LOCALIZATION THEOREMS BY SYMPLECTIC CUTS

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Abstract. Given a compact symplectic manifold M with the Hamiltonian action of a torus T, let zero be a regular value of the moment map, and M_0 the symplectic reduction at zero. Denote by κ_0 the Kirwan map $H_T^*(M) \to H^*(M_0)$. For an equivariant cohomology class $\eta \in H_T^*(M)$ we present new localization formulas which express $\int_{M_0} \kappa_0(\eta)$ as sums of certain integrals over the connected components of the fixed point set M^T. To produce such a formula we apply a residue operation to the Atiyah-Bott-Berline-Vergne localization formula for an equivariant form on the symplectic cut of M with respect to a certain cone, and then, if necessary, iterate this process using other cones. When all cones used to produce the formula are one-dimensional we recover, as a special case, the localization formula of Guillemin and Kalkman [GK]. Using similar ideas, for a special choice of the cone (whose dimension is equal to that of T) we give a new proof of the Jeffrey-Kirwan localization formula [JK1].

This paper is dedicated to Alan Weinstein on the occasion of his 60th birthday.

1. Introduction

Assume we are given a compact symplectic manifold M with the Hamiltonian action of a torus T. There are two kinds of localization theorems which express the integral over M of an equivariant cohomology class $\eta \in H_T^*(M)$ and the integral over the reduced space M_0 of the Kirwan map $\kappa_0(\eta)$ as sums of certain terms which involve integration over the connected components of the fixed point set M^T. In particular, these localization theorems say that both $\int_M \eta$ and $\int_{M_0} \kappa_0(\eta)$ depend only on the restriction of η to M^T.

More specifically, the localization theorem of Atiyah-Bott [AB] and Berline-Vergne [BV] (or just the ABBV localization theorem) expresses the integral over M (that is, the pushforward in equivariant cohomology with respect to the map $M \to \mathrm{pt}$) of a class $\eta \in H_T^*(M, \mathbb{C})$ as a sum of integrals over the connected components F of the fixed point set M^T of the restriction of η to F divided by the equivariant Euler class of the normal bundle of F:

$$\int_M \eta = \sum_F \int_F \frac{\iota_F^*(\eta)}{e(\nu(F))},$$

where ι_F is the natural inclusion $F \to M$ and $e(\nu(F))$ is the equivariant Euler class of the normal bundle $\nu(F)$.

To treat the other kind of localization theorems, let zero be a regular value of the moment map $\mu : M \to \mathfrak{t}$, (M_0, ω_0) the symplectic reduction at zero and $\kappa_0 : H_T^*(M) \to H^*(M_0)$ the Kirwan map. Then the Jeffrey-Kirwan [JK1] and Guillemin-Kalkman [GK] localization theorems express the integral over M_0 of classes $\kappa_0(\eta)e^{\omega_0}$ and $\kappa_0(\eta)$ respectively as sums over some connected components of M^T of certain terms similar to those appearing in the ABBV localization theorem. We postpone the precise statement of the Guillemin-Kalkman theorem until Section 4. However the Jeffrey-Kirwan localization theorem applied to the case of abelian group actions states the following under the above assumptions.

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Theorem A. [JK1] *For $\eta \in H_T^*(M)$ we have*

$$\int_{M_0} \kappa_0(\eta) e^{\omega_0} = c \cdot \mathrm{Res}^{\Lambda} \left(\sum_F e^{i(\mu(F))(X)} \int_F \frac{\iota_F^*(\eta(X)e^{\omega})}{e(\nu(F))(X)} [dX] \right) \tag{1.1}$$

where c is a non-zero constant, X is a variable in $\mathfrak{t} \otimes \mathbb{C}$ so that a T-equivariant cohomology class can be evaluated at X, and Res^{Λ} is a multi-dimensional residue with respect to the cone $\Lambda \subset \mathfrak{t}$ defined in Section 3.

The similarity of the ABBV localization theorem to the Jeffrey-Kirwan localization theorems is transparent, and is of course not a coincidence. In the case of Hamiltonian circle actions, Lerman [Le] showed that it is possible to deduce the Jeffrey-Kirwan and Guillemin-Kalkman theorems from ABBV localization using the techniques of symplectic cutting. The idea, which was also present, though not explicitly stated, in [GK], is to use symplectic cutting to produce a Hamiltonian space whose connected components of the T-fixed set are either the reduced space M_0 or some connected components of the set M^T. Then the ABBV localization theorem on this symplectic cut yields a formula which relates integration over some of the connected components of M^T and over the reduced space M_0. To arrive at the Jeffrey-Kirwan and Guillemin-Kalkman theorems for the case of a circle action, it remains to apply residues to both sides of the ABBV localization formula for the symplectic cut.

The goal of this paper is to illustrate that an analogous approach works for higher dimensional torus actions as well. Let us outline the main idea. Lerman's original definition of symplectic cutting was given for the case of circle actions. However it can be generalized to multidimensional torus actions, when the symplectic cut is defined using any rational convex polytope. We will only consider the special case of symplectic cutting with respect to a cone. Let Σ be a convex rational polyhedral cone (centered at the origin) in the dual $\mathfrak{t}^*$ of the Lie algebra $\mathfrak{t}$. If σ is an open face of Σ (that is, the interior of a face of Σ), let T^σ be the subtorus of T whose Lie algebra is annihilated by σ. Then, as a topological space, the symplectic cut M_Σ is $\mu^{-1}(\Sigma)/\sim$, where $p \sim q$ if $\mu(p) = \mu(q) \in \sigma$, for some open face σ of Σ and p, q lying in the same T^σ orbit. As shown in [LMTW], for a generic choice of Σ, the cut space M_Σ is a symplectic orbifold with a Hamiltonian T action. The moment map image of M_Σ is just the intersection $\mu(M) \cap \Sigma$. Moreover, any equivariant cohomology class $\eta \in H_T^*(M)$ naturally descends to an equivariant class η_Σ on M_Σ.

Some connected components of the fixed point set M_Σ^T may be identical to those of M^T; we call them the *old* connected components (see Definition 2.1). One of the connected components of M_Σ^T is always the reduced space M_0. Hence if we apply ABBV localization to the class η_Σ on the symplectic cut M_Σ, we will get a formula which relates the integration over M_0 to the integration over the connected components of M_Σ^T, some of which are the connected components of M^T. We will show that we can apply the iterated residue operation to both sides of this ABBV localization formula, so that the term corresponding to M_0 simplifies. More specifically, this term becomes a constant times the integral of $\kappa_0(\eta)$ over M_0.

The major difference with the circle case is that in the formula just described, besides a contribution from the term corresponding to M_0 and the terms corresponding to the old connected components, there will be contributions coming from the *new* connected components of M_Σ^T, which are neither part of M^T nor part of M_0 (see Definition 2.1). However, using the ideas of [GK] we can iterate this process to get rid of these terms. Namely for each new connected component F', we can symplectically cut a certain submanifold M' of M with respect to some cone Σ', so that F' is a connected component of the T action on $M'_{\Sigma'}$. Then we can apply the ABBV theorem and the residue operation again to $M'_{\Sigma'}$ to express the integral over F' in terms of integrals over the connected components of the fixed point set $M'^T_{\Sigma'}$. As we will show this process can be iterated until all the terms coming from the new fixed points disappear. So the integration of the Kirwan map over M_0 can be expressed as a sum of terms which involve integration only over the connected components of M^T.

The objects which carry the information about which cones are chosen in this process are called *dendrites*, and are defined in Section 4.3. Every dendrite gives a localization formula. If all the cones used in a dendrite are one-dimensional, then we recover the Guillemin-Kalkman [GK] localization theorem. However, if we choose higher dimensional cones, in other words multi-dimensional dendrites, we get new localization formulas.

Notice that the Jeffrey-Kirwan localization theorem does not involve any iteration. Nevertheless, it fits into the framework just described. While for actions of tori of dimension greater than one any symplectic cut with respect to a cone always has new connected components of M^T_Σ (the connected components which are neither in M^T nor in M_0), it is plausible that their contribution to the ABBV localization theorem becomes zero after taking residues. In this case the iterative process described above would stop at step one. We were not able to use precisely this argument to give a proof of the Jeffrey-Kirwan localization theorem. However, we will show that for a good choice of the cone Σ, a very similar argument which involves taking residues of the ABBV formula for symplectic cuts yields the Jeffrey-Kirwan formula given in Theorem A.

The paper is organized as follows. In Section 2 we carefully review well-known objects from the theory of Hamiltonian group actions, such as symplectic reductions, symplectic cuts, equivariant cohomology and the Kirwan map. We also recall the orbifold version of the ABBV localization theorem. Section 3 is devoted to the residue operation. The definitions of residues are based on the theory of complex variables, do not involve any symplectic geometry and are independent of the material summarized in Section 2. In Section 4 we present the generalization of the Guillemin-Kalkman localization theorem to the case of multi-dimensional dendrites. Finally, using an analogous approach, in Section 5 we give a new proof of the Jeffrey-Kirwan localization theorem.

2. Symplectic cuts and other preliminaries

In this section we recall the construction of symplectic cuts with respect to cones and other results in symplectic geometry. All of the results in this section (with only one exception: Proposition 2.2) have appeared in the literature, so we state them without proof.

2.1. **Symplectic reduction.** Let (M, ω) be a symplectic manifold with the action of a Hamiltonian torus T and a moment map $\mu : M \to \mathfrak{t}^*$. For $p \in \mathfrak{t}^*$ *the symplectic reduction* $M_p = M/\!/_p T$ *of* M *at* p is defined to be $\mu^{-1}(p)/T$. Whenever p is a regular value of the moment map, the symplectic reduction M_p is an orbifold.

