LOCALIZATION OF PERIODIC ORBITS
OF AUTONOMOUS SYSTEMS BASED ON
HIGH-ORDER EXTREMUM CONDITIONS

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This paper gives localization and nonexistence conditions of periodic orbits in some subsets of the state space. Mainly, our approach is based on high-order extremum conditions, on high-order tangency conditions of a nonsingular solution of a polynomial system with an algebraic surface, and on some ideas related to algebraically-dependent polynomials. Examples of the localization analysis of periodic orbits are presented including the Blasius equations, the generalized mass action (GMA) system, and the mathematical model of the chemical reaction with autocatalytic step.

1. Introduction

The study of periodic orbits has a significant impact on understanding the dynamics of multidimensional systems. Apart from its purely theoretical importance, this research area has various engineering applications, for example, in chemical engineering, electrical engineering, and so forth. One of the interesting problems of qualitative theory of ordinary differential equations

$$\dot{x} = f(x), \tag{1.1}$$

with $x = (x_1,\dots,x_n)^T \in \mathbb{R}^n$, $f(x) = (f_1(x),\dots,f_n(x))^T \in C^\infty(\mathbb{R}^n)$, consists in finding a domain in the state space containing all periodic orbits. It is well known that this problem is difficult, for example, in the case of a search for unstable periodic orbits of multidimensional systems (1.1). Though many papers devoted to this topic for two-dimensional systems have been published, see, for example, the references in [18, 30], one can give only a short list of publications concerning periodic orbits of three- and higher-dimensional systems. Here, one can recall the papers [1, 2, 4, 7, 8, 10, 11, 14, 22, 23, 24, 27]. It is worthwhile to mention that the localization of periodic orbits is closely related to the similar problem for attractors. The idea of the characterization of a chaotic attractor **A** by means of infinitely many unstable periodic orbits embedded in **A** has been used in a number of articles, see, for example, [5] concerning the Lorenz system and the references therein.

Thus the localization of domains with periodic orbits can provide information respecting a location of domains with chaotic attractors.

Let $h \in C^{\infty}(\mathbb{R}^n)$ and $L_f h$ be the Lie derivative of h along the vector field f. Also, $L_f^i h = L_f(L_f^{i-1}h)$, $i \geq 1$, $L_f^0 h = h$. The function h will be used in the solution of the localization problem. If it is not specified explicitly, we assume that f is a polynomial vector field and h is a polynomial. The main objective of this paper is to find some sets K in $\mathbb{R}^n$ in terms of conditions imposed on f and h such that (1) any periodic orbit of (1.1) is contained in K, with K a domain, or (2) any periodic orbit has common points with K; here K is a surface. Further, by using these sets, we propose nonexistence conditions of periodic orbits in some domains of the state space. Here, by saying that a periodic orbit Γ is not contained in some set U, we mean that Γ is not *totally* contained in U.

This paper is the extended version of the short conference paper [28].

The structure of this paper is as follows. Section 2 presents the main localization/nonexistence result of periodic orbits and some relevant assertions. In Section 3, we are concerned with the relationship between invariant sets and periodic orbits. The localization of periodic orbits for a class of systems (1.1) satisfying one triangularizability condition is discussed in Section 4. Section 5 contains a refinement of the main result under one condition expressed in terms of equilibrium points. In Section 6, we address applications of our results to the Blasius equations, the generalized mass action (GMA) system, and the mathematical model of the chemical reaction with autocatalytic step. General comments respecting a solution of the localization problem are contained in Section 7. Section 8 includes our concluding remarks.

2. The main result and some corollaries

By $\varphi(x,t)$ we denote a solution of system (1.1) such that $\varphi(x,0) = x$. Let f_s, $s = 1,\ldots,n$, and $h \in C^{\infty}(\mathbb{R}^n)$. We introduce

(1) sets $K_1 = \{x \in \mathbb{R}^n \mid L_f h(x) = 0\}$, $K_1^+ = \{x \in \mathbb{R}^n \mid L_f h(x) > 0\}$, $K_1^- = \{x \in \mathbb{R}^n \mid L_f h(x) < 0\}$;

(2) sets K_j defined with the help of the following system of equations:

$$L_f^i h(x) = 0, \quad i = 1,\ldots, j; \tag{2.1}$$

(3) sets K_k^+ (K_k^-) defined with the help of (2.1) for $j = k - 1$ and one inequality $L_f^k h(x) > 0$ ($L_f^k h(x) < 0$ appropriately); $K_k^{\pm} = K_k^+ \cup K_k^-$, $k \geq 2$;

(4) sets

$$\begin{aligned} S_k^+ &= K_2^+ \cup K_3^{\pm} \cup \cdots \cup K_{2j-1}^{\pm} \cup K_{2j}^+, \quad k = 2j, \\ S_k^+ &= K_2^+ \cup K_3^{\pm} \cup \cdots \cup K_{2j-1}^{\pm} \cup K_{2j}^+ \cup K_{2j+1}^{\pm}, \quad k = 2j+1. \end{aligned} \tag{2.2}$$

Similarly, sets S_k^- are defined with the help of a simultaneous replacement of all "+" by "−" in (2.2). Though all sets introduced here depend on h, this dependence will be usually suppressed.

Further, by $C\{V\}$ we denote the complement of the set V in the space $\mathbb{R}^n$. If g is some function of n real variables, then we denote by $g|_{\{V\}}$ the restriction of g on V. Let S be any

set in $\mathbb{R}^n$. Then by $\mathrm{Inv}(S)$ we denote the maximal (with respect to the inclusion) invariant subset of S of the phase flow of system (1.1).

