

## LOCALIZED HARDY SPACES $H^1$ RELATED TO ADMISSIBLE FUNCTIONS ON RD-SPACES AND APPLICATIONS TO SCHRÖDINGER OPERATORS

DACHUN YANG AND YUAN ZHOU

ABSTRACT. Let  $\mathcal{X}$  be an RD-space, which means that  $\mathcal{X}$  is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling property holds in  $\mathcal{X}$ . In this paper, the authors first introduce the notion of admissible functions  $\rho$  and then develop a theory of localized Hardy spaces  $H_\rho^1(\mathcal{X})$  associated with  $\rho$ , which includes several maximal function characterizations of  $H_\rho^1(\mathcal{X})$ , the relations between  $H_\rho^1(\mathcal{X})$  and the classical Hardy space  $H^1(\mathcal{X})$  via constructing a kernel function related to  $\rho$ , the atomic decomposition characterization of  $H_\rho^1(\mathcal{X})$ , and the boundedness of certain localized singular integrals on  $H_\rho^1(\mathcal{X})$  via a finite atomic decomposition characterization of some dense subspace of  $H_\rho^1(\mathcal{X})$ . This theory has a wide range of applications. Even when this theory is applied, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on  $\mathbb{R}^n$ , or to the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups, some new results are also obtained. The Schrödinger operators considered here are associated with nonnegative potentials satisfying the reverse Hölder inequality.

### 1. INTRODUCTION

The theory of Hardy spaces on the Euclidean space  $\mathbb{R}^n$  plays an important role in various fields of analysis and partial differential equations; see, for example, [48, 18, 5, 47, 20]. One of the most important applications of Hardy spaces is that they are good substitutes of Lebesgue spaces when  $p \in (0, 1]$ . For example, when  $p \in (0, 1]$ , it is well known that Riesz transforms are not bounded on  $L^p(\mathbb{R}^n)$ , however, they are bounded on Hardy spaces. A localized version of Hardy spaces on  $\mathbb{R}^n$  was first introduced by Goldberg [24]. These classical Hardy spaces are essentially related to the Laplace operator  $\Delta \equiv -\sum_{j=1}^n (\frac{\partial}{\partial x_j})^2$  on  $\mathbb{R}^n$ .

On the other hand, the studies of Schrödinger operators with nonnegative potentials satisfying the reverse Hölder inequality obtain an increasing interest; see, for example, [17, 57, 46, 33, 10, 11, 30, 31, 12, 14, 15, 9, 16, 34, 2]. In particular, Fefferman [17], Shen [46] and Zhong [57] established some basic results, including

---

Received by the editors August 26, 2008.

2010 *Mathematics Subject Classification*. Primary 42B30; Secondary 42B20, 42B25, 42B35, 42B37.

*Key words and phrases*. Space of homogeneous type, Heisenberg group, connected and simply connected Lie group, admissible function, Hardy space, maximal function, atom, Schrödinger operator, reverse Hölder inequality, Riesz transform.

The first author was supported by the National Natural Science Foundation (Grant No. 10871025) of China.

©2010 American Mathematical Society  
Reverts to public domain 28 years from publication

estimates of the fundamental solutions and the boundedness on Lebesgue spaces of Riesz transforms, for the Schrödinger operator  $\mathcal{L} \equiv \Delta + V$  on  $\mathbb{R}^n$  with  $n \geq 3$  and the nonnegative potential  $V$  satisfying the reverse Hölder inequality. Lu [35] extended part of these results to the sub-Laplace Schrödinger operator on stratified groups, and Li [33] on connected and simply connected nilpotent Lie groups. Kurata and Sugano [30] extended some of these results to the degenerate Schrödinger operator on  $\mathbb{R}^n$  with  $n \geq 3$ . On the other hand, Dziubański and Zienkiewicz [11] first characterized the Hardy space  $H_{\mathcal{L}}^1(\mathbb{R}^n)$  for Schrödinger operators via atoms, the maximal function defined by the semigroup generated by  $\mathcal{L}$  and the Riesz transforms  $\nabla \mathcal{L}^{-1/2}$ , which were further generalized by C. Lin, H. Liu and Y. Liu [34] to Heisenberg groups. Also, Duong and Yan [9] established the Lusin-area function and molecular characterizations of Hardy spaces  $H_{\mathcal{L}}^1(\mathbb{R}^n)$  associated to the operator  $\mathcal{L}$  with heat kernel bounds, which includes the Schrödinger operator with nonnegative potential as an example. Dziubański [15] further obtained the atomic characterization and the maximal function characterization of the semigroup generated by  $\mathcal{L}$  for Hardy spaces  $H_{\mathcal{L}}^1(\mathbb{R}^n)$  associated with the degenerate Schrödinger operator  $\mathcal{L}$  on  $\mathbb{R}^n$  via a theory of Hardy spaces on spaces of homogeneous type with the additional assumption that the measure of any ball is equivalent to its radius in [7, 37, 51].

Recently, a theory of Hardy spaces on so-called RD-spaces was established in [26, 27, 21, 22]. A space  $\mathcal{X}$  of homogeneous type in the sense of Coifman and Weiss is called an RD-space if  $\mathcal{X}$  has the additional property that a reverse doubling property holds in  $\mathcal{X}$  (see [27]). It is well known that a connected space of homogeneous type is an RD-space. Typical examples of RD-spaces include Euclidean spaces, Euclidean spaces with weighted measures satisfying the doubling property, Heisenberg groups, Lie groups of polynomial growth ([52, 53]) and the boundary of an unbounded model polynomial domain in  $\mathbb{C}^2$  ([40, 41]), or more generally, Carnot-Carathéodory spaces with doubling measures ([42, 27]). Throughout this paper, we only consider those RD-spaces with infinitely total measures.

Motivated by the properties of nonnegative potentials satisfying the reverse Hölder inequality in the aforementioned Schrödinger operators, in this paper, we first introduce a class of admissible functions  $\rho$  on  $\mathcal{X}$ . Via establishing some basic properties of  $\rho$ , we develop a theory of Hardy spaces  $H_{\rho}^1(\mathcal{X})$  associated to admissible functions  $\rho$ , which includes several maximal function characterizations of  $H_{\rho}^1(\mathcal{X})$ , the relations between  $H_{\rho}^1(\mathcal{X})$  and the classical Hardy space  $H^1(\mathcal{X})$  via constructing a kernel function related to  $\rho$ , the atomic decomposition characterization of  $H_{\rho}^1(\mathcal{X})$ , and the boundedness of certain localized singular integrals on  $H_{\rho}^1(\mathcal{X})$  via a finite atomic decomposition characterization of some dense subspace of  $H_{\rho}^1(\mathcal{X})$ . Since these results hold for any admissible function  $\rho$  and any RD-space  $\mathcal{X}$ , they have a wide range of applications. Moreover, even when this theory is applied, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on  $\mathbb{R}^n$ , or to the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups, we also obtain some new results. Precisely, this paper is organized as follows.

In Section 2, we first recall some notation and notions from [27]. Then we introduce the notions of admissible functions  $\rho$ , localized Hardy spaces  $H_{\rho}^1(\mathcal{X})$  defined by the grand maximal functions, and atomic Hardy spaces  $H_{\rho}^{1,q}(\mathcal{X})$ . Some properties of admissible functions are also presented, which are used throughout

the entire paper. We also recall some results on classical Hardy spaces  $H^1(\mathcal{X})$  from [26, 27, 21, 22].

One key step of this paper is to construct a kernel function on  $\mathcal{X} \times \mathcal{X}$  associated to any given admissible function  $\rho$  in Proposition 3.1 below by subtly exploiting some ideas originally from Coifman [8]. A suitable variant of this kernel function actually yields an approximation of the identity related to  $\rho$ . This may be very useful in establishing a theory of Besov and Triebel-Lizorkin spaces, including Hardy spaces  $H^p(\mathcal{X})$  when  $p \leq 1$  but near to 1 and fractional Sobolev spaces; see [26, 27]. Using this kernel function, in Section 3 of this paper, we establish the relations between  $H_\rho^1(\mathcal{X})$  and  $H^1(\mathcal{X})$  (see Theorem 3.1 below), and as an application, we further obtain an atomic decomposition characterization of  $H_\rho^1(\mathcal{X})$  via  $(1, q)_\rho$ -atoms with  $q \in (1, \infty]$  (see Theorem 3.2 (i) below). Moreover, for a certain dense subspace of  $H_\rho^1(\mathcal{X})$ , we establish its finite atomic decomposition characterization via  $(1, q)_\rho$ -atoms with  $q < \infty$  and continuous  $(1, \infty)_\rho$ -atoms (see Theorem 3.2 (ii) below). As an application of this result, we establish a general boundedness criterion for sublinear operators on  $H_\rho^1(\mathcal{X})$  via atoms (see Proposition 3.2 below), and then we obtain the boundedness on  $H_\rho^1(\mathcal{X})$  of certain localized singular integrals (see Proposition 3.3 below), which is useful in establishing the boundedness of Riesz transforms related to Schrödinger operators in Section 5.

In Section 4, we establish a radial maximal function characterization of  $H_\rho^1(\mathcal{X})$ ; see Theorem 4.1 below. For the sake of applications, we also characterize  $H_\rho^1(\mathcal{X})$  via a variant of the radial maximal functions, which is closely related to the considered admissible function  $\rho$ ; see Theorem 4.2 below. We should point out that the method used to obtain the radial maximal function characterization of  $H_\rho^1(\mathcal{X})$  is totally different from the method used by Dziubański and Zienkiewicz in [10, 11, 12, 15] to obtain a similar result on  $\mathbb{R}^n$ . The method in [10, 11, 12, 15] strongly depends on an existing theory of localized Hardy spaces  $h^1$ , on  $\mathbb{R}^n$  or on spaces of homogeneous type with the additional assumption that the measure of any ball is equivalent to its radius, in the sense of Goldberg [24]. We successfully avoid this via the discrete Calderón reproducing formula from [27, 21] and a subtle split of dyadic cubes of Christ in [4].

In Section 5, we apply the results obtained in Sections 3 and 4, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on  $\mathbb{R}^n$ , to the sub-Laplace Schrödinger operator on Heisenberg groups or on connected and simply connected nilpotent Lie groups. The nonnegative potentials of these Schrödinger operators are assumed to satisfy the reverse Hölder inequality. Even for these special cases, our results further complement the results in [11, 12, 15, 34]. Especially for the sub-Laplace Schrödinger operator on connected and simply connected nilpotent Lie groups, Theorem 5.1 through Theorem 5.4 below seem to be previously unknown.

Moreover, in forthcoming papers, we will develop a dual theory for  $H_\rho^1(\mathcal{X})$  and also apply these results obtained in this paper to the sub-Laplace Schrödinger operator with nonnegative potentials satisfying the reverse Hölder inequality on the boundary of an unbounded model polynomial domain in  $\mathbb{C}^2$  which appeared in [40, 41].

We finally make some conventions. Throughout this paper, we always use  $C$  to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as  $C_1$ ,

do not change in different occurrences. If  $f \leq Cg$ , we then write  $f \lesssim g$  or  $g \gtrsim f$ , and if  $f \lesssim g \lesssim f$ , we then write  $f \sim g$ . We also denote  $\max\{\beta, \gamma\}$  and  $\min\{\beta, \gamma\}$ , respectively, by  $\beta \vee \gamma$  and  $\beta \wedge \gamma$ . For any set  $E \subset \mathcal{X}$ , set  $E^c \equiv (\mathcal{X} \setminus E)$ .

## 2. PRELIMINARIES

We first recall the notions of spaces of homogeneous type in the sense of Coifman and Weiss [6, 7] and RD-spaces in [27].

**Definition 2.1.** Let  $(\mathcal{X}, d)$  be a metric space with a regular Borel measure  $\mu$  such that all balls defined by  $d$  have finite and positive measure. For any  $x \in \mathcal{X}$  and  $r > 0$ , set the ball  $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$ .

(i) The triple  $(\mathcal{X}, d, \mu)$  is called a space of homogeneous type if there exists a constant  $C_1 \geq 1$  such that for all  $x \in \mathcal{X}$  and  $r > 0$ ,

$$\mu(B(x, 2r)) \leq C_1 \mu(B(x, r)) \quad (\text{doubling property}).$$

(ii) The triple  $(\mathcal{X}, d, \mu)$  is called an RD-space if there exist constants  $0 < \kappa \leq n$  and  $C_2 \geq 1$  such that for all  $x \in \mathcal{X}$ ,  $0 < r < \text{diam}(\mathcal{X})/2$  and  $1 \leq \lambda < \text{diam}(\mathcal{X})/(2r)$ ,

$$(2.1) \quad (C_2)^{-1} \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C_2 \lambda^n \mu(B(x, r)),$$

where  $\text{diam}(\mathcal{X}) = \sup_{x, y \in \mathcal{X}} d(x, y)$ .

*Remark 2.1.* (i) Obviously, an RD-space is a space of homogeneous type. Conversely, a space of homogeneous type automatically satisfies the second inequality of (2.1). Moreover, it was proved in [27, Remark 1.1] that if  $\mu$  is doubling, then  $\mu$  satisfies (2.1) if and only if there exist constants  $a_0 > 1$  and  $C_0 > 1$  such that for all  $x \in \mathcal{X}$  and  $0 < r < \text{diam}(\mathcal{X})/a_0$ ,

$$\mu(B(x, a_0 r)) \geq C_0 \mu(B(x, r)) \quad (\text{reverse doubling property})$$

(if  $a_0 = 2$ , this is the classical reverse doubling condition), and equivalently, for all  $x \in \mathcal{X}$  and  $0 < r < \text{diam}(\mathcal{X})/a_0$ ,  $(B(x, a_0 r) \setminus B(x, r)) \neq \emptyset$ , which, as pointed out to us by the referee, is known in the topology as uniform perfectness. For more equivalent characterizations of RD-spaces, see [56].

(ii) Let  $d$  be a quasi-metric, which means that there exists  $A_0 \geq 1$  such that for all  $x, y, z \in \mathcal{X}$ ,  $d(x, y) \leq A_0(d(x, z) + d(z, y))$ . Recall that Macías and Segovia [36, Theorem 2] proved that there exists an equivalent quasi-metric  $\tilde{d}$  such that all balls corresponding to  $\tilde{d}$  are open in the topology induced by  $\tilde{d}$ , and there exist constants  $\tilde{A}_0 > 0$  and  $\theta \in (0, 1)$  such that for all  $x, y, z \in \mathcal{X}$ ,

$$\left| \tilde{d}(x, z) - \tilde{d}(y, z) \right| \leq \tilde{A}_0 \left[ \tilde{d}(x, y) \right]^\theta \left[ \tilde{d}(x, z) + \tilde{d}(y, z) \right]^{1-\theta}.$$

It is known that the approximation of the identity as in Definition 2.3 below also exists for  $\tilde{d}$ ; see [27]. Obviously, all results in this section and Sections 3 and 4 are invariant on equivalent quasi-metrics. From these facts, it follows that all conclusions of this section and Sections 3 and 4 are still valid for quasi-metrics (especially for so-called  $d$ -spaces of Triebel; see [50, p. 189]).

Throughout the entire paper, we always assume that  $\mathcal{X}$  is an RD-space and  $\mu(\mathcal{X}) = \infty$ . In what follows, for any  $x, y \in \mathcal{X}$  and  $r \in (0, \infty)$ , we set  $V_r(x) \equiv \mu(B(x, r))$  and  $V(x, y) \equiv \mu(B(x, d(x, y)))$ .

