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Localization of Bounded Sets in Tensor Products

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ABSTRACT. The problem of topologies of Grothendieck is considered for complete tensor products of Fréchet spaces endowed with the topology defined by an arbitrary tensor norm. Some consequences on the stability of certain locally convex properties in spaces of operators are also given.

Grothendieck, in his work about Fréchet and (DF)-spaces and tensor products (see [12], [13]), studied locally convex properties of function spaces. Many examples of spaces of vector-valued functions can be represented as tensor products and this motivated the study of the topological structure of tensor products of Fréchet and (DF)-spaces.

The "probléme des topologies" of Grothendieck asks if every bounded set B of the complete projective tensor product $E\hat{\otimes}_\pi F$ of two Fréchet spaces E and F can be "localized", i.e. if there are bounded subsets C, D of E and F respectively with $B \subset \overline{\Gamma(C \otimes D)}$. Due to

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the important work of Taskinen (see [22]) we know that the answer is negative in general. Bonet, Díaz and Taskinen [6], inspired by Bonet and Díaz [5] and Taskinen [22,23], introduced the classes of (FG) and (DFG)-spaces as a general frame in which the "probléme des topologies" of Grothendieck (see [13]) and some related dual questions have a positive answer. The purpose of this paper is to extend the results of Bonet, Díaz and Taskinen for arbitrary tensor norms and show that the bounded sets of the α-tensor product $E\hat{\otimes}_\alpha F$ of two (FG)-spaces E and F can be localized in a "canonical sense". Analogously it is proved that the α-tensor product $G \otimes_\alpha H$ of two (DFG)-spaces G and H is always a (DF)-space. This also extends the corresponding results for the Lapresté's α_{pq}-tensor products due to López Molina and Rivera [18] and Junek [16]. It is also shown that, if $(\mathcal{A}, \alpha)$ is a normed ideal of operators in the sense of Pietsch [21], E is an (FG)-space and G is a (DFG)-space, then $\mathcal{A}^w(E, F)$ is a (DF)-space, where $\mathcal{A}^w$ denotes the largest (weak) extension of $\mathcal{A}$. The classes of (FG) and (DFG)-spaces are large as was shown in [5], [6], [9]. Concrete examples are mentioned after Definition 1. Some consequences on the distinguishedness of $E\hat{\otimes}_\alpha F$ and the barrelledness of $G\hat{\otimes}_\alpha H$ for E, F Fréchet spaces and G, H (DF)-spaces are also indicated.

Some of the results here are contained in the Doctoral Thesis of the first author written under the guidance of J. Bonet.

Our notation is standard, we refer to [15], [17] and [19]. If E is a locally convex space (l.c.s.), $\mathcal{U}_0(E)$ and $\mathcal{B}(E)$ stand for the families of all absolutely convex (abx.) 0-neighbourhoods and abx. bounded sets in E respectively. If E is a Fréchet space, we will say that $(U_n)_{n\in\mathbb{N}}$ is a *0-basis* in E if it is a decreasing basis of abx. closed 0-neighbourhoods in E. The absolutely convex hull of a subset A of E is denoted by $\Gamma(A)$. If A is an abx. subset of E, we denote by p_A the Minkowski functional associated with A and $E_A := [A]/\ker p_A$ endowed with the norm induced by p_A. We mean by E'_b the strong dual of E.

If E and F are l.c.s., we denote by $\mathcal{L}(E, F)$ the space of all linear mappings from E into F and $L(E, F)$ is the subspace of continuous linear mappings.

We recall the definitions of (FG)-spaces and (DFG)-spaces given by Bonet, Díaz and Taskinen [6]. The definitions given below are slightly different from the original ones. The reason for this modification is that

these are less restrictive than the original definitions, all the known results for these classes remain valid (see [6, Lemma 6]) even with clearer proofs and, surprisingly, it is possible to obtain more stability properties. Since Bonet, Díaz and Taskinen will use in the future these new definitions for the (FG)-spaces and (DFG)-spaces (personal communication), it will not cause any confusion.

Definition 1. *A Fréchet space E is said to be an (FG)-space if there is a 0-basis $(U_n)_{n\in\mathbb{N}}$ in E such that for every sequence $(\alpha_k)_{k\in\mathbb{N}}$, $\alpha_k \geq 1$ $(k \in \mathbb{N})$ there are a sequence of operators $(P_k)_{k\in\mathbb{N}}$ in $L(E,E)$ and a bounded subset $B \subset E$ satisfying*

(1) $x = \sum_{j\in\mathbb{N}} P_j(x)$, $\forall x \in E$,
(2) $P_n(\alpha_n U_n) \subset B$, $n \in \mathbb{N}$.

A (DF)-space (G,t) is said to be a (DFG)-space if there is an increasing fundamental sequence $(B_k)_{k\in\mathbb{N}}$ of closed abx. bounded sets in G and there is a locally convex topology s in G weaker than t such that (G,t) has a basis of s-closed abx. 0-neighbourhoods and, for every sequence $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, there are a sequence of operators $(Q_k)_{k\in\mathbb{N}}$ in $L((G,t),(G,t))$ and an s-closed abx. 0-neighbourhood U in (G,t) such that

(1) $x = \sum_{j\in\mathbb{N}} Q_j(x)$, $\forall x \in G$, *where the series converges for the topology s,*
(2) $U \subset Q_k^{-1}(\alpha_k B_k)$ *for every* $k \in \mathbb{N}$.