It follows from the construction of sets K_s and local analyticity of the solution that

$$K_1 \supset K_2 \supset \cdots \supset K_s \supset \cdots \supset f^{-1}(0) \tag{2.3}$$

and $\cap_{s=1}^{\infty} K_s = \mathrm{Inv}\, K_1$.

Further, the following fact plays the key role.

LEMMA 2.1. *Let f in (1.1) and $h \in C^{\infty}(\mathbb{R}^n)$. If the point x_0 is contained in the periodic orbit Γ and $h(\varphi(x_0,t))$ is not constant as a time function, then there are, at least, two instants $t_{\max}, t_{\min} \geq 0$ and positive integers k_1 and k_2 such that $\varphi(x_0,t_{\min}) \in K_{2k_1}^+$ and $\varphi(x_0,t_{\max}) \in K_{2k_2}^-$.*

Proof. We notice that $h(\varphi(x_0,t))$ is a periodic time function with a positive period. Therefore one can find instants $t_{\max}$ and $t_{\min}$ such that

$$\begin{aligned} h(\varphi(x_0,t_{\max})) &= \max_t h(\varphi(x_0,t)), \\ h(\varphi(x_0,t_{\min})) &= \min_t h(\varphi(x_0,t)). \end{aligned} \tag{2.4}$$

By applying high-order extremum conditions, we obtain the desirable conclusion.   □

Lemma 2.1 entails the following proposition.

PROPOSITION 2.2. *Let f_s, $s = 1,\ldots,n$, and $h \in C^{\infty}(\mathbb{R}^n)$. If the set $\mathrm{Inv}(K_1) = K_1$, then all periodic orbits are contained in K_1.*

Proof. By Lemma 2.1, each periodic orbit has common points with K_1. The latter implies the desirable result because of the invariance of K_1.   □

For example, we consider the system $\dot{x} = A_0 x + g(x) \cdot A_1 x$. Assume that there is a vector row C such that $CA_1 = 0$ and C is a right eigenvector of A_0 corresponding to some nonzero real eigenvalue. We take the function $h(x) = Cx$. Then $\mathrm{Ker}\, C = K_1$ and K_1 is the invariant subspace containing all periodic orbits.

We define the following sets: $M_1 = \mathrm{Inv}(K_1) \cap C\{f^{-1}(0)\}$ and $M_2 = K_1 - \mathrm{Inv}(K_1)$. Now we are in position to state the following theorem.

THEOREM 2.3. *Let f and $h \in C^{\infty}(\mathbb{R}^n)$. Assume that the set K_1 is a union of a finite number of infinitely differentiable hypersurfaces in $\mathbb{R}^n$. If Γ is a periodic orbit, then one of the following cases is realized: (1) $\Gamma \subset M_1$; (2) $\Gamma \cap \mathrm{Inv}(K_1) = \varnothing$, $\Gamma \cap S_1^+ \neq \varnothing$, $\Gamma \cap S_1^- \neq \varnothing$, and*

$$\mathrm{card}\left\{ \Gamma \cap \left(\cup_{s=1}^{\infty} K_{2s}^+ \right) \right\} = \mathrm{card}\left\{ \Gamma \cap \left(\cup_{s=1}^{\infty} K_{2s}^- \right) \right\} > 0. \tag{2.5}$$

In addition, suppose that f and h are polynomials. Let $\alpha = \max_{s=1,\ldots,n} \deg f_s \geq 2$, $\deg h = p$, $\max(p;n) \leq \beta$, and

$$N = \frac{1}{2} \sum_{k=0}^{n} \left(2\beta + 2k(p-1) \right)^{2n+2}. \tag{2.6}$$

Then the formula

$$\text{card}\left\{\Gamma \cap \left(\cup_{2s \le N} K_{2s}^+ \right)\right\} = \text{card}\left\{\Gamma \cap \left(\cup_{2s \le N} K_{2s}^- \right)\right\} > 0 \tag{2.7}$$

is valid.

Proof. Firstly, we note that M_1 is invariant. Suppose that Γ is not contained in the set M_1. Then Γ is contained in the set $C\{\text{Inv}(K_1)\}$. It is easy to see that $h(\varphi(z,t))$ is not constant as a time function for any point $z \in C\{\text{Inv}(K_1)\}$. We take $x \in \Gamma$. Then the period T of the function $L_f h(\varphi(x,t))$ is a positive number. We can meet the following cases of the location of Γ: (1) $\Gamma \subset ((L_f h)^{-1}((0,\infty)) \cup (L_f h)^{-1}((-\infty,0)))$, (2) Γ has exactly one common point with the set M_2, (3) Γ intersects the set M_2 in a few points.

(1) Firstly, we assume that

$$\Gamma \subset (L_f h)^{-1}((-\infty,0)) \tag{2.8}$$

and take $x_0 \in \Gamma$. Let $T > 0$ be a minimal period of Γ. Since Γ is compact, we can replace h by $h + c$, where c is a constant such that $h + c$ is nonnegative on Γ. So $h + c$ is a Lyapunov function on Γ. Besides, the restriction $L_f h|_\Gamma$ is bounded above and below. Therefore we can apply the LaSalle theorem [13, Theorem 1] to Γ. As a result, we obtain that the distance of points $\varphi(x_0, t_m)$ from the set S_1 vanishes as $t_m \to \infty$, $m \to \infty$. By taking $t_m = mT$, we deduce that $x_0 \in S_1$. This contradicts (2.8). Now we assume that $\Gamma \subset (L_f h)^{-1}((0,\infty))$. Since the change of time $t \to -t$ preserves periodic orbits, we obtain that this case is examined in the same way.

