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Localization Length and Inverse Participation Ratio of Two Dimensional Electron in the Quantized Hall Effect

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The inverse participation ratio, whose nonvanishing value represents the localized state, is calculated for the two dimensional lowest Landau level in the presence of a random potential. In the band tail, an exact value for the inverse participation ratio is obtained for the Gaussian white noise random potential and the state becomes localized. By the large order perturbational analysis, the exponent for the localization length around the band center is estimated. A $1/N$ expansion is also considered for the inverse participation ratio.

§ 1. Introduction

Recently, the quantum Hall effect has attracted much interest. The two dimensional quantum Hall effect is closely related to the localization phenomenon at the region of the energy where the Hall conductivity σ_{xy} is quantized and it takes a value of e^2/h multiplied by integer, since at this region of the energy the diagonal conductivity is vanishing.

The localization at the band tail is Anderson localization due to the impurity scattering under a strong magnetic field. In previous papers, the diffusion constant has been calculated by the perturbational method for a white noise random potential[1),2)] and also for a spatially correlated random potential.[3)] It has been suggested that the extended state appears at the energy of the band center in the lowest Landau level and the value of the diagonal conductivity σ_{xx} has been calculated by a Borel-Padé method. Recently, Singh and Chakravarty extended the same calculation to the higher order.[4)] This finite conductivity at the band center under a strong magnetic field has also been studied by a numerical simulation[5),6)] and by a self-consistent diagrammatic treatment.[7)] Other theoretical investigations also give the conclusion that the extended state appears only at the band center.[8)~10)]

In this paper, we consider the inverse participation ratio for the lowest Landau level.[11)] The inverse participation ratio is the average of the fourth power of the wave function and it is a convenient quantity which distinguishes the extended state from the localized state.[12)~15)] This inverse participation ratio is thus related to the localization length.[14)] The purpose of this paper is to calculate the localization length via the inverse participation ratio and to obtain the value of the critical exponent for the localization length near the band center. Also we investigate the $1/N$ expansion for the inverse participation ratio and derive a logarithmic term which indicate the localization. The $1/N$ expansion has been calculated for the diffusion constant.[2)] We consider N-orbital electron as a generalization of a single electron in the lowest Landau level.

In § 2, we firstly discuss the perturbational treatment for the one-body Green function and derive the exact recurrence equation for the coefficients of the self-energy for Gaussian white noise random potential. The inverse participation ratio is considered and the perturbational series for the participation ratio is calculated in § 3. The exact value of the participation ratio at the band tail is derived in § 4. In § 5, the $1/N$ expansion is investigated for the participation ratio and the logarithmic terms in $1/N^2$ order are discussed. The relations to the diffusion constant and to the nonlinear σ model are considered from the renormalization group point of view. The value of the inverse participation ratio is estimated near the band center of the lowest Landau level by the Borel summation. The localization length and its critical exponent is estimated in § 6. Section 7 is devoted to the discussion.

§ 2. Exact perturbational series for the one particle Green function

The density of state of the two dimensional Landau level has a broadening due to the impurity scattering. Recently, for the white noise Gaussian random potential, the density of state of the lowest Landau level in the strong magnetic field has been investigated by Wegner[16] and the exact expression has been derived.

In this section, we consider the exact recurrence equation for the coefficients of the perturbational series. Although the exact expression for the density of state is known, the exact perturbational series in all orders is meaningful since it gives the general behavior of the perturbational treatment and also its large order behavior gives a clue for the behavior of the two particle Green function. Previously, the coefficients for the one particle Green function were known up to ninth order.[4]

Hamiltonian for the two dimensional electron under a strong magnetic field in a white noise Gaussian random potential is given by

$$H=\frac{1}{2m}(P-eA)^2+V(r)\,, \tag{2·1}$$

where the vector potential A is written as $A=B(-y/2, x/2, 0)$. The random potential satisfies

$$\langle V(r)\rangle_{\mathrm{av}}=0\,, \tag{2·2a}$$

$$\langle V(r)V(r')\rangle_{\mathrm{av}}=W\delta(r-r') \tag{2·2b}$$

and $\langle\cdots\rangle_{\mathrm{av}}$ means the average over the randomness.