If the topology s can be taken equal to t in the definition we will say that (G,t) is a strong (DFG)-space.

The following are examples of (FG)-spaces [6]:

(i) Banach spaces and countable products of Banach spaces.
(ii) Banach valued Köthe echelon spaces of order p, $1 \leq p < \infty$ or $p = 0$, $\lambda_p(A,(X_i)_{i\in\mathbb{N}})$ with X_i Banach $(i \in \mathbb{N})$.
(iii) Generalized Dubinsky echelon spaces with decreasing steps which are Montel.
(iv) The Fréchet spaces of measurable functions introduced by Reiher, $L_\rho(A)$ with absolutely continuous ρ. In particular the spaces $L_p((\mu_n)_{n\in\mathbb{N}})$ of Grothendieck, $1 \leq p < \infty$, where μ_i are σ-finite measures $(i \in \mathbb{N})$.

(v) Fréchet Schwartz spaces with a finite dimensional decomposition and a continuous norm.

(vi) The Fréchet space of continuous functions $CA_0(X)$.

Moreover, it is easy to see that (with the above definition) complemented subspaces of (FG)-spaces are also (FG)-spaces, then every Fréchet Schwartz space with the bounded approximation property and a continuous norm is an (FG)-space (see [5]).

The following are examples of (DFG)-spaces [6]:

(i) Normed spaces and countable direct sums of normed spaces.

(ii) Strong duals of (FG)-spaces.

(iii) The inductive limit of continuous functions $\operatorname{ind}_n C(v_n)_0(X)$ and the projective hulls $C\bar{V}(X)$, $C\bar{V}_0(X)$.

1. TENSOR PRODUCTS OF FRÉCHET AND (DF)-SPACES

We need some definitions and a technical lemma on tensor norms. We refer to [19] and [8] for the notations.

If E and F are l.c.s. and α is a tensor norm, the topology of $E \otimes_\alpha F$ is given by the system of seminorms $(p_U \otimes_\alpha p_V)(z) := \alpha((\Phi_U \otimes \Phi_V)(z); E_U, F_V)$, $z \in E \otimes F$, $U \in \mathcal{U}_0(E)$, $V \in \mathcal{U}_0(F)$, where $\Phi_U : E \longrightarrow E_U$, $\Phi_V : F \longrightarrow F_V$ are the canonical maps.

If A and B are abx. subsets of E and F, respectively, we will denote by $\alpha(A,B) := \{x \in E \otimes_\alpha F \ / \ (p_A \otimes_\alpha p_B)(x) \leq 1\} = \{x \in [A] \otimes [B] \ / \ \alpha((\Phi_A \otimes \Phi_B)(x); E_A, E_B) \leq 1\}$.

Definition 2. *If E and F are l.c.s., α is a tensor norm and B is a bounded subset of $E\hat{\otimes}_\alpha F$, we will say that B is localizable if there are $C \in \mathcal{B}(E)$, $D \in \mathcal{B}(F)$ such that $B \subset \overline{\alpha(C,D)}$.*

This concept of "localization" of bounded sets in α-tensor products coincides with the usual ones when $\alpha = \pi$ ($\overline{\pi(C,D)} = \overline{\Gamma(C \otimes D)}$) or $\alpha = \alpha_{p,q}$, the Lapresté tensor product (see [18], [16]).

The following lemma is necessary to obtain our main result

Lemma 3.([20]) *If E_i and F_i are l.c.s., $T_i \in \mathcal{L}(E_i, F_i)$, $i = 1,2$; then*

$$(T_1 \otimes T_2)(\alpha(A,B)) \subset \alpha(T_1(A), T_2(B))$$

for every $A \subset E_1$, $B \subset E_2$ *absolutely convex.*

The following proposition improves [6, Corollary 4.6] and [18, Theorem 2].

Proposition 4. *If E and F are (FG)-spaces and α is a tensor norm, then the bounded sets of $E\hat{\otimes}_\alpha F$ are localizable and $E\hat{\otimes}_\alpha F$ is an (FG)-space.*

Proof. By hypothesis, there are $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ 0-basis in E and F, respectively, such that for every sequence $(\lambda_k)_{k\in\mathbb{N}}$, $1 \le \lambda_k < \infty$ ($k \in \mathbb{N}$), there are $B \in \mathcal{B}(E)$, $C \in \mathcal{B}(F)$, $Q_i \in L(E,E)$, and $R_i \in L(F,F)$, $i \in \mathbb{N}$, satisfying:

(a) $\forall i \in \mathbb{N}$

$$Q_i(\lambda_i U_i) \subset \frac{1}{2^i}B, \qquad R_i(\lambda_i V_i) \subset \frac{1}{2^i}C,$$

(b)

$$\sum_1^\infty Q_i = I_E; \quad \sum_1^\infty R_i = I_F \text{ pointwisely.}$$

Defining $P_i := \sum_{j=1}^i Q_i \otimes R_j + \sum_{j=1}^{i-1} Q_j \otimes R_i$, $i \in \mathbb{N}$, we easily obtain $\sum_1^m P_i = (\sum_1^m Q_i)\otimes(\sum_1^m R_i)$ and the sum converges pointwisely to $I_{E\otimes F}$ when m tends to infinity. On the other hand, the equicontinuity of $(P_i)_{i\in\mathbb{N}}$ follows from the equicontinuity of $(Q_i)_{i\in\mathbb{N}}$ and $(R_i)_{i\in\mathbb{N}}$, and this implies that $\sum_1^\infty \hat{P}_i$ converges pointwisely to $I_{E\hat{\otimes}_\alpha F}$, where $\hat{P}_i$ is the extension to $E\hat{\otimes}_\alpha F$ of P_i, $i \in \mathbb{N}$.