**2.1. Admissible functions.** We first introduce the notion of admissible functions.

**Definition 2.2.** A positive function  $\rho$  on  $\mathcal{X}$  is called admissible if there exist positive constants  $C_3$  and  $k_0$  such that for all  $x, y \in \mathcal{X}$ ,

$$(2.2) \quad \rho(y) \leq C_3[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x, y)]^{k_0/(1+k_0)}.$$

Obviously, if  $\rho$  is a constant function, then  $\rho$  is admissible. Another non-trivial class of admissible functions is given by the well-known reverse Hölder class  $\mathcal{B}_q(\mathcal{X}, d, \mu)$  (see, for example, [23, 39, 46] for its definition on  $\mathbb{R}^n$  and [49] for its definition on spaces of homogenous type). Recall that a nonnegative potential  $U$  is said to belong to  $\mathcal{B}_q(\mathcal{X}, d, \mu)$  (for short,  $\mathcal{B}_q(\mathcal{X})$ ) with  $q \in (1, \infty]$  if there exists a positive constant  $C$  such that for all balls  $B$ ,

$$\left\{ \frac{1}{\mu(B)} \int_B [U(y)]^q d\mu(y) \right\}^{1/q} \leq C \frac{1}{\mu(B)} \int_B U(y) d\mu(y)$$

with the usual modification when  $q = \infty$ . It was proved in [49, pp. 7-8] that if  $U \in \mathcal{B}_q(\mathcal{X})$  for some  $q \in (1, \infty]$  and the measure  $U(z) d\mu(z)$  has the doubling property, then  $U$  is an  $\mathcal{A}_p(\mathcal{X}, d, \mu)$ -weight for some  $p \in [1, \infty)$  in the sense of Muckenhoupt and also  $U \in \mathcal{B}_{q+\epsilon}(\mathcal{X})$  for some  $\epsilon > 0$ . Thus  $\mathcal{B}_q(\mathcal{X}) = \bigcup_{q_1 > q} \mathcal{B}_{q_1}(\mathcal{X})$ . Here it should be pointed out that, generally,  $U \in \mathcal{B}_q(\mathcal{X})$  cannot imply the doubling property of  $U(z)d\mu(z)$ , but when  $\mu(B(x, r))$  increasing continuous respect to  $r$  for all  $x \in \mathcal{X}$ ,  $U \in \mathcal{B}_q(\mathcal{X})$  does not imply the doubling property of  $U(z)d\mu(z)$  by [49, p. 9, Theorem 17]. Following [46], for all  $x \in \mathcal{X}$ , set

$$(2.3) \quad \rho(x) \equiv \sup \left\{ r > 0 : \frac{r^2}{V_r(x)} \int_{B(x, r)} U(y) d\mu(y) \leq 1 \right\},$$

where we recall that  $V_r(x) \equiv \mu(B(x, r))$  for all  $x \in \mathcal{X}$  and  $r > 0$ . Then we have the following conclusion.

**Proposition 2.1.** Let  $q \in (1 \vee (n/2), \infty]$  and  $U \in \mathcal{B}_q(\mathcal{X})$ . If the measure  $U(z)d\mu(z)$  has the doubling property, then  $\rho$  as in (2.3) is an admissible function.

*Proof.* For any fixed  $y \in \mathcal{X}$  and  $0 < r < R < \infty$ , by the Hölder inequality,  $U \in \mathcal{B}_q(\mathcal{X})$  and the doubling property of  $\mu$ , we have

$$(2.4) \quad \begin{aligned} \frac{r^2}{V_r(y)} \int_{B(y, r)} U(z) d\mu(z) &\lesssim r^2 \left\{ \frac{1}{V_r(y)} \int_{B(y, r)} [U(z)]^q d\mu(z) \right\}^{1/q} \\ &\lesssim r^2 \left[ \frac{V_R(y)}{V_r(y)} \right]^{1/q} \frac{1}{V_R(y)} \int_{B(y, R)} U(z) d\mu(z) \\ &\lesssim \left( \frac{r}{R} \right)^{2-n/q} \frac{R^2}{V_R(y)} \int_{B(y, R)} U(z) d\mu(z). \end{aligned}$$

By the assumption that the measure  $U(z) d\mu(z)$  has the doubling property, so there exist positive constants  $C$  and  $n_1 > \{(\kappa - n/q) \vee 0\}$  such that for all  $\lambda > 1$ ,  $r > 0$  and  $x \in \mathcal{X}$ ,

$$(2.5) \quad \int_{B(x, \lambda r)} U(z) d\mu(z) \leq C\lambda^{n_1} \int_{B(x, r)} U(z) d\mu(z).$$

By (2.4) and the fact that  $q > n/2$ , there exists at least one  $r > 0$  such that

$$\frac{r^2}{V_r(y)} \int_{B(y,r)} U(z) d\mu(z) \leq 1$$

and

$$\lim_{R \rightarrow \infty} \frac{R^2}{V_R(y)} \int_{B(y,R)} U(z) d\mu(z) = \infty,$$

which imply that  $0 < \rho(y) < \infty$ . Thus, from (2.5), it further follows that

$$(2.6) \quad \frac{[\rho(y)]^2}{V_{\rho(y)}(y)} \int_{B(y,\rho(y))} U(z) d\mu(z) \sim 1.$$

Now we prove that  $\rho$  satisfies (2.2). For any fixed  $x, y \in \mathcal{X}$ , if  $d(x, y) < \rho(y)$ , then by the doubling property of  $\mu$  and (2.6) we have

$$\frac{[\rho(y)]^2}{V_{\rho(y)}(x)} \int_{B(x,\rho(y))} U(z) d\mu(z) \sim \frac{[\rho(y)]^2}{V_{\rho(y)}(y)} \int_{B(y,\rho(y))} U(z) d\mu(z) \sim 1.$$

This together with (2.4) implies that  $\rho(y) \sim \rho(x)$ , and hence (2.2) holds in this case. If  $d(x, y) \geq \rho(y)$ , then there exists  $j \in \mathbb{N}$  such that  $2^{j-1}\rho(y) \leq d(x, y) < 2^j\rho(y)$ . Thus for any integer  $k > j$ , if we choose  $r_k \equiv 2^{j-k}\rho(y) \in (0, \rho(y))$ , then by (2.4), (2.5) and (2.6), we have

$$\begin{aligned} \frac{r_k^2}{V_{r_k}(x)} \int_{B(x,r_k)} U(z) d\mu(z) &\lesssim 2^{kn/q} \frac{r_k^2}{V_{2^k r_k}(x)} \int_{B(x,2^k r_k)} U(z) d\mu(z) \\ &\lesssim 2^{kn/q} 2^{2(j-k)} \frac{[\rho(y)]^2}{V_{2^j \rho(y)}(y)} \int_{B(y,2^j \rho(y))} U(z) d\mu(z) \\ &\lesssim 2^{-k(2-n/q)} 2^{-j(\kappa-n_1-2)}. \end{aligned}$$

Notice that  $q > n/2$  and  $n_1 > \kappa - n/q$  imply that  $n_1 + 2 - \kappa > 2 - n/q > 0$ . Let  $k$  be the maximal positive integer no more than  $1 + j(n_1 + 2 - \kappa)/(2 - n/q)$ . Then

$$\frac{r_k^2}{V_{r_k}(x)} \int_{B(x,r_k)} U(z) d\mu(z) \lesssim 1,$$

which together with (2.4) implies that

$$\rho(x) \gtrsim r_1 \sim 2^{j-k}\rho(y) \sim 2^{-j\{(n_1+2-\kappa)/(2-n/q)-1\}}\rho(y).$$

Let  $k_0 \equiv (n_1 + 2 - \kappa)/(2 - n/q) - 1$ . Then  $k_0 > 0$  and

$$\rho(y) \lesssim [\rho(x)]^{1/(1+k_0)} [2^j \rho(y)]^{k_0/(1+k_0)} \lesssim [\rho(x)]^{1/(1+k_0)} [d(x, y)]^{k_0/(1+k_0)},$$

which also implies that (2.2) holds in this case, and hence completes the proof of Proposition 2.1.

We now establish some properties of admissible functions.

**Lemma 2.1.** *Let  $\rho$  be an admissible function. Then*

- (i) *for any  $\tilde{C} > 0$ , there exists a positive constant  $C$ , depending on  $\tilde{C}$ , such that if  $d(x, y) \leq \tilde{C}\rho(x)$ , then  $C^{-1}\rho(y) \leq \rho(x) \leq C\rho(y)$ ;*

- (ii) there exists a positive constant  $C$  such that for all  $x, y \in \mathcal{X}$ ,
 
$$C^{-1}[\rho(x) + d(x, y)] \leq \rho(y) + d(x, y) \leq C[\rho(x) + d(x, y)];$$
- (iii) there exists a positive constant  $C_4$  such that for all  $x, y \in \mathcal{X}$ ,
 
$$\rho(y) \geq C_4[\rho(x)]^{1+k_0}[\rho(x) + d(x, y)]^{-k_0}.$$

*Proof.* If  $d(x, y) \leq \tilde{C}\rho(x)$ , then by (2.2),  $\rho(y) \lesssim \rho(x)$ . By (2.2) and exchanging  $x$  and  $y$  again, we have  $\rho(x) \lesssim [\rho(y)]^{1/(1+k_0)}[\rho(x)]^{k_0/(1+k_0)}$ , which implies that  $\rho(x) \lesssim \rho(y)$ . Thus (i) holds.

To prove (ii), if  $\rho(x) \leq d(x, y)$ , then it is easy to see that  $\rho(x) + d(x, y) \lesssim \rho(y) + d(x, y)$ . If  $\rho(x) > d(x, y)$ , then by (i),  $\rho(y) \sim \rho(x)$ , which implies that

$$\rho(x) + d(x, y) \sim \rho(y) + d(x, y).$$

By symmetry, we have (ii).

To prove (iii), by (2.2) exchanging  $x$  and  $y$ , and (ii), we have

$$\rho(x) \lesssim [\rho(y)]^{1/(1+k_0)}[\rho(x) + d(x, y)]^{k_0/(1+k_0)},$$

which gives (iii). This finishes the proof of Lemma 2.1.

For each  $m \in \mathbb{Z}$ , let  $\mathcal{X}_m \equiv \{x \in \mathcal{X} : 2^{-(m+1)/2} < \rho(x)/8 \leq 2^{-m/2}\}$ . Then, obviously,  $\mathcal{X} = \bigcup_{m \in \mathbb{Z}} \mathcal{X}_m$ . Moreover, using some ideas from [11] on  $\mathbb{R}^n$ , we have the following results.

**Lemma 2.2.** *There exists a positive constant  $C_5$  such that for all  $R \geq 2$  and  $m, m' \in \mathbb{Z}$ , if  $x \in \mathcal{X}_m$  and  $(\mathcal{X}_{m'} \cap B(x, 2^{-m/2}R)) \neq \emptyset$ , then  $|m' - m| \leq C_5 \log R$ .*

*Proof.* If  $x \in \mathcal{X}_m$  and  $y \in (\mathcal{X}_{m'} \cap B(x, 2^{-m/2}R))$ , then by (2.2) and Lemma 2.1 (iii), we have

$$R^{-k_0}2^{-m/2} \lesssim \rho(y) \lesssim R^{k_0/(1+k_0)}2^{-m/2},$$

which implies that

$$R^{-k_0}2^{-m/2} \lesssim 2^{-m'/2} \lesssim R^{k_0/(1+k_0)}2^{-m/2},$$

namely,  $R^{-k_0} \lesssim 2^{(m-m')/2} \lesssim R^{k_0/(1+k_0)}$ . Thus,  $|m' - m| \lesssim \log R$ , which completes the proof of Lemma 2.2.

**Lemma 2.3.** *There exist a positive constant  $C$  and a subset  $\{x_{(m,k)} : x_{(m,k)} \in \mathcal{X}_m\}_{m \in \mathbb{Z}, k}$  such that for all  $R \geq 2$  and  $m \in \mathbb{Z}$ ,  $\mathcal{X}_m \subset [\bigcup_k B(x_{(m,k)}, 2^{-m/2})]$  and*

$$\#\{(m', k') : (B(x_{(m,k)}, R2^{-m/2}) \cap B(x_{(m',k')}, R2^{-m'/2})) \neq \emptyset\} \leq R^C,$$

where  $\#E$  denotes the cardinality of any set  $E$ .

*Proof.* For each fixed  $m \in \mathbb{Z}$ , since  $\mathcal{X}_m \subset [\bigcup_{x \in \mathcal{X}_m} B(x, \frac{1}{5}2^{-m/2})]$ , using the standard 5-covering theorem (see, for example, Theorem 1.2 in [25]) we obtain a subset  $\{x_{(m,k)}\}_k$  of  $\mathcal{X}_m$  such that

$$\mathcal{X}_m \subset \left\{ \bigcup_{x \in \mathcal{X}_m} B\left(x, \frac{1}{5}2^{-m/2}\right) \right\} \subset \left\{ \bigcup_k B(x_{(m,k)}, 2^{-m/2}) \right\}$$

and  $\{B(x_{(m,k)}, \frac{1}{5}2^{-m/2})\}_k$  are disjointed.

Assume that  $(B(x_{(m,k)}, R2^{-m/2}) \cap B(x_{(m',k')}, R2^{-m'/2})) \neq \emptyset$ . If  $m \leq m'$ , then

$$B(x_{(m,k)}, 2R2^{-m/2}) \cap \mathcal{X}_{m'} \neq \emptyset,$$

and if  $m > m'$ , then  $(B(x_{(m', k')}, 2R2^{-m'/2}) \cap \mathcal{X}_m) \neq \emptyset$ . Thus, by Lemma 2.2,

$$(2.7) \quad |m - m'| \leq C_5 \log(2R).$$

Moreover, for any fixed  $m'$ , if  $y \in B(x_{(m', k')}, R2^{-m'/2})$ , then

$$d(x_{(m, k)}, y) \leq d(x_{(m, k)}, x_{(m', k')}) + d(x_{(m', k')}, y) \leq [R + (2R)^{1+C_5/2}]2^{-m/2}.$$

This implies that

$$\begin{aligned} B(x_{(m', k')}, R2^{-m'/2}) &\subset B(x_{(m, k)}, [R + (2R)^{1+C_5/2}]2^{-m/2}) \\ &\subset B(x_{(m', k')}, R^{4+2C_5}2^{-m'/2}). \end{aligned}$$

Set  $\tilde{C} \equiv 4 + C_5/2$ . Observe that by the doubling property of  $\mu$ , we have

$$\begin{aligned} \mu \left( B \left( x_{(m', k')}, \frac{1}{5}2^{-m'/2} \right) \right) &\geq \frac{1}{C_2} \left( \frac{5R^{\tilde{C}}2^{-m/2}}{2^{-m'/2}} \right)^{-n} \mu(B(x_{(m', k')}, R^{\tilde{C}}2^{-m'/2})) \\ &\geq R^{C'} \mu(B(x_{(m, k)}, [R + (2R)^{1+C_5/2}]2^{-m/2})) \end{aligned}$$

for some positive constant  $C'$  independent of  $R, m, m'$  and  $k$ . Thus, for fixed  $m'$ , by the disjointness of  $\{B(x_{(m', k')}, \frac{1}{5}2^{-m'/2})\}_{k'}$ , we have

$$\#\{k' : (B(x_{(m, k)}, R2^{-m/2}) \cap B(x_{(m', k')}, R2^{-m'/2})) \neq \emptyset\} \leq R^C$$

for some positive constant  $C$  independent of  $R, m, m'$  and  $k$ . This together with (2.7) implies that

$$\#\{(m', k') : (B(x_{(m, k)}, R2^{-m/2}) \cap B(x_{(m', k')}, R2^{-m'/2})) \neq \emptyset\} \leq C_5 R^C \log(2R),$$

which completes the proof of Lemma 2.3.