For a subtorus $H \subseteq T$, denote by M^H the fixed point set of the H action on M and by M^H_i the connected components of M^H. It is well known that every M^H_i is a symplectic manifold with a Hamiltonian action of the torus T/H. The convexity theorem of Atiyah [A] and Guillemin-Sternberg [GS] states that if M is compact and connected then $\mu(M)$ is a convex polytope. In particular, every $\mu(M^H_i)$ is a convex polytope inside $\mu(M)$; we call it *a wall of* μ.

Let T be a product of two subtori $H \times S$. Denote by π the natural projection $\mathfrak{t}^* \to \mathfrak{s}^*$. Then the composition $\pi \circ \mu$ is a moment map for the S action on M. Moreover, the action of S on M restricts to a Hamiltonian action on M^H whose moment map is again given by $\pi \circ \mu$. Because of this, for $p \in \mu(M^H)$ we call the space

$$M^H/\!/_p S = (\mu^{-1}(p) \cap M^H)/T = (\mu^{-1}(p) \cap M^H)/S$$

the symplectic reduction of M^H at p.

If M is compact and $q \in \mathfrak{s}^*$ then the set $\pi^{-1}(q) \cap \mu(M^H)$ contains finitely many points p_i. It is easy to see that the fixed point set of the H action on $M/\!/_q S$ is the union of spaces $M^H/\!/_{p_i} S$.

2.2. **Cutting with respect to a cone.** Given a linearly independent set $\beta = \{\beta_1, \ldots, \beta_k\}$ of weights of T, define the cone $\Sigma = \Sigma_\beta \subset \mathfrak{t}^*$ to be given by all nonnegative linear combinations of the weights of β, namely $\Sigma = \{\sum s_i \beta_i | s_i \geq 0\}$. (Note that we allow k to be less than $\dim T$.) For a set I of indices between 1 and k, denote by Σ^I the open face of Σ given by positive linear combinations of weights indexed by the elements of I, that is, $\Sigma^I = \{\sum_{i\in I} s_i\beta_i | s_i > 0\}$. It will be convenient to denote the subtorus of T perpendicular to Σ^I by T^I.

Assume that (M, ω) carries a Hamiltonian T action with moment map μ. For simplicity assume that the T action is effective so that $\mu(M)$ is a polytope of dimension equal to $\dim T$. As a topological space the symplectic cut with respect to a cone $\Sigma = \Sigma_\beta$ is the space

$$M_\Sigma = \mu^{-1}(\Sigma)/\sim,$$

where $x \sim x'$ if $\mu(x) = \mu(x') \in \Sigma^I_\beta$ and x and x' lie in the same T^I orbit. Clearly, the torus action on M descends to an action on M_Σ. Notice that the action on M_Σ is not effective unless $\dim\Sigma = \dim T$, since the subtorus T^Σ of T perpendicular to Σ acts trivially on M_Σ.

In the case of a circle action, when every cone is just a ray, Lerman [Le] realized that under certain mild conditions this space is an orbifold, it carries a natural symplectic form, and the residual torus action is Hamiltonian. As shown in [LMTW] similar results hold for symplectic cuts with respect to cones. Even more generally, symplectic cutting has been extended to cutting with respect to polytopes [LMTW], cutting for nonabelian groups [Me] and Kähler cutting [BGL]. For the purposes of this paper we only need to consider symplectic cutting with respect to cones.

To recall the results of [LMTW] let us give another construction of M_Σ. Let $\mathbb{C}_{\beta_i}$ be a complex line on which T acts with weight β_i. Let $\mathbb{C}_\beta = \mathbb{C}_{\beta_1} \times \cdots \times \mathbb{C}_{\beta_k}$. Then the T action on $\mathbb{C}_\beta$ is Hamiltonian with respect to the symplectic form $\omega_\mathbb{C} = \sqrt{-1}\sum dz_i \wedge d\bar{z}_i$, where z_i is the standard complex coordinate on $\mathbb{C}_{\beta_i}$. Its moment map is $\psi(z) = \sum \beta_i |z_i|^2$, so that the image of $\mathbb{C}_\beta$ under the moment map is the cone Σ. Consider the symplectic form $(\omega, -\omega_\mathbb{C})$ on $M \times \mathbb{C}_\beta$. Then the diagonal torus $\Delta T \subset T \times T$ acts on this space in a Hamiltonian fashion and its moment map ϕ is given by $\phi(x, z) = \mu(x) - \psi(z)$. As shown in [LMTW] the symplectic cut M_Σ is homeomorphic to the symplectic reduction $(M \times \mathbb{C}_\beta)/\!/_0 \Delta T$.

Hence, to guarantee that M_Σ is a symplectic orbifold it is enough to assume that zero is a regular value of ϕ. It is easy to see that this is equivalent to the following

($\star$) Every Σ^I is transverse to every wall W of $\mu(M)$, that is, $\dim(\Sigma^I \cap W) \le |I| + \dim W - \dim T$.

If ($\star$) holds we say that Σ is *transverse to* μ and from now on we consider only the cones satisfying ($\star$).

We can write $T \times T$ as the product of the first copy of T (that is, $T \times e$) and ΔT. Hence after reducing $M \times \mathbb{C}_\beta$ with respect to ΔT, the action of the first copy of T descends to an action on the reduction M_Σ. As shown in [LMTW], this action is Hamiltonian and there exists a moment map μ_Σ on M_Σ whose moment map image is the intersection $\mu(M) \cap \Sigma$.

Let us describe the connected components of the fixed point set M^T_Σ. We will separate them into three sets: the old fixed points, the new fixed points, and the fixed points at zero.

Definition 2.1. The old fixed points exist only when $\dim\Sigma = \dim T$; they are all the connected components F_i of M^T_Σ for which $\mu_\Sigma(F_i)$ is in the interior of Σ. The set of fixed points at zero is defined to be the connected component $\mu_\Sigma^{-1}(0)$ of M^T_Σ, which is just the symplectic reduction M_0. The new fixed points are all the other connected components F'_i of M^T_Σ.

It is not difficult to see that every F_i is also a connected component of the fixed point set M^T, while F'_i do not correspond to any fixed points on M.

2.3. **Kirwan map.** We recall the definition of equivariant cohomology $H^*_T(M)$ with complex coefficients using the Cartan model. Denote

$$\Omega^*_T(M) = S(\mathfrak{t}^*) \otimes \Omega^*(M)^T,$$

where $S(\mathfrak{t}^*)$ denotes the algebra of polynomials on $\mathfrak{t}$ and $\Omega^*(M)^T$ denotes all T invariant differential forms. So, if $f \in S(\mathfrak{t}^*)$, $\alpha \in \Omega^*(M)^T$ and $X \in \mathfrak{g}$ we set $(f \otimes \alpha)(X) = f(X)\alpha$. The T-equivariant differential on $\Omega^*_T(M)$ which defines the equivariant cohomology is given by

$$d_T(f \otimes \alpha) = f \otimes d\alpha - \sum x_i f \otimes \imath_{\xi^i_M}\alpha$$

where $\xi_1, \ldots, \xi_d$ is a basis of $\mathfrak{g}$; $x_1, \ldots, x_d$ is the dual basis of $\mathfrak{t}^*$; and ξ^i_M is the vector field generated by the action of ξ^i. If the equivariant form $\beta \in \Omega^*_T(M)$ is closed, that is $d_T\beta = 0$, we denote by $[\beta]$ the cohomology class it represents.

If T acts locally freely on M then it is well known that

$$H^*_T(M) = H^*(M/T),$$

and if the action is trivial then

$$H^*_T(M) = H^*(M) \otimes S(\mathfrak{t}^*) = H^*(M) \otimes H^*_T(pt).$$

For $p \in \mathfrak{t}^*$ let i_p be the inclusion $\mu^{-1}(p) \hookrightarrow M$. If p is a regular value of the moment map then T acts locally freely on $\mu^{-1}(p)$ so that

$$H^*_T(\mu^{-1}(p)) \cong H^*(\mu^{-1}(p)/T) = H^*(M/\!/_pT) \tag{2.1}$$

Composition of the pullback i^*_p with the isomorphism (2.1) defines the Kirwan map

$$\kappa_p : H^*_T(M) \to H^*(M/\!/_pT).$$

Kirwan [K] showed that if M is compact this map is surjective. In the presence of another Hamiltonian action on M by a torus T' which commutes with the action of T, the Kirwan map generalizes to its equivariant version:

$$\kappa_p : H^*_{T\times T'}(M) \to H^*_{T'}(M/\!/_pT).$$

Kirwan surjectivity was generalized to this case in [Go].

Analogously, in the case when $T = H \times S$ and $p \in \mu(M^H)$ we can define the Kirwan map

$$\kappa^H_p : H^*_S(M^H) \to H^*(M^H/\!/_pS).$$

We will mostly be interested in the equivariant version of this map. Let us take account of the trivial action of H on both M^H and $M^H/\!/_pS$. So in the rest of the paper κ^H_p will be the map

$$\kappa^H_p : H^*_T(M^H) \to H^*_H(M^H/\!/_pS). \tag{2.2}$$

Let us now apply the equivariant version of the Kirwan map to symplectic cuts with respect to cones. Given a cone $\Sigma = \Sigma_\beta$, the product $T \times T$ acts on $M \times \mathbb{C}_\beta$ and the symplectic cut M_Σ is produced by reducing $M \times \mathbb{C}_\beta$ at 0. So if we think of $T \times T$ as the product of $T \times e$ and ΔT, then the equivariant version of the Kirwan map produces the map

$$\kappa_\Sigma : H^*_{T\times T}(M \times \mathbb{C}_\beta) \to H^*_{T\times e}(M_\Sigma) = H^*_T(M_\Sigma).$$

As mentioned before the action of T on M_Σ might not be effective, since the torus T^Σ orthogonal to the cone Σ acts trivially on M_Σ. In particular,

$$H^*_T(M_\Sigma) = H^*_{T/T^\Sigma}(M_\Sigma) \otimes H^*_{T^\Sigma}(pt).$$

For $\eta \in H^*_T(M)$ denote by η_Σ the class $\kappa_\Sigma(\eta \otimes 1) \in H^*_T(M_\Sigma)$. It is an easy exercise to see that $\eta_\Sigma \in H^*_{T/T^\Sigma}(M_\Sigma) \otimes 1$.