(2) Here, by using necessary extremum conditions applied to the time function $h(\varphi(x_0, t))$, we get that Γ has at least two common points with the set S_1. So this case is not possible.

(3) Consider the third case. Assume that the function $h(\varphi(x_0,t))$ has no local extremum at the point $t = 0$ (we can get $h(\varphi(x_0,t))$ with this property by a small time shift if it is necessary). We notice that the function $L_f h(\varphi(x_0,t))$ has the same period T just as $h(\varphi(x_0,t))$. We take the function $L_f h(\varphi(x,t))$ restricted on the open interval $(0,T)$. Then it follows from Pólya and Szegő [20, Example 22, page 39] that $L_f h(\varphi(x_0,t))$ has an even number of sign reversals on $(0,T)$. Now, by using sign extremum conditions applied to each of the points of sign reversals, we establish that there is an equal number of minimum points and maximum points of the function $h(\varphi(x,t))$ restricted on $(0,T)$. Therefore, by applying high-order extremum conditions to each of these points, we derive (2.5).

It remains to consider the polynomial case of (f,h). We use the Gabrielov-Khovanskii theorem, see [6, Theorem 3]. Namely, assume that x is not an equilibrium point. Then $h(\varphi(x,t))$ is not constant as a time function if and only if there is an integer i not exceeding N given in formula (2.6) for which $L_f^i h(x) \neq 0$.

So, by using this theorem, we conclude that $K_i = f^{-1}(0)$ for $i > 2N$, which entails $K_{2s}^\pm = \varnothing$ with $s > N$. As a result, we come to formula (2.7).   □

Theorem 2.3 is a generalization of the classical result of Poincaré concerning tangential curves, see, for example, [30, Theorem 1.9]. Indeed, let $n = 2$ and let U be a domain

in $\mathbb{R}^2$ in which $L_f h \leq 0$ or $L_f h \geq 0$. Assume that $K_1 \cap U$ does not contain any periodic orbit. By Theorem 2.3, the unique possibility for Γ to be located in U is to intersect both of the sets $U \cap K_1^+$ and $U \cap K_1^-$. But this leads to a contradiction since the function $L_f h$ is sign-definite in U.

Let f_s, $s = 1,\dots,n$, and $h \in C^\infty(\mathbb{R}^n)$. Now we come to a formulation of nonexistence conditions of periodic orbits which are based on the algebraic relationship between h, $L_f h$, and $L_f^2 h$.

COROLLARY 2.4. *Let $L_f h(x) = a + g_1(x)h(x)$ with a constant $a \neq 0$. In addition, assume that (1) $L_f^2 h|_{\{h(x)>0\}} > 0$ and $L_f^2 h|_{\{h(x)<0\}} < 0$, or (2) $L_f^2 h|_{\{h(x)<0\}} > 0$ and $L_f^2 h|_{\{h(x)>0\}} < 0$. Then system (1.1) has no periodic orbits.*

Proof. Indeed, in these cases, the surface $h(x) = 0$ is permeable only in one direction by the phase flow. So each periodic orbit $\Gamma \subset \{h(x) > 0\}$ or $\Gamma \subset \{h(x) < 0\}$. Therefore one of sets K_2^+ and K_2^- is empty, which entails the desirable result. $\square$

COROLLARY 2.5. *Let $L_f^2 h(x) = a + g_2(x)L_f h(x) + \mathrm{sgn}(a)g_3(x)$ with a constant $a \neq 0$ and a nonnegative function g_3 on $\mathbb{R}^n$. Then system (1.1) has no periodic orbits.*

Proof. Analogously, one of the sets K_2^+ and K_2^- is empty, which entails the desirable result. $\square$

We present one sufficient nonexistence condition of periodic orbits stated in terms of sets S_{2N}^+, S_{2N}^-.

PROPOSITION 2.6. *Let f_s, $s = 1,\dots,n$, and $h \in C^\infty(\mathbb{R}^n)$. Assume that the function h is chosen in such a way that $h(\varphi(x,t))$ is not constant as a time function for any point $x \notin f^{-1}(0)$. Besides, suppose that one of the following conditions is realized: (1) $S_{2N}^+ \cup f^{-1}(0) = K_1$; (2) $S_{2N}^- \cup f^{-1}(0) = K_1$. Then system (1.1) has no periodic orbits.*

Proof. Let $x_0 \in \Gamma$, where Γ is a periodic orbit. Assume that condition (1) or condition (2) holds. Then the function $h(\varphi(x_0,t))$ has no minimum points (no maximum points) appropriately. This contradicts Lemma 2.1. $\square$

As an illustrating example, we consider system (1.1) with $n = 2$ and $f_1(x_1,x_2) = a_1 x_1 + a_2 x_2$, $f_2(x_1,x_2) = p(x_1,x_2)$. Here, p is a homogeneous form of even degree. Let $a_2 \neq 0$ and $p(1,-a_1 a_2^{-1}) > 0$. Then this system has no periodic orbits. Indeed, we take $h(x) = x_1$ and compute that $K_2^+ \cup (0,0) = K_1$.

Also, we note that if $K_1 = f^{-1}(0)$, then system (1.1) has no periodic orbits. Here, the classical example is a gradient system. The case $K_n = f^{-1}(0)$ will be discussed below.