Setting $H := \overline{\alpha(B,C)}^{E\hat{\otimes}_\alpha F}$, we obtain

$$\hat{P}_i(\lambda_i\alpha(U_i,V_i)) \subset \left(\sum_{j=1}^i Q_i\hat{\otimes}R_j\right)(\alpha(\lambda_i U_i, \lambda_j V_j)) + \left(\sum_{j=1}^{i-1} Q_j\hat{\otimes}R_i\right)(\alpha(\lambda_j U_j, \lambda_i V_i))$$

$$\subset \frac{1}{2^i}\sum_{j=1}^{i}\frac{1}{2^j}H + \frac{1}{2^i}\sum_{j=1}^{i-1}\frac{1}{2^j}H \subset \frac{1}{2^{i-1}}H,$$

$i = 1,\ldots,\infty$. Therefore $E\hat{\otimes}_\alpha F$ is an (FG)-space and, moreover, if $x \in \bigcap_1^\infty \lambda_i \overline{\alpha(U_i,V_i)}^{E\hat{\otimes}_\alpha F}$, then

$$\left(\sum_1^m \hat{P}_i\right)(x) \in \left(\sum_1^m \frac{1}{2^{i-1}}\right)H.$$

Thus, since $x = \lim_{m\to\infty}\sum_1^m \hat{P}_i(x) \in 2H$,

$$\bigcap_1^\infty \lambda_i \overline{\alpha(U_i,V_i)}^{E\hat{\otimes}_\alpha F} \subset 2H,$$

which implies that the bounded sets of $E\hat{\otimes}_\alpha F$ are localizable. ■

The following technical lemma is essential to show that $G \otimes_\alpha H$ and $\mathcal{A}^w(E,G)$ are (DF)-spaces whenever E is an (FG)-space and G, H are (DFG)-spaces. It provides a useful property which is very close to (DFop) introduced in [7].

Lemma 5. *Let E be an (FG)-space and G a (DFG)-space. There are two sequences: $(V_n)_{n\in\mathbb{N}}$ a 0-basis in E and $(B_n)_{n\in\mathbb{N}}$ a fundamental sequence of bounded sets in G such that*

(a) For every sequences $(\epsilon_k)_{k\in\mathbb{N}}$ of strictly positive escalars and $(C_n)_{n\in\mathbb{N}}$ of bounded sets in E, there are $C \in \mathcal{B}(E)$ and $(Q_k)_{k\in\mathbb{N}} \subset L(E,E)$ such that

$$Q_k(V_k) \subset \epsilon_k C, \quad (I - \sum_{k=1}^{n} Q_k)(C_n) \subset C, \quad k,n \in \mathbb{N}.$$

(b) For every sequences $(\epsilon_k)_{k\in\mathbb{N}}$ of strictly positive escalars and $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in G, there are $U \in \mathcal{U}_0(G)$ and $(R_k)_{k\in\mathbb{N}} \subset L(G,G)$ such that

$$R_k(U) \subset \epsilon_k B_k, \quad (I - \sum_{k=1}^{n} R_k)(U) \subset U_n, \quad k,n \in \mathbb{N}.$$

Proof. We will show (b). The proof of (a) is analogous.

Since (G,t) is a (DFG)-space, it has a fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex bounded subsets and there is a locally convex topology s on G, coarser than t, such that (G,t) has a basis of s-closed absolutely convex 0-neighbourhoods and for every sequence $(\delta_k)_{k\in\mathbb{N}}$ of strictly positive escalars there is a sequence $(R_k)_{k\in\mathbb{N}} \subset L(G,G)$ such that

(i) $x = \sum_{k\in\mathbb{N}} R_k(x)\ \forall x \in G$, where the series converges for the topology s,

(ii) $\bigcap_{k\in\mathbb{N}} R_k^{-1}(\delta_k B_k) \in \mathcal{U}_0(G)$.

Given two sequences: $(\epsilon_k)_{k\in\mathbb{N}}$ of strictly positive numbers and $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in G (which we can suppose s-closed), we can find another sequence $(\delta_k)_{k\in\mathbb{N}}$ such that $0 < \delta_k < \epsilon_k$, $k \in \mathbb{N}$, and $\sum_{k=n}^{m} \delta_k B_k \subset U_n$, $n \in \mathbb{N}$, $m \geq n$. Now choose $(R_k)_{k\in\mathbb{N}}$ satisfying (i) and (ii), and define $U := \bigcap_{k\in\mathbb{N}} R_k^{-1}(\delta_k B_k)$. Then we have

$$x - \sum_{k=1}^{n} R_k(x) = s - \lim_{m\to\infty} \sum_{k=n+1}^{m} R_k(x) \in U_n,$$

for all $x \in U$, since U_n is s-closed, $n \in \mathbb{N}$. ■

We also need an easy but key lemma due to J. Bonet and the first author (see e.g. [11, Theorem 5.8]). A similar argument can be found in [15].

Lemma 6. *Let E be a l.c.s. and $(B_n)_{n\in\mathbb{N}}$ a sequence of absolutely convex bounded sets in E with $2B_n \subset B_{n+1}$, $n \in \mathbb{N}$, which satisfies the following property:*

For every sequence $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in E there is $U \in \mathcal{U}_0(E)$ such that

$$U \subset \bigcap_{n=1}^{\infty} (U_n + B_n).$$

Then $(\bar{B}_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in E.