Let us also notice that if F_i is a connected component for the fixed point set of both M and M_Σ with $\mu(F_i) = \mu_\Sigma(F_i) \in \mathrm{Int}(\Sigma)$, then

$$\iota^*_{F_i}\eta = \iota^*_{F_i}\eta_\Sigma$$

where by abuse of notation we denote by ι_{F_i} both the inclusion of F_i into M and that into M_Σ. (Because Σ is transverse to $\mu(M)$, the existence of such F_i implies that the dimension of the cone is at least the dimension of $\mu(M)$.) It is also easy to see that

$$\iota_{M_0}\eta_\Sigma = \kappa_0(\eta) \otimes 1 \in H^*(M_0) \otimes 1 \subset H^*_T(M_0),$$

where, since T acts trivially on M_0, we know that $H^*_T(M_0) = H^*(M_0) \otimes S(\mathfrak{t}^*)$.

2.4. **ABBV localization theorem.** Suppose that M is compact, oriented and carries the action of a torus T. The pushforward map from $H_T^*(M)$ to $H_T^*(pt)$ is the integration over M and is denoted by $\int$. The theorem of Atiyah-Bott and Berline-Vergne (or just the ABBV theorem) [AB, BV] states that for $\eta \in H_T^*(M)$

$$\int_M \eta = \sum_{F_i} \int_{F_i} \frac{\iota_{F_i}^*(\eta)}{e(\nu(F_i))}. \tag{2.3}$$

Here F_i are the connected components of the fixed point set M^T, $\iota_{F_i} : F_i \hookrightarrow M$ are their inclusions, $\nu(F_i)$ are the normal bundles of F_i, and $e(\nu(F_i))$ are their T-equivariant Euler classes.

We need to know more about these T-equivariant Euler classes. Because of the splitting principle [BT] we can assume without loss of generality that $e(\nu(F_i))$ splits as a sum of line bundles $L_1 \otimes \cdots \otimes L_k$. Assume T acts on the fibers of L_i with weight λ_i. Then the T-equivariant Euler class is

$$e(\nu(F_i)) = \prod(\lambda_i + c_1(L_i)), \tag{2.4}$$

where $c_1(L_i)$ is the first Chern class of L_i.

The ABBV localization formula was generalized to orbifolds by Meinrenken [Me]. We refer to [Me] for details. Let us just mention that the only difference with (2.3) is the appearance of constants before each term of the formula:

$$\frac{1}{d_M}\int_M \eta = \sum_{F_i} \frac{1}{d_{F_i}} \int_{F_i} \frac{\iota_{F_i}^*(\eta)}{e(\nu(F_i))}, \tag{2.5}$$

where for a connected orbifold X, the size of the finite stabilizer at a generic point of X is equal to d_X. Moreover (2.4) is still a valid formula for orbifolds, where the λ_i (when properly interpreted) are rational weights of the action on the normal bundle.

Another important property of the equivariant Euler classes is the following generalization of [GK, Proposition 3.1].

Proposition 2.2. *Assume $T = S \times H$, $\pi : \mathfrak{t}^* \to \mathfrak{s}^*$ is the natural projection, $p \in \mu(M^H)$, and $q = \pi(p)$. Then $\kappa_p^H : H_T^*(M^H) \to H_H^*(M^H/\!/_p S)$ takes the T-equivariant Euler class of the normal bundle of M^H onto the H-equivariant Euler class of the normal bundle of $M^H/\!/_p S$ in $M/\!/_q S$.*

Proof. The argument is almost identical to the one used in [GK]. Namely, let $Z = \mu^{-1}(p) \cap M^H$ and let ρ be the projection from Z to $M^H/\!/_p S$ and $i : Z \hookrightarrow M^H$. Then

$$i^*\nu(M^H) = \rho^*\nu(M^H/\!/_p S).$$

The proposition follows from functoriality of the Euler class as a map from oriented vector bundles to cohomology. $\square$

3. Residues

In this section we define the residue operations and discuss their basic properties.

3.1. **Residues of meromorphic 1-forms in one variable.** Think of the Riemann sphere as the one point compactification $\mathbb{C} \cup \{\infty\}$ with the complex coordinate z on $\mathbb{C}$. Let $f(z)$ be a meromorphic function on the Riemann sphere with values in a topological vector space V which can be written as the finite sum

$$f(z) = \sum_j g_j(z) e^{i\lambda_j z},$$

where $g_j(z)$ are rational functions of z and $\lambda_j \in \mathbb{R}$. Then in the case when all $\lambda_j \neq 0$ we define

$$\mathrm{Res}(f\ dz) = \sum_{\lambda_j > 0} \sum_{b \in \mathbb{C}} \mathrm{res}(g_j(z) e^{i\lambda_j z}; z = b)$$

as was done in [JK2]. The other case we will be interested in is when $\lambda_j = 0$ for all j; then we define

$$\operatorname{Res}(f\ dz) = \lim_{s\to 0^+} \operatorname{Res} f(z) e^{is\lambda z} dz \tag{3.1}$$

for some $\lambda > 0$. It is easy to see that in this case $\operatorname{Res}(f\ dz)$ is just the sum of all residues on $\mathbb{C}$ and since the sum of all residues of a meromorphic function is zero we conclude that

$$\operatorname{Res}(f\ dz) = -\operatorname{res}_{z=\infty}(f\ dz).$$

Given a linear map $\psi : V \to W$ between two topological spaces, the residue commutes with it

$$\psi(\operatorname{Res}(f)dz) = \operatorname{Res}(\psi(f)dz). \tag{3.2}$$

In the case when V carries an algebra structure, an example which will be important for us is the residue of the function of the form

$$f = \frac{g(z)}{cz+a},$$

where $a \in V$, $c \in \mathbb{C} - \{0\}$, and $g(z)$ is a polynomial in z with values on V. For $A = \frac{a}{z}$ use

$$\frac{1}{1+A} = 1 - A + A^2 - \dots, \tag{3.3}$$

to rewrite f as a sum $\sum_{j=-\infty}^{m_0} \gamma_j z^j$ for $\gamma_j \in V$, which converges for $|\frac{a}{z}| < 1$.

Let $w = \frac{1}{z}$ be another coordinate on the Riemann sphere. Then

$$\sum_{j=-m_0}^{\infty} \gamma_{-j} w^j \left(\frac{-dw}{w^2}\right)$$

is the Taylor expansion of $f dz$ at $w = 0$. Hence

$$\operatorname{Res}(f\ dz) = -\operatorname{res}_{z=\infty}\Big(\sum_{j=-\infty}^{m_0} \gamma_j z^j\ dz\Big) = \operatorname{res}_{w=0}\Big(\sum_{j=-m_0}^{\infty} \gamma_{-j} w^{j-2}\ dw\Big) = \gamma_{-1}. \tag{3.4}$$

In the case when $g(z)$ is just a constant $g_0 \in V$, we get

$$\operatorname{Res}(f\ dz) = \frac{g_0}{c}. \tag{3.5}$$

We emphasize that the residue is well defined only as a function of meromorphic 1-forms, not functions; the residue at 0 of the 1-form $f(z)dz$ is independent of the choice of coordinate z (invariant under a change of variables $z \mapsto g(z)$ provided that $g(z)$ is a meromorphic function of z and $dg(0) = 0$ whereas this is not true of usual definition of the residue at 0 of a meromorphic function).

3.2. **Residues of functions of several variables.** Let us now consider a function f of several complex variables with values in a topological space V. More precisely, we assume f is defined on the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ of the torus T, and f is a linear combination of functions of the form

$$h(X) = \frac{q(X) e^{i\lambda(X)}}{\prod_{j=1}^{k} \alpha_j(X)} \tag{3.6}$$

for some polynomials $q(X)$ of $X \in \mathfrak{t}_{\mathbb{C}}$, with values in V, $\lambda \in \mathfrak{t}^*$ and some $\alpha_1, \dots, \alpha_k \in \mathfrak{t}^* - \{0\}$.

Choose a coordinate system $X_1, \dots, X_m$ on $\mathfrak{t}$, and denote by the same letters the complexified coordinates which provide a coordinate system of $\mathfrak{t}_{\mathbb{C}}$. Let $\mathfrak{t}_\ell$ be the subspace of $\mathfrak{t}$ given by zeros of $X_1, \dots, X_\ell$. Define

$$\operatorname{Res}_m(h\ dX_m) = \operatorname{Res}(h\ dX_m),$$

where the variables $X_1, \dots, X_{m-1}$ are held constant while calculating this residue. As explained in Remark 3.5(1) of [JK2], in the case $\lambda \neq 0$ the residue Res_m is well defined only for a generic choice of coordinates $X_1, \dots, X_m$ (the precise condition on the coordinates being specified in this Remark).

Moreover, by Remark 3.5(2) of [JK2], $\mathrm{Res}_m(f\ dX_m)$ is a linear combination of functions of the form

$$\tilde{h}(X) = \frac{\tilde{q}(X)e^{i\lambda(X)}}{\prod_{j=1}^{k-1}\tilde{\alpha}_j(X)} \tag{3.7}$$

where $\tilde{q}(X)$ is a polynomial on the complexification of $\mathfrak{t}/\mathfrak{t}_{m-1}$ and $\tilde{\alpha}_j$ are in the dual of $\mathfrak{t}/\mathfrak{t}_{m-1}$.