3. Invariant sets and the localization of periodic orbits

In this section, we study the set $\mathrm{Inv}(K_1)$ and present new localization conditions of periodic orbits.

Consider the equality

$$L_f^{m+1} h(x) = \sum_{s=1}^m a_s(x) L_f^s h(x), \qquad (3.1)$$

where $a_s(x)$, $s = 1,\dots,m$, are rational functions, that is, $a_s = p_s/q_s$, with all p_s, q_s polynomials. In addition, we suppose that m is minimal with this property. Let G be a set of zeroes of the polynomial $\prod_{j=1}^{m} q(x)$. Firstly, we give the following lemma.

LEMMA 3.1. *If (3.1) is valid, then $K_m \cap C\{G\} \subset \operatorname{Inv} K_1$.*

Proof. We take any point $x \in K_m \cap C\{G\}$. It follows from (3.1) that $L_f^{m+1} h(x) = 0$. Now, by applying iteratively the Lie operator L_f to formula (3.1) and making substitutions of equalities (3.1) into expressions for $L_f^s h$, with $s \geq m + 2$, we come to the formula $L_f^{m+j} h(x) = \sum_{s=1}^{m} a_{sj}(x) L_f^s h(x) = 0$, $j = 2, 3,\dots$, in which all a_{sj} are some rational functions with denominators having zeroes in the set G. Since the solution φ is locally analytic, the latter formula entails the desirable result. $\square$

Obviously, if all a_s are polynomials in (3.1), then $\operatorname{Inv} K_1 = K_m$.

We address the case of $m = 1$ in (3.1), where we are concerned with its relation with classical results. We recall one definition.

Definition 3.2. Suppose that there exists a polynomial h of n variables satisfying the equality

$$L_f h(x) = a_0 h(x) \tag{3.2}$$

for some polynomial a_0 of n variables. Then h is called the Darboux polynomial and a_0 is called a cofactor, see, for example, [15].

It is well known that these polynomials are useful in studying integrability property of polynomial systems, and up to now, there have been published many results respecting finding Darboux polynomials for the Lorenz system [17], the Rikitake system [15], the Rössler system [16], and so forth. Our new example is the Sprott system [25, Example K, Table 11] $\dot{x}_1 = x_1 x_2 - x_3$, $\dot{x}_2 = x_1 - x_2$, $\dot{x}_3 = x_1 + ax_3$, in which the coefficient 0.3 is replaced by a nonzero parameter a. We have found by the method of indefinite coefficients that the list of all quadratic Darboux polynomials consists of two polynomials, $h(x) = x_1 - x_2^2/2 - 2x_3$ for $a = -5/2$ and $h(x) = (x_2 - x_3)^2$ for $a = -1$. The latter means that all periodic orbits are located in the set $x_1 - x_2^2/2 - 2x_3 = 0$ for $a = -5/2$ and in the plane $x_2 - x_3 = 0$ for $a = -1$.

Below, we state the relationship between Darboux polynomials and the formula for $\operatorname{Inv} K_1$.

LEMMA 3.3. *If h is a Darboux polynomial with a cofactor a_0 for system (1.1), then*

$$\operatorname{Inv} K_1 \supset K_1 \cap C\{a_0^{-1}(0)\}. \tag{3.3}$$

Proof. We compute

$$L_f^2 h = L_f h \cdot (a_0^{-1} L_f a_0 + a_0). \tag{3.4}$$

Let $b_2 = L_f a_0 + a_0^2$. By applying L_f to this formula and using (3.4), we get the similar formula for $L_f^3 h$: $L_f^3 h = a_0^{-2} b_3 L_f h$ for some polynomial b_3, and so forth. As a result, there

is a sequence of polynomials $\{b_s\}$ such that $L_f^s h = b_s a_0^{1-s} L_f h$, $s = 2,\ldots,\infty$. After this, we argue as above. $\qquad\square$

If a_0 is constant in (3.2), then $\operatorname{Inv} K_1 = K_1$.

In the general case, we meet the problem of finding h satisfying (3.1) for some natural number m and functional coefficients a_s. This problem is difficult and leads to a solution of a partial differential equation of a high unknown degree. Instead of going in this direction, we choose another way related to results of the author which are based on using algebraically-dependent polynomials, see [26].

We take a polynomial h of n real variables and then form a sequence of polynomials $\{L_f^s h\}$, $s = 1,2,\ldots$. It follows from the Perron theorem, see the modern version in [19, Theorem 3.1], that there is a natural number m such that the polynomials $\{L_f^s h\}$, $s = 1,\ldots,m$, are algebraically independent while the polynomials $\{L_f^s h\}$, $s = 1,\ldots,m+1$, are algebraically dependent, with $m \le n$. This means that one can find a nonzero polynomial $\Phi(w_1,\ldots,w_{m+1})$ such that

$$\Phi\big(L_f^s h(x), \ s = 1,\ldots,m+1\big) \equiv 0. \tag{3.5}$$

Here, without loss of generality, one can assume additionally that the degree $\deg\Phi(w)$ is minimal with this property. Applying the Lie derivative L_f to (3.5), we get the identity

$$L_f^{m+2} h(x) \cdot q_{m+2}\big(L_f^s h(x), \ s = 1,\ldots,m+1\big) + p\big(L_f^s h(x), \ s = 1,\ldots,m+1\big) \equiv 0, \tag{3.6}$$

where p, q_{m+2} are two polynomials of $m+1$ real variables $w = (w_1,\ldots,w_{m+1})$. It is proved, see [26, Lemma 1], that $\deg q_{m+2}(w) < \deg\Phi(w)$ and $q_{m+2}(L_f^s h(x), \ s = 1,\ldots,m+1)$ is a nonzero polynomial. So the set $S := \{x \in \mathbb{R}^n \mid q_{m+2}(L_f^s h(x), \ s = 1,\ldots,m+1) = 0\}$ is a proper subset in $\mathbb{R}^n$, that is, $S \subsetneq \mathbb{R}^n$. Then we apply L_f to (3.6), and so forth.