Proof. Let us suppose that there is $B \in \mathcal{B}(E)$ such that $B \not\subset n\bar{B}_n$, $\forall n \in \mathbb{N}$. Then we can find a bounded sequence $(z_n)_{n\in\mathbb{N}}$ in E with

$z_n \notin n\bar{B}_n$ and, hence, there exists $U_n \in \mathcal{U}_0(E)$, $n \in \mathbb{N}$, such that $z_n \notin n(B_n + U_n)$. But $\bigcap_{n\in\mathbb{N}}(B_n + U_n)$ is a neighbourhood of 0 in E, which contradicts the fact that $(z_n)_{n\in\mathbb{N}}$ is bounded.

Thus there is $n_0 \in \mathbb{N}$ with $B \subset n_0\bar{B}_{n_0} \subset \bar{B}_{2n_0}$. ■

From this lemma and the characterization of (DF)-spaces given by Bierstedt and Bonet in [3, 5.A.Lemma], we deduce (see [11, 5.8]) that a l.c.s. E is a (DF)-space if and only if it has a sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex bounded subsets such that for every sequences $(\epsilon_k)_{k\in\mathbb{N}}$ of strictly positive numbers and $(U_n)_{n\in\mathbb{N}}$ of absolutely convex 0-neighbourhoods, there is $U \in \mathcal{U}_0(E)$ such that

$$U \subset \bigcap_{n\in\mathbb{N}} (\sum_{k=1}^{n} \epsilon_k B_k + U_n).$$

This will be useful in the following

Proposition 7. *If α is a tensor norm and G, H are (DFG)-spaces (respectively, strong (DFG)-spaces) then $G \otimes_\alpha H$ is a (DF)-space (resp., a strong (DFG)-space).*

Proof. Let $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ be the fundamental sequences of bounded sets in G and H, respectively, which satisfy the property (DFG).

Given a sequence of strictly positive escalars $(\epsilon_k)_{k\in\mathbb{N}}(\epsilon_k < 1, k \in \mathbb{N})$ and a sequence $(W_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in $G \otimes_\alpha H$, find two decreasing sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in G and H, respectively, such that

$$(i) \qquad \alpha(U_n, V_n) \subset W_n,$$

$$(ii) \qquad \alpha(\epsilon_j B_j, V_n) \subset W_n,$$

$$(iii) \qquad \alpha(U_n, \epsilon_j C_j) \subset W_n,$$

$n \in \mathbb{N}$, $j = 1 \ldots n$. By Lemma 5, there are two sequences $(Q_k)_{k\in\mathbb{N}} \subset L(G,G)$ and $(R_k)_{k\in\mathbb{N}} \subset L(H,H)$ and there are $U \in \mathcal{U}_0(G)$ and $V \in \mathcal{U}_0(H)$ such that

$$(j) \qquad Q_k(U) \subset \frac{\epsilon_k}{2^k} B_k; \quad R_k(V) \subset \frac{\epsilon_k}{2^k} C_k,$$

$$(jj) \quad (I - \sum_{k=1}^{n} Q_k)(U) \subset \frac{1}{3} U_n; \quad (I - \sum_{k=1}^{n} R_k)(V) \subset \frac{1}{3} V_n,$$

$n, k \in \mathbb{N}$. Now we claim that

$$\alpha(U,V) \subset \bigcap_{n \in \mathbb{N}} (\sum_{k=1}^{n} \epsilon_k \alpha(B_k, C_k) + W_n)$$

To show this, take $x \in \alpha(U,V)$. Given $n \in \mathbb{N}$, define

$$x_k := \left(\sum_{j=1}^{k} Q_k \otimes R_j + \sum_{j=1}^{k-1} Q_j \otimes R_k \right)(x), \quad k = 1 \ldots n.$$

Since $Q_k(U) \subset \frac{\epsilon_k}{2^k} B_k$ and $R_j(V) \subset \frac{\epsilon_j}{2^j} C_j$, we have that $(Q_k \otimes R_j)(x) \in \frac{\epsilon_k \epsilon_j}{2^{k+j}} \alpha(B_k, C_j)$, $j, k \in \mathbb{N}$. Hence

$$x_k \in \sum_{j=1}^{k} \frac{\epsilon_k}{2^{1+j}} \alpha(B_k, C_j) + \sum_{j=1}^{k-1} \frac{\epsilon_k}{2^{1+j}} \alpha(B_j, C_k)$$

$$\subset \epsilon_k \left(\sum_{j=1}^{k} \frac{1}{2^{1+j}} \alpha(B_k, C_j) + \sum_{j=1}^{k-1} \frac{1}{2^{1+j}} \alpha(B_j, C_k) \right) \subset \epsilon_k \alpha(B_k, C_k),$$