To consider the case when $\lambda = 0$, in other words

$$h^0(X) = \frac{q(X)}{\prod_{j=1}^{k}\alpha_j(X)}, \tag{3.8}$$

we may make a choice of $\lambda^0 \in \mathfrak{t}^*$ for which $\lambda^0(X) = \sum_{j=1}^m \lambda_j^0 X_j$ with $\lambda_m^0 > 0$ and define $\mathrm{Res}_m(h^0 dX_m)$ as

$$\mathrm{Res}_m(h^0 dX_m) = \lim_{s\to 0^+}\mathrm{Res}_m(h^0 e^{is\lambda^0}dX_m). \tag{3.9}$$

We can easily check that the residue Res_m is a continuous function of $s \in \mathbb{R}^+$ and this limit exists, so we can define $\mathrm{Res}_m(hdX_m)$ even when $\lambda = 0$. It is still true in this situation that $\mathrm{Res}_m(fdX_m)$ is a linear combination of functions of the form given in (3.8) with $\lambda = 0$.

In the case when V is an algebra, an important generalization of (3.5) is the following.

Lemma 3.1.

$$\mathrm{Res}_m\Big(\frac{g_0}{\sum_i c_iX_i + v}\ dX_m\Big) = \frac{g_0}{c_m},$$

where $g_0, v \in V$ *and* $c_i \in \mathbb{C}$.

Proof. The proof follows from (3.4) after setting $c = c_m$ and $a = \sum_{i=1}^{m-1} c_iX_i + v$. □

3.3. **Iterated residues.** We again consider functions f which are linear combinations of functions of the form (3.6). As was just explained, $\mathrm{Res}_m(fdX_m)$ is a linear combination of functions of the form (3.7), which allows us to take the residue of this function again. So we set $\mathrm{Res}_m^m = \mathrm{Res}_m$ and by induction define the iterated residue

$$\mathrm{Res}_\ell^m(f\ [dX]_\ell^m) = \mathrm{Res}(\mathrm{Res}_{\ell+1}^m(f\ [dX]_{\ell+1}^m)\ dX_\ell),$$

where $[dX]_\ell^m$ stands for the form $dX_\ell\wedge\cdots\wedge dX_m$ and, as above, the coordinates $X_1, \dots, X_{\ell-1}$ are held constant while calculating this residue. Clearly $\mathrm{Res}_\ell^m(f\ [dX]_\ell^m)$ is a function on the complexification of $\mathfrak{t}/\mathfrak{t}_{\ell-1}$.

In the case V is an algebra, a generalization of Lemma 3.1 states the following.

Lemma 3.2. *For a generic choice of coordinates* $X_1, \dots, X_m$

$$\mathrm{Res}_\ell^m\Big(\frac{g_0}{\prod_{i=1}^{m-\ell+1}(\alpha_i(X)+v_i)}[dX]_\ell^m\Big) = \frac{g_0}{\det(\tilde{\alpha})}$$

where $g_0, v_i \in V$, $\alpha_i \in \mathfrak{t}^* - \{0\}$, *and* $\tilde{\alpha}$ *is the the matrix* $\{a_{ij}\}_{1\le i\le m-\ell+1;\ell\le j\le m}$, *where* $\alpha_i(X) = \sum a_{ij}X_j$.

Proof. The proof is analogous to the proof of property (iv) for the iterated residue in Proposition 3.4 of [JK2]. Notice that for h^0 defined in (3.8) the definition of $\mathrm{Res}_\ell^m(h^0[dX]_\ell^m)$ is given in [JK2] as the limit as $s \to 0^+$ of $\mathrm{Res}_\ell^m(h^0e^{is\lambda(X)}[dX]_\ell^m)$ where $\lambda(X) = \sum_j \lambda_jX_j$ and we choose λ so that all the λ_j satisfy $\lambda_j > 0$. □

Given a linear map $\psi : V \to W$ between two topological spaces, the iterated residue commutes with this map:

$$\psi(\mathrm{Res}_\ell^m(f[dX]_\ell^m)) = \mathrm{Res}_\ell^m(\psi(f)[dX]_\ell^m). \tag{3.10}$$

3.4. **Residues with respect to cones.** Iterated residues depend on the choice of coordinates on $\mathfrak{t}$. Fix an inner product on $\mathfrak{t}$. Let us define residues which depend only on this inner product and a choice of a certain cone Λ in $\mathfrak{t}$.

We introduce a function f which is a linear combination of functions of the form (3.6). We consider the set where none of the functions α_j appearing in the denominators of functions h become zero, namely the set

$$\{X \in \mathbf{t}_\ell : \alpha_j(X) \neq 0, \text{ for all } \alpha_j\}. \tag{3.11}$$

Let Λ be an open cone, which is a connected component of this set.

Then for a generic choice of coordinate system $X = (X_1, \dots, X_m)$ on $\mathfrak{t}_\mathbb{C}$ for which $(0, \dots, 0, 1) \in \Lambda$ define the residue with respect to the cone Λ by

$$\mathrm{Res}^\Lambda(h\ [dX]) = \triangle \mathrm{Res}_1^m(h\ [dX]) \tag{3.12}$$

where $[dX] = [dX]_1^m$ and $\triangle$ is the determinant of any $(m) \times (m)$ matrix whose columns are the coordinates of an orthonormal basis of $\mathfrak{t}$ defining the same orientation on $\mathfrak{t}$ as the chosen coordinate system.

To guarantee that $\mathrm{Res}^\Lambda(h\ [dX])$ is well defined (where h is of the form (3.6)), we need to make one additional assumption: we assume that λ is not in any proper subspace of $\mathfrak{t}$ spanned by some α_i's. It was shown in [JK2] that under the above assumptions $\mathrm{Res}^\Lambda(h\ [dX])$ is well defined, does not depend on the choice of the coordinates but only on the choice of the cone Λ and the inner product on $\mathfrak{t}$.

Originally, the residue Res^Λ was introduced in [JK1] as a generalization of a certain integral over a vector space. In [JK2, Proposition 3.4], it was shown that the definition (3.12) coincides with the original definition of Res^Λ from [JK1]. In [JK2, Proposition 3.2] it was shown that certain properties together with linearity uniquely define Res^Λ. Let us recall these properties:

(1) Let $\alpha_1, \dots, \alpha_v \in \Lambda^*$ be vectors in the dual cone. Suppose that λ is not in any cone of dimension $m-1$ or less spanned by a subset of the $\{\alpha_i\}$. If $J = (j_1, \dots, j_m)$ is a multi-index and $X^J = X_1^{j_1} \dots X_m^{j_m}$ then

$$\mathrm{Res}^\Lambda\left(\frac{X^J e^{i\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)}\right) = 0$$

unless all of the following properties are satisfied:
   (a) $\{\alpha_i\}_{i=1}^v$ span $\mathfrak{t}^*$ as a vector space,
   (b) $v - (j_1 + \dots + j_m) \geq m$,
   (c) $\lambda \in \langle \alpha_1, \dots, \alpha_v \rangle^+$, the positive span of the vectors $\{\alpha_i\}$.

(2) If properties (1)(a)-(c) above are satisfied, then

$$\mathrm{Res}^\Lambda\left(\frac{X^J e^{i\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)}\right) = \sum_{k\geq 0} \lim_{s\to 0^+} \mathrm{Res}^\Lambda\left(\frac{X^J (i\lambda(X))^k e^{is\lambda(X)}[dX]}{k!\prod_{i=1}^v \alpha_i(X)}\right),$$

and all but one term in this sum are 0 (the non-vanishing term being that with $k = v - (j_1 + \dots + j_m) - m$).

(3) The residue is not identically 0. If properties (1) $(a)-(c)$ are satisfied with $\alpha_1, \dots, \alpha_m$ linearly independent in $\mathfrak{t}^*$, then

$$\mathrm{Res}^\Lambda\left(\frac{e^{i\lambda(X)}[dX]}{\prod_{i=1}^m \alpha_i(X)}\right) = \frac{1}{\det(\overline{\alpha})},$$

where $\overline{\alpha}$ is the nonsingular matrix whose columns are the coordinates of $\alpha_1, \dots, \alpha_m$ with respect to any orthonormal basis of $\mathfrak{t}$ defining the same orientation.

When λ is of the form $\sum_{i=1}^k s_i\alpha_i$ where fewer than m of the s_i are nonzero, then we define

$$\mathrm{Res}^\Lambda\left(\frac{e^{i\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)}\right) = \lim_{s\to 0^+} \mathrm{Res}^\Lambda\left(\frac{e^{i(\lambda(X)+s\rho(X))}[dX]}{\prod_{i=m}^v \alpha_i(X)}\right) \tag{3.13}$$

where $\rho \in \mathbf{t}^*$ is chosen so that $\rho(\xi) > 0$ for all $\xi \in \Lambda$, and for small s, $\lambda + s\rho$ does not lie in any cone of dimension $m-1$ or less spanned by a subset of the $\{\beta_j\}$.

We will need another property of residues.

Lemma 3.3. *Let $f(X_1, \ldots, X_m)$ be a function on $\mathfrak{t}$ given by a linear combination of functions of the form (3.6) with $\lambda \neq 0$. Moreover, assume that for every set of values $a_1, \ldots, a_{m-1}$ of the variables $X_1, \ldots, X_{m-1}$ the function $g(z) = f(a_1, \ldots, a_{m-1}, z)$ is holomorphic. Let Λ be an appropriate choice of cone such that $\mathrm{Res}^\Lambda(f[dX])$ is defined, in particular Λ contains the point $(0, \ldots, 0, 1)$. Then*

$$\mathrm{Res}^\Lambda(f[dX]) = \mathrm{Res}^{-\Lambda}(f[dY]) \tag{3.14}$$

where $Y_1, \ldots, Y_m$ is a set of coordinates such that $(0, \ldots, 0, 1) \in -\Lambda$.