Our next result shows explicitly the location of $\operatorname{Inv}(K_1)$ in the polynomial case. Namely, we have the following theorem.

THEOREM 3.4. *Define the set*

$$\begin{aligned} \mathcal{M}_1 = \big\{x \in \mathbb{R}^n \mid &\ L_f^s h(x) = 0, \ s = 1,\ldots,m+1, \\ &\ q_{m+2}\big(L_f^s h(x), \ s = 1,\ldots,m+1\big) \ne 0\big\}. \end{aligned} \tag{3.7}$$

Then $\mathcal{M}_1 \subset \operatorname{Inv}(K_1) \subset K_{n+1}$.

Proof. By computations taken from [26, Lemma 1], we obtain that $L_f^{m+j} h(x)$ is expressed as a rational function due to (3.6) for some polynomial q_{m+j}, while its denominator has the same zero set S for any positive integer j. Now let $L_f^s h(x_*) = 0$, $s = 1,\ldots,m+1$ and $x_* \notin S$. Then $L_f^s h(x_*) = 0$ for any $s = m+2,\ldots,\infty$. Since the solution of (1.1) is locally analytic, then by using the theorem on continuation of a solution, we get that $L_f h(\varphi(x_*,t)) = 0$ for any t from the maximal interval of the existence of the solution. Now let $x_* \in \operatorname{Inv}(K_1)$. Then, by local analyticity of a solution, we conclude that $L_f^i h(x_*) = 0$ for $i = 1,\ldots,m+1$ and $x_* \in K_{m+1}$. $\qquad\square$

To summarize, there are the following cases of locating periodic orbits inside the set $C\{S\}$: (1) each periodic orbit Γ is contained in the set K_{m+1}, or (2) Γ is not contained in K_1, but Γ has common points with both of the sets K_1^+ and K_1^- and, besides, $\text{card}\{\Gamma \cap (\cup_{2s<m+1} K_{2s}^+)\} = \text{card}\{\Gamma \cap (\cup_{2s<m+1} K_{2s}^-)\} > 0$. By applying the ideas of Proposition 2.6 in this situation, if $S_{m+1}^+ \cup f^{-1}(0) = K_1$ or $S_{m+1}^- \cup f^{-1}(0) = K_1$, then the system (1.1) has no periodic orbits contained in the set $C\{S\}$.

4. Periodic orbits of triangularizable systems

In this section, we describe new localization conditions of periodic orbits which are based on Proposition 2.6. Suppose that the vector field f is analytic and $f(0) = 0$. Then f can be written as $f(x) = A(x)x$ for some analytic matrix-valued function $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, or as $f(x) = \sum_{j \in J} x^j A_j x$ in the standard multinomial notation. By Z we denote the set $\{x \in \mathbb{R}^n \mid \det A(x) = 0\}$. Assume that the set of matrices $\{A_j \mid j \in J\}$ is triangularizable. This means that there is a basis of $\mathbb{R}^n$ such that the matrix $A(x)$ is upper triangular. We mention here that many results concerning triangularizability conditions of the set of matrices $\{A_j \mid j \in J\}$ are contained in [21]. Now we are in position to state the following proposition.

PROPOSITION 4.1. *Suppose that the matrix $A(x)$ is given in the upper triangular form and the set Z is invariant. Then all periodic orbits of system (1.1) are located in Z.*

Proof. By $\mathcal{S}_j$ we denote the system obtained from (1.1) by taking the restriction of (1.1) on the plane $\Pi_j : x_j = \cdots = x_n = 0$. It follows from the triangular structure of the matrix $A(x)$ that $\Pi_j \cup \{a_{jj}(x_1,\ldots,x_{j-1},0,\ldots,0) = 0 \mid (x_1,\ldots,x_{j-1})^T \in \mathbb{R}^{j-1}\}$ is the invariant set of the system $\mathcal{S}_j$. Now, with the help of Proposition 2.6 successively applied to systems (1.1), $\mathcal{S}_j$, $j = n, n-1,\ldots,2$, we get that periodic orbits are not contained in Π_j, $j = 1,2,\ldots,n$.

Hence, since $\det A(x) = \prod_{j=1}^n a_{jj}(x)$ and Z is invariant, we get that any periodic orbit is located in the set Z. $\qquad \square$

For example, suppose that the matrix $A(x)$ is upper triangular with constant diagonal elements. Applying the last result to system (1.1), we get that, in this case, system (1.1) has no periodic orbits.