In what follows, we set

$$(2.8) \quad \begin{aligned} \eta &\in \mathcal{C}^1(\mathbb{R}), \quad \eta(t) \in [0, 1] \text{ for all } t \in \mathbb{R}, \\ \eta(t) &= 1 \text{ when } |t| \leq 1 \text{ and } \eta(t) = 0 \text{ when } |t| \geq 2. \end{aligned}$$

**Lemma 2.4.** *There exist a constant  $C > 0$  and functions  $\{\psi_{(m, k)}\}_{m \in \mathbb{Z}, k}$  such that*

- (i)  $\text{supp } \psi_{(m, k)} \subset B(x_{(m, k)}, \rho(x_{(m, k)})/2)$  and  $0 \leq \psi_{(m, k)}(x) \leq 1$  for all  $x \in \mathcal{X}$ ;
- (ii)  $|\psi_{(m, k)}(x) - \psi_{(m, k)}(y)| \leq Cd(x, y)[\rho(x_{(m, k)})]^{-1}$  for all  $x, y \in \mathcal{X}$ ;
- (iii)  $\sum_{m \in \mathbb{Z}, k} \psi_{(m, k)}(x) = 1$  for all  $x \in \mathcal{X}$ .

*Proof.* Let  $\eta$  be as in (2.8). For each  $m \in \mathbb{Z}$  and  $k$  and for all  $x \in \mathcal{X}$ , set  $\eta_{(m, k)}(x) \equiv \eta(2^{m/2}d(x_{(m, k)}, x))$  and

$$\psi_{(m, k)}(x) \equiv \frac{\eta_{(m, k)}(x)}{\sum_{m' \in \mathbb{Z}, k'} \eta_{(m', k')}(x)}.$$

Then it is easy to show that  $\{\psi_{(m, k)}\}_{m \in \mathbb{Z}, k}$  satisfies (i) through (iii), which completes the proof of Lemma 2.4.

In what follows, we always simply denote  $\psi_{(m, k)}$  and  $B(x_{(m, k)}, \rho(x_{(m, k)})/2)$ , respectively, by  $\psi_\alpha$  and  $B_\alpha$ .



**2.2. Hardy spaces  $H^1(\mathcal{X})$  and their localized variants.** The following notion of approximations of the identity on RD-spaces was first introduced in [27], whose existence was given in Theorem 2.6 of [27]. Recall that  $V_r(x) \equiv \mu(B(x, r))$  and  $V(x, y) \equiv \mu(B(x, d(x, y)))$  for all  $x, y \in \mathcal{X}$  and  $r > 0$ .

**Definition 2.3.** Let  $\epsilon_1 \in (0, 1]$ ,  $\epsilon_2 > 0$  and  $\epsilon_3 > 0$ . A sequence  $\{S_k\}_{k \in \mathbb{Z}}$  of bounded linear integral operators on  $L^2(\mathcal{X})$  is called an approximation of the identity of order  $(\epsilon_1, \epsilon_2, \epsilon_3)$  (for short,  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI) if there exists a positive constant  $C_6$  such that for all  $k \in \mathbb{Z}$  and  $x, x', y, y' \in \mathcal{X}$ ,  $S_k(x, y)$ , the integral kernel of  $S_k$ , is a measurable function from  $\mathcal{X} \times \mathcal{X}$  into  $\mathbb{C}$  satisfying

- (i)  $|S_k(x, y)| \leq C_6 \frac{1}{V_{2^{-k}}(x)+V(x, y)} \left[ \frac{2^{-k}}{2^{-k}+d(x, y)} \right]^{\epsilon_2}$ ;
- (ii)  $|S_k(x, y) - S_k(x', y)| \leq C_6 \left[ \frac{d(x, x')}{2^{-k}+d(x, y)} \right]^{\epsilon_1} \frac{1}{V_{2^{-k}}(x)+V(x, y)} \left[ \frac{2^{-k}}{2^{-k}+d(x, y)} \right]^{\epsilon_2}$  for  $d(x, x') \leq [2^{-k} + d(x, y)]/2$ ;
- (iii) property (ii) also holds with  $x$  and  $y$  interchanged;
- (iv)  $||S_k(x, y) - S_k(x, y')| - [S_k(x', y) - S_k(x', y')]| \leq C_6 \left[ \frac{d(x, x')}{2^{-k}+d(x, y)} \right]^{\epsilon_1} \left[ \frac{d(y, y')}{2^{-k}+d(x, y)} \right]^{\epsilon_1} \times \frac{1}{V_{2^{-k}}(x)+V(x, y)} \left[ \frac{2^{-k}}{2^{-k}+d(x, y)} \right]^{\epsilon_3}$  for  $d(x, x') \leq [2^{-k} + d(x, y)]/3$  and  $d(y, y') \leq [2^{-k} + d(x, y)]/3$ ;
- (v)  $\int_{\mathcal{X}} S_k(x, z) d\mu(z) = 1 = \int_{\mathcal{X}} S_k(z, y) d\mu(z)$ .

*Remark 2.2.* (i) In [27], for any  $N > 0$ , it was proved that there exists  $(1, N, N)$ -AOTI  $\{S_k\}_{k \in \mathbb{Z}}$  with bounded support in the sense that  $S_k(x, y) = 0$  when  $d(x, y) > \tilde{C}2^{-k}$ , where  $\tilde{C}$  is a fixed positive constant independent of  $k$ . In this case,  $\{S_k\}_{k \in \mathbb{Z}}$  is called a 1-AOTI with bounded support; see [27].

(ii) If a sequence  $\{\tilde{S}_t\}_{t>0}$  of bounded linear integral operators on  $L^2(\mathcal{X})$  satisfies (i) through (v) of Definition 2.3 with  $2^{-k}$  replaced by  $t$ , then  $\{\tilde{S}_t\}_{t>0}$  is called a continuous approximation of the identity of order  $(\epsilon_1, \epsilon_2, \epsilon_3)$  (for short, a continuous  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI). For example, if  $\{S_k\}_{k \in \mathbb{Z}}$  is an  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI and if we set  $\tilde{S}_t(x, y) \equiv S_k(x, y)$  for  $t \in (2^{-k-1}, 2^{-k}]$  with  $k \in \mathbb{Z}$ , then  $\{\tilde{S}_t\}_{t>0}$  is a continuous  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI.

(iii) If  $S_k$  (resp.  $\tilde{S}_t$ ) satisfies (i), (ii), (iii) and (v) of Definition 2.3, then  $S_k S_k$  (resp.  $\tilde{S}_t \tilde{S}_t$ ) satisfies conditions (i) through (v) of Definition 2.3; see [26].

The following spaces of test functions play an important role in the theory of function spaces on a space of homogeneous type; see [26, 27].

**Definition 2.4.** Let  $x \in \mathcal{X}$ ,  $r > 0$ ,  $\beta \in (0, 1]$  and  $\gamma > 0$ . A function  $f$  on  $\mathcal{X}$  is said to belong to the space of test functions,  $\mathcal{G}(x, r, \beta, \gamma)$ , if there exists a positive constant  $C_f$  such that

- (i)  $|f(y)| \leq C_f \frac{1}{V_r(x)+V(x, y)} \left[ \frac{r}{r+d(x, y)} \right]^\gamma$  for all  $y \in \mathcal{X}$ ;
- (ii)  $|f(y) - f(y')| \leq C_f \left[ \frac{d(y, y')}{r+d(x, y)} \right]^\beta \frac{1}{V_r(x)+V(x, y)} \left[ \frac{r}{r+d(x, y)} \right]^\gamma$  for all  $y, y' \in \mathcal{X}$  satisfying the fact that  $d(y, y') \leq [r + d(x, y)]/2$ .

Moreover, for any  $f \in \mathcal{G}(x, r, \beta, \gamma)$ , its norm is defined by

$$\|f\|_{\mathcal{G}(x, r, \beta, \gamma)} \equiv \inf \{C_f : \text{(i) and (ii) hold}\}.$$

It is easy to see that  $\mathcal{G}(x, r, \beta, \gamma)$  is a Banach space. Let  $\epsilon \in (0, 1]$  and  $\beta, \gamma \in (0, \epsilon]$ . For applications, we further define the space  $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$  to be the completion of the set  $\mathcal{G}(x, r, \epsilon, \epsilon)$  in  $\mathcal{G}(x, r, \beta, \gamma)$ . For  $f \in \mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$ ,

define  $\|f\|_{\mathcal{G}_0^\epsilon(x,r,\beta,\gamma)} \equiv \|f\|_{\mathcal{G}(x,r,\beta,\gamma)}$ . Let  $(\mathcal{G}_0^\epsilon(x,r,\beta,\gamma))'$  be the set of all continuous linear functionals on  $\mathcal{G}_0^\epsilon(x,r,\beta,\gamma)$ , and as usual, endow  $(\mathcal{G}_0^\epsilon(x,r,\beta,\gamma))'$  with the weak  $*$ -topology. Throughout the entire paper we fix  $x_1 \in \mathcal{X}$  and write  $\mathcal{G}(\beta,\gamma) \equiv \mathcal{G}(x_1,1,\beta,\gamma)$  and  $(\mathcal{G}_0^\epsilon(\beta,\gamma))' \equiv (\mathcal{G}_0^\epsilon(x_1,1,\beta,\gamma))'$ .

The following results concerning approximations of the identity were proved in [27, Proposition 2.7] and in Lemma 3.5 through Lemma 3.7 and Proposition 3.8 in [21].

**Lemma 2.5.** *Let  $\epsilon_1 \in (0, 1]$ ,  $\epsilon_2, \epsilon_3 > 0$ ,  $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$  and  $\{S_k\}_{k \in \mathbb{Z}}$  be an  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI.*

(i) *If  $p \in [1, \infty]$ , then  $\{S_k\}_{k \in \mathbb{Z}}$  is a sequence of bounded operators on  $L^p(\mathcal{X})$  uniformly in  $k$ . Moreover, for any  $p \in [1, \infty)$  and  $f \in L^p(\mathcal{X})$ ,  $\|S_k(f) - f\|_{L^p(\mathcal{X})} \rightarrow 0$  as  $k \rightarrow \infty$ .*

(ii) *If  $\beta, \gamma \in (0, \epsilon)$ , then  $\{S_k\}_{k \in \mathbb{Z}}$  is a sequence of bounded operators on  $\mathcal{G}_0^\epsilon(\beta, \gamma)$  uniformly in  $k$ . Moreover, for any  $f \in \mathcal{G}_0^\epsilon(\beta, \gamma)$ ,  $\|S_k(f) - f\|_{\mathcal{G}_0^\epsilon(\beta, \gamma)} \rightarrow 0$  as  $k \rightarrow \infty$ , and for any  $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$ ,  $S_k(f)$  converges to  $f$  in the weak  $*$ -topology of  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$  as  $k \rightarrow \infty$ .*

**Definition 2.5.** Let  $\epsilon_1 \in (0, 1]$ ,  $\epsilon_2, \epsilon_3 > 0$ ,  $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$  and  $\{S_k\}_{k \in \mathbb{Z}}$  be an  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI. Let  $\rho$  be an admissible function. For any  $\beta, \gamma \in (0, \epsilon)$ ,  $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$  and  $x \in \mathcal{X}$ , define

- (i) the radial maximal function  $S^+(f)$  by  $S^+(f)(x) \equiv \sup_{k \in \mathbb{Z}} |S_k(f)(x)|$ ;
- (ii) the radial maximal function  $S_\rho^+(f)$  associated to  $\rho$  by

$$S_\rho^+(f)(x) \equiv \sup_{\{k \in \mathbb{Z}, 2^{-k} < \rho(x)\}} |S_k(f)(x)|;$$

- (iii) the grand maximal function  $G^{(\epsilon, \beta, \gamma)}(f)$  by

$$G^{(\epsilon, \beta, \gamma)}(f)(x) \equiv \sup \{ |\langle f, \varphi \rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leq 1 \text{ for some } r > 0 \};$$

- (iv) the grand maximal function  $G_\rho^{(\epsilon, \beta, \gamma)}(f)$  associated to  $\rho$  by

$$G_\rho^{(\epsilon, \beta, \gamma)}(f)(x) \equiv \sup \{ |\langle f, \varphi \rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leq 1 \text{ for some } r \in (0, \rho(x)) \}.$$

When there exists no ambiguity, we simply write  $G^{(\epsilon, \beta, \gamma)}(f)$  and  $G_\rho^{(\epsilon, \beta, \gamma)}(f)$  as  $G(f)$  and  $G_\rho(f)$ , respectively. Notice that  $\|S_k(x, \cdot)\|_{\mathcal{G}_0^\epsilon(x, 2^{-k}, \beta, \gamma)} \leq C_6$  for all  $x \in \mathcal{X}$  and  $\beta, \gamma \in (0, \epsilon)$ . It is easy to see that for all  $x \in \mathcal{X}$ ,  $S_\rho^+(f)(x) \leq S^+(f)(x) \leq C_6 G(f)(x)$  and

$$(2.9) \quad S_\rho^+(f)(x) \leq C_6 G_\rho(f)(x) \leq C_6 G(f)(x).$$

**Definition 2.6.** Let  $\epsilon \in (0, 1)$ ,  $\beta, \gamma \in (0, \epsilon)$  and  $\rho$  be an admissible function.

- (i) The Hardy space  $H^1(\mathcal{X})$  is defined by

$$H^1(\mathcal{X}) \equiv \{f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H^1(\mathcal{X})} \equiv \|G(f)\|_{L^1(\mathcal{X})} < \infty\}.$$

- (ii) The Hardy space  $H_\rho^1(\mathcal{X})$  associated to  $\rho$  is defined by

$$H_\rho^1(\mathcal{X}) \equiv \left\{ f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H_\rho^1(\mathcal{X})} \equiv \|G_\rho(f)\|_{L^1(\mathcal{X})} < \infty \right\}.$$

**Definition 2.7.** Let  $q \in (1, \infty]$ .

- (i) A measurable function  $a$  is called a  $(1, q)$ -atom associated to the ball  $B(x, r)$  if
  - (A1)  $\text{supp } a \subset B(x, r)$  for some  $x \in \mathcal{X}$  and  $r > 0$ ,
  - (A2)  $\|a\|_{L^q(\mathcal{X})} \leq [\mu(B(x, r))]^{1/q-1}$ ,
  - (A3)  $\int_{\mathcal{X}} a(x) d\mu(x) = 0$ .
- (ii) A measurable function  $a$  is called a  $(1, q)_\rho$ -atom associated to the ball  $B(x, r)$  if  $r < \rho(x)$  and  $a$  satisfies (A1) and (A2), and when  $r < \rho(x)/4$ ,  $a$  also satisfies (A3).

**Definition 2.8.** Let  $\epsilon \in (0, 1)$ ,  $\beta, \gamma \in (0, \epsilon)$  and  $q \in (1, \infty]$ .

(i) The space  $H^{1,q}(\mathcal{X})$  is defined to be the set of all  $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$  in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ , where  $\{a_j\}_{j \in \mathbb{N}}$  are  $(1, q)$ -atoms and  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  such that  $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$ . For any  $f \in H^{1,q}(\mathcal{X})$ , define  $\|f\|_{H^{1,q}(\mathcal{X})} \equiv \inf\{\sum_{j \in \mathbb{N}} |\lambda_j|\}$ , where the infimum is taken over all the above decompositions of  $f$ .