Proof. For fixed $a_1, \ldots, a_{m-1}$, let

$$g(z) = \sum g_j(z) e^{i\lambda_j z}.$$

Then

$$\mathrm{Res}(g(z)dz) = \mathrm{res}^+(g(z)dz) = \sum_{\lambda_j > 0} \sum_{b \in \mathbb{C}} \mathrm{res}(g_j(z) e^{i\lambda_j z}; z = b).$$

Define

$$\mathrm{res}^-(g(z)dz) = \sum_{\lambda_j < 0} \sum_{b \in \mathbb{C}} \mathrm{res}(g_j(z) e^{i\lambda_j z}; z = b).$$

Since $g(z)$ is holomorphic we have

$$\mathrm{res}^+(g(z)dz) + \mathrm{res}^-(g(z)dz) = 0. \tag{3.15}$$

Since $\mathrm{Res}^{-\Lambda}$ does not depend on the choice of coordinates as long as $(0, \ldots, 0, 1) \in -\Lambda$, we may choose $Y_i = X_i$ for $1 \leq i \leq m-1$, and $Y_m = -X_m$. Then

$$\begin{aligned}\mathrm{Res}_m(f(Y_1, \ldots, Y_m)dY_m) &= \mathrm{Res}_m(-f(X_1, \ldots, X_{m-1}, -X_m)dX_m) = \mathrm{res}^-(-f(X_1, \ldots, X_m)dX_m) \\ &=^\dagger \mathrm{res}^+(f(X_1, \ldots, X_m)dX_m) = \mathrm{Res}_m(f(X_1, \ldots, X_m)dX_m),\end{aligned}$$

where (†) holds because of (3.15). Now (3.14) follows immediately from the definition of Res^Λ using iterated residues. □

4. A generalization of Guillemin-Kalkman localization.

In this section we discuss how to obtain the localization formula of Guillemin and Kalkman [GK] by applying ABBV localization and then residue operations on symplectic cuts along certain cones of dimension one. The same approach but with cones of higher dimension and iterated residues provides a generalization of Guillemin-Kalkman localization.

4.1. **Guillemin-Kalkman localization.** Let (M, ω) be a compact symplectic manifold with an effective Hamiltonian T action. Let $\mu : M \to \mathfrak{t}^*$ be the moment map. Pick a one dimensional cone Σ transverse to μ. Assume the cone Σ is generated by a single weight β. If $\dim T = m$, consider all $(m-1)$ dimensional walls of μ which Σ intersects. Every such wall W_i is an image $\mu(M_i)$ of a connected component M_i of M^{H_i} for some one dimensional subtorus H_i of T. If p_i is the intersection $W_i \cap \Sigma$, then $G_i = M_i/\!/_{p_i} T$ is a connected component of the fixed point set of the T/T^Σ action on M_Σ. The only other connected component of this fixed point set is M_0, the reduction of M at zero by T. (If $\dim T = 1$, then G_i are also connected components of the fixed point set M^T, and using the notation introduced in Definition 2.1, G_i would be called the old connected components of the fixed point set. If $\dim T > 1$, then in terms of the same notation G_i are the new fixed points.)

Recall that any equivariant cohomology class $\eta \in H_T^*(M)$ descends to a class η_Σ on M_Σ:

$$\eta_\Sigma = \kappa_\Sigma(\eta \otimes 1) \in H^*_{T/T^\Sigma}(M_\Sigma) \otimes 1 \subset H_T^*(M_\Sigma).$$

Think of η_Σ as a class in $H^*_{T/T^\Sigma}(M_\Sigma)$ and apply the orbifold version of the ABBV localization theorem to it:

$$\frac{1}{d_{M_\Sigma}}\int_{M_\Sigma}\eta_\Sigma = \frac{1}{d_{M_0}}\int_{M_0}\frac{\kappa_0(\eta)}{e(\nu(M_0))} + \sum_{G_i}\frac{1}{d_{G_i}}\int_{G_i}\frac{\iota^*_{G_i}(\eta_\Sigma)}{e(\nu(G_i))}. \tag{4.1}$$

where, as usual, κ_0 is the Kirwan map at zero and ι_{G_i} are the inclusions of G_i into M^Σ.

Apply the residue operation to both sides of (4.1). For this we have to specify a coordinate X_1 on the one dimensional Lie algebra of T/T^Σ. Clearly the dual of this Lie algebra can be identified with the line in $\mathfrak{t}^*$ passing through the cone Σ. So β defines a coordinate on the Lie algebra of T/T_Σ, and we define $X_\beta = X_1$ to be this coordinate.

Using X_1 we get the formula

$$\mathrm{Res}_1\frac{dX_\beta}{d_{M_\Sigma}}\int_{M_\Sigma}\eta_\Sigma = \mathrm{Res}_1\frac{dX_\beta}{d_{M_0}}\int_{M_0}\frac{\kappa_0(\eta)}{e(\nu(M_0))} + \sum_{G_i}\mathrm{Res}_1\frac{dX_\beta}{d_{G_i}}\int_{G_i}\frac{\iota^*_{G_i}(\eta_\Sigma)}{e(\nu(G_i))}. \tag{4.2}$$

Since $\int_{M_\Sigma}\eta_\Sigma$ is just a polynomial function on the Lie algebra of T/T^Σ, by (3.4) the left hand side of (4.2) is zero.

The normal bundle $\nu(M_0)$ is just a line bundle whose Euler class is given by

$$e(\nu(M_0)) = X_\beta + c_1(\nu(M_0)). \tag{4.3}$$

where $c_1(\nu(M_0))$ is the first Chern class of $\nu(M_0)$. Thus by (3.10) and (3.5)

$$\mathrm{Res}_1\frac{dX_\beta}{d_{M_0}}\int_{M_0}\frac{\kappa_0(\eta)}{e(\nu(M_0))} =^{\dagger} \frac{1}{d_{M_0}}\int_{M_0}\mathrm{Res}_1\Big(\frac{\kappa_0(\eta)\; dX_\beta}{X_\beta + c_1(\nu(M_0))}\Big) = \frac{1}{d_{M_0}}\int_{M_0}\kappa_0(\eta).$$

Remark 4.1. To show that (†) follows from (3.10) think of $\frac{\kappa_0(\eta)}{e(\nu(M_0))}$ as a function of $\mathfrak{t}$ with values in $H^*(M)$. If we define $\psi : H^*(M) \to \mathbb{C}$ to be the usual integration on M, then (3.10) applied to this ψ yields (†). □

Hence (4.2) yields a formula for the the integral of $\kappa_0(\eta)$ over M_0 of the Kirwan map of η in terms of the residues of certain integrals over the G_i. To put this formula in the form in which it appeared in [GK] let us transform the terms corresponding to the G_i in (4.2) by commuting Res_1 with the integration and the Kirwan map.

By (3.10) and Proposition 2.2 we have

$$\mathrm{Res}_1\Big(\frac{dX_\beta}{d_{G_i}}\int_{G_i}\frac{\iota^*_{G_i}(\eta_\Sigma)}{e(\nu(G_i))}\Big) = \frac{1}{d_{G_i}}\int_{G_i}\mathrm{Res}_1\Big(\kappa^{H_i}_{p_i}\Big(\frac{\iota^*_{M_i}\eta}{e(\nu(M_i))}\Big)dX_\beta\Big)$$

where ι_{M_i} is just the inclusion of M_i into M, and as defined in (2.2) $\kappa^{H_i}_{p_i}$ is the Kirwan map from $H^*_T(M_i)$ to $H^*_{T/T^\Sigma}(G_i)$.

Notice that each M_i is fixed by the one-dimensional torus H_i, so that the Lie algebra $\mathfrak{h}_i$ is a subspace of $\mathfrak{t}$. Pick a coordinate system $X^i_1,\dots,X^i_m$ on $\mathfrak{t}$ such that $X^i_m = \beta$ and $X^i_1,\dots,X^i_{m-1}$ vanish on $\mathfrak{h}_i$, so that $\mathfrak{t}_{m-1} = \mathfrak{h}_i$. Then it is not difficult to see that

$$\mathrm{Res}_1\circ\kappa^{H_i}_{p_i} = \kappa^{H_i}_{p_i}\circ\mathrm{Res}_{m,i},$$

where $\mathrm{Res}_{m,i}$ is Res_m defined using the coordinate system $\{X^i_j\}$ (note that the definition of Res_m was given in (3.9). This allows us to obtain the following restatement of [GK, Theorem 3.1].

Theorem B. *For $\eta \in H^*_T(M)$*

$$\int_{M_0}\kappa_0(\eta) = -\sum_{G_i}\frac{d_{M_0}}{d_{G_i}}\int_{G_i}\kappa^{H_i}_{p_i}\mathrm{Res}_{m,i}\Big(\frac{\iota^*_{M_i}\eta}{e(\nu(M_i))}d\beta\Big). \tag{4.4}$$

Guillemin and Kalkman iterated this result using certain combinatorial objects called dendrites to produce a formula for $\int_{M_0}\kappa_0(\eta)$ in terms of the integration over the connected components of the fixed point set M^T of the original T action on M. We will discuss dendrites and their generalizations in Section 4.3.

4.2. **Higher dimensional generalization of Guillemin-Kalkman localization.** We now generalize the formula of the previous section to the case when we cut with a cone of any dimension.