Now we take the three-dimensional system with the upper triangular matrix $A(x)$:

$$\begin{aligned} \dot{x}_1 &= a_{11}(x_1,x_2,x_3)x_1 + a_{12}(x_1,x_2,x_3)x_2 + a_{13}(x_1,x_2,x_3)x_3, \\ \dot{x}_2 &= a_{22}(x_1,x_2,x_3)x_2 + a_{23}(x_1,x_2,x_3)x_3, \\ \dot{x}_3 &= a_{33}(x_1,x_2,x_3)x_3, \end{aligned} \tag{4.1}$$

and take the function $h(x) = x_3$. By A_{33} we denote the set defined by the equation $a_{33}(x) = 0$. Below, by $\mathcal{P}_i$, we denote the hyperplane $x_i = 0$ in $\mathbb{R}^3$. We notice that the hyperplane $\mathcal{P}_3$ is the invariant set and $S_1 = \mathcal{P}_3 \cup A_{33}$. Assume that $L_f a_{33}(x) > 0$ for $x_3 \neq 0$. We take any periodic orbit Γ outside $\mathcal{P}_3$. Then the function $L_f a_{33}|_{\{\Gamma\}}$ cannot get different signs since Γ cannot cross $\mathcal{P}_3$. Thus $L_f^2 h|_{\{\Gamma\}}$ cannot change its sign. As a result, there are no periodic

orbits outside $\mathcal{P}_3$. So it remains to study periodic orbits of the two-dimensional system

$$\begin{aligned}
\dot{x}_1 &= a_{11}(x_1, x_2, 0)x_1 + a_{12}(x_1, x_2, 0)x_2, \\
\dot{x}_2 &= a_{22}(x_1, x_2, 0)x_2,
\end{aligned} \tag{4.2}$$

which can be fulfilled by any of the known methods of the study of periodic orbits of two-dimensional systems.

5. The reduction of the main result in the case of $K_n = f^{-1}(0)$

Below, our goal is to find conditions leading to $K_n = f^{-1}(0)$. In this case, formula (2.5) is reduced to card$\{\Gamma \cap (\cup_{2s \le n} K_{2s}^+)\}$ = card$\{\Gamma \cap (\cup_{2s \le n} K_{2s}^-)\} > 0$. This allows us to localize the position of periodic orbits more efficiently. Firstly, we notice that

$$\left(L_f^s h(x), \, s = 1, \ldots, n\right)^T = T(x)f(x), \tag{5.1}$$

with $T(x)$ a polynomial matrix. We take $x_0 \in K_n$. Then it follows from (5.1) that $x_0 \in f^{-1}(0)$, provided the determinant $\det T(x_0) \ne 0$. Hence, we obtain that

$$K_n \cap \{x \mid \det T(x) \ne 0\} \subset f^{-1}(0). \tag{5.2}$$

The last formula will be applied to a polynomial system with $n - 1$ linear components f_i, $i = 1, \ldots, n - 1$; f_n is a nonlinear polynomial.

   We recall two classical definitions borrowed from control theory. Consider a linear control system $\dot{w} = Kw + Bu$ with a solution γ and an observation function $y = Cw$. Here, K is a constant $n \times n$ matrix, C is a constant vector row. The pair (K, C) is called *observable* if $C\exp(Kt)w$ is not equal to zero as a time function for any nonzero state vector w. The minimal number i for which $\partial^i C\gamma/\partial t^i$ depends on a control u explicitly is called the relative degree of the triple (K, B, C). We establish the following lemma.

LEMMA 5.1. *Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be an $n \times n$ matrix representing the linear part of f and let A_{n-1} be an $(n-1) \times (n-1)$ submatrix got from A by deleting the last column and the last row. Suppose that* (1) $a_{n-1\,n} \ne 0$, *and* (2) *the relative degree of the triple (A_{n-1}, l, C), with $l = (0, \ldots, 0, a_{n-1\,n})^T \in \mathbb{R}^{n-1}$, is equal to $n - 1$. Take the function $h(x) = C(x_1, \ldots, x_{n-1})^T$. Let the pair (A_{n-1}, C) be observable. Introduce the matrix $T(x)$ by formula (5.1) for this choice of h. Then $K_n = f^{-1}(0)$.*

Proof. Indeed, since $L_f^s h(x) = CA^s x$, $s \le n - 1$, we deduce from the observability rank condition that T is a constant nonsingular matrix. Therefore $K_n = f^{-1}(0)$. □

   For example, the Rössler system with the function $h(x) = x_2$ satisfies the conditions of Lemma 5.1 with

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & a^2 - 1 & -1 \end{pmatrix}, \tag{5.3}$$

a is a parameter of the Rössler system. Some of the chaotic Sprott systems [25] with one nonlinear monomial term satisfy the conditions of the last lemma for a proper choice of h.

6. Applications

Now we demonstrate how the results stated above can work.

(1) We consider the Blasius equations taken from the theory of fluid boundary layers, see, for example, [29, page 129]:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -x_1 x_3.
\end{aligned}
\tag{6.1}
$$

We use the function $h(x) = x_1$. Since the plane $\mathcal{P}_3$ is invariant, we obtain that any periodic orbit Γ is located in the semispace $x_3 > 0$ or in the semispace $x_3 < 0$. So we get that either $\Gamma \cap K_2^+ = \varnothing$ or $\Gamma \cap K_2^- = \varnothing$. Now we compute that $K_3 = f^{-1}(0)$. Thus we get that $K_j^+ = \varnothing$ and $K_j^- = \varnothing$ for $j > 3$. Hence the Blasius equations have no periodic orbits.

(2) Even if the matrix $A(x)$ is not given in the triangular form, it is possible in some cases to establish the nonexistence of periodic orbits. We will show how to derive this result in the case of the GMA model for biochemical reactions, see [3, 9].