(ii) The space  $H_{\text{fin}}^{1,q}(\mathcal{X})$  is defined to be the set of all  $f = \sum_{j=1}^N \lambda_j a_j$ , where  $N \in \mathbb{N}$ ,  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ , and  $\{a_j\}_{j=1}^N$  are  $(1, q)$ -atoms when  $q < \infty$  or continuous  $(1, \infty)$ -atoms when  $q = \infty$ . For any  $f \in H_{\text{fin}}^{1,q}(\mathcal{X})$ , define  $\|f\|_{H_{\text{fin}}^{1,q}(\mathcal{X})} \equiv \inf\{\sum_{j=1}^N |\lambda_j|\}$ , where the infimum is taken over all the above finite decompositions of  $f$ .

(iii) The space  $H_\rho^{1,q}(\mathcal{X})$  is defined as in (i) with  $(1, q)$ -atoms replaced by  $(1, q)_\rho$ -atoms.

(iv) The space  $H_{\rho, \text{fin}}^{1,q}(\mathcal{X})$  is defined as in (ii) with  $(1, q)$ -atoms replaced by  $(1, q)_\rho$ -atoms.

The atomic Hardy spaces  $H^{1,q}(\mathcal{X})$  were originally introduced in [7]. Moreover, in [21, 22], the following results were established.

**Theorem 2.1.** (i) Let  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ . Then the following are equivalent:

- (a)  $f \in H^1(\mathcal{X})$ ;
- (b)  $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$  and  $\|S^+(f)\|_{L^1(\mathcal{X})} < \infty$ ;
- (c)  $f \in H^{1,q}(\mathcal{X})$  with  $q \in (1, \infty]$ .

Moreover, for any fixed  $q \in (1, \infty]$  and all  $f \in H^1(\mathcal{X})$ ,

$$\|f\|_{H^1(\mathcal{X})} \sim \|S^+(f)\|_{L^1(\mathcal{X})} \sim \|f\|_{H^{1,q}(\mathcal{X})}.$$

(ii) If  $q \in (1, \infty]$ , then for all  $f \in H_{\text{fin}}^{1,q}(\mathcal{X})$ ,  $\|f\|_{H_{\text{fin}}^{1,q}(\mathcal{X})} \sim \|f\|_{H^1(\mathcal{X})}$ .

We finally point out that by Definitions 2.6 and 2.8 above, the spaces  $H^1(\mathcal{X})$ ,  $H_\rho^1(\mathcal{X})$ ,  $H^{1,q}(\mathcal{X})$  and  $H_\rho^{1,q}(\mathcal{X})$  seem to depend on the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ . However, in Remark 3.1 below, we show that all these spaces are independent of the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ , which is the reason why we omit the parameters  $\epsilon, \beta$  and  $\gamma$  when mentioning them.

### 3. ATOMIC DECOMPOSITION CHARACTERIZATIONS OF $H_\rho^1(\mathcal{X})$

We begin with the following relations concerning the Hardy spaces in Definition 2.6 and Definition 2.8 and the Lebesgue space  $L^1(\mathcal{X})$ . Recall that the symbol  $\subset$  means continuous embedding.

**Lemma 3.1.** *Let  $q \in (1, \infty]$ . Then*

- (i)  $H^{1,q}(\mathcal{X}) \subset H_\rho^{1,q}(\mathcal{X}) \subset H_\rho^1(\mathcal{X}) \subset L^1(\mathcal{X})$ ;
- (ii)  $H_\rho^{1,q}(\mathcal{X}) = H_\rho^{1,\infty}(\mathcal{X})$  with equivalent norms independent of  $\rho$ .

*Proof.* To see  $H^{1,q}(\mathcal{X}) \subset H_\rho^{1,q}(\mathcal{X})$ , we only need to prove that if  $a$  is a  $(1, q)$ -atom supported in  $B(x_0, r_0)$  with  $r_0 \geq \rho(x_0)$ , then  $a \in H_\rho^{1,q}(\mathcal{X})$ . In fact, by Lemma 2.4, we write  $a \equiv \sum_\alpha \psi_\alpha a$  pointwise. Recall that  $\{\psi_\alpha\}_\alpha$  is as in Lemma 2.4. From Lemma 2.3, it is easy to see that  $a = \sum_\alpha \psi_\alpha a$  holds in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$  with  $\epsilon, \beta, \gamma$  as in Definition 2.8. Let

$$\lambda_\alpha \equiv [\mu(B_\alpha)]^{1-1/q} \|\psi_\alpha a\|_{L^q(\mathcal{X})}.$$

If  $\lambda_\alpha = 0$ , set  $a_\alpha \equiv 0$ , and if  $\lambda_\alpha \neq 0$ , set  $a_\alpha \equiv (\lambda_\alpha)^{-1} \psi_\alpha a$ . Notice that by Lemma 2.1 (i), if  $(B_\alpha \cap B(x_0, r_0)) \neq \emptyset$ , then  $B_\alpha \subset B(x_0, Cr_0)$ . Thus,  $a_\alpha$  is a  $(1, q)_\rho$ -atom associated to the ball  $B_\alpha \equiv B(x_\alpha, \rho(x_\alpha)/2)$ , and by the Hölder inequality and Lemma 2.3, we have

$$\sum_\alpha \lambda_\alpha \lesssim \|a\|_{L^q(\mathcal{X})} \left[ \sum_\alpha \mu(B_\alpha) \right]^{1/q'} \lesssim [\mu(B(x_0, r_0))]^{1/q-1} [\mu(B(x_0, Cr_0))]^{1/q'} \lesssim 1.$$

This means that  $a \in H_\rho^{1,q}(\mathcal{X})$  and  $\|a\|_{H_\rho^{1,q}(\mathcal{X})} \lesssim 1$ . Thus,  $H^{1,q}(\mathcal{X}) \subset H_\rho^{1,q}(\mathcal{X})$ .

To prove  $H_\rho^{1,q}(\mathcal{X}) \subset H_\rho^1(\mathcal{X})$ , by the definition of  $G_\rho$ , it suffices to prove that for all  $(1, q)_\rho$ -atoms  $a$ ,  $\|G_\rho(a)\|_{L^1(\mathcal{X})} \lesssim 1$ . In fact, if  $a$  is a  $(1, q)$ -atom, then it is known that  $\|G_\rho(a)\|_{L^1(\mathcal{X})} \lesssim \|G(a)\|_{L^1(\mathcal{X})} \lesssim 1$ . If  $\int_{\mathcal{X}} a(x) d\mu(x) \neq 0$ , assuming that  $\text{supp } a \subset B(x_0, r_0)$ , then  $\rho(x_0)/4 \leq r_0 < \rho(x_0)$ . Since  $G_\rho(a) \lesssim M(a)$ , where  $M$  is the Hardy-Littlewood maximal operator on  $\mathcal{X}$ , then  $G_\rho$  is bounded on  $L^q(\mathcal{X})$  (see [6, 7]); then by the Hölder inequality, we have

$$\|G_\rho(a)\|_{L^1(B(x_0, 4\rho(x_0)))} \lesssim [\mu(B(x_0, \rho(x_0)))]^{1/q'} \|a\|_{L^q(B(x_0, 4\rho(x_0)))} \lesssim 1.$$

Where and in what follows, for any set  $E \subset \mathcal{X}$ , we write

$$\|f\|_{L^q(E)} \equiv \left\{ \int_E |f(x)|^q d\mu(x) \right\}^{1/q}.$$

For  $x \notin B(x_0, 4\rho(x_0))$ , since for any  $\psi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$  with  $\|\psi\|_{\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)} \leq 1$  and  $r < \rho(x)$ , we have

$$\left| \int_{\mathcal{X}} a(y) \psi(y) d\mu(y) \right| \lesssim \int_{\mathcal{X}} |a(y)| \frac{1}{V(x, y)} \left[ \frac{\rho(x)}{d(x, y)} \right]^\gamma d\mu(y) \lesssim \frac{1}{V(x, x_0)} \left[ \frac{\rho(x)}{d(x, x_0)} \right]^\gamma,$$

which together with (2.2) implies that

$$G_\rho(a)(x) \lesssim \frac{1}{V(x, x_0)} \left[ \frac{\rho(x)}{d(x, x_0)} \right]^\gamma \lesssim \frac{1}{V(x, x_0)} \left[ \frac{\rho(x_0)}{d(x, x_0)} \right]^{\gamma/(1+k_0)}.$$

Thus,  $\|G_\rho(a)\|_{L^1(\mathcal{X} \setminus B(x_0, 4\rho(x_0)))} \lesssim 1$  and, therefore,  $\|G_\rho(a)\|_{L^1(\mathcal{X})} \lesssim 1$ . This shows that  $H_\rho^{1,q}(\mathcal{X}) \subset H_\rho^1(\mathcal{X})$ .

To prove  $H_\rho^1(\mathcal{X}) \subset L^1(\mathcal{X})$ , assume that  $f \in H_\rho^1(\mathcal{X})$ . By (2.9) and Definition 2.8, we have that  $\|S_\rho^+(f)\|_{L^1(\mathcal{X})} \lesssim \|f\|_{H^1(\mathcal{X})}$ , which means that  $\{S_k(f)\chi_{\{2^{-k} < \rho(\cdot)\}}\}_{k \in \mathbb{Z}}$  is a bounded set in  $L^1(\mathcal{X})$ . Thus, by the proof of [54, Theorem III. C. 12],  $\{S_k(f)\chi_{\{2^{-k} < \rho(\cdot)\}}\}_{k \in \mathbb{Z}}$  is relatively weakly compact in  $L^1(\mathcal{X})$ . This, together with

the Eberlein-Šmulian theorem (see, for example, [54, Theorem II. C. 3]), implies that there exist a subsequence  $\{S_{k_j}(f)\chi_{\{2^{-k_j} < \rho(\cdot)\}}\}_{j \in \mathbb{N}}$  of  $\{S_k(f)\chi_{\{2^{-k} < \rho(\cdot)\}}\}_{k \in \mathbb{Z}}$  and a measurable function  $g \in L^1(\mathcal{X})$  such that  $\{S_{k_j}(f)\chi_{\{2^{-k_j} < \rho(\cdot)\}}\}_{j \in \mathbb{N}}$  weakly converges to  $g$  in  $L^1(\mathcal{X})$  and hence in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$  with  $\epsilon, \beta$  and  $\gamma$  as in Definition 2.6. From this, it is easy to follow that

$$\|g\|_{L^1(\mathcal{X})} \leq \|S_\rho^+(f)\|_{L^1(\mathcal{X})} \lesssim \|f\|_{H_\rho^1(\mathcal{X})}.$$

Denote by  $\mathcal{G}_{0,b}^\epsilon(\beta, \gamma)$  the set of functions in  $\mathcal{G}_0^\epsilon(\beta, \gamma)$  with bounded support. For any  $\psi \in \mathcal{G}_{0,b}^\epsilon(\beta, \gamma)$ , assume that  $\text{supp } \psi \subset B(x_1, r)$ . By Lemma 2.1 (iii), there exists  $j_1 \in \mathbb{N}$  such that  $2^{-j_1} \leq \inf_{y \in B(x_1, r)} \rho(y)$ . Therefore, by Lemma 2.5 (ii),

$$\langle g, \psi \rangle = \lim_{j \rightarrow \infty} \langle S_{k_j}(f)\chi_{\{2^{-k_j} < \rho(\cdot)\}}, \psi \rangle = \lim_{j \rightarrow \infty} \langle S_{k_j}(f), \psi \rangle = \langle f, \psi \rangle.$$

On the other hand, it is easy to show that  $\mathcal{G}_{0,b}^\epsilon(\beta, \gamma)$  is dense in  $\mathcal{G}_0^\epsilon(\beta, \gamma)$ , which further implies that  $f = g$  in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ . In this sense, we say  $f \in L^1(\mathcal{X})$ . Thus (i) holds.

To prove (ii), by Definition 2.8, obviously  $H_\rho^{1,\infty}(\mathcal{X}) \subset H_\rho^{1,q}(\mathcal{X})$ , and the inclusion is continuous. Conversely, it suffices to prove that if  $a$  is any  $(1, q)_\rho$ -atom supported in  $B \equiv B(x_0, r_0)$ , then  $a \in H_\rho^{1,\infty}(\mathcal{X})$  and  $\|a\|_{H_\rho^{1,\infty}(\mathcal{X})} \lesssim 1$ . In fact, if  $a$  is a  $(1, q)$ -atom, then by (i) of this lemma and Theorem 2.1 (i),  $a \in H^{1,q}(\mathcal{X}) = H^{1,\infty}(\mathcal{X}) \subset H_\rho^{1,\infty}(\mathcal{X})$  and  $\|a\|_{H_\rho^{1,\infty}(\mathcal{X})} \lesssim \|a\|_{H^{1,\infty}(\mathcal{X})} \lesssim 1$ . If  $\int_{\mathcal{X}} a(x) d\mu(x) \neq 0$ , then  $[a - a_B\chi_B]/2$  is a  $(1, q)$ -atom, where  $a_B \equiv \frac{1}{\mu(B)} \int_B a(y) d\mu(y)$ . Thus  $a - a_B\chi_B \in H_\rho^{1,\infty}(\mathcal{X})$  with  $\|a - a_B\chi_B\|_{H_\rho^{1,\infty}(\mathcal{X})} \lesssim 1$ . Since  $|a_B| \leq [\mu(B)]^{-1}$  and  $\rho(x_0)/4 \leq r < \rho(x_0)$ , we know that  $a_B\chi_B$  is a  $(1, \infty)_\rho$ -atom. Thus,  $a \in H_\rho^{1,\infty}(\mathcal{X})$  and  $\|a\|_{H_\rho^{1,\infty}(\mathcal{X})} \lesssim 1$ . This gives (ii), which completes the proof of Lemma 3.1.

*Remark 3.1.* (i) Observe that in Definition 2.8, if  $\sum_{j \in \mathbb{N}} \lambda_j a_j$  converges to  $f$  in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ , where  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  such that  $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$  and  $\{a_j\}_{j \in \mathbb{N}}$  are  $(1, q)$ -atoms or  $(1, q)_\rho$ -atoms, then by Lemma 3.1,  $\sum_{j \in \mathbb{N}} \lambda_j a_j$  also converges to  $\tilde{f}$  in  $L^1(\mathcal{X})$ , where  $f$  and  $\tilde{f}$  coincide in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ . By identifying  $f$  with  $\tilde{f}$ , if we replace the distribution space  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$  in Definition 2.8 with  $L^1(\mathcal{X})$ , we still obtain the same atomic Hardy spaces, which further implies that the spaces  $H^{1,q}(\mathcal{X})$  and  $H_\rho^{1,q}(\mathcal{X})$  are independent of the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ . This is the reason why we omit the parameters  $\epsilon, \beta, \gamma$  when we mention the atomic Hardy spaces  $H^{1,q}(\mathcal{X})$  and  $H_\rho^{1,q}(\mathcal{X})$ .

(ii) Notice that Theorem 2.1 shows that  $H^{1,q}(\mathcal{X}) = H^1(\mathcal{X})$ . By (i) of this remark, we know that the spaces  $H^1(\mathcal{X})$ , whose definitions seem to depend on the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ , are actually equivalent. Thus, the space  $H^1(\mathcal{X})$  is independent of the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ .

(iii) Similarly, if we can prove  $H_\rho^{1,q}(\mathcal{X}) = H_\rho^1(\mathcal{X})$ , then the space  $H_\rho^1(\mathcal{X})$  is also independent of the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ . We do prove  $H_\rho^{1,q}(\mathcal{X}) = H_\rho^1(\mathcal{X})$  in Theorem 3.2 below without using the fact that the space  $H_\rho^1(\mathcal{X})$  is independence of the choices of  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$ .

To obtain an atomic decomposition characterization of  $H_\rho^1(\mathcal{X})$ , we first construct a kernel function on  $\mathcal{X} \times \mathcal{X}$  by subtly developing some ideas of Coifman presented in [8] (see also [27]).