As before, let (M,ω) be a compact symplectic manifold with an effective Hamiltonian T action. Let $\mu : M \to \mathfrak{t}^*$ be the moment map. Pick any cone Σ transverse to μ. Assume Σ is generated by the linearly independent weights $\beta_1,\dots,\beta_k$. Recall that in Definition 2.1 we distinguished three kinds of fixed points of M_Σ: the old fixed points (whose connected components are denoted by F_i), the new fixed points (whose connected components are denoted by F_i'), and the symplectic reduction M_0.

As in the case of one-dimensional cones any equivariant cohomology class $\eta \in H_T^*(M)$ descends to a class η_Σ on M_Σ

$$\eta_\Sigma = \kappa_\Sigma(\eta\otimes 1) \in H^*_{T/T^\Sigma}(M_\Sigma)\otimes 1 \subset H_T^*(M_\Sigma).$$

As before, we think of η_Σ as a class in $H^*_{T/T^\Sigma}(M_\Sigma)$ and apply the ABBV localization theorem to it:

$$\frac{1}{d_{M_\Sigma}}\int_{M_\Sigma}\eta_\Sigma = \frac{1}{d_{M_0}}\int_{M_0}\frac{\kappa_0(\eta)}{e(\nu(M_0))} + \sum_{F_i}\frac{1}{d_{F_i}}\int_{F_i}\frac{\iota^*_{F_i}(\eta_\Sigma)}{e(\nu(F_i))} + \sum_{F_i'}\frac{1}{d_{F_i'}}\int_{F_i'}\frac{\iota^*_{F_i'}(\eta_\Sigma)}{e(\nu(F_i'))}. \tag{4.5}$$

Here, κ_0 is the Kirwan map at zero and $\iota_{F_i},\iota_{F_i'}$ are the inclusions of F_i, F_i' into M_Σ.

Let us apply the iterated residue operation to both sides of (4.5). As in the one-dimensional case the dual of the Lie algebra of T/T^Σ can be identified with the subspace in $\mathfrak{t}^*$ passing through the cone Σ. Consider the set of weights $\{\alpha_i\}$ which appear in the isotropy representation of the T/T^Σ action at the connected components of the fixed point set of the T/T_Σ action on M_Σ. Notice that the weights $\tilde\beta_1,\dots,\tilde\beta_k$ of the isotropy representation of T/T^Σ at M_0 are the images of the weights $\beta_1,\dots,\beta_k$ under the map $(\mathfrak{t}/\mathfrak{t}^\Sigma)^* \to \mathfrak{t}^*$. In particular, the set $\{\alpha_i\}$ contains the weights $\tilde\beta_i$. Let $\{X_1,\dots,X_k\}$ be a generic coordinate system on T/T^Σ such that the determinant of the matrix which expresses $\tilde\beta_i$ as a linear combination of X_j is one.

Then by applying Res_1^k to both sides of (4.5) we get the formula

$$0 = \mathrm{Res}_1^k\frac{[dX]_1^k}{d_{M_0}}\int_{M_0}\frac{\kappa_0(\eta)}{e(\nu(M_0))} + \sum_{F_i}\mathrm{Res}_1^k\frac{[dX]_1^k}{d_{F_i}}\int_{F_i}\frac{\iota^*_{F_i}(\eta_\Sigma)}{e(\nu(F_i))} + \sum_{F_i'}\mathrm{Res}_1^k\frac{[dX]_1^k}{d_{F_i'}}\int_{F_i'}\frac{\iota^*_{F_i'}(\eta_\Sigma)}{e(\nu(F_i'))}, \tag{4.6}$$

where the left hand side is zero, since $\int_{M_\Sigma}\eta_\Sigma$ is just a polynomial function on the Lie algebra of T/T^Σ.

The weights of the isotropy representation of T/T_Σ on the normal bundle $\nu(M_0)$ are just $\tilde\beta_i$. Hence $\nu(M_0)$ splits as a direct sum of line bundles $\oplus_i L_i$, where T/T^Σ acts on the fibers of L_i by the weights $\tilde\beta_i$. Then the Euler class of $\nu(M_0)$ is just

$$e(\nu(M_0)) = \prod_i(\tilde\beta_i + c_1(L_i)). \tag{4.7}$$

where $c_1(L_i)$ are the first Chern classes of L_i. Thus by (3.10) (see also Remark 4.1) and Lemma 3.2 we have

$$\mathrm{Res}_1^k\frac{[dX]_1^k}{d_{M_0}}\int_{M_0}\frac{\kappa_0(\eta)}{e(\nu(M_0))} = \frac{1}{d_{M_0}}\int_{M_0}\kappa_0(\eta). \tag{4.8}$$

By analogy with the one-dimensional case, we should now simplify the second and third terms of the right hand side of (4.6). But the second term is already in the form we want, since it is written in terms of the fixed points of M.

To simplify the last term of (4.6), first consider the situation when a point $p_i' = \mu_\Sigma(F_i')$ is an intersection of the interior of Σ with a wall of $\mu(M)$ of dimension $r>0$. Assume that this wall is the moment map image of M_i, which is stabilized by $H_i \subset T$. Then by (3.10) and Proposition 2.2 we have

$$\frac{1}{d_{F_i'}}\mathrm{Res}_1^k\Big([dX]_1^k\int_{F_i'}\frac{\iota^*_{F_i'}(\eta_\Sigma)}{e(\nu(F_i'))}\Big) = \frac{1}{d_{F_i'}}\int_{F_i}\mathrm{Res}_1^k\Big([dX]_1^k\kappa^{H_i}_{p_i'}\big(\frac{\iota^*_{M_i}\eta}{e(\nu(M_i))}\big)\Big) \tag{4.9}$$

where ι_{M_i} is just the inclusion of M_i into M, and as defined before $\kappa^{H_i}_{p'_i}$ is the Kirwan map from $H^*_T(M_i)$ to $H^*_{T/T^\Sigma}(F'_i)$.

For the more general situation when p_i is an intersection of any face of Σ with a wall of $\mu(M)$ of complementary dimension, formula (4.9) becomes a little more complicated. Namely the Euler class of F'_i is no longer a Kirwan map image of the Euler class of M_i, but rather this multiplied by another class ν_i. The class ν_i can be understood as follows. Take the symplectic cut $M_{\Sigma,i}$ of M_i with respect to the cone Σ. This symplectic cut sits naturally inside M_Σ. Then ν_i is just the Euler class of the normal bundle of F'_i inside $M_{\Sigma,i}$. While this class can be quite complicated, by the Kirwan surjectivity theorem we know that there exists a class τ_i on M_i such that $\kappa^{H_i}_{p'_i}(\tau_i) = \nu_i$. So we conclude that

$$\frac{1}{d_{F'_i}}\mathrm{Res}^k_1\Big([dX]^k_1\int_{F'_i}\frac{\iota^*_{F'_i}(\eta_\Sigma)}{e(\nu(F'_i))}\Big) = \frac{1}{d_{F'_i}}\int_{F'_i}\mathrm{Res}^k_1\Big([dX]^k_1\kappa^{H_i}_{p'_i}\big(\frac{\iota^*_{M_i}\eta}{e(\nu(M_i))\tau_i}\big)\Big).$$

To simplify this term even further, let us choose a set of coordinates $\{X^i_1,\dots,X^i_m\}$ on $\mathfrak{t}$ as follows. Each M_i is fixed by a subtorus H_i of T, so that its Lie algebra $\mathfrak{h}_i$ lies inside $\mathfrak{t}$. (Notice that $\dim H_i \le \dim \Sigma$ and this inequality might be strict.). Choose any generic coordinates $X^i_1,\dots X^i_{m-k}$ which vanish on $\mathfrak{h}_i$ and coordinates $X^i_{m-k+1},\dots,X^i_m$, which are the images of $X_1,\dots,X_k$ under the map $(\mathfrak{t}/\mathfrak{t}^\Sigma)^* \to \mathfrak{t}^*$.

Then we conclude that

$$\mathrm{Res}^k_1 \circ \kappa^{H_i}_{p'_i} = \kappa^{H_i}_{p'_i} \circ \mathrm{Res}^m_{m-k+1},$$

where Res^m_{m-k+1} on the right hand side is take with respect to coordinates $X^i_1,\dots,X^i_m$. Thus we get the following generalization of [GK, Theorem 3.1] to the case of cones of arbitrary dimension.

Theorem C. *For $\eta \in H^*_T(M)$*
(4.10)

$$\int_{M_0}\kappa_0(\eta) = -\sum_{F_i}\frac{d_{M_0}}{d_{F_i}}\int_{F_i}\mathrm{Res}^k_1[dX]^k_1\frac{\iota^*_{F_i}(\eta_\Sigma)}{e(\nu(F_i))} - \sum_{F'_i}\frac{d_{M_0}}{d_{F'_i}}\int_{F'_i}\kappa^{H_i}_{p'_i}\mathrm{Res}^m_{m-k+1}[dX^i]^m_{m-k+1}\frac{\iota^*_{M_i}\eta}{e(\nu(M_i))\tau_i}.$$

4.3. **Dendrites and their generalizations.** In addition to Theorem B, Guillemin and Kalkman [GK] proved a formula which uses Theorem B iteratively to express $\int_{M_0}\kappa_0(\eta)$ in terms of integration over the connected components of M^T. They called the combinatorial objects responsible for how the iteration is performed dendrites.