Firstly, we take the polynomial system (1.1) of the form

$$
\begin{aligned}
\dot{x}_1 &= l_1 x_1 - a_1 x_1^{b_3} x_2^{b_1}, \\
\dot{x}_2 &= -l_2 x_2 + a_2 x_1^{b_1} x_3^{b_2}, \\
\dot{x}_3 &= -l_3 x_3 + a_3 x_2^{b_1}.
\end{aligned}
\tag{6.2}
$$

Here, x_s, $s = 1,2,3$, are concentrations (e.g., of a hormone or a protein), a_s, b_s, l_s, $s = 1,2,3$, are some parameters. Let $l_1 l_2 l_3 \neq 0$, $a_1 a_2 a_3 \neq 0$, and $b_3 \geq 1$. We notice that $\mathcal{P}_1$ is the invariant set. We use the function $h(x) = x_1$ and apply Proposition 2.6. It is easy to see that the restriction of system (6.2) on $\mathcal{P}_1$ has no periodic orbits. We introduce the set S_1 defined by the equation $l_1 - a_1 x_1^{b_3-1} x_2^{b_1} = 0$. Then $K_1 = \mathcal{P}_1 \cup S_1$. Our goal is to compute $K_3 \cap C(\mathcal{P}_1)$. Applying the Lie derivative to $l_1 - a_1 x_1^{b_3-1} x_2^{b_1}$, we obtain some polynomial η, and then we compute the restriction $\eta|_{\{S_1\}}$. As a result, we get that $\eta|_{\{S_1 \cap C\{\mathcal{P}_1\}\}}(x) = 0$ if and only if $x_2^{b_1}(l_2 x_2 - a_2 x_1^{b_1} x_3^{b_2}) = 0$. Again, by similar routine computations carried out for the polynomial $x_2^{b_1}(l_2 x_2 - a_2 x_1^{b_1} x_3^{b_2})$, we obtain that the set $K_3 \cap C\{\mathcal{P}_1\}$ is described by the system of equations

$$
\begin{aligned}
l_1 - a_1 x_1^{b_3-1} x_2^{b_1} &= 0, \\
l_2 x_2 - a_2 x_1^{b_1} x_3^{b_2} &= 0, \\
l_3 x_3 - a_3 x_2^{b_1} &= 0.
\end{aligned}
\tag{6.3}
$$

So, $K_3 = f^{-1}(0)$. Now we suppose that the sets K_2^+ and K_2^- are crossed by some periodic orbit Γ. In this case, there are two points p and $q \in \Gamma$ such that

$$
\begin{aligned}
p_2^{b_1} &= l_1 a_1^{-1} p_1^{1-b_3}, \\
p_3^{b_2} &> l_1^{1/b_1} a_1^{-1/b_1} l_2 a_2^{-1} p_1^{b_*}, \\
q_2^{b_1} &= l_1 a_1^{-1} q_1^{1-b_3}, \\
q_3^{b_2} &< l_1^{1/b_1} a_1^{-1/b_1} l_2 a_2^{-1} q_1^{b_*},
\end{aligned}
\tag{6.4}
$$

with $b_* = -b_1 + (1 - b_3)/b_1$. We notice that (1) if b_1 is even, b_3 is odd and $l_1 a_1 < 0$, or (2) if b_2 is even and $l_1 l_2 a_1 a_2 < 0$, then (6.4) is not satisfied for p and $q \in \Gamma$. Indeed, in case of (1), real p_2 and q_2 do not exist. In case (2), the signs of p_1 and q_1 are the same because $\mathcal{P}_1$ is not crossed by Γ. So, the inequalities in (6.4) are always incompatible with p and $q \in \Gamma$. This means that there are no periodic orbits for system (6.2) provided that condition (1) or (2) is fulfilled. Now we suppose that instead of (2) we have (3): b_2 is a negative rational number, $b_2 = b_{21}/b_{22}$, where b_{21}/b_{22} is noncancelable and b_{21} is an even numerator. By the same computations, we find that there are no periodic orbits outside the hyperplane $\mathcal{P}_3$ provided that one of conditions (1) and (3) is valid.

(3) Consider the mathematical model of the chemical reaction with autocatalytic step, taken from [12]:

$$
\begin{aligned}
\dot{x}_1 &= \gamma - x_1 - \lambda x_1 x_3, \\
\dot{x}_2 &= x_1 - x_2 x_3, \\
\dot{x}_3 &= x_2 x_3 - \lambda x_1 x_3,
\end{aligned}
\tag{6.5}
$$

