent geometric proofs of the theorems appearing in the work of Andersen, Jantzen and Soergel [AJS], leading to the proof of Lusztig's conjecture about multiplicities in characteristic p and also extending them to nontrivial central characters. Our work defines a perverse t -structure on \mathcal{O} -modules in zero characteristic (root of unity), where [BMR] defines such a structure in positive characteristic. By showing that in specializing the root of unity t -structure one would get (at least for big p) the positive characteristic t -structure, one would be able to deduce that the quantum modules and the positive characteristic modules have the same multiplicity formulas, as shown in [AJS]. This will appear in future work.

ACKNOWLEDGEMENTS

We thank Joseph Bernstein and David Kazhdan for many useful conversations.

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