We use the same notation as Refs. 1) and 2) hereafter. The one particle Green function becomes

$$\left\langle\!\!\left\langle r\left|\frac{1}{E-H\pm\frac{i}{2}\varepsilon}\right|r\right\rangle\!\!\right\rangle_{\mathrm{av}}=\frac{eB}{2\pi\left(E-\frac{1}{2}\hbar\omega_c\pm\frac{i}{2}\varepsilon-\frac{1}{2\pi}\Sigma_+\right)}\,, \tag{2·3}$$

where Σ_+ is the self-energy. As discussed before,[1),2)] we write this one particle Green function as

$$\Gamma_\pm=\frac{1}{A_1\pm iA_2}=Ce^{\pm i\theta} \tag{2·4}$$

since the Green function reduces to the complex number in this strong magnetic field. We put $eB=1$ for simplicity in the following sections. The quantities A_1 and A_2 are

$$A_1=2\pi\left(E-\frac{1}{2}\hbar\omega_c-\frac{\Sigma_+{}^R}{2}\right), \tag{2·5a}$$

$$A_2=2\pi\left(\frac{\varepsilon}{2}-\frac{\Sigma_+{}^I}{2}\right), \tag{2·5b}$$

where $\Sigma_+{}^R=\mathrm{Re}\Sigma_+$ and $\Sigma_+{}^I=\mathrm{Im}\Sigma_+$.

From (2·5b), the infinitesimal small quantity ε is expanded in the power series about x which is defined as

$$x=2\pi WC^2, \tag{2·6}$$

where the quantity C is given in (2·4) and W is the strength of the random potential. By the calculation of order by order of the self-energy diagram Σ_+, we have[1),2)]

$$\frac{\pi\varepsilon}{A_2}=1-x-\frac{1}{2}x^2\frac{\sin 3\theta}{\sin\theta}-\frac{5}{4}x^3\frac{\sin 5\theta}{\sin\theta}-\frac{41}{8}x^4\frac{\sin 7\theta}{\sin\theta}-\cdots$$
$$=1-\sum_{n=0}^{\infty}L_n x^{n+1}\frac{\sin(2n+1)\theta}{\sin\theta}. \tag{2·7}$$

For the white noise Gaussian random potential, this coefficient L_n is obtained in arbitrary order by the following exact recurrence equation,

$$L_n=\left(n+\frac{1}{2}\right)L_{n-1}+\sum_{l=1}^{n-2}(2l+1)L_l L_{n-l-1}, \quad (n\geq 3) \tag{2·8a}$$

$$L_0=1, \qquad L_1=\frac{1}{2}, \qquad L_2=\frac{5}{4}. \tag{2·8b}$$

The derivation of this recurrence equation will be discussed in § 3 since a particular relation between the one particle and the two particle Green functions gives this recurrence equation. This recurrence equation gives the consistent result which has been known before up to L_8.[4)] The large order behavior for L_n is written as

$$L_n\sim n!\,a^n n^b c. \tag{2·9}$$

The values of a and b become

$$a=1, \qquad b=1.5. \tag{2·10}$$

The large order behavior for the perturbational series has been investigated for other several cases. Particularly, the unharmonic oscillator eigenvalue has been calculated in large order with respect to the coupling constant[17)] and to the $1/N$ expansion[18)] by the help of the exact recurrence equations. Our series is very close to that of such cases. The coefficient L_n becomes a very large number according to (2·9). For example, L_{45} becomes 5.5×10^{57}. To obtain the value of x for the fixed value of θ, it is useful to apply the complex Borel-Padé method which has been discussed before.[1)] For the spatially correlated Gaussian random potential,[3),4)] we have no simple recurrence equation as that for this white noise Gaussian random potential.