Consider a finite collection D of tuples $(\Sigma(q), W)$, where $\Sigma(q)$ is a one-dimensional shifted cone (that is, if $\Sigma \subset \mathfrak{t}^*$ is a cone centered at the origin and $q \in \mathfrak{t}^*$, then $\Sigma(q) = \{x + q \mid x \in \Sigma\}$) and W is a wall of $\mu(M)$ such that $\Sigma(q)$ lies inside the affine space spanned by W. Notice that according to our conventions every W uniquely defines a subtorus H_W of T and a connected component M_W of M^{H_W} such that $W = \mu(M_W)$. Assume that D contains a tuple $(\Sigma_0(0), \mu(M))$ (we will treat Σ_0 as the cone used in Theorem B). Then we say that D is a dendrite if the following conditions hold

(1) For $(\Sigma(q), W) \in D$ the cone $\Sigma(q)$ is transverse to $\mu(M_W)$,
(2) For $(\Sigma(q), W) \in D$ if $\Sigma(q)$ intersects a codimension one wall W' of $\mu(M_W)$ at a point q', then there is a unique cone Σ' such that $(\Sigma'(q'), W') \in D$.

Obviously it is possible to construct a dendrite which contains a tuple $(\Sigma_0(0), \mu(M))$ as long as Σ_0 is transverse to $\mu(M)$.

Now consider formula (4.4) of Theorem B produced using the cone Σ_0. Assume we are given a dendrite D containing $(\Sigma_0(0), \mu(M))$. Every term in the summation on the right hand side of (4.4) corresponds to an intersection $p_i = \mu_\Sigma(G_i)$ of Σ_0 with a codimension one wall W_i. By definition of the dendrite, there exists a unique cone Σ_i such that $(\Sigma_i(p_i), W_i) \in D$. Set $H_i = H_{W_i}$ and $M_i = M_{W_i}$ and notice that $G_i = M_i/\!/_{p_i}(T/H_i)$. Moreover, let us shift the moment map on M_i by defining $\mu_i = \mu - p_i$. Then, we can apply Theorem B to the T/H_i action on M_i and the cone Σ_i to compute

$$\int_{M_i//_{p_i}(T/H_i)} \kappa_{p_i}^{H_i} \mathrm{Res}_m\Big(dX_m \frac{\iota_{M_i}^* \eta}{e(\nu(M_i))}\Big)$$

as a sum of integrals over symplectic reductions of the form $N/\!/(T/H)$, where H is a two dimensional subtorus of T and N is a connected component of M^H.

Repeating this process we can express $\int_{M_0} \kappa_0(\eta)$ as a summation of integrals over the connected components of the fixed point set M^T. More specifically, for a dendrite D we say that the sequence of tuples from D

$$\big((\Sigma_0(q_0), W_0), (\Sigma_1(q_1), W_1), \ldots, (\Sigma_k(q_k), W_k)\big)$$

is a path P if $(\Sigma_0, \mu(M)) = (\Sigma_0(q_0), W_0)$; W_k is zero-dimensional, so that there is a connected component F of M^T with $\mu(F) = W_k$ (we will denote this F by F_P); each q_j is an intersection of $\Sigma_{j-1}(q_{j-1})$ with a codimension one wall W_j of $\mu(M_{W_{j-1}})$. For a path P, let $k_j = \dim W_j$. Then for $\gamma \in H^*_{T/H_{W_{j-1}}}(M_{W_{j-1}})$ and an appropriately chosen set of coordinates on the Lie algebra of $T/H_{W_{j-1}}$ define

$$Q_j(\gamma) = -\mathrm{Res}_{k_j}\Big(dX_{k_j} \frac{\iota_{M_{W_j}}^* \gamma}{e(\nu(M_{W_j}))}\Big).$$

For a path P define

$$Q_P = Q_k \circ \cdots \circ Q_1.$$

The above discussion proves the following.

Theorem D. *For a dendrite D and $\eta \in H_T^*(M)$*

$$\int_{M_0} \kappa_0(\eta) = d_{M_0} \sum_P \int_{F_P} Q_P(\eta)$$

where the above sum is taken over all possible paths in D.

Now consider a collection D of tuples $(\Sigma(q), W)$, where $\Sigma(q)$ is a shifted cone of any dimension and W is a wall of $\mu(M)$ such that $\Sigma(q)$ lies inside the affine space spanned by W. Again, every W uniquely determines a subtorus H_W of T and a connected component M_W of M^{H_W} such that $W = \mu(M_W)$. Assume that D contains a tuple $(\Sigma_0(0), \mu(M))$. Then we say that D is a multi-dimensional dendrite if the following conditions hold

1. For $(\Sigma(q), W) \in D$ the cone $\Sigma(q)$ is transverse to $\mu(M_W)$,
2. For $(\Sigma(q), W) \in D$ if a face of $\Sigma(q)$ of dimension $r \leq \dim W$ intersects a wall W' of $\mu(M_W)$ of dimension $\dim W - r$ at a point q', then there is a unique cone Σ' such that $(\Sigma'(q'), W') \in D$.

In analogy with the case of dendrites of one-dimensional cones, consider formula (4.10) of Theorem C produced using the cone Σ_0. Assume we are given a multi-dimensional dendrite D containing $(\Sigma_0(0), \mu(M))$. Every term in the second summation on the right hand side of (4.10) corresponds to a new connected component F_i' of M_Σ^T. Moreover, its moment map image $p_i' = \mu_\Sigma(F_i')$ is always an intersection of a face of Σ with a wall W_i of complementary dimension. By definition of the multi-dimensional dendrite, there exists a unique cone Σ_i such that $(\Sigma_i(p_i'), W_i) \in D$. (Notice that M_{W_i} and H_{W_i} are just M_i and H_i in the notation of Theorem C.) As in the case of one-dimensional cones, we know that $F_i' = M_i/\!/_{p_i}(T/H_i)$, so we can apply Theorem C to the T/H_i action on W_i and the cone $\Sigma_i(p_i')$ to compute

$$\int_{M_i//_{p_i}(T/H_i)} \kappa_{p_i'}^{H_i} \mathrm{Res}_{m-k+1}^m\Big([dX^i]_{m-k+1}^m \frac{\iota_{M_i}^* \eta}{e(\nu(M_i))\tau_i}\Big)$$

as a sum of integrals over symplectic reductions of the form $N/\!/(T/H)$, where H is a subtorus of T of dimension at least 2 and N is a connected component of M^H.

Using the multi-dimensional dendrite we can iterate the process and express $\int_{M_0} \kappa_0(\eta)$ as a summation of integrals over the connected components of M^T as follows. For a multi-dimensional dendrite D we say that the sequence of tuples

$$\big((\Sigma_0(q_0), W_0), (\Sigma_1(q_1), W_1), \ldots, (\Sigma_k(q_k), W_k)\big)$$

is a path P if $(\Sigma_0, \mu(M)) = (\Sigma_0(q_0), W_0)$; W_k is zero-dimensional, so that there exists a connected component F of M^T with $\mu(F) = W_k$ (again we denote $F_P = F$); each q_j is the intersection of the interior of a face σ of $\Sigma_{j-1}(q_{j-1})$ with a wall W_j of $\mu(M_{W_{j-1}})$ such that $\dim\sigma + \dim W_j = \dim W_{j-1}$. For a path P, let $k_j = \dim W_j$ and $m_j = \dim \Sigma_j$. Then for $\gamma \in H^*_{T/H_{W_{j-1}}}(M_{W_{j-1}})$ and an appropriately chosen set of coordinates on the Lie algebra of $T/H_{W_{j-1}}$ define

$$Q_j(\gamma) = -\mathrm{Res}^{k_j}_{k_j-m_j+1}\Big([dX]^{k_j}_{k_j-m_j+1}\frac{\iota^*_{M_{W_j}}\gamma}{e(\nu(M_{W_j}))}\Big).$$

For a path P define

$$Q_P = Q_k \circ \cdots \circ Q_1.$$

It is clear that Theorem D holds in the case when D is a multi-dimensional dendrite.

5. A new proof of the Jeffrey-Kirwan localization formula.

As explained in the previous section it is possible to iterate the formulas of Theorems B and C using (multi-dimensional) dendrites to express integration of $\kappa_0(\eta)$ over M_0 as a sum of integrals of certain forms over the connected components of M^T. In this section we present another way of writing such a formula. The rough idea is to choose a very wide cone Σ, whose dual is inside a certain cone Λ, then apply the residue operation with respect to Λ to the ABBV formula for $\eta e^{i\tilde{\omega}}$ (where $\tilde{\omega}$ is the equivariant symplectic form) in such a way that the terms corresponding to the new fixed points F_i' of M_Σ are zero. As a result we will obtain a new proof of the the Jeffrey-Kirwan localization theorem [JK1].

5.1. **The choice of the cone.** In this section we explain how to choose the cone Σ. It is convenient to think of Σ as a very wide cone, in other words a cone which is close to being a hyperplane.

Given an effective Hamiltonian action of T on a compact manifold M, let $\{\alpha_i\}$ be the set of all weights appearing in the isotropy representation of T at the fixed points. Pick a connected component Λ of the set

$$\{\xi \in \mathfrak{t} | \alpha_i(\xi) \neq 0 \text{ for all } i\}.$$

Consider the dual cone $\Lambda^* = \{X \in \mathfrak{t}^* | X(\xi) \geq 0 \text{ for all } \xi \in \Lambda\}$. Now pick a cone Σ transverse to $\mu(M)$ spanned by $m = \dim \mathfrak{t}$ weights $\beta_1, \ldots, \beta_m$ such that Σ satisfies

(1) $\Lambda^* \subseteq \Sigma$, or in other words $\Sigma^* \subseteq \Lambda$,
(2) For every wall W of $\mu(M)$ if $W \cap \Sigma$ is not empty, then there exists $\xi_W \in \Sigma^*$ such that the maximum of $q(\xi_W)$ for $q \in W \cap \Sigma$ is attained at a vertex of W.