where x_s, $s = 1, 2, 3$, are concentrations of chemicals and $\gamma, \lambda > 0$ are parameters. We use the function $h(x) = x_3$. Then we introduce the plane S_2 which is defined by the equation $h_1(x) := x_2 - \lambda x_1 = 0$. We observe that $\mathcal{P}_3$ is the invariant plane and we get that each periodic orbit has no common points with $\mathcal{P}_3$. Also, $K_1 = \mathcal{P}_3 \cup S_2$. We obtain that $L_f h_1 = -\gamma\lambda - x_1 - x_2 x_3 + \lambda x_1 + \lambda^2 x_1 x_3$. Let the set S_3 be defined by $h_1(x) = L_f h_1(x) = 0$. Then we compute that S_3 is given by the equations $x_2 = \lambda x_1$ and $\gamma\lambda = (\lambda - 1)(1 + \lambda x_3)x_1$. Also, $L_f^2 h_1|_{\{S_3 \cap C(\mathcal{P}_1)\}} = 0$ is given by $(\lambda^2 - \lambda^4)x_3^2 + x_3(-2\gamma\lambda^3 - 2\gamma\lambda^2) + \gamma - 2\gamma\lambda^2 - \gamma\lambda = 0$. So, if $\lambda \neq 0, \pm 1$, then the maximal invariant subset in S_2 can contain not more than two points. One of them is the equilibrium point $(\gamma/2, \lambda\gamma/2, \lambda^{-1})^T$. So, all periodic orbits must cross both of the sets $K_2^+ \cap C(\mathcal{P}_3)$ and $K_2^- \cap C(\mathcal{P}_3)$. Since $f(x) \neq 0$ for any $x \in \mathcal{P}_1$, we deduce that each periodic orbit has no common points with $\mathcal{P}_1$. Now let $0 < \lambda < 1$. Then there are no periodic orbits in $(0, \infty) \times \mathbb{R}^1 \times (0, \infty) \cup \mathcal{P}_1 \cup \mathcal{P}_3$. In addition, if $\Gamma \subset (-\infty, 0) \times \mathbb{R}^1 \times (0, \infty)$, or $\Gamma \subset (-\infty, 0) \times \mathbb{R}^1 \times (-\infty, 0)$, or $\Gamma \subset (0, \infty) \times \mathbb{R}^1 \times (-\infty, 0)$, then S_2 is crossed by Γ. Now let $\lambda > 1$. Similarly, there are no periodic orbits in $(-\infty, 0) \times \mathbb{R}^1 \times (0, \infty) \cup \mathcal{P}_1 \cup \mathcal{P}_3$. In addition, if $\Gamma \subset (0, \infty) \times \mathbb{R}^1 \times (0, \infty)$, or $\Gamma \subset (-\infty, 0) \times \mathbb{R}^1 \times (-\infty, 0)$, or $\Gamma \subset (0, \infty) \times \mathbb{R}^1 \times (-\infty, 0)$, then S_2 is crossed by Γ. At last, let $\lambda = 1$. Then $L_f h_1|_{\{S_2\}} = -\gamma x_3$. Since any periodic orbit Γ has no common points with $\mathcal{P}_3$, we get that $L_f h_1|_{\{S_2 \cap \Gamma\}}$ is a function with only positive or negative values. So, in this case, our system has no periodic orbits in $\mathbb{R}^3$.

7. A few comments on the localization of periodic orbits

Information on a location of periodic orbits obtained by our approach crucially depends on a choice of functions h used for the localization. The first important problem consists in the choice of some function h for which one can get the "best possible" localization set of periodic orbits. Here, intuitively, the "best possible" means the smallest set with respect to inclusions. We believe that the clever choice of functions h includes classes of Lyapunov functions, Darboux polynomials, and functions for which the corresponding set $K_s = f^{-1}(0)$ for some s, $1 \le s \le n$. However, it may have significance for attempts to localize periodic orbits by using functions h of more general types than mentioned here.

By taking a proper Lyapunov function, it is possible in some cases to localize all periodic orbits in a compact domain, see the example of the Lorenz system in [1, 11]. Darboux polynomials can provide the localization of periodic orbits in invariant surfaces. However, the latter type of localization is valid only for specific values of parameters of the system. Further, we notice that analysis of sets K_j, $j > n$, is not required in our examples, which leads to simple computations. Therefore, it is worthwhile to describe a class of systems for which we can avoid computations of sets K_j, $j > n$, by a proper choice of the function h.

The localization of periodic orbits by means of their intersections with some surface is easy to obtain, see the computations in our examples. Usually, it has a reasonable value when it is reformulated as a nonexistence condition in the complementary set to this surface.

One more interesting question arising in the process of the localization concerns the idea of "minimizing" in some sense a localization set containing all periodic orbits by a proper choice of a few different functions h. For example, if we know that all periodic orbits are located in different sets U_1 and U_2 in the state space, then they are contained in $U_1 \cap U_2$ as well. We expect that in this case there is a possibility to play with different Lyapunov functions.

8. Conclusions

In this paper, we described sets in the state space of a nonlinear system, which do not contain periodic orbits, contain all periodic orbits, or have common points with any periodic orbit. These sets are given in terms of f and a differentiable, usually polynomial function h used in the process of the localization. Mainly, the case of polynomial systems was examined. It has been established that for a polynomial system and a polynomial function h, these sets can be described by means of a finite number k_1 (k_2) of polynomial equations (polynomial inequalities) appropriately. Upper bounds for k_j, $j = 1, 2$, were provided in different cases. The role of invariant sets in the localization problem of periodic orbits was discussed. Our approach is based on using high-order extremum conditions, high-order tangency conditions of a nonsingular solution with an algebraic set in the state space, and some ideas related to algebraically-dependent polynomials. A few examples were considered including the Blasius equations, the GMA system, and the mathematical model of the chemical reaction with autocatalytic step. Finally, we remark that the results of this paper can be applied to a solution of the localization problem of homoclinic and heteroclinic phase curves.
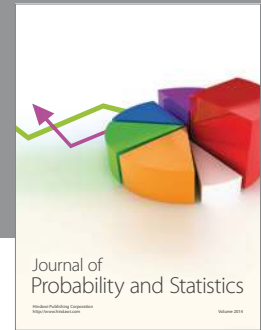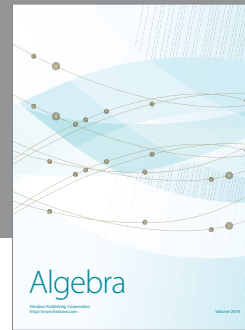
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