**Proposition 3.1.** *Let  $\rho$  be an admissible function. There exist a nonnegative function  $K_\rho$  on  $\mathcal{X} \times \mathcal{X}$  and a positive constant  $C$  such that*

(i)  $K_\rho(x, y) = 0$  if  $d(x, y) > C[\rho(x) \wedge \rho(y)]$  and  $K_\rho(x, y) \leq C \frac{1}{V_{\rho(x)}(x) + V_{\rho(y)}(y)}$  for all  $x, y \in \mathcal{X}$ ;

(ii)  $K_\rho(x, y) = K_\rho(y, x)$  for all  $x, y \in \mathcal{X}$ ;

(iii)  $|K_\rho(x, y) - K_\rho(x, y')| \leq C \frac{d(y, y')}{\rho(x)} \frac{1}{V_{\rho(x)}(x) + V_{\rho(y)}(y)}$  for all  $x, y, y' \in \mathcal{X}$  with  $d(y, y') \leq [\rho(x) + d(x, y)]/2$ ;

(iv)

$$\begin{aligned} & |[K_\rho(x, y) - K_\rho(x, y')] - [K_\rho(x', y) - K_\rho(x', y')]| \\ & \leq C \frac{d(x, x')}{\rho(x)} \frac{d(y, y')}{\rho(x)} \frac{1}{V_{\rho(x)}(x) + V_{\rho(y)}(y)} \end{aligned}$$

for all  $x, x', y, y' \in \mathcal{X}$  with  $d(x, x') \leq [\rho(y) + d(x, y)]/3$  and  $d(y, y') \leq [\rho(x) + d(x, y)]/3$ ;

(v)  $\int_{\mathcal{X}} K_\rho(x, y) d\mu(x) = 1$  for all  $y \in \mathcal{X}$ .

*Proof.* Let  $\eta$  be as in (2.8) and  $h(t) \equiv \eta(2t)$  for all  $t \in \mathbb{R}$ . For any locally integrable function  $f$  on  $\mathcal{X}$  and  $u \in \mathcal{X}$ , define

$$(3.1) \quad T_\rho(f)(u) \equiv \int_{\mathcal{X}} h\left(\frac{d(u, w)}{\rho(w)}\right) f(w) d\mu(w)$$

and

$$\tilde{T}_\rho(f)(u) \equiv \int_{\mathcal{X}} h\left(\frac{d(u, w)}{\rho(u)}\right) f(w) d\mu(w).$$

We first claim that for any fixed constant  $\tilde{C} > 1$ , if  $d(x, u) \leq \tilde{C}\rho(x)$ , then

$$(3.2) \quad T_\rho(1)(u) \sim V_{\rho(x)}(x) \sim \tilde{T}_\rho(1)(u),$$

where the equivalent constants depend only on  $\tilde{C}$ ,  $C_2$  and  $C_3$ . To see that, notice that for any  $u \in \mathcal{X}$ , by (2.8) and (2.1) it is easy to see that  $V_{\rho(u)}(u) \sim \tilde{T}_\rho(1)(u)$ . Since  $\rho(w) \sim \rho(u)$  for all  $w \in \mathcal{X}$  with  $d(x, u) < \rho(w)$  via Lemma 2.1 (i), by (2.8) and the doubling property of  $\mu$  we also have  $T_\rho(1)(u) \sim V_{\rho(u)}(u)$ . Moreover, if  $d(x, u) \leq \tilde{C}\rho(x)$ , then by Lemma 2.1 (i) we have  $\rho(x) \sim \rho(u)$ , which together with (2.1) implies that  $V_{\rho(x)}(x) \sim V_{\rho(x)}(u) \sim V_{\rho(u)}(u)$ . This shows the above claim (3.2).

Moreover, for all  $z \in \mathcal{X}$ , since  $h(\frac{d(z, w)}{\rho(z)}) \neq 0$  implies that  $d(z, w) < \rho(z)$ , by (3.2) we have

$$(3.3) \quad \tilde{T}_\rho\left(\frac{1}{T_\rho(1)}\right)(z) = \int_{\mathcal{X}} h\left(\frac{d(z, w)}{\rho(z)}\right) \frac{1}{T_\rho(1)(w)} d\mu(w) \sim \tilde{T}_\rho(1)(z) \frac{1}{V_{\rho(z)}(z)} \sim 1.$$

For all  $x, y \in \mathcal{X}$ , define

$$(3.4) \quad \begin{aligned} K_\rho(x, y) \equiv & \frac{1}{T_\rho(1)(x)} \int_{\mathcal{X}} h\left(\frac{d(x, z)}{\rho(z)}\right) \frac{1}{\tilde{T}_\rho\left(\frac{1}{T_\rho(1)}\right)(z)} \\ & \times h\left(\frac{d(z, y)}{\rho(z)}\right) d\mu(z) \frac{1}{T_\rho(1)(y)}. \end{aligned}$$

Then  $K_\rho$  satisfies (i) through (v) of Proposition 3.1.

It is easy to see that  $K_\rho(x, y) = K_\rho(y, x)$  for all  $x, y \in \mathcal{X}$ , which yields (ii).

To see (v), by (3.1) we have

$$\begin{aligned} \int_{\mathcal{X}} K_{\rho}(x, y) d\mu(x) &= \int_{\mathcal{X}} \left\{ \int_{\mathcal{X}} \frac{1}{T_{\rho}(1)(x)} h\left(\frac{d(x, z)}{\rho(z)}\right) d\mu(x) \right\} \\ &\quad \times \frac{1}{\tilde{T}_{\rho}\left(\frac{1}{T_{\rho}(1)}\right)(z)} h\left(\frac{d(z, y)}{\rho(z)}\right) d\mu(z) \frac{1}{T_{\rho}(1)(y)} \\ &= \int_{\mathcal{X}} h\left(\frac{d(z, y)}{\rho(z)}\right) d\mu(z) \frac{1}{T_{\rho}(1)(y)} = 1. \end{aligned}$$

To prove (i), notice that  $h(d(x, z)/\rho(z))h(d(z, y)/\rho(z)) \neq 0$  implies that  $d(x, z) \leq \rho(z)$  and  $d(z, y) \leq \rho(z)$ , which together with Lemma 2.1 (i) yields that  $\rho(x) \sim \rho(z) \sim \rho(y)$  and  $d(x, y) \lesssim [\rho(x) \wedge \rho(y)]$ . From these estimates, (3.3) and (3.4), it follows that

$$K_{\rho}(x, y) \lesssim \frac{1}{V_{\rho(x)}(x)} \int_{d(x, z) \lesssim \rho(x)} d\mu(z) \frac{1}{V_{\rho(y)}(y)} \lesssim \frac{1}{V_{\rho(x)}(x)} \lesssim \frac{1}{V_{\rho(x)}(x) + V_{\rho(y)}(y)}.$$

Moreover, by (3.4) it is easy to see that  $K_{\rho}(x, y) \neq 0$  if and only if

$$h(d(x, z)/\rho(x))h(d(z, y)/\rho(z)) \neq 0$$

for some  $z$ , which implies that  $d(x, y) \lesssim \rho(x)$ . Thus  $\text{supp } K_{\rho}(x, \cdot) \subset B(x, C\rho(x))$ , which establishes (i).

To obtain (iii), if  $d(x, z) < \rho(z)$ ,  $d(y, y') < [\rho(x) + d(x, y)]/2$  and  $d(z, y) < \rho(z)$  or  $d(z, y') < \rho(z)$ , by Lemma 2.1 (i) we then have  $d(y, y') \lesssim \rho(x) + \rho(z) \lesssim \rho(x)$ ,  $d(x, y) \lesssim \rho(x)$  and  $\rho(x) \sim \rho(z) \sim \rho(y) \sim \rho(y')$ . By this and (3.3) we have

$$\begin{aligned} (3.5) \quad & \left| \frac{1}{T_{\rho}(1)(y)} h\left(\frac{d(z, y)}{\rho(z)}\right) - \frac{1}{T_{\rho}(1)(y')} h\left(\frac{d(z, y')}{\rho(z)}\right) \right| \\ & \leq \frac{1}{T_{\rho}(1)(y)} \left| h\left(\frac{d(z, y)}{\rho(z)}\right) - h\left(\frac{d(z, y')}{\rho(z)}\right) \right| + \frac{|T_{\rho}(1)(y) - T_{\rho}(1)(y')|}{T_{\rho}(1)(y)T_{\rho}(1)(y')} \\ & \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)} + \frac{1}{[V_{\rho(x)}(x)]^2} \int_{\mathcal{X}} \left| h\left(\frac{d(w, y)}{\rho(w)}\right) - h\left(\frac{d(w, y')}{\rho(w)}\right) \right| d\mu(w) \\ & \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)} + \frac{1}{[V_{\rho(x)}(x)]^2} \int_{|w-y| < \rho(w) \text{ or } |w-y'| < \rho(w)} \frac{d(y, y')}{\rho(w)} d\mu(w) \\ & \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)} + \frac{1}{[V_{\rho(x)}(x)]^2} \frac{d(y, y')}{\rho(x)} \int_{|w-y| \lesssim \rho(x) \text{ or } |w-y'| \lesssim \rho(x)} d\mu(w) \\ & \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)}. \end{aligned}$$

This, together with (3.3), implies that for all  $x, y, y' \in \mathcal{X}$  with  $d(y, y') < [\rho(x) + d(x, y)]/2$ ,

$$\begin{aligned} & |K_{\rho}(x, y) - K_{\rho}(x, y')| \\ & \leq \frac{1}{T_{\rho}(1)(x)} \int_{\mathcal{X}} h\left(\frac{d(x, z)}{\rho(z)}\right) \frac{1}{\tilde{T}_{\rho}\left(\frac{1}{T_{\rho}(1)}\right)(z)} \\ & \quad \times \left| \frac{1}{T_{\rho}(1)(y)} h\left(\frac{d(z, y)}{\rho(z)}\right) - \frac{1}{T_{\rho}(1)(y')} h\left(\frac{d(z, y')}{\rho(z)}\right) \right| d\mu(z) \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)}, \end{aligned}$$

which shows (iii).

To prove (iv), for all  $x, x', y, y' \in \mathcal{X}$  with  $d(x, x') \leq [\rho(y) + d(x, y)]/3$  and  $d(y, y') \leq [\rho(x) + d(x, y)]/3$ , by (3.5) we have

$$\begin{aligned} & |[K_\rho(x, y) - K_\rho(x, y')] - [K_\rho(x', y) - K_\rho(x', y')]| \\ & \lesssim \int_{\mathcal{X}} \left| \frac{1}{T_\rho(1)(x)} h\left(\frac{d(x, z)}{\rho(z)}\right) - \frac{1}{T_\rho(1)(x')} h\left(\frac{d(x', z)}{\rho(z)}\right) \right| \\ & \quad \times \frac{1}{\tilde{T}_\rho\left(\frac{1}{T_\rho(1)}\right)(z)} \left| \frac{1}{T_\rho(1)(y)} h\left(\frac{d(z, y)}{\rho(z)}\right) - \frac{1}{T_\rho(1)(y')} h\left(\frac{d(z, y')}{\rho(z)}\right) \right| d\mu(z) \\ & \lesssim \frac{d(x, x')}{\rho(y)} \frac{d(y, y')}{\rho(x)} \frac{1}{V_{\rho(x)}(x) + V_{\rho(y)}(y)}, \end{aligned}$$

which yields (iv) and hence completes the proof of Proposition 3.1.

**Theorem 3.1.** *Let  $\rho$  be an admissible function and  $K_\rho$  as in Proposition 3.1. If  $f \in H_\rho^1(\mathcal{X})$ , then  $f - K_\rho(f) \in H^1(\mathcal{X})$ , where*

$$K_\rho(f)(x) = \int_{\mathcal{X}} K_\rho(x, y) f(y) d\mu(y)$$

for all  $x \in \mathcal{X}$ . Moreover, there exists a positive constant  $C$  such that for all  $f \in H_\rho^1(\mathcal{X})$ ,

$$\|f - K_\rho(f)\|_{H^1(\mathcal{X})} \leq C \|f\|_{H_\rho^1(\mathcal{X})}.$$

*Proof.* Let  $\{S_k\}_{k \in \mathbb{Z}}$  be a 1-AOTI with bounded support as in Remark 2.2 (i) and  $f \in H_\rho^1(\mathcal{X})$ . Then

$$\begin{aligned} \|f - K_\rho(f)\|_{H^1(\mathcal{X})} & \leq \left\| \sup_{\{k: 2^{-k} < \rho(\cdot)\}} |S_k(f)(\cdot)| \right\|_{L^1(\mathcal{X})} \\ & \quad + \left\| \sup_{\{k: 2^{-k} < \rho(\cdot)\}} |S_k(K_\rho(f))(\cdot)| \right\|_{L^1(\mathcal{X})} \\ & \quad + \left\| \sup_{\{k: 2^{-k} \geq \rho(\cdot)\}} |S_k(f)(\cdot) - S_k(K_\rho(f))(\cdot)| \right\|_{L^1(\mathcal{X})} \\ & \equiv I_1 + I_2 + I_3. \end{aligned}$$

By (2.9), it is easy to see that

$$I_1 \leq \|S_\rho^+(f)\|_{L^1(\mathcal{X})} \lesssim \|G_\rho(f)\|_{L^1(\mathcal{X})} \sim \|f\|_{H_\rho^1(\mathcal{X})}.$$

By Lemma 3.1,  $f \in L^1(\mathcal{X})$ , and moreover, for all  $x \in \mathcal{X}$ ,

$$S_k(K_\rho(f))(x) = \int_{\mathcal{X}} \int_{\mathcal{X}} S_k(x, z) K_\rho(z, y) f(y) d\mu(z) d\mu(y).$$

For all  $x, y \in \mathcal{X}$ , let

$$\varphi(x, y) \equiv \int_{\mathcal{X}} S_k(x, z) K_\rho(z, y) d\mu(z).$$

To obtain that  $I_2 \lesssim \|f\|_{H_\rho^1(\mathcal{X})}$ , by the definition of  $H_\rho^1(\mathcal{X})$  it suffices to prove that  $\varphi(x, \cdot) \in \mathcal{G}(\epsilon, \epsilon)$  and  $\|\varphi(x, \cdot)\|_{\mathcal{G}_\delta^s(x, \ell, \epsilon, \epsilon)} \lesssim 1$  for some  $0 < \ell < \rho(x)$  and  $\epsilon \in (0, 1)$  as in Definition 2.6. To this end, notice that by Proposition 3.1 (i),  $K_\rho(z, y) \neq 0$  if



and only if  $d(y, z) \lesssim \rho(z)$ . Then if  $2^{-k} < \rho(x)$ ,  $d(x, z) \lesssim 2^{-k}$  and  $d(z, y) < \rho(z)$ , by Lemma 2.1 (i) we have

$$(3.6) \quad \rho(z) \sim \rho(y) \sim \rho(x)$$

and  $d(x, z) \lesssim \rho(x)$ , which further implies that  $\text{supp } \varphi(x, \cdot) \subset B(x, C\rho(x))$ . Also, by Proposition 3.1 (i) and (3.6),  $|\varphi(x, y)| \lesssim [V_{\rho(x)}(x)]^{-1}$ . On the other hand, if  $2^{-k} < \rho(x)$ ,  $d(x, z) \lesssim 2^{-k}$ ,  $d(x, y) \lesssim \rho(x)$  and  $d(y', y) \leq [\rho(x) + d(x, y)]/2$ , by Lemma 2.1 (i) we have  $\rho(x) \sim \rho(z)$ ,  $d(x, z) \lesssim \rho(x)$  and  $d(y', y) \lesssim \rho(z) + d(z, y)$ . Therefore,

$$|K_\rho(z, y) - K_\rho(z, y')| \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)}.$$

This implies that

$$|\varphi(x, y) - \varphi(x, y')| \leq \int_{\mathcal{X}} |S_k(x, z)| |K_\rho(z, y) - K_\rho(z, y')| d\mu(z) \lesssim \frac{1}{V_{\rho(x)}(x)} \frac{d(y, y')}{\rho(x)}.$$

Thus, letting  $\ell \equiv \rho(x)/2$ , we then have  $\varphi(x, \cdot) \in \mathcal{G}(\epsilon, \epsilon)$  and  $\|\varphi\|_{\mathcal{G}(x, \ell, \epsilon, \epsilon)} \lesssim 1$ , which is desired.