Let us explain why such a cone Σ always exists. Pick a rational vector $\xi_0 \in \Lambda$ and let H be the hyperplane in $\mathfrak{t}^*$ annihilated by ξ_0. For a weight p in the interior of Λ^* define $H_p = H + p$. Then $H_p \cap \Lambda^*$ is a rational polytope. In particular, there exists a simplex $S \subset H_p$ with rational vertices $\tilde{\beta}_1, \ldots, \tilde{\beta}_m$ which contains the polytope $H_p \cap \Lambda^*$. Take the weights $\beta_1, \ldots, \beta_m$, which define Σ, to be multiples of $\tilde{\beta}_1, \ldots, \tilde{\beta}_m$. Clearly Σ satisfies (1). Moreover, by increasing the size of S, that is by making Σ wider, we can always guarantee that Σ satisfies (2). Since there are infinitely many choices of simplices S, there is a choice of S for which Σ is transverse to $\mu(M)$.

Now consider the symplectic cut M_Σ and the set $\{\gamma_i\}$ of all the weights which appear as weights of isotropy representations at fixed points of M_Σ. Consider the set

$$\Sigma^* \cap \{\xi \in \mathfrak{t} | \gamma_i(\xi) \neq 0 \text{ for all } i\},$$

and let the cone $\bar{\Lambda}$ be a connected component of this set.

For each connected component F of M_Σ^T, let $\{\gamma_j^F\}$ be the set of weights of the isotropy representation of T at F. Polarize the weights $\{\gamma_j^F\}$ using $\bar{\Lambda}$, that is if $\gamma_j^F(\xi) < 0$ for all $\xi \in \bar{\Lambda}$ then set $\bar{\gamma}_j^F = \gamma_j^F$, otherwise let $\bar{\gamma}_j^F = -\gamma_j^F$. Define C_F to be the cone containing all the points of the form $\mu_\Sigma(F) + \sum s_j \bar{\gamma}_j^F$ with s_j being nonnegative real numbers.

If F is an old connected component of the fixed point set, then the polarization of the weights $\{\gamma_j^F\}$ using $\bar{\Lambda}$ is the same as using Λ, since $\bar{\Lambda} \subset \Lambda$. If $F = M_0$, then $C_F = -\Sigma$. For the new fixed points let us prove the following important fact.

Lemma 5.1. *Let F be a new connected component of the fixed point set M_Σ^T. Then the cone C_F does not intersect the interior of Σ.*

Proof. Let $p = \mu_\Sigma(F)$. Since Σ intersects $\mu(M)$ transversely, there is a subtorus H of T and a connected component M' of M^H for which F is the symplectic reduction of M' at p. Moreover, p is just the intersection of the wall $W = \mu(M')$ of dimension k and an open face σ of Σ of dimension $m-k$.

Let us order the weights $\{\gamma_j^F\}_{j=1}^n$ of the isotropy representation of T at F in a special way. Assume that each of the first k' weights is parallel to an intersection $W \cap \sigma'$, where σ' is an open face of Σ such that $\dim \sigma' = \dim \sigma + 1$ and the closure of σ' contains σ. Assume also that each of the last $n-k'$ weights is parallel to an intersection $W' \cap \sigma$, where W' is a wall with $\dim W' = \dim W + 1$ and $W \subset W'$. (It is easy to see that every γ_j^F falls into one of these categories.)

Let C_F^1 be the cone spanned by the first k' polarized weights centered at the origin and let C_F^2 be the cone spanned by the other weights, also centered at the origin. Then $C_F = p + C_F^1 + C_F^2$. Since the cone $p + C_F^2$ lies in the affine space passing through σ, it remains to show that $p + C_F^1$ does not intersect the interior of Σ.

If two of the weights γ_j^F are parallel, then we can remove one of them without changing the polarized cone Σ_F. Since the first k' weights point in exactly k different directions (there are precisely k faces σ' of Σ with $\sigma \subset \bar{\sigma}'$ and $\dim \sigma' = \dim \sigma + 1$), we can assume without loss of generality that $k' = k$.

Let the first k weights span a cone $\tilde{C}_F$ centered at the origin. It is clear that in a small neighborhood of p the polytope $\Sigma \cap W$ is equal to the cone $p + \tilde{C}_F$. Moreover, since the first k weights are linearly independent, the cones $\tilde{C}_F$ and C_F^1 are either the same or do not have common points in the interior. So it remains to show that $\tilde{C}_F$ and C_F^1 are different, which is the same as showing that during polarization at least one of the first k weights changes sign.

So, we must show that it is impossible to have $\gamma_j^F(\xi) < 0$ for $1 \le j \le k$ for every $\xi \in \bar{\Lambda}$. If this happens then the maximum of $q(\xi)$ for $q \in W \cap \Sigma$ is attained at $\mu_\Sigma(F)$, which is impossible, by property (2) of the cone Σ. $\square$

5.2. **Jeffrey-Kirwan localization.** Let cones $\Lambda, \bar{\Lambda}$ and Σ be defined as in the previous section. Consider the symplectic cut M_Σ. Let $p \in \Sigma$ be a point close to the origin. Then $\mu_\varepsilon = \mu_\Sigma - \varepsilon p$ for $\varepsilon > 0$ is a moment map on M_Σ. Define $\tilde{\omega}_\varepsilon = \omega_\Sigma + i\mu_\varepsilon$ to be an equivariant symplectic form on M_Σ, where ω_Σ is the symplectic form on M_Σ.

For a form $\eta \in H_T^*(M)$, there corresponds a form $\eta_\Sigma \in H_T^*(M_\Sigma)$. Apply the ABBV localization theorem to get

$$\int_{M_\Sigma} \eta_\Sigma e^{\tilde{\omega}_\epsilon} = I_{M_0}^\varepsilon + \sum I_{F_i}^\varepsilon + \sum I_{F_i'}^\varepsilon \tag{5.1}$$

where

$$I_F^\varepsilon = \frac{1}{d_F} e^{i(\mu_\varepsilon(F))} \int_F \frac{\iota_F^*(\eta_\Sigma e^\omega)}{e(\nu(F))} = \frac{1}{d_F} e^{i(\mu_\Sigma(F) - \varepsilon p)} \int_F \frac{\iota_F^*(\eta_\Sigma e^\omega)}{e(\nu(F))}.$$

Let us now apply Lemma 3.3 to the function $\int_{M_\Sigma} e^{\tilde{\omega}_\varepsilon}$ and cones $\bar{\Lambda}$ and $-\bar{\Lambda}$:

$$\operatorname{Res}^{\bar{\Lambda}}([dX] \int_{M_\Sigma} \eta_\Sigma e^{\tilde{\omega}_\varepsilon}) = \operatorname{Res}^{-\bar{\Lambda}}([dX] \int_{M_\Sigma} \eta_\Sigma e^{\tilde{\omega}_\varepsilon}).$$

Hence by (5.1) we get

$$\mathrm{Res}^{\bar\Lambda}([dX](I^\varepsilon_{M_0}+\sum I^\varepsilon_{F_i}+\sum I^\varepsilon_{F'_i}))=\mathrm{Res}^{-\bar\Lambda}([dX](I^\varepsilon_{M_0}+\sum I^\varepsilon_{F_i}+\sum I^\varepsilon_{F'_i})) \tag{5.2}$$

Let us show that four out six terms of (5.2) are zeros.

Indeed by Lemma 5.1 the cones $C_{F'_i}$ do not contain εp. Hence by property (1) of the residue, we get

$$\mathrm{Res}^{\bar\Lambda}(I^\varepsilon_{F'_i})=0.$$

Analogously, since the cone C_{M_0} does not contain εp

$$\mathrm{Res}^{\bar\Lambda}(I^\varepsilon_{M_0})=0.$$

Similarly, for small enough ε the cones $-C_{F'_i}$ and $-C_{F'_j}$ do not contain εp. Hence by property (1) of the residue we have

$$\mathrm{Res}^{-\bar\Lambda}(I^\varepsilon_{F'_i})=0,\quad \mathrm{Res}^{-\bar\Lambda}(I^\varepsilon_{F_j})=0$$

for all i and j.

Hence only two terms of (5.2) are not zero. Moreover, since $\bar\Lambda$ and Λ define the same polarization at each F_i one of these terms can be modified using

$$\mathrm{Res}^{\bar\Lambda}(I^\varepsilon_{F_i})=\mathrm{Res}^{\Lambda}(I^\varepsilon_{F_i}).$$

Hence these computations transform (5.2) into

$$\mathrm{Res}^{-\bar\Lambda}(I^\varepsilon_{M_0})=\sum\mathrm{Res}^{\Lambda}(I^\varepsilon_{F_i}). \tag{5.3}$$

Let us now take the limit of both sides of (5.3) as $\varepsilon\to 0$. By property (3) of the residue map, and by equations (3.13) and (4.7) we have:

$$\lim_{\varepsilon\to 0^+}\mathrm{Res}^{-\bar\Lambda}(I^\varepsilon_{M_0})=c'\int_{M_0}\kappa_0(\eta e^\omega),$$

for some constant c'. Hence the limit of (5.3) gives

$$\int_{M_0}\kappa_0(\eta e^\omega)=c\sum_i\mathrm{Res}^{\Lambda}\Big(e^{i\mu_\Sigma(F_i)}\int_{F_i}\frac{\iota^*_{F_i}(\eta_\Sigma e^\omega)}{e(\nu(F_i))}\Big)$$

for some constant c.

To finish the proof of Theorem A, remember that F_i are the old connected components of the fixed point set M^T_Σ, so that $F_i\subset M^T$ and $\mu(F_i)=\mu_\Sigma(F_i)$. In particular, $\iota^*_{F_i}(\eta_\Sigma e^\omega)$ is the same as the restriction of $\eta e^\omega\in H^*_T(M)$ to F_i. Moreover, if for a connected component F of M^T its moment map image $\mu(F)$ is not inside Σ, then the cone at F polarized with respect to Λ does not contain the origin, and by property (1) of the residue the term which corresponds to F in the formula (1.1) is zero.

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