$k = 1 \ldots n$. Define now $y_n := x - \sum_{k=1}^{n} x_k$. It easily follows that

$$y_n = \left(\sum_{j=1}^{n} (I - \sum_{k=1}^{n} Q_k) \otimes R_j \right)(x) + \left(\sum_{j=1}^{n} Q_j \otimes (I - \sum_{k=1}^{n} R_k) \right)(x)$$

$$+\Big((I-\sum_{k=1}^{n}Q_k)\otimes(I-\sum_{k=1}^{n}R_k)\Big)(x).$$

Finally, from (i), (ii), (iii), (j) and (jj), we obtain

$$(1)\quad ((I-\sum_{k=1}^{n}Q_k)\otimes R_j)(x)\in\frac{1}{3}\cdot\frac{1}{2^j}\alpha(U_n,\epsilon_jC_j)\subset\frac{1}{3}\cdot\frac{1}{2^j}W_n,$$

$$(2)\quad (Q_j\otimes(I-\sum_{k=1}^{n}R_k))(x)\in\frac{1}{3}\cdot\frac{1}{2^j}\alpha(\epsilon_jB_j,V_n)\subset\frac{1}{3}\cdot\frac{1}{2^j}W_n,$$

$$(3)\quad \Big((I-\sum_{k=1}^{n}Q_k)\otimes(I-\sum_{k=1}^{n}R_k)\Big)(x)\in\frac{1}{9}\alpha(U_n,V_n)\subset\frac{1}{3}W_n,$$

$j=1\dots n$, which implies that $y_n\in W_n$. This shows that $G\otimes_\alpha H$ is a (DF)-space by lemma 6.

If G and H are strong (DFG)-spaces, to prove that $G\otimes_\alpha H$ is a strong (DFG)-space, it remains to show that $\lim_n\sum_{k=1}^{n}x_k=x$. But this is a consequence of the fact that $(\sum_{k=1}^{n}Q_k)_{n\in\mathbb{N}}$ and $(\sum_{k=1}^{n}R_k)_{n\in\mathbb{N}}$ converge pointwisely to the identity of G and H, respectively. ■

Remarks: (1) For finitely generated tensor norms α, it is possible to give a shorter and more elegant argument to show that $G\otimes_\alpha H$ is a (DF)-space whenever G and H are (DFG)-spaces. The proof was kindly provided by Andreas Defant: By [8, 29.2, 29.7, 29.8 and 35.3], there is a Banach space X and a surjective topological homomorphism

$$\Psi:(G\otimes_\varepsilon X^*)\otimes_\pi(X\otimes_\alpha H)\longrightarrow G\otimes_\alpha H.$$

Since G is a (DFG)-space, the space $G\otimes_\epsilon X^*$ is a (DF)-space (see [6, Proposition 9]). By the same argument, $H\otimes_\alpha X$ is a quotient of $(H\otimes_\varepsilon X^*)\otimes_\pi(X\otimes_\alpha X)$, which is a (DF)-space since H is a (DFG)-space

and the projective tensor product of two (DF)-spaces is a (DF)-space. This implies that $G \otimes_\alpha H$ is also a (DF)-space.

(2) Note that, by Lemma 6, we have shown in the above proposition that the bounded subsets of $G \otimes_\alpha H$ are localizable if G and H are (DFG)-spaces.

In order to obtain more properties, such as distinguishedness for α-tensor products of Fréchet spaces or barrelledness for α-tensor products of (DF)-spaces, we need some definitions and notation.

The *density condition*, (DC), was introduced by Heinrich [14] and it was thoroughly studied by Bierstedt and Bonet (see [1], [2], [3], [4]) in the context of Fréchet spaces. A Fréchet space E has the density condition if and only if the bounded sets of E'_b are metrizable [1]. Quasinormable spaces and Fréchet Montel spaces have the (DC). Every Fréchet space E with (DC) is distinguished, that is E'_b is bornological. Bierstedt and Bonet also introduced the *dual density condition*, (DDC), (see [2]). A (DF)-space G has the (DDC) if and only if its bounded sets are metrizable, and every (DF)-space with (DDC) is quasibarrelled (see [2]).

The (DC) and (DDC) are stable under projective and injective tensor products in the context of (FG)-spaces and barrelled strong (DFG)-spaces, respectively (see [4], [9]). Our purpose is to generalize these results for arbitrary tensor norms. We need the following

Definition 8. *Let E be a Fréchet space and $(U_n)_{n\in\mathbb{N}}$ a 0-basis in E. E is said to satisfy the density condition by operators, briefly (DCo), if there is $B \in \mathcal{B}(E)$ such that for all $m \in \mathbb{N}$ and $B' \in \mathcal{B}(E)$, there exist $\lambda > 0$ and $P \in L(E,E)$ such that*

(i) $P(B') \subset U_m$,
(ii) $(I-P)(B') \subset \lambda B$.

Let G be a (DF)-space and $(B_n)_{n\in\mathbb{N}}$ a fundamental sequence of bounded sets in G. G is said to satisfy the dual density condition by operators, briefly (DDCo), if there is $U \in \mathcal{U}_0(G)$ such that for all $m \in \mathbb{N}$ and $V \in \mathcal{U}_0(G)$, there are $\varepsilon > 0$ and $P \in L(G,G)$ such that

(i) $P(B_m) \subset V$,
(ii) $(I-P)(\varepsilon U) \subset V$.

Remark: It easily follows from these definitions and the characterizations of (DC) and (DDC) due to Bierstedt and Bonet in the context of Fréchet spaces and (DF)-spaces, that every Fréchet space E (respectively, (DF)-space G) with (DCo) (resp., (DDCo)) satisfies the (DC) (resp., (DDC)).