To estimate  $I_3$ , by Proposition 3.1 (v) we write

$$\begin{aligned} & S_k(f)(x) - S_k(K_\rho(f))(x) \\ &= \int_{\mathcal{X}} \left\{ \int_{\mathcal{X}} [S_k(x, z) - S_k(x, u)] K_\rho(u, z) d\mu(u) \right\} f(z) d\mu(z). \end{aligned}$$

For all  $x, z \in \mathcal{X}$ , let

$$\psi(x, z) \equiv \int_{\mathcal{X}} [S_k(x, z) - S_k(x, u)] K_\rho(u, z) d\mu(u).$$

Notice that by Proposition 3.1 (i), if  $K_\rho(u, z) \neq 0$ , then  $d(u, z) < C'\rho(u)$ , and by Remark 2.2 (i), if  $S_k(x, u) \neq 0$ , then  $d(x, u) < \tilde{C}2^{-k}$ . We first claim that there exists a positive constant  $\tilde{C}_0 \geq 2\tilde{C}$  such that if  $d(x, u) \geq \tilde{C}_02^{-k}$  and  $d(u, z) < C'\rho(u)$ , then  $d(x, z) \geq \tilde{C}2^{-k}$ , and hence  $S_k(x, z) = 0$ .

In fact, by (2.2) and  $\rho(x) \leq 2^{-k}$ , we have

$$\begin{aligned} d(x, z) &\geq d(x, u) - d(u, z) \geq d(x, u) - C'\rho(u) \\ &\geq d(x, u) - C'C_3[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x, u)]^{k_0/(1+k_0)} \\ &\geq \left\{ 1 - C'C_3(\tilde{C}_0)^{-1/(1+k_0)}(1 + 1/\tilde{C}_0)^{k_0/(1+k_0)} \right\} d(x, u). \end{aligned}$$

Choosing  $\tilde{C}_0$  large enough such that  $\tilde{C}_0 \geq 2\tilde{C}$  and

$$\left\{ 1 - C'C_3(\tilde{C}_0)^{-1/(1+k_0)}(1 + 1/\tilde{C}_0)^{k_0/(1+k_0)} \right\} \geq 1/2,$$

we then have  $d(x, z) \geq \tilde{C}2^{-k}$ .

Thus  $\psi(x, z) \neq 0$  only when there exists a  $u \in \mathcal{X}$  such that  $d(x, u) < \tilde{C}_02^{-k}$  and  $d(u, z) < C'\rho(u)$ . Based on this observation, set

$$W_1 \equiv \{u \in \mathcal{X} : d(z, u) \leq [2^{-k} + d(z, x)]/2, d(z, u) < C'\rho(u), d(x, u) < \tilde{C}_02^{-k}\}$$

and

$$W_2 \equiv \{u \in \mathcal{X} : d(z, u) > [2^{-k} + d(z, x)]/2, d(z, u) < C'\rho(u), d(x, u) < \tilde{C}_02^{-k}\}.$$

Then

$$|\psi(x, z)| \leq \left[ \int_{W_1} + \int_{W_2} \right] |S_k(x, z) - S_k(x, u)| K_\rho(u, z) d\mu(u) \equiv I_4 + I_5.$$

If  $u \in W_1$ , then  $\rho(u) \sim \rho(z)$  and

$$d(x, z) \leq d(x, u) + d(z, u) < \tilde{C}_0 2^{-k} + [2^{-k} + d(x, z)]/2 \leq (\tilde{C}_0 + 1/2) 2^{-k} + d(x, z)/2,$$

which implies that  $d(x, z) \lesssim 2^{-k}$ . Therefore, noticing  $2^{-k} \gtrsim \rho(x) + d(x, z) \sim \rho(z) + d(x, z)$  via Lemma 2.1 (iii), by the regularity of  $S_k$  and Proposition 3.1 (i), we have

$$I_4 \lesssim 2^k \rho(z) \frac{\mu(W_1)}{[V_{2^{-k}}(x) + V_{2^{-k}}(z)] V_{\rho(z)}(z)} \lesssim \frac{\rho(z)}{d(x, z) + \rho(z)} \frac{1}{V(x, z) + V_{\rho(z)}(z)}.$$

If  $u \in W_2$ , then  $\rho(u) \sim \rho(z)$  and  $d(x, z) \leq 2d(z, u) \lesssim \rho(z)$ , which implies that  $\rho(x) \sim \rho(z)$ . Hence by Proposition 3.1 (i) and (v), and Definition 2.3 (i), we obtain

$$I_5 \lesssim \frac{1}{V_{\rho(z)}(z)} \lesssim \frac{\rho(z)}{d(x, z) + \rho(z)} \frac{1}{V(x, z) + V_{\rho(z)}(z)}.$$

Thus,

$$|\psi(x, z)| \lesssim \frac{\rho(z)}{d(x, z) + \rho(z)} \frac{1}{V(x, z) + V_{\rho(z)}(z)},$$

which implies that

$$|S_k(f)(x) - S_k(K_\rho(f))(x)| \lesssim \int_{\mathcal{X}} \frac{\rho(z)}{d(x, z) + \rho(z)} \frac{1}{V(x, z) + V_{\rho(z)}(z)} |f(z)| d\mu(z).$$

Therefore, by Lemma 3.1 (i), we have

$$\begin{aligned} I_3 &\leq \int_{\mathcal{X}} \int_{\mathcal{X}} \frac{\rho(z)}{\rho(z) + d(x, z)} \frac{1}{V_{\rho(z)}(z) + V(x, z)} |f(z)| d\mu(z) d\mu(x) \\ &\lesssim \|f\|_{L^1(\mathcal{X})} \lesssim \|f\|_{H_\rho^1(\mathcal{X})}, \end{aligned}$$

which completes the proof of Theorem 3.1.

**Theorem 3.2.** *Let  $\rho$  be an admissible function and  $q \in (1, \infty]$ . Then*

- (i)  $H_\rho^1(\mathcal{X}) = H_\rho^{1,q}(\mathcal{X})$  with equivalent norms;
- (ii)  $\|\cdot\|_{H_\rho^1(\mathcal{X})}$  and  $\|\cdot\|_{H_{\rho, \text{fin}}^{1,q}(\mathcal{X})}$  are equivalent norms on  $H_{\rho, \text{fin}}^{1,q}(\mathcal{X})$ .

*Proof.* We first show (i). Let  $f \in H_\rho^1(\mathcal{X})$ . Then by Theorem 3.1,  $f - K_\rho(f) \in H^1(\mathcal{X})$ . By the atomic decomposition of  $H^1(\mathcal{X})$  in Theorem 2.1 (i), there exist  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $(1, q)$ -atoms  $\{a_j\}_{j \in \mathbb{N}}$  such that  $f - K_\rho(f) = \sum_{j \in \mathbb{N}} \lambda_j a_j$  in  $L^1(\mathcal{X})$  and

$$\sum_{j \in \mathbb{N}} |\lambda_j| \lesssim \|f - K_\rho(f)\|_{H^1(\mathcal{X})},$$

which, together with Theorem 3.1, implies that  $\sum_{j \in \mathbb{N}} |\lambda_j| \lesssim \|f\|_{H_\rho^1(\mathcal{X})}$ .

Now we decompose  $K_\rho(f)$  as a summation of  $(1, q)_\rho$ -atoms. Let  $\{\psi_\alpha\}_\alpha$  be as in Lemma 2.4 and  $\lambda_\alpha \equiv [\mu(B_\alpha)]^{1-1/q} \|\psi_\alpha K_\rho(f)\|_{L^q(\mathcal{X})}$ . If  $\lambda_\alpha = 0$ , set  $a_\alpha \equiv 0$ ; if  $\lambda_\alpha > 0$ , set  $a_\alpha \equiv (\lambda_\alpha)^{-1} \psi_\alpha K_\rho(f)$ . Obviously,  $a_\alpha$  is a  $(1, q)_\rho$ -atom and

$$(3.7) \quad K_\rho(f) = \sum_{\alpha} \lambda_\alpha a_\alpha.$$

Recall that  $\text{supp } \psi_\alpha \subset B_\alpha \equiv B(x_\alpha, \rho(x_\alpha)/2)$ . Notice that by Lemma 2.1 (i), it is easy to see that if  $x \in B_\alpha$ , then  $V_{\rho(x)}(x) \sim \mu(B_\alpha)$ . This, together with  $\text{supp } K_\rho(x, \cdot) \subset B(x, C\rho(x))$ , Lemma 2.3 and Lemma 3.1 (i), yields that

$$\sum_\alpha \lambda_\alpha \lesssim \sum_\alpha \mu(B_\alpha) \|f \chi_{B(x_\alpha, C\rho(x_\alpha))}\|_{L^1(\mathcal{X})} \sup_{x \in B_\alpha} \frac{1}{V_{\rho(x)}(x)} \lesssim \|f\|_{L^1(\mathcal{X})} \lesssim \|f\|_{H_\rho^1(\mathcal{X})}.$$

On the other hand, assume that  $\text{supp } a_j \subset B(x_j, r_j)$ . If  $r_j < \rho(x_j)$ , then  $a_j$  is a  $(1, q)_\rho$ -atom. If  $r_j \geq \rho(x_j)$ , then by Lemma 2.3, there exist finite many  $\alpha_j$  such that  $(B_{\alpha_j} \cap B(x_j, r_j)) \neq \emptyset$ ; namely,  $a_j = \sum_{\alpha_j} \psi_{\alpha_j} a_j$  is a finite summation. Let

$$\lambda_{j, \alpha_j} \equiv [\mu(B_{\alpha_j})]^{1-1/q} \|\psi_{\alpha_j} a_j\|_{L^q(\mathcal{X})}.$$

If  $\lambda_{j, \alpha_j} = 0$ , then set  $a_{j, \alpha_j} \equiv 0$ ; if  $\lambda_{j, \alpha_j} > 0$ , then set  $a_{j, \alpha_j} \equiv (\lambda_{j, \alpha_j})^{-1} \psi_{\alpha_j} a_j$ . Then  $a_{j, \alpha_j}$  is a  $(1, q)_\rho$ -atom, and by the Hölder inequality and Lemmas 2.1 and 2.4, we have

$$\begin{aligned} \sum_{\alpha_j} |\lambda_{j, \alpha_j}| &\lesssim \left\{ \sum_{B_{\alpha_j} \cap B(x_j, r_j) \neq \emptyset} \mu(B_{\alpha_j}) \right\}^{1-1/q} \left\{ \sum_{B_{\alpha_j} \cap B(x_j, r_j) \neq \emptyset} \|\psi_{\alpha_j} a_j\|_{L^q(\mathcal{X})}^q \right\}^{1/q} \\ &\lesssim [\mu(B(x_j, r_j))]^{1/q-1} \|a_j\|_{L^q(\mathcal{X})} \lesssim 1. \end{aligned}$$

Thus,

$$(3.8) \quad f = \sum_{r_j < \rho(x_j)} \lambda_j a_j + \sum_{r_j \geq \rho(x_j)} \sum_{\alpha_j} \lambda_j \lambda_{j, \alpha_j} a_{j, \alpha_j} + \sum_\alpha \lambda_\alpha a_\alpha,$$

where  $\{a_j\}_{r_j < \rho(x_j)}$ ,  $\{a_{j, \alpha_j}\}_{r_j \geq \rho(x_j), \alpha_j}$  and  $\{a_\alpha\}_\alpha$  are  $(1, q)_\rho$ -atoms and

$$\sum_{r_j < \rho(x_j)} |\lambda_j| + \sum_{r_j \geq \rho(x_j)} \sum_{\alpha_j} |\lambda_j \lambda_{j, \alpha_j}| + \sum_\alpha \lambda_\alpha \lesssim \sum_j |\lambda_j| + \sum_\alpha \lambda_\alpha \lesssim \|f\|_{H_\rho^1(\mathcal{X})}.$$

That is,  $f \in H_\rho^{1, q}(\mathcal{X})$  and  $\|f\|_{H_\rho^{1, q}(\mathcal{X})} \lesssim \|f\|_{H_\rho^1(\mathcal{X})}$ , which, together with Lemma 3.1, implies (i).

To prove (ii), if  $f \in H_{\rho, \text{fin}}^{1, q}(\mathcal{X})$ , then  $f \in L^q(\mathcal{X})$  with bounded support when  $q < \infty$  and  $f \in C_c(\mathcal{X})$  when  $q = \infty$ , and so is  $K_\rho(f)$  by (i), (ii) and (iii) of Proposition 3.1. By Proposition 3.1 (v),  $f - K_\rho(f) \in H_{\text{fin}}^{1, q}(\mathcal{X})$ . From Theorem 2.1 (ii), it follows that there exist  $N \in \mathbb{N}$ ,  $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ , and  $(1, q)_\rho$ -atoms  $\{a_j\}_{j=1}^N$  when  $q < \infty$  and continuous  $(1, \infty)_\rho$ -atoms  $\{a_j\}_{j=1}^N$  when  $q = \infty$  such that  $f - K_\rho(f) = \sum_{j=1}^N \lambda_j a_j$  and  $\sum_{j=1}^N |\lambda_j| \lesssim \|f - K_\rho(f)\|_{H^1(\mathcal{X})}$ , which together with Theorem 3.1 implies that  $\sum_{j=1}^N |\lambda_j| \lesssim \|f\|_{H_\rho^1(\mathcal{X})}$ . Observe that by Lemma 2.3, (3.7) in this case is a finite summation of  $(1, q)_\rho$ -atoms when  $q < \infty$  and continuous  $(1, \infty)_\rho$ -atoms when  $q = \infty$ . This, together with the above argument in the proof of (i), implies that (3.8) in this case is also a finite summation of  $(1, q)_\rho$ -atoms when  $q < \infty$  and continuous  $(1, \infty)_\rho$ -atoms when  $q = \infty$ , and

$$\|f\|_{H_{\rho, \text{fin}}^{1, q}(\mathcal{X})} \lesssim \sum_{r_j < \rho(x_j)} |\lambda_j| + \sum_{r_j \geq \rho(x_j)} \sum_{\alpha_j} |\lambda_j \lambda_{j, \alpha_j}| + \sum_\alpha \lambda_\alpha \lesssim \|f\|_{H_\rho^1(\mathcal{X})}.$$

On the other hand, obviously  $\|f\|_{H_\rho^1(\mathcal{X})} \lesssim \|f\|_{H_{\rho, \text{fin}}^{1, q}(\mathcal{X})}$ , which completes the proof of Theorem 3.2.

We point out that an interesting application of finite atomic decomposition characterizations as in Theorem 3.2 is to obtain a general criterion for the boundedness of certain sublinear operators on Hardy spaces via atoms; see [38, 55, 21] for similar results.