§ 3. Inverse participation ratio

We consider the following two particle Green function given by

$$K(x_1', x_1, x_2', x_2; E, \varepsilon)= \left\langle\!\!\left\langle x_1'\left|\frac{1}{E-H+i\varepsilon}\right|x_1\right\rangle \left\langle x_2'\left|\frac{1}{E-H-i\varepsilon}\right|x_2\right\rangle\!\!\right\rangle_{\rm av}. \tag{3·1}$$

The density-density correlation function $\Gamma(x_1-x_2, E)$ is written as

$$\Gamma(x_1-x_2, E)=\frac{1}{\pi\rho}\lim_{\varepsilon\to 0}\varepsilon K(x_1, x_1, x_2, x_2; E, \varepsilon), \tag{3·2}$$

where the quantity ρ is the density of states at the energy E. The inverse participation ratio,[12)~15)] which is a convenient quantity for distinguishing a localized state from an extended state, is defined as

$$\begin{aligned} P&=\Gamma(0, E) \\ &=\frac{1}{\pi\rho}\lim_{\varepsilon\to 0}\varepsilon K(x, x, x, x; E, \varepsilon). \end{aligned} \tag{3·3}$$

We denote simply this two particle Green function $K(x, x, x, x; E, \varepsilon)$ by K. The inverse participation ratio is fourth power of the wave function, and it takes a nonvanishing value for the localized state and becomes zero for the extended state. The infinitesimal small quantity ε has already been expanded in (2·17). For the perturbational series of P, a new expansion for K is needed. The calculation for the diagram of two particle Green function is reduced to the evaluation of the determinant and it is also related to the counting of the number of Euler trails. In the same notation of Ref. 1), this number of Euler trails is represented by ν'. For example, we consider the two particle Green function K given by Fig. 1(a), which represents one impurity scattering. We put one dotted line between two r as Fig. 1(b), and insert a starting point S. Contracting two dotted lines, we have Euler trails starting from the point S shown as Fig. 1(c). The number of the different Euler trails in Fig. 1(c) is two and therefore, ν' becomes two. Extending this counting rule for the more complicated diagram, we easily obtain the coefficients of the series for K. The two particle Green function K is expanded by

$$\frac{2\pi K}{A_2}=\sum\frac{1}{\nu'}x^n e^{i(m_1-m_2)\theta}, \tag{3·4}$$

where m_1 and m_2 are numbers of retarded Green function and advanced Green function in the graphs. We denote $2\pi K/A_2$ by $\tilde{K}$. The number ν' is also calculated as a determinant of the corresponding adjacent matrix.[2)] Therefore, this coefficient $\tilde{K}$ is easily calculated by computer. We obtain the following series expansion for $\tilde{K}$ up to order x^9,

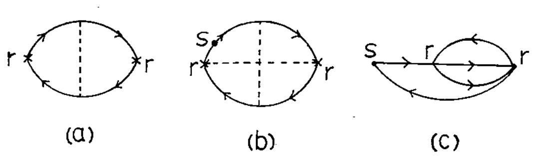

Fig. 1. One impurity scattering graph and a counting rule of Euler trail for the two particle Green function K.

$$\begin{aligned} \tilde{K}=1&+\frac{1}{2}x+\left(\frac{2}{3}\cos 2\theta+\frac{7}{12}\right)x^2 \\ &+\left(\frac{49}{30}\cos 4\theta+\frac{11}{5}\cos 2\theta+\frac{31}{24}\right)x^3 \end{aligned}$$

$$+(6.7\cos 6\theta+7.914718614\cos 4\theta+8.769336218\cos 2\theta$$
$$+4.6784451659)x^4$$
$$+(36.7352773\cos 8\theta+40.4582477\cos 6\theta+42.6582514\cos 4\theta$$
$$+44.98596114\cos 2\theta+22.943512402)x^5$$
$$+(246.1449616\cos 10\theta+259.350926\cos 8\theta+265.2285108\cos 6\theta$$
$$+274.0007122\cos 4\theta+279.8272036\cos 2\theta+141.27581043)x^6$$
$$+(1923.773318\cos 12\theta+1965.837265\cos 10\theta+1970.233463\cos 8\theta$$
$$+2004.855486\cos 6\theta+2032.751576\cos 4\theta+2055.04628\cos 2\theta$$
$$+1030.635419)x^7$$
$$+(17057.99374\cos 14\theta+17042.80586\cos 12\theta+16842.93568\cos 10\theta$$
$$+16955.60114\cos 8\theta+17079.93038\cos 6\theta+17212.53177\cos 4\theta$$
$$+17287.44067\cos 2\theta+8660.764655)x^8$$
$$+(168523.7170\cos 16\theta+165486.8678\cos 14\theta+161895.5511\cos 12\theta$$
$$+161720.9802\cos 10\theta+162095.0736\cos 8\theta+162823.8406\cos 6\theta$$
$$+163349.6823\cos 4\theta+163751.2059\cos 2\theta+81922.80186)x^9+\cdots, \qquad (3\cdot5)$$

where the angle θ is related to the energy and θ is expressed by the density of state ρ and the real part of the one particle Green function ζ[1),2)] as

$$\sin\theta=-\frac{\rho}{\sqrt{\rho^2+\zeta^2}}=-\frac{A_2}{\sqrt{A_1{}^2+A_2{}^2}}. \qquad (3\cdot6)$$

The inverse participation ratio P is obtained by multiplying the expansion for ε to the expansion for K. We have

$$P=1-\frac{x}{2}-\left(\frac{1}{6}\frac{\sin 3\theta}{\sin\theta}+\frac{1}{4}\right)x^2+\cdots. \qquad (3\cdot7)$$

In the limit of vanishing ε, the quantity x is calculated exactly from the exact expressions for ρ and ζ. It is also possible to write the series of (3·7) in the power series of a new variable y, which is defined as

$$y=1-\frac{\pi\varepsilon}{A_2}$$
$$=x+\frac{1}{2}x^2\frac{\sin 3\theta}{\sin\theta}+\cdots. \qquad (3\cdot8)$$

In this variable y, the critical value of y becomes always one in the limit of vanishing ε. We have applied two different estimations by x-variable expansion and y-variable expansion for the value of P.

§ 4. Exact value of the inverse participation ratio at the band tail

At the end of the band tail, the angle θ takes a value of zero. From the expressions for ε and $\tilde{K}$, the inverse participation ratio P is expanded as

$$P=1-\frac{1}{2}x-\frac{3}{4}x^2-\frac{25}{8}x^3-\frac{287}{16}x^4-\cdots. \tag{4·1}$$

Comparing this expression with the series expansion of ε (2·7), we have exactly as ($\tilde{\varepsilon}=\pi\varepsilon/A_2$)

$$P=\frac{1}{2}+\frac{1}{2}\tilde{\varepsilon}. \tag{4·2}$$

In the limit of vanishing ε, the value of P is one-half and the state at the tail of the Landau level becomes localized.

This exact relation (4·2) at $\theta=0$ gives the recurrence equation which we have discussed in § 2. We write the series expansions for $\tilde{K}$ and $\tilde{\varepsilon}$ at $\theta=0$ as

$$\tilde{K}=\sum\tilde{K}_n x^n, \tag{4·3}$$

$$\tilde{\varepsilon}=1-\sum\tilde{\varepsilon}_n x^n. \tag{4·4}$$

From the expressions (2·7) and (3·5), the two coefficients are known to be related as

$$\tilde{K}_n=\frac{\tilde{\varepsilon}_{n+1}}{(2n+1)}. \tag{4·5}$$

By using relations (4·2) and (4·5), we have a recurrence equation as

$$\tilde{K}_1-\tilde{\varepsilon}_1=-\frac{1}{2}\tilde{\varepsilon}_1,$$

$$\tilde{K}_2-\tilde{\varepsilon}_1\tilde{K}_1-\tilde{\varepsilon}_2=-\frac{\tilde{\varepsilon}_2}{2},$$

$$\tilde{K}_n-\tilde{\varepsilon}_1\tilde{K}_{n-1}-\tilde{\varepsilon}_2\tilde{K}_{n-2}-\cdots-\tilde{\varepsilon}_{n-1}\tilde{K}_1=-\frac{\tilde{\varepsilon}_n}{2}. \tag{4·6}$$

Since $\tilde{\varepsilon}_n$ is related to $\tilde{K}_{n-1}$, (4·5), we have

$$\tilde{K}_n=\left(n+\frac{1}{2}\right)\tilde{K}_{n-1}+\sum_{l=1}^{n-2}(2l+1)\tilde{K}_l\tilde{K}_{n-l-1}. \tag{4·7}$$

The coefficient of ε, L_n is equal to $\tilde{\varepsilon}_{n+1}/(2n+1)$. Therefore we have also a recurrence equation for L_n as (2·8).

§ 5. 1/N expansion for the inverse participation ratio

In a strong magnetic field case, the $1/N$ expansion has been investigated for the conductivity of the N-orbital lowest Landau level and the logarithmic term which leads to the localization in $1/N^2$ order has been calculated.[2),19)] The consistency with the prediction of the renormalization group analysis by the unitary nonlinear σ model[20)] has been checked. According to the nonperturbative theory based upon a topological term,[8),9)] the band center is predicted to remain extended. Although the $1/N$ expansion is a perturbational calculation and it is not sufficient for the topological term, it may be valid for the localized state except the band center. We consider the inverse participation ratio P in the $1/N$ expansion and investigate the relation to the diffusion constant.

In the large N limit, the contribution to K comes from the ladder diagram. As discussed before,[2)] the expansion for ε becomes

$$\frac{\pi\varepsilon}{NA_2}=1-Nx-\frac{x^2}{2}\frac{\sin 3\theta}{\sin\theta}-\frac{5}{4}Nx^3\frac{\sin 5\theta}{\sin\theta}-\cdots. \tag{5·1}$$

By the change of variable $x=\tilde{x}/N$, we have in the large N limit as

$$\frac{\pi\varepsilon}{NA_2}=1-\tilde{x}. \tag{5·2}$$

The two particle Green function K in this large N limit is calculated with the summation of the ladder diagrams as Fig. 2

$$K=\sum_{n=0}\frac{\tilde{x}^n}{n+1}=-\frac{1}{\tilde{x}}\ln(1-\tilde{x}). \tag{5·3}$$

Therefore, the inverse participation ratio P in this limit becomes

$$\begin{aligned} P&=\tilde{\varepsilon}\tilde{K} \\ &=-\frac{\tilde{\varepsilon}\ln\tilde{\varepsilon}}{(1-\tilde{\varepsilon})}+O\left(\frac{1}{N^2}\right). \end{aligned} \tag{5·4}$$

All states are extended since the inverse participation ratio P becomes zero in the vanishing ε limit. This result is consistent with the calculation of the conductivity in the large N limit.

The graphs of order $1/N^2$ have the structure represented by Fig. 3(a), in which the vertex part Γ is irreducible graph of order $1/N^2$. In Figs. 3 (b) and (c), for example, graphs of order $1/N^2$ are presented. Up to order $1/N^2$, the quantity K is expanded as

$$\begin{aligned} \tilde{K}=&-\frac{\ln(1-\tilde{x})}{\tilde{x}}+\frac{1}{N^2}\left(\frac{C_0}{1-\tilde{x}}+C_1\ln^3(1-\tilde{x})\right. \\ &\left.+C_2\ln^2(1-\tilde{x})+C_3\ln(1-\tilde{x})+C_4\right)+O\left(\frac{1}{N^4}\right). \end{aligned} \tag{5·5}$$

The divergent term $C_0/(1-\tilde{x})$ is absorbed in the first logarithmic term as a shift of the

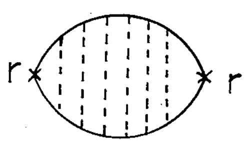

Fig. 2. A ladder graph which contributes in the large N limit.

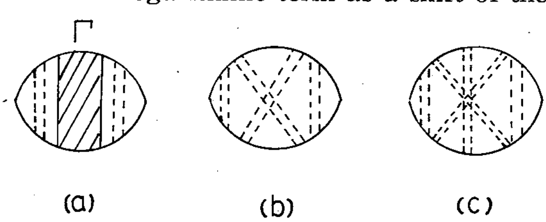

Fig. 3. (a) Γ is a vertex part of the two particle Green function K. (b), (c) Examples of the graphs of order $1/N^2$.

critical point $\tilde{x}=1-C_0/N^2$. The first term becomes $-\ln(1-\tilde{x}-C_0/N^2)$.

The coefficients of the term of order $1/N^2$ have the following structure,

$$f=\frac{1}{\nu'+(l_1+l_2)\nu}, \tag{5·6}$$

where l_1 and l_2 are numbers of the parallel lines in Fig. 3(a). The number ν is the number of Euler trails for the vertex part Γ and ν' is the number of Euler trails in the case $l_1=l_2=0$.

The leading divergent term is obtained by the approximation for f as

$$f\sim\frac{1}{(l_1+l_2)\nu}. \qquad (l_1+l_2\gg 1) \tag{5·7}$$

The contribution of this leading term becomes

$$\begin{aligned}\frac{1}{N^2}\sum\frac{1}{(l_1+l_2)\nu}\tilde{x}^{l_1+l_2+l}e^{im\theta}&=\frac{1}{N^2}\sum_l\left(\frac{1}{1-\tilde{x}}\right)\frac{1}{\nu}\tilde{x}^l\frac{\sin(2l-1)\theta}{\sin\theta}\\&=\frac{C_0}{N^2}\frac{1}{1-\tilde{x}}.\end{aligned} \tag{5·8}$$

Up to $1/N^2$ order, we have[2)]

$$\begin{aligned}\tilde{\varepsilon}&=1-\tilde{x}-\frac{1}{N^2}\sum\frac{\tilde{x}^l}{\nu}\frac{\sin(2l-1)\theta}{\sin\theta}\\&=1-\tilde{x}-\frac{C_0}{N^2}+O\left(\frac{1}{N^4}\right).\end{aligned} \tag{5·9}$$

Therefore, the term of (5·8) is interpreted as a shift of the critical exponent. The term of the next order is obtained by the expansion of f as

$$f\cong\frac{1}{(l_1+l_2)\nu}-\frac{\nu'}{(l_1+l_2)^2\nu^2}+\cdots. \tag{5·10}$$

This second term gives the logarithmic term as

$$\frac{1}{N^2}\sum_l\sum_{l_1+l_2\gg 1}\left(-\frac{\nu'}{(l_1+l_2)^2\nu^2}\right)\tilde{x}^{l_1+l_2+l}e^{im\theta}=\frac{1}{N^2}C_2\ln^2(1-\tilde{x}). \tag{5·11}$$

The coefficient C_2 has appeared in the calculation of the diffusion constant D.[2)] For the band center, $\theta=-\pi/2$, we have $C_2=1/4$.

For the diagram of $l_1=l_2=0$, the factor f becomes $1/\nu'$ and in this case, there is a possibility of divergence of order $\ln^3(1-x)$. However, it is easily seen that the cancellation of this $\ln^3(1-x)$ term occurs in the summation of three graphs represented in Ref. 2) as b_3, b_5 and b_6. Therefore, there is no $\ln^3(1-x)$ term. Up to order $1/N^2$, concerning with the logarithmic terms, we have the following relation between P and D as

$$P=-\frac{\tilde{\varepsilon}\ln\tilde{\varepsilon}}{D}. \tag{5·12}$$

If the diffusion constant is finite, the inverse participation ratio becomes zero and the state becomes extended. The result of the $1/N$ expansion for the diffusion constant and for the inverse participation ratio indicates the localization. In the next section, we investigate numerically the value of the inverse participation ratio P near the band center.

§ 6. Estimation for the inverse participation ratio near the band center by Borel summation

The inverse participation ratio P behaves near the mobility edge E_c[14] as

$$P \sim |E-E_c|^{\pi_2}. \tag{6·1}$$

In our two dimensional case, E_c is $\hbar\omega_c/2$ if only the band center is extended. In this case, we write

$$P \sim \left|\theta+\frac{\pi}{2}\right|^{\pi_2} \sim \xi^{-2}, \tag{6·2}$$

where ξ is the localization length and θ takes a value $-\pi/2$ at the band center. Here we assume the power law behavior of P as (6·2), and we estimate the exponent ν of the localization length ξ which is defined as

$$\xi^{-1} \sim \left|E-\frac{1}{2}\hbar\omega_c\right|^{\nu} \tag{6·3}$$

with $\nu=\pi_2/2$. At the band center, $\theta=-\pi/2$, the inverse participation ratio P is expanded as

$$P=1-\frac{1}{2}x-\frac{1}{12}x^2-\frac{23}{120}x^3+0.8571609x^4-5.264329x^5+37.51252x^6$$
$$-305.6137x^7+2791.830x^8-28210.78x^9+\cdots. \tag{6·4}$$

The value of x becomes exactly $4/\pi=1.273240$ at the band center. The large order behavior in this asymptotic expansion is analyzed as (2·9). For this inverse participation ratio, the value of a is estimated as one and also the value of b is estimated as one,

$$P=\sum_{n=0} C_n x^n, \tag{6·5a}$$

$$C_n \sim n!\,a^n b^n c \cos(2n\theta). \tag{6·5b}$$

To make the series (6·4) convergent, we apply the Borel summation with a conformal transformation, which has been employed in the study of the critical phenomena.[21] We write P in the integral form as

$$P=\int_0^\infty z^B e^{-z/x}\frac{1}{x^{B+1}}(\Sigma u_n t^n)dz, \tag{6·6}$$

where t is given by

$$t=\frac{\sqrt{1+az}-1}{\sqrt{1+az}+1}, \qquad z=\frac{4}{a}\frac{t}{(1-t)^2}. \tag{6·7}$$

The coefficient u_n is determined by the comparison with (6·5a). The value of a is the same as the value of the large order (6·5b) and we put it to one. The parameter B is a variational parameter and it is chosen in such a way for making a smooth extraporation

Table I. Partial sum P_n and estimation of P for several values of energy $\tilde{\nu}$ are presented. x_c is calculated from the exact expressions (5).

$\tilde{\nu}$	0	0.3	0.5	0.7
B	3	3.5	4.5	6
x_c	1.27323	1.25867	1.23168	1.18887
$n=1$	0.79895	0.81240	0.83436	0.85908
$n=2$	0.61831	0.635	0.665	0.70259
$n=3$	0.48	0.49833	0.52972	0.56898
$n=4$	0.37935	0.39621	0.42764	0.46674
$n=5$	0.30701	0.32205	0.35008	0.38918
$n=6$	0.25500	0.26888	0.29334	0.32787
$n=7$	0.21746	0.22955	0.25314	0.28194
$n=8$	0.19005	0.20093	0.22192	0.25197
$n=9$	0.16978	0.18019	0.19773	0.22922
P	0	0.0015	0.009	0.035

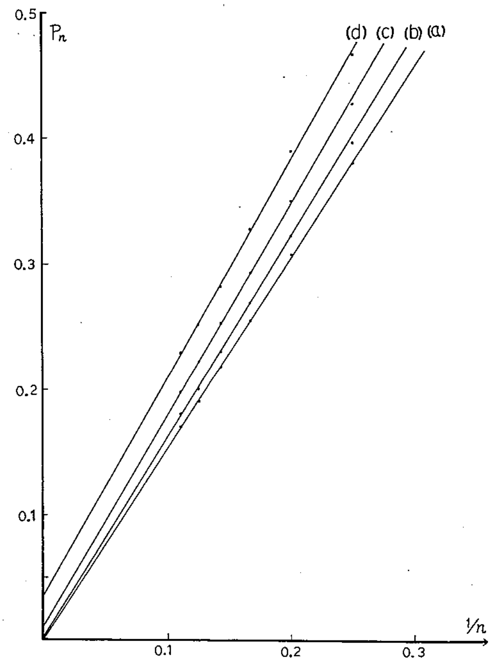

Fig. 4. Partial sum P_n vs $1/n$. (a) $\tilde{\nu}=0, B=3$ (b) $\tilde{\nu}=0.3$, $B=3.5$ (c) $\tilde{\nu}=0.5, B=4.5$ (d) $\tilde{\nu}=0.7, B=6$.

Table II. Coefficient C_n for several values of $\tilde{\nu}$ and the value of θ.

	$\tilde{\nu}=0.2$	$\tilde{\nu}=0.3$	$\tilde{\nu}=0.5$	$\tilde{\nu}=0.7$
$n=1$	-0.5	-0.5	-0.5	-0.5
$n=2$	-0.0943929	-0.1081008	-0.1510881	-0.2130011
$n=3$	-0.1272173	-0.0526314	0.1432471	0.3240458
$n=4$	0.3740726	-0.1331182	-1.129585	-1.277006
$n=5$	-1.155712	2.608879	6.914217	1.350932
$n=6$	-0.9027124	-30.05558	-34.65367	36.66949
$n=7$	82.76412	308.4514	71.23052	-441.7981
$n=8$	-1408.428	-3032.922	1443.928	2395.122
$n=9$	20009.04	28704.80	-29690.68	9284.568
θ	1.44164	1.37684	1.24634	1.11406

of the partial sum P_n, which is obtained by the summation up to the order of n in (6·6). In our problem, we find that it is convenient to plot the partial sum P_n of (6·6) against the value of $1/n$, since the curve becomes a straight line and the estimation of P becomes easy. In Table I, we present the value of the partial sums with the best value of B and the estimation for P. The energy $\tilde{\nu}$ is measured from the band center as

$$\tilde{\nu}=\sqrt{\frac{2\pi}{W}}\left(E-\frac{1}{2}\hbar\omega_c\right). \tag{6·8}$$

In Fig. 4, the partial sum P_n is plotted vs $1/n$. The large order behavior shows an oscillatory behavior with a period $\cos(2n\theta)$. Therefore, the precision of our estimation becomes worse for large value of $\tilde{\nu}$. The oscillatory behavior can be seen in Table II, in

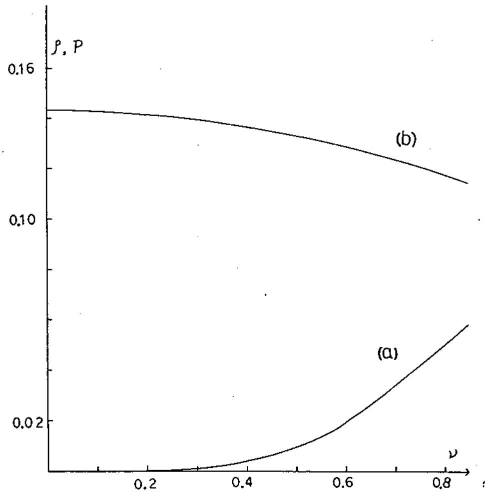

Fig. 5.(a) Estimated value of P vs energy $\tilde{\nu}$.
(b) Exact density of state ρ for $W=1$.

which the values of C_n (6·5a) are presented. The estimation of P near the band center and the value of the density of state are plotted in Fig. 5. It is found by a log-log plot that the curve of P is well approximated for the value less than $\tilde{\nu}=0.6$ by

$$P \sim 0.14\,\tilde{\nu}^{3.8}\,. \tag{6·9}$$

From the relation between the inverse participation ratio P and the localization length ξ, the critical exponent ν for the localization length is estimated as $\nu=1.9(\pm 0.2)$. We have also tried the estimation based upon the representation of y-variable (3·8). The result is almost the same. The Borel-*Padé* analysis which has been employed for the analysis of diffusion constant[2),4)] is not enough for inverse participation ratio since there appears a spurious pole. Our Borel summation is superior to the Borel-Padé method. Our result shows clearly that there is an extended state at the band center.

§ 7. Discussion

In this paper, we have investigated the behavior of the inverse participation ratio for the Gaussian white noise random potential. We have also studied the spatially correlated random potential.[3)] In this case, we have a small parameter η related to the range of the potential.[3)] This small parameter η plays the role of $1/N$. The η-expansion for the spatially correlated random potential gives the same result as is written in (5·12). For the Borel-transformation of the perturbational series, which we have calculated up to x^8, the extrapolation becomes more difficult in the spatially correlated random potential. However, the behavior of the localization length near the band center is quite similar to the case of the Gaussian white noise random potential.

Our conclusion about the existence of the extended state at the band center in the case of the white noise random potential coincides with the result of other studies.[5)~10)] For the critical exponent ν, our value is very close to the result by Ando,[5)] but different from the percolation exponent.[10)] Recently, Affleck[22)] has discussed the exponent ν based upon the unitary nonlinear σ model and using the exactly solvable model, he has concluded that the exponent ν is one-half. Our value $\nu=1.9\pm 0.2$ is far from 0.5. The reason of this discrepancy is not clear up to now. In a separate paper, we will discuss the Hall conductivity by the perturbational method and discuss the field theoretical formulation for the Hall conductivity. It is possible to extend the perturbational calculation in the higher order. Up to ninth order x^9, we have calculated 10^8 different diagrams and the calculation takes almost ten hours in the computer of the University of Tokyo.

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