Proposition 9. *Every (FG)-space with the (DC) satisfies property (DCo).*

Proof. Let E be an (FG)-space with a 0-basis $(U_n)_{n\in\mathbb{N}}$ and $(P_n)_{n\in\mathbb{N}}$ any sequence of bounded operators in E associated to some sequence of strictly positive scalars by property (FG). Then, as a consequence of [10], $(P_n)_{n\in\mathbb{N}}$ is a resolution of the identity of E with property (M) of Díaz and Miñarro, that is, for every $B \in \mathcal{B}(E)$ and every $U \in \mathcal{U}_0(E)$, we have

$$\lim_{n\to\infty} \sup_{x\in B} p_U \left(\sum_{j=n}^{\infty} P_j(x) \right) = 0.$$

Now, since $(P_n)_{n\in\mathbb{N}}$ is a sequence of bounded operators, we can find $B \in \mathcal{B}(E)$ such that $P_n^{-1}(B) \in \mathcal{U}_0(E)$, $\forall n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $B' \in \mathcal{B}(E)$. Then, by property (M), there is $n_0 \in \mathbb{N}$ such that, for $P := \sum_{n_0+1}^{\infty} P_n$, the following is satisfied

$$P(B') = \sum_{n_0+1}^{\infty} P_n(B') \subset U_m.$$

Moreover,

$$(I-P)^{-1}(B) = \left(\sum_{1}^{n_0} P_n \right)^{-1} (B) \in \mathcal{U}_0(E),$$

which yields the result. ■

Proposition 10. *If G is a barrelled strong (DFG)-space with (DDC), then it satisfies (DDCo).*

Proof. Let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded subsets in G and $(P_n)_{n\in\mathbb{N}}$ any sequence of bounded operators in G associated

to some sequence of strictly positive scalars by property (DFG). Then, as a consequence of [9], $(P_n)_{n\in\mathbb{N}}$ is a resolution of the identity of G with property (M).

Now, since $(P_n)_{n\in\mathbb{N}}$ is a sequence of bounded operators, we can find $U \in \mathcal{U}_0(G)$ such that $P_n(U) \in \mathcal{B}(G)$, $\forall n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $V \in \mathcal{U}_0(G)$. Then, by property (M), there is $n_0 \in \mathbb{N}$ such that, for $P := \sum_1^{n_0} P_n$, the following is satisfied

$$(I - P)(B_m) = \sum_{n_0+1}^{\infty} P_n(B_m) \subset V.$$

On the other hand,

$$P(U) = \sum_{1}^{n_0} P_n(U) \in \mathcal{B}(G),$$

and we obtain the result. ■

Díaz, López Molina and Rivera showed in [9] that the strong duals of (FG)-spaces with (DC) are examples of barrelled strong (DFG)-spaces with (DDC).

Proposition 11. *If α is a tensor norm, E and F are Fréchet spaces (respectively, (DF)-spaces) with the (DCo) (resp., (DDCo)) and the bounded subsets of $E\hat{\otimes}_\alpha F$ are localizable, then $E\hat{\otimes}_\alpha F$ also satisfies (DCo) (resp., (DDCo)).*

Proof. We will show the result for (DCo). The proof for (DDCo) is analogous.

Let $(U_k)_{k\in\mathbb{N}}$ and $(V_k)_{k\in\mathbb{N}}$ 0-basis in E and F, respectively. If E and F satisfy (DCo), then there are $B \in \mathcal{B}(E)$ and $C \in \mathcal{B}(F)$ such that for all $m, m' \in \mathbb{N}$ and for all $B' \in \mathcal{B}(E)$, $C' \in \mathcal{B}(F)$, there are $\lambda, \lambda' > 0$ and $P \in L(E,E)$, $Q \in L(F,F)$ which satisfy

(i) $P(B') \subset U_m, \quad (I - P)(B') \subset \lambda B.$
(ii) $Q(C') \subset V_{m'}, \quad (I - Q)(C') \subset \lambda C.$

Let $n \in \mathbb{N}$, $B' \in \mathcal{B}(E)$ and $C' \in \mathcal{B}(F)$. Find $m \in \mathbb{N}$ $(m \geq n)$ with $2U_m \subset \varepsilon U_n$, where $\varepsilon > 0$ satisfies $\varepsilon C' \subset V_n$, and find $m' \in \mathbb{N}$ $(m' \geq n)$ with $2V_{m'} \subset \varepsilon' V_n$, where $\varepsilon' > 0$ satisfies $\varepsilon'\lambda B \subset U_n$. Then we have

(a)

$$[(I-P)\otimes(I-Q)](\alpha(B',C')) \subset \alpha(U_m,V_{m'}) \subset \alpha(U_n,V_n).$$

(b)

$$[I-(I-P)\otimes(I-Q)](\alpha(B',C')) = [(I-P)\otimes Q + P\otimes I](\alpha(B',C')) \subset$$

$$\alpha(\lambda B,V_{m'}) + \alpha(U_m,C') \subset \frac{1}{2}\alpha(U_n,V_n) + \frac{1}{2}\alpha(U_n,V_n) = \alpha(U_n,V_n).$$

From this we conclude that $E\hat{\otimes}_\alpha F$ satisfies (DCo). ■

Corollary 12. *If α is a tensor norm and E, F are (FG)-spaces with the (DC) (respectively, barrelled strong (DFG)-spaces with the (DDC)), then $E\hat{\otimes}_\alpha F$ is a Fréchet space with (DC), hence distinguished (respectively, a barrelled (DF)-space with (DDC)).*

It is worth to mention that the Fréchet spaces *quasinormable by operators* (respectively, the (DF)-spaces with property *(QNo)'*), introduced and studied in [20], are examples of Fréchet spaces with (DCo) (resp., (DF)-spaces with (DDCo)).