Let  $B$  be a Banach space with the norm  $\|\cdot\|_B$  and  $\mathcal{Y}$  be a linear space. An operator  $T$  from  $\mathcal{Y}$  to  $B$  is called  $B$ -sublinear if for all  $f, g \in \mathcal{Y}$  and numbers  $\lambda, \nu \in \mathbb{C}$ , we have

$$\|T(\lambda f + \nu g)\|_B \leq |\lambda| \|T(f)\|_B + |\nu| \|T(g)\|_B$$

and  $\|T(f) - T(g)\|_B \leq \|T(f - g)\|_B$ ; see [55]. Obviously, if  $T$  is linear, then  $T$  is  $B$ -sublinear. Moreover, if  $B \equiv L^r(\mathcal{X})$  with  $r \geq 1$ ,  $T$  is sublinear in the classical sense and  $T(f) \geq 0$  for all  $f \in \mathcal{Y}$ , then  $T$  is also  $B$ -sublinear. Using Theorem 3.2 (ii), we immediately obtain the following result.

**Proposition 3.2.** *Let  $\rho$  be an admissible function,  $q \in (1, \infty)$ ,  $B$  be a Banach space and  $T$  be a  $B$ -sublinear operator from  $H_{\rho, \text{fin}}^{1, q}(\mathcal{X})$  to  $B$ . If*

$$(3.9) \quad \sup\{\|T(a)\|_B : a \text{ is any } (1, q)_{\rho}\text{-atom}\} < \infty$$

for some  $q \in (1, \infty)$  or

$$(3.10) \quad \sup\{\|T(a)\|_B : a \text{ is any continuous } (1, \infty)_{\rho}\text{-atom}\} < \infty,$$

then  $T$  uniquely extends to a bounded  $B$ -sublinear operator from  $H_{\rho}^1(\mathcal{X})$  to  $B$ .

*Proof.* For any  $f \in H_{\rho, \text{fin}}^{1, q}(\mathcal{X})$ , by Theorem 3.2 (ii) there exist an  $N \in \mathbb{N}$ ,  $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ , and  $(1, q)_{\rho}$ -atoms  $\{a_j\}_{j=1}^N$  when  $q < \infty$  and continuous  $(1, \infty)_{\rho}$ -atoms when  $q = \infty$  such that  $f = \sum_{j=1}^N \lambda_j a_j$  pointwise and  $\sum_{j=1}^N |\lambda_j| \lesssim \|f\|_{H_{\rho}^1(\mathcal{X})}$ . Then by the assumption (3.9), we have that  $\|T(f)\|_B \lesssim \sum_{j=1}^N |\lambda_j| \lesssim \|f\|_{H_{\rho}^1(\mathcal{X})}$ . Since  $H_{\rho, \text{fin}}^{1, q}(\mathcal{X})$  is dense in  $H_{\rho}^1(\mathcal{X})$ , a density argument gives the desired conclusion, which completes the proof of Proposition 3.2.

*Remark 3.2.* (i) It is obvious that if  $T$  is a bounded  $B$ -sublinear operator from  $H^1(\mathcal{X})$  to  $B$ , then  $T$  maps all  $(1, q)_{\rho}$ -atoms when  $q \in (1, \infty)$  and continuous  $(1, \infty)_{\rho}$ -atoms when  $q = \infty$  into uniformly bounded elements of  $B$ . Thus, in Proposition 3.2, the assumption that the uniform boundedness of  $T$  on all  $(1, q)_{\rho}$ -atoms when  $q \in (1, \infty)$  and all continuous  $(1, \infty)_{\rho}$ -atoms when  $q = \infty$  is actually necessary.

(ii) Even when  $B \equiv H_{\rho}^1(\mathcal{X})$  or  $B \equiv L^r(\mathcal{X})$  with  $r \geq 1$ , to apply Proposition 3.2 it is not necessary to know the continuity of the considered operator  $T$  from any space of test functions to its dual space, or the boundedness of  $T$  in  $L^2(\mathcal{X})$  or in  $L^p(\mathcal{X})$  for certain  $p \in (1, \infty)$ , which may be convenient in applications.

(iii) Suppose that

$$(3.11) \quad \sup\{\|T(a)\|_B : a \text{ is any } (1, \infty)_{\rho}\text{-atom}\} < \infty.$$

Denote by  $T_0$  the restriction of  $T$  in  $H_{\rho, \text{fin}}^{1, \infty}(\mathcal{X})$ . Then  $T_0$  satisfies (3.10). By Proposition 3.2,  $T_0$  has an extension, denoted by  $\tilde{T}_0$ , such that  $\tilde{T}_0$  is bounded from  $H_{\rho}^1(\mathcal{X})$  to  $L^1(\mathcal{X})$ . However,  $\tilde{T}_0$  may not coincide with  $T$  on all  $(1, \infty)_{\rho}$ -atoms. See [3, 55, 38, 21] for further details and examples.

(iv) As a replacement of (iii) of this remark, we point out that if  $B \equiv L^q(\mathcal{X})$  for some  $q \in [1, \infty)$ ,  $T$  is bounded from  $L^{p_1}(\mathcal{X})$  to  $L^{q_1}(\mathcal{X})$  for some  $p_1, q_1 \in [1, \infty)$ ,

and  $T$  satisfies (3.11), then  $T$  and  $\tilde{T}_0$  coincide on all  $(1, \infty)_\rho$ -atoms. In fact, since  $T$  is bounded from  $L^{p_1}(\mathcal{X})$  to  $L^{q_1}(\mathcal{X})$ , from Lemma 2.5 (i) with a 1-AOTI with bounded support, it is easy to deduce this conclusion. Therefore, in this case, to obtain the boundedness of  $T$  from  $H_\rho^1(\mathcal{X})$  to  $L^q(\mathcal{X})$ , it is enough to prove (3.11).

As an application of Proposition 3.2, we obtain the boundedness in  $H_\rho^1(\mathcal{X})$  of certain localized singular integrals, which are closely related to  $\rho$  and motivated by the Riesz transforms associated to the Schrödinger operators with nonnegative potentials satisfying the reverse Hölder inequality. In what follows,  $L_b^\infty(\mathcal{X})$  denotes the space of functions  $f \in L^\infty(\mathcal{X})$  with bounded support.

Let  $T$  be a linear operator bounded on  $L^q(\mathcal{X})$  for some  $q \in (1, \infty)$ . In addition, suppose that  $T$  is associated with a kernel  $K$  satisfying the fact that there exist constants  $\epsilon \in (0, 1]$  and  $\tilde{C}, C > 0$  such that

$$(K1) \quad |K(x, y)| \leq C \frac{1}{V(x, y)} \text{ for all } x, y \in \mathcal{X} \text{ with } x \neq y;$$

$$(K2) \quad |K(x, y) - K(x, y')| \leq C \frac{1}{V(x, y)} \left[ \frac{d(y, y')}{d(x, y)} \right]^\epsilon \text{ for all } x, y, y' \in \mathcal{X} \text{ with } x \neq y \text{ and } d(y, y') \leq d(x, y)/2;$$

$$(K3) \quad \text{for all } f \in L_b^\infty(\mathcal{X}) \text{ and almost all } x \notin \text{supp } f,$$

$$(3.12) \quad T(f)(x) = \int_{\mathcal{X}} K(x, y) \eta \left( \frac{d(x, y)}{\tilde{C}\rho(x)} \right) f(y) d\mu(y),$$

where  $\eta$  is as in (2.8). Then we have the following result.

**Proposition 3.3.** *The operator  $T$  as in (3.12) is bounded from  $H_\rho^1(\mathcal{X})$  to  $L^1(\mathcal{X})$ .*

*Proof.* By Proposition 3.2, to show Proposition 3.3 it suffices to prove that for all  $(1, 2)_\rho$ -atoms  $a$ ,  $\|T(a)\|_{L^1(\mathcal{X})} \lesssim 1$ . To this end, assume that the atom  $a$  is supported in  $B(x_0, r)$  with  $r < \rho(x_0)$ . We first claim that  $\text{supp } T(a) \subset B(x_0, C\rho(x_0))$ , where  $C \equiv 1 + 2C_3(C_4)^{-1}(1 + 2\tilde{C})^{k_0}$ .

In fact, if  $x \notin B(x_0, C\rho(x_0))$  and  $d(x, y) > 2\tilde{C}\rho(x)$  for all  $y \in B(x_0, r)$ , then the support assumption of  $\eta$ , together with (3.12), implies that  $T(a)(x) = 0$ . If  $x \notin B(x_0, C\rho(x_0))$  and there exists some  $y \in B(x_0, r)$  such that  $d(x, y) \leq 2\tilde{C}\rho(x)$ , then by Lemma 2.1 (iii) and (2.2),

$$\rho(x) \leq (C_4)^{-1}(1 + 2\tilde{C})^{k_0} \rho(y) \leq 2(C_4)^{-1}C_3(1 + 2\tilde{C})^{k_0} \rho(x_0).$$

Thus,

$$d(x_0, y) \geq d(x_0, x) - d(x, y) \geq C\rho(x_0) - 2(C_4)^{-1}C_3(1 + 2\tilde{C})^{k_0} \rho(x_0) \geq \rho(x_0),$$

which is a contradiction with  $y \in B(x_0, r)$ . Thus,  $T(a)(x) = 0$  also in this case, and this shows the claim.

If  $r \geq \rho(x_0)/4$ , from the Hölder inequality and the  $L^q(\mathcal{X})$ -boundedness of  $T$ , it then follows that

$$\|T(a)\|_{L^1(\mathcal{X})} = \|T(a)\|_{L^1(B(x_0, C\rho(x_0)))} \lesssim [V_r(x_0)]^{1-1/q} \|a\|_{L^q(B(x_0, r))} \lesssim 1.$$

If  $r < \rho(x_0)/4$ , then by the Hölder inequality and the  $L^q(\mathcal{X})$ -boundedness of  $T$ , we have

$$\|T(a)\|_{L^1(B(x_0, Cr))} \lesssim [V_r(x_0)]^{1-1/q} \|T(a)\|_{L^q(B(x_0, Cr))} \lesssim 1.$$

For any  $x \in (B(x_0, C\rho(x_0)) \setminus B(x_0, Cr))$ , by  $\int_{\mathcal{X}} a(y) d\mu(y) = 0$ , (K1) and (K2), together with Lemma 2.1 (i), we have

$$\begin{aligned} |T(a)(x)| &= \int_{\mathcal{X}} \left| K(x, y)\eta\left(\frac{d(x, y)}{\rho(x)}\right) - K(x, x_0)\eta\left(\frac{d(x, x_0)}{\rho(x)}\right) \right| |a(y)| d\mu(y) \\ &\leq \int_{\mathcal{X}} |K(x, y) - K(x, x_0)| \eta\left(\frac{d(x, y)}{\rho(x)}\right) |a(y)| d\mu(y) \\ &\quad + \int_{\mathcal{X}} |K(x, x_0)| \left| \eta\left(\frac{d(x, y)}{\rho(x)}\right) - \eta\left(\frac{d(x, x_0)}{\rho(x)}\right) \right| |a(y)| d\mu(y) \\ &\lesssim \int_{B(x_0, r)} \frac{1}{V(x, x_0)} \left[ \frac{d(x_0, y)}{d(x_0, x)} \right]^\epsilon |a(y)| d\mu(y) \\ &\quad + \int_{B(x_0, r)} \frac{1}{V(x, x_0)} \frac{d(x_0, y)}{\rho(x)} |a(y)| d\mu(y) \\ &\lesssim \frac{1}{V(x, x_0)} \left[ \frac{r}{d(x_0, x)} \right]^\epsilon + \frac{1}{V(x, x_0)} \frac{r}{\rho(x_0)} \lesssim \frac{1}{V(x, x_0)} \left[ \frac{r}{d(x_0, x)} \right]^\epsilon. \end{aligned}$$

Thus, assuming that  $2^{j_0}r \leq \rho(x_0) < 2^{j_0+1}r$  for certain  $j_0 \in \mathbb{N}$ , we obtain

$$\begin{aligned} &\int_{B(x_0, C\rho(x_0)) \setminus B(x_0, Cr)} |T(a)(x)| d\mu(x) \\ &\lesssim \int_{B(x_0, C\rho(x_0)) \setminus B(x_0, Cr)} \frac{1}{V(x, x_0)} \left[ \frac{r}{d(x_0, x)} \right]^\epsilon d\mu(x) \lesssim \sum_{j=0}^{j_0} 2^{-j\epsilon} \lesssim 1, \end{aligned}$$

which completes the proof of Proposition 3.3.

*Remark 3.3.* We should point out that Proposition 3.2 is used in Section 5.4 to prove the boundedness on Hardy spaces of Riesz transforms associated to Schrödinger operators with potentials satisfying the reverse Hölder inequality on connected and simply connected nilpotent Lie groups. Moreover, there exist many examples of such localized singular integrals as in (3.12). For example, if  $x \in \mathbb{R}^n$  and

$$T(f)(x) \equiv \text{p. v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} \eta\left(\frac{x - y}{\rho(x)}\right) f(y) dy,$$

then  $T$  is an operator as in (3.12). Let  $\tilde{T}$  be a linear operator bounded on  $L^q(\mathcal{X})$  for some  $q \in (1, \infty)$ , and for all  $x \in \mathcal{X}$ ,

$$\tilde{T}(f)(x) \equiv \text{p. v.} \int_{\mathcal{X}} K(x, y) f(y) d\mu(y),$$

with  $K$  and  $K^t$  satisfying (K1) and (K2), where  $K^t(x, y) = K(y, x)$  for all  $x, y \in \mathcal{X}$ . Define  $T$  by setting, for all  $x \in \mathcal{X}$ ,

$$T(f)(x) \equiv \text{p. v.} \int_{\mathcal{X}} K(x, y)\eta\left(\frac{d(x, y)}{\rho(x)}\right) f(y) d\mu(y).$$

Since the maximal operator  $\tilde{T}_*$ , which is defined by setting, for all  $x \in \mathcal{X}$ ,

$$\tilde{T}_*(f)(x) = \sup_{\epsilon > 0} \left| \int_{d(x, y) > \epsilon} K(x, y) f(y) d\mu(y) \right|,$$

is bounded on  $L^q(\mathcal{X})$  for certain  $q \in (1, \infty)$  (see, for example, [47]), then  $T$  is an operator as in (3.12).

4. RADIAL MAXIMAL FUNCTION CHARACTERIZATIONS OF  $H^1_\rho(\mathcal{X})$

In this section, we establish a radial maximal function characterization of  $H^1_\rho(\mathcal{X})$  as follows.

**Theorem 4.1.** *Let  $\rho$  be admissible and let  $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0, \epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$  and  $\{S_k\}_{k \in \mathbb{Z}}$  be an  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI. Then  $f \in H^1_\rho(\mathcal{X})$  if and only if  $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$  for some  $\beta, \gamma \in (0, \epsilon)$  and  $\|S_\rho^+(f)\|_{L^1(\mathcal{X})} < \infty$ . Moreover, there exists a positive constant  $C$  such that for all  $f \in H^1_\rho(\mathcal{X})$ ,*

$$(4.1) \quad C^{-1} \|S_\rho^+(f)\|_{L^1(\mathcal{X})} \leq \|f\|_{H^1_\rho(\mathcal{X})} \leq C \|S_\rho^+(f)\|_{L^1(\mathcal{X})}.$$

For the sake of applications, we need the following characterization of  $H^1_\rho(\mathcal{X})$  via a variant of the radial maximal function.

**Theorem 4.2.** *Let  $\rho$  be an admissible function. Assume that  $\{T_t\}_{t>0}$  is a family of linear operators bounded on  $L^2(\mathcal{X})$  with integrable kernels  $\{T_t(x, y)\}_{t>0}$  satisfying the fact that there exist a continuous  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI  $\{\tilde{T}_t\}_{t>0}$  for some  $\epsilon_1 \in (0, 1]$  and  $\epsilon_2, \epsilon_3 > 0$ , and constants  $C > 0, \delta_2 \in (0, \epsilon_2]$  and  $\delta_1, \delta_3 > 0$  such that for all  $x, y \in \mathcal{X}$ ,*

$$(i) \quad |T_t(x, y)| \leq C \frac{1}{V_t(x)+V(x, y)} \left[\frac{t}{t+d(x, y)}\right]^{\delta_2} \left[\frac{\rho(x)}{t+\rho(x)}\right]^{\delta_3};$$

$$(ii) \quad |T_t(x, y) - \tilde{T}_t(x, y)| \leq C \left[\frac{t}{t+\rho(x)}\right]^{\delta_1} \frac{1}{V_t(x)+V(x, y)} \left[\frac{t}{t+d(x, y)}\right]^{\delta_2}.$$

Then the following are equivalent:

- (a)  $f \in H^1_\rho(\mathcal{X})$ ;
- (b)  $f \in L^1(\mathcal{X})$  and  $\|T^+(f)\|_{L^1(\mathcal{X})} < \infty$ ;
- (c)  $f \in L^1(\mathcal{X})$  and  $\|T_\rho^+(f)\|_{L^1(\mathcal{X})} < \infty$ .

Moreover, for all  $f \in L^1(\mathcal{X})$ ,

$$\|f\|_{H^1_\rho(\mathcal{X})} \sim \|T^+(f)\|_{L^1(\mathcal{X})} \sim \|T_\rho^+(f)\|_{L^1(\mathcal{X})},$$

where  $T^+(f)(x) \equiv \sup_{t>0} |T_t(f)(x)|$  and  $T_\rho^+(f)(x) \equiv \sup_{0 < t < \rho(x)} |T_t(f)(x)|$  for all  $x \in \mathcal{X}$ .

*Remark 4.1.* (i) If  $\{T_t\}_{t>0}$  and  $\{\tilde{T}_t\}_{t>0}$  satisfy the assumptions of Theorem 4.2, then it is easy to see that for all  $f \in L^1_{loc}(\mathcal{X})$  and  $x \in \mathcal{X}$ ,

$$T^+(f)(x) \lesssim \tilde{T}^+(f)(x) + M(f)(x) \lesssim M(f)(x),$$

and thus  $T^+$  is bounded on  $L^p(\mathcal{X})$  for  $p \in (1, \infty]$  and bounded from  $L^1(\mathcal{X})$  to weak- $L^1(\mathcal{X})$ . Moreover, for all  $f \in L^1(\mathcal{X})$ , and observing that for almost all  $x \in \mathcal{X}$ , by (ii) of Theorem 4.2,

$$|f(x)| = \lim_{t < \rho(x), t \rightarrow 0} |\tilde{T}_t(f)(x)| \leq T_\rho^+(f)(x) + C \lim_{t \rightarrow 0} \left[\frac{t}{\rho(x)}\right]^{\delta_3} M(f)(x) \leq T_\rho^+(f)(x),$$

we have that  $\|f\|_{L^1(\mathcal{X})} \leq \|T_\rho^+(f)\|_{L^1(\mathcal{X})} \leq \|T^+(f)\|_{L^1(\mathcal{X})}$ .

(ii) Let  $\{\tilde{T}_t\}_{t>0}$  be as in Theorem 4.2. Then  $\{\tilde{T}_t(x, y)\chi_{\{t \leq C\rho(x)\}}(x)\}_{t>0}$  satisfies (i) and (ii) of Theorem 4.2 with  $\delta_2 = \epsilon_2$  and any  $\delta_1, \delta_3 > 0$ . Moreover, let  $\delta_3 > 0$ , and for all  $t > 0$  and  $x, y \in \mathcal{X}$ , define

$$T_t(x, y) \equiv \tilde{T}_t(x, y) \frac{[\rho(x)]^{\delta_3}}{t^{\delta_3} + [\rho(x)]^{\delta_3}}.$$

Then it is easy to verify that  $\{T_t\}_{t>0}$  satisfies (i) and (ii) of Theorem 4.2 with  $\delta_1 \equiv \delta_3$  and  $\delta_2 \equiv \epsilon_2$ .

To prove Theorem 4.1, we need a variant of the inhomogeneous discrete Calderón reproducing formula established in [27]. This variant was established in [21]. To state this variant, we first recall the dyadic cubes on spaces of homogeneous type constructed by Christ [4].

**Lemma 4.1.** *Let  $\mathcal{X}$  be a space of homogeneous type. Then there exists a collection  $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$  of open subsets of  $\mathcal{X}$  (where  $I_k$  is some index set) and the constants  $\delta \in (0, 1)$  and  $\tilde{C}_1, \tilde{C}_2 > 0$  such that*

- (i)  $\mu(\mathcal{X} \setminus \bigcup_\alpha Q_\alpha^k) = 0$  for each fixed  $k$  and  $(Q_\alpha^k \cap Q_\beta^k) = \emptyset$  if  $\alpha \neq \beta$ ;
- (ii) for any  $\alpha, \beta, k, \ell$  with  $\ell \geq k$ , either  $Q_\beta^\ell \subset Q_\alpha^k$  or  $(Q_\beta^\ell \cap Q_\alpha^k) = \emptyset$ ;
- (iii) for each  $(k, \alpha)$  and  $\ell < k$ , there exists a unique  $\beta$  such that  $Q_\alpha^k \subset Q_\beta^\ell$ ;
- (iv)  $\text{diam}(Q_\alpha^k) \leq \tilde{C}_1 \delta^k$ , where  $\text{diam}(Q_\alpha^k) \equiv \sup\{d(x, y) : x, y \in Q_\alpha^k\}$ ;
- (v) each  $Q_\alpha^k$  contains some ball  $B(z_\alpha^k, \tilde{C}_2 \delta^k)$ , where  $z_\alpha^k \in \mathcal{X}$ .

In fact, we can think of  $Q_\alpha^k$  as being a dyadic cube with diameter roughly  $\delta^k$  centered at  $z_\alpha^k$ . In what follows, for simplicity, we always assume that  $\delta = 1/2$ ; see [27] on how to remove this restriction.

For any  $j \in \mathbb{N}, k \in \mathbb{Z}$  and  $\tau \in I_k$ , denote by  $Q_\tau^{k,\nu}, \nu = 1, 2, \dots, N(k, \tau)$ , the set of all cubes  $Q_{\tau'}^{k+j} \subset Q_\tau^k$ . We also denote by  $z_\tau^{k,\nu}$  the center of  $Q_\tau^{k,\nu}$  and by  $y_\tau^{k,\nu}$  any point of  $Q_\tau^{k,\nu}$ . For  $\ell \in \mathbb{Z}$  and  $j \in \mathbb{N}$ , set

$$(4.2) \quad \mathcal{D}(\ell, j) \equiv \{y_\tau^{k,\nu} \in Q_\tau^{k,\nu} : k = \ell, \dots, \infty, \tau \in I_k, \nu = 1, \dots, N(k, \tau)\}.$$

In what follows, for any set  $E$  and locally integrable function  $f$ , set

$$m_E(f) \equiv \frac{1}{\mu(E)} \int_E f(z) d\mu(z).$$

Let  $j_0 \in \mathbb{N}$  such that

$$(4.3) \quad 2^{-j_0} \tilde{C}_1 < 1/3.$$

The following Calderón reproducing formula comes from [21].

**Lemma 4.2.** *Let  $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0, \epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$  and  $\{S_k\}_{k \in \mathbb{Z}}$  be an  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI. Then there exists  $j_1 > j_0$  with  $j_0$  as in (4.3) such that for any  $\ell \in \mathbb{Z}$  and  $\mathcal{D}(\ell + 1, j_1)$  as in (4.2), there exist operators  $\{\tilde{D}_k\}_{k=\ell}^\infty$  with kernels  $\{\tilde{D}_k(x, y)\}_{k=\ell}^\infty$  such that for any  $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$  with  $\beta, \gamma \in (0, \epsilon)$ ,*

$$\begin{aligned} f(x) &= \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \int_{Q_\tau^{\ell, \nu}} \tilde{D}_\ell(x, y) d\mu(y) m_{Q_\tau^{\ell, \nu}}(S_\ell(f)) \\ &\quad + \sum_{k=\ell+1}^\infty \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_\tau^{k, \nu}) \tilde{D}_k(x, y_\tau^{k, \nu}) D_k(f)(y_\tau^{k, \nu}), \end{aligned}$$

where  $D_k \equiv S_k - S_{k-1}$  for any  $k \geq \ell + 1$  and the series converges in  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ . Moreover, for any  $\epsilon' \in [\epsilon, \epsilon_1 \wedge \epsilon_2)$ , there exists a positive constant  $C_{\epsilon'}$ , depending on  $\epsilon'$  but not on  $\ell, j_1$  and  $\mathcal{D}(\ell + 1, j_1)$ , such that  $\tilde{D}_k$  for  $k \geq \ell$  satisfies (i) and (ii) of Definition 2.3 with  $\epsilon_1$  and  $\epsilon_2$  replaced by  $\epsilon'$  and the constant  $C_6$  replaced by  $C_{\epsilon'}$ , and  $\int_{\mathcal{X}} \tilde{D}_k(z, y) d\mu(z) = \int_{\mathcal{X}} \tilde{D}_k(x, z) d\mu(z) = 1$  when  $k = \ell$ , and  $= 0$  when  $k \geq \ell + 1$ .



The following estimate is a variant of Lemma 5.3 in [27], which is also used in the proof of Theorem 4.1.

**Lemma 4.3.** *Let  $\epsilon > 0$  and  $r \in (n/(n+\epsilon), 1]$ . Then there exists a positive constant  $C$  such that for all  $k, k' \in \mathbb{Z}$ ,  $a_\tau^{k,\nu} \in \mathbb{C}$ ,  $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ ,  $\tilde{Q}_\tau^{k,\nu} \subset Q_\tau^{k,\nu}$  with  $\tau \in I_k$  and  $\nu = 1, \dots, N(k, \tau)$ , and  $x \in \mathcal{X}$ ,*

$$\begin{aligned} & \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(\tilde{Q}_\tau^{k,\nu}) \frac{|a_\tau^{k,\nu}|}{V_{2^{-(k' \wedge k)}} + V(x, y_\tau^{k,\nu})} \left[ \frac{2^{-(k' \wedge k)}}{2^{-(k' \wedge k)} + d(x, y_\tau^{k,\nu})} \right]^\epsilon \\ & \leq C 2^{[(k' \wedge k) - k]n(1-1/r)} \left\{ M \left( \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} |a_\tau^{k,\nu}|^r \chi_{\tilde{Q}_\tau^{k,\nu}} \right) (x) \right\}^{1/r}. \end{aligned}$$

We point out that if  $\tilde{Q}_\tau^{k,\nu} = Q_\tau^{k,\nu}$ , then this is just Lemma 5.3 of [27]. The proof of Lemma 4.3 is a slight modification of the proof of [27, Lemma 5.3]. We omit the details.

We point out that the following approach used in the proof of Theorem 4.1 is totally different from that used by Dziubański and Zienkiewicz in their papers [10, 11, 12, 15] to obtain a similar result on  $\mathbb{R}^n$ . The method in [10, 11, 12, 15] strongly depends on an existing theory of localized Hardy spaces  $h^1$  in the sense of Goldberg [24]. Our method successfully avoids this via the discrete Calderón reproducing formula, Lemma 4.2.

*Proof of Theorem 4.1.* By (2.9), to prove Theorem 4.1 we only need to prove the second inequality in (4.1). To this end, let  $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$  with  $\epsilon, \beta, \gamma$  as in Definition 2.8 such that  $\|S_\rho^+(f)\|_{L^1(\mathcal{X})} < \infty$ . Then by the proof of Lemma 3.1,  $f \in L^1(\mathcal{X})$  in the sense of  $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$  and  $\|f\|_{L^1(\mathcal{X})} \leq \|S_\rho^+(f)\|_{L^1(\mathcal{X})}$ . For any  $x \in \mathcal{X}$ , there exists an  $\ell \in \mathbb{Z}$  such that  $2^{-\ell} < \rho(x) \leq 2^{-\ell+1}$ . We first claim that for any  $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$  satisfying  $\int_{\mathcal{X}} \varphi(x) d\mu(x) = 0$  and  $\|\varphi\|_{\mathcal{G}_0^\epsilon(x, 2^{-k'}, \beta, \gamma)} \leq 1$  for some  $k' \geq \ell + 1$ , we have

$$\begin{aligned} (4.4) \quad |\langle f, \varphi \rangle| & \lesssim \{M([\mathcal{S}_\rho^+(f)]^r)(x)\}^{1/r} \\ & + \left\{ M \left( \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} [m_{Q_\tau^{\ell, \nu}}(|\mathcal{S}_\rho^+(f)|)]^r \chi_{Q_\tau^{\ell, \nu}} \right) (x) \right\}^{1/r} \\ & + \int_{\mathcal{X}} \frac{1}{V_{\rho(z)}(z) + V(x, z)} \left( \frac{\rho(z)}{\rho(z) + d(x, z)} \right)^{\gamma/(1+k_0)} |f(z)| d\mu(z), \end{aligned}$$

where  $r \in (n/(n+\epsilon), 1)$ .

Assume that the above claim holds temporarily. Notice that for any function  $\phi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$  with  $\|\phi\|_{\mathcal{G}_0^\epsilon(x, 2^{-k'}, \beta, \gamma)} \leq 1$  for some  $k' \geq \ell + 1$ , we have

$$\sigma \equiv \left| \int_{\mathcal{X}} \phi(y) d\mu(y) \right| \leq \int_{\mathcal{X}} \frac{1}{V_{2^{-k'}}(x) + V(x, y)} \left( \frac{2^{-k'}}{2^{-k'} + d(x, y)} \right)^\gamma d\mu(y) \lesssim 1.$$

Set  $\varphi(y) \equiv \frac{1}{1+\sigma C_6} [\phi(y) - \sigma S_{k'}(x, y)]$  for all  $y \in \mathcal{X}$ . Obviously,  $\int_{\mathcal{X}} \varphi(y) d\mu(y) = 0$  and  $\|\varphi\|_{\mathcal{G}_0^\epsilon(x, 2^{-k'}, \beta, \gamma)} \leq 1$ . Then from (4.4),

$$|\langle f, \phi \rangle| \leq \sigma |S_{k'}(f)(x)| + (1 + \sigma C_6) |\langle f, \varphi \rangle|$$

and

$$|S_{k'}(f)(x)| \leq S_{\rho}^{+}(f)(x) \leq \{M([S_{\rho}^{+}(f)]^r)(x)\}^{1/r}$$

for almost all  $x \in \mathcal{X}$ , it follows that (4.4) still holds with  $|\langle f, \varphi \rangle|$  replaced by  $G_{\rho}(f)$  as in Definition 2.5. This, together with the boundedness on  $L^{1/r}(\mathcal{X})$  of the Hardy-Littlewood operator  $M$  and  $\|f\|_{L^1(\mathcal{X})} \leq \|S_{\rho}^{+}(f)\|_{L^1(\mathcal{X})}$ , implies that

$$\begin{aligned} \|G_{\rho}(f)\|_{L^1(\mathcal{X})} &\lesssim \|S_{\rho}^{+}(f)\|_{L^1(\mathcal{X})} + \left\| \sum_{\tau \in I_{\ell}} \sum_{\nu=1}^{N(\ell, \tau)} m_{Q_{\tau}^{\ell, \nu}}