2. IDEALS OF OPERATORS AND (DF)-SPACES

Following Piestch [21], the following properties define a *normed ideal* $(\mathcal{A},\alpha)$ of operators on the class of Banach spaces:

(I1) $\mathcal{F}(X,Y) \subset \mathcal{A}(X,Y) \subset L(X,Y)$, for all Banach spaces X, Y, where $\mathcal{F}(X,Y)$ denotes the space of all finite rank operators.

(I2) If $S \in \mathcal{A}(X,Y)$, $T \in L(Z_1,X)$ and $R \in L(Y,Z_2)$, then $R\circ S\circ T \in \mathcal{A}(Z_1,Z_2)$.

(I3) α is a complete norm on each component $\mathcal{A}(X,Y)$ satisfying $\alpha(x'\otimes y) = \| x' \| \| y \|$, $x' \in X^*, y \in Y$, and $\alpha(R\circ S\circ T) \leq \| R \| \| T \| \alpha(S)$ for all operators R,S,T as in (I2).

Among the several possibilities to extend an ideal $\mathcal{A}$ of operators on Banach spaces to the category of locally convex spaces, we will concentrate on the largest extension $\mathcal{A}^w$ (see [21]). More precisely, if E and F are locally convex spaces (E quasi-complete) and $S \in L(E,F)$,

then $S \in \mathcal{A}^w(E,F)$ if, and only if, for all Banach spaces X, Y and for all operators $T \in L(X,E)$, $R \in L(F,Y)$ the product $R \circ S \circ T$ belongs to $\mathcal{A}(X,Y)$. That is, for all $B \in \mathcal{B}(E)$ and $U \in \mathcal{U}_0(F)$ the operator $\Phi_U \circ S \circ \Phi_B$ belongs to $L(\hat{E}_B, \hat{F}_U)$, where $\Phi_B : \hat{E}_B \to E$ and $\Phi_U : F \to \hat{F}_U$ are the canonical mappings.

The space $\mathcal{A}^w(E,F)$ admits a natural topology induced by proj $\mathcal{A}(\hat{E}_B, \hat{F}_U)$. In order to give a description of a basis of 0-neighbourhoods, let us establish the following notation: If $A \subset E$, $B \subset F$ are absolutely convex and $S \in L(E,F)$ satisfies $S(A) \subset B$, we denote by $S_{A,B} : \hat{E}_A \to \hat{F}_B$ the induced map. Now we write

$$\mathcal{W}(A,B) := \{S \in \mathcal{A}^w(E,F) \ / \ S(A) \subset B, \ \ S_{A,B} \in \mathcal{A}(\hat{E}_A, \hat{F}_B)$$

$$\text{and} \ \ \alpha(S_{A,B}) \leq 1\}.$$

With this notation, the family $\{\mathcal{W}(B,U) \ / \ B \in \mathcal{B}(E), \ U \in \mathcal{U}_0(F)\}$ is a basis of 0-neighbourhoods in $\mathcal{A}^w(E,F)$.

For instance, if $\mathcal{A}$ is the ideal of all bounded operators and α is the operator norm, then $\mathcal{A}^w(E,F)$ is the space $L(E,F)$ of all continuous linear operators endowed with the topology of uniform convergence on the bounded subsets of E. And, for $\mathcal{A} = \mathcal{K}$ the ideal of compact operators, $\mathcal{K}^w(E,F)$ coincides with the space of Montel operators from E into F, that is, the operators that send each bounded subset of E into a precompact set in F.

Remark: As a consequence of (I2) and (I3), it is easily seen that, for $A_1, A_2 \subset E$ and $B_1, B_2 \subset F$ absolutely convex, $S \in \mathcal{W}(A_2, B_1)$ and $T \in L(E,E)$, $R \in L(F,F)$ satisfying $T(A_1) \subset A_2$ and $R(B_1) \subset B_2$, the product $R \circ S \circ T$ belongs to $\mathcal{W}(A_1, B_2)$. This will be used later on.

The following theorem should be compared with [6, Theorem 4.5], where Bonet, Díaz and Taskinen show that, under the same hypothesis, the space $L_b(E,G)$ is a (DF)-space using a similar argument.

Theorem 13. *Let $(\mathcal{A}, \alpha)$ be a normed ideal of operators in Banach spaces, E an (FG)-space and G a (DFG)-space. Then $\mathcal{A}^w(E,G)$ is a (DF)-space.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be the 0-basis and fundamental sequence of bounded sets in E and G respectively, which satisfy the corresponding properties (FG) and (DFG).

Given a sequence of strictly positive escalars $(\epsilon_k)_{k\in\mathbb{N}}(\epsilon_k < 1, k \in \mathbb{N})$ and a sequence $(W_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in $\mathcal{A}^\omega(E,G)$, find a sequence $(C_n)_{n\in\mathbb{N}}$ of bounded sets in E and a sequence $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in G such that

$$(i) \qquad \mathcal{W}(C_n, U_n) \subset W_n,$$

$$(ii) \qquad \mathcal{W}(\epsilon_j^{-1}V_j, U_n) \subset W_n,$$

$$(iii) \qquad \mathcal{W}(C_n, \epsilon_j B_j) \subset W_n,$$

$n \in \mathbb{N}$, $j = 1 \ldots n$. By Lemma 5, there are two sequences $(Q_k)_{k\in\mathbb{N}} \subset L(E,E)$ and $(R_k)_{k\in\mathbb{N}} \subset L(G,G)$ and there are $C \in \mathcal{B}(E)$ and $U \in \mathcal{U}_0(G)$ such that

$$(j) \qquad Q_k(V_k) \subset \frac{\epsilon_k}{2^k}C; \quad R_k(U) \subset \frac{\epsilon_k}{2^k}B_k,$$

$$(jj) \quad (I - \sum_{k=1}^{n} Q_k)(C_n) \subset \frac{1}{3}C; \quad (I - \sum_{k=1}^{n} R_k)(U) \subset \frac{1}{3}U_n,$$

$n, k \in \mathbb{N}$. Now we claim that

$$\mathcal{W}(C,U) \subset \bigcap_{n\in\mathbb{N}} (\sum_{k=1}^{n} \epsilon_k \mathcal{W}(V_k, B_k) + W_n)$$

To show this, take $f \in \mathcal{W}(C,U)$. Given $n \in \mathbb{N}$, define

$$f_k := \sum_{j=1}^{k} R_k \circ f \circ Q_j + \sum_{j=1}^{k-1} R_j \circ f \circ Q_k, \qquad k = 1 \ldots n.$$

Since $R_k(U) \subset \frac{\epsilon_k}{2^k}B_k$ and $Q_j(V_j) \subset \frac{\epsilon_j}{2^j}C$, we have that $R_k \circ f \circ Q_j \in \frac{\epsilon_k \epsilon_j}{2^{k+j}}\mathcal{W}(V_j, B_k)$, $j, k \in \mathbb{N}$. Hence

$$f_k \in \sum_{j=1}^{k} \frac{\epsilon_k}{2^{1+j}}\mathcal{W}(V_j, B_k) + \sum_{j=1}^{k-1} \frac{\epsilon_k}{2^{1+j}}\mathcal{W}(V_k, B_j)$$

$$\subset \epsilon_k \left(\sum_{j=1}^{k} \frac{1}{2^{1+j}} \mathcal{W}(V_j, B_k) + \sum_{j=1}^{k-1} \frac{1}{2^{1+j}} \mathcal{W}(V_k, B_j) \right) \subset \epsilon_k \mathcal{W}(V_k, B_k),$$

$k = 1 \ldots n$. Define now $g_n := f - \sum_{k=1}^{n} f_k$. It easily follows that

$$g_n = \sum_{j=1}^{n} (I - \sum_{k=1}^{n} R_k) \circ f \circ Q_j + \sum_{j=1}^{n} R_j \circ f \circ (I - \sum_{k=1}^{n} Q_k)$$

$$+ (I - \sum_{k=1}^{n} R_k) \circ f \circ (I - \sum_{k=1}^{n} Q_k).$$

Finally, from (i), (ii), (iii), (j) and (jj), we obtain

$$(1) \quad (I - \sum_{k=1}^{n} R_k) \circ f \circ Q_j \in \frac{1}{3} \cdot \frac{1}{2^j} \mathcal{W}(\epsilon_j^{-1} V_j, U_n) \subset \frac{1}{3} \cdot \frac{1}{2^j} W_n,$$

$$(2) \quad R_j \circ f \circ (I - \sum_{k=1}^{n} Q_k) \in \frac{1}{3} \cdot \frac{1}{2^j} \mathcal{W}(C_n, \epsilon_j B_j) \subset \frac{1}{3} \cdot \frac{1}{2^j} W_n,$$

$$(3) \quad (I - \sum_{k=1}^{n} R_k) \circ f \circ (I - \sum_{k=1}^{n} Q_k) \in \frac{1}{9} \mathcal{W}(C_n, U_n) \subset \frac{1}{3} W_n,$$

$j = 1 \ldots n$, which implies that $g_n \in W_n$. ■

Remark: Given $(\mathcal{A}, \alpha)$ a normed ideal of operators, there is another interesting extension of $\mathcal{A}$ to the category of locally convex spaces, the smallest (strong) extension $\mathcal{A}^s$ (see [21]): If E, F are l.c.s. with F quasicomplete and $S \in L(E, F)$, then $S \in \mathcal{A}^s(E, F)$ if, and only if, there is some factorization $S = R_0 \circ S_0 \circ T_0$, where $T_0 \in L(E, X_0)$, $R_0 \in L(Y_0, F)$ and $S_0 \in \mathcal{A}(X_0, Y_0)$ for certain Banach spaces X_0, Y_0. The Banach spaces X_0, Y_0 can be choosen to be the canonical Banach spaces $\hat{E_U}$, $\hat{F_B}$ for some $U \in \mathcal{U}_0(E)$, $B \in \mathcal{B}(F)$. Since

$$\mathcal{A}^s(E,F) = \bigcup_{\substack{U\in\mathcal{U}_0(E)\\ B\in\mathcal{B}(F)}} \mathcal{A}(\hat{E}_U,\hat{F}_B),$$

the natural topology on $\mathcal{A}^s(E,F)$ is the locally convex inductive topology ind $\mathcal{A}(\hat{E}_U,\hat{F}_B)$.

Obviously $\mathcal{A}^s(E,F)$ is continuously embeded in $\mathcal{A}^w(E,F)$. Now, following the proof of the above theorem, if E is an (FG)-space and G is a quasicomplete (DFG)-space, the bounded sets of $\mathcal{A}^w(E,G)$ are contained in the closure (in $\mathcal{A}^w(E,G)$) of the bounded subsets of $\mathcal{A}^s(E,G)$ by Lemma 6. This can have consequences on the algebraic (or even topological) equality $\mathcal{A}^s(E,G) = \mathcal{A}^w(E,G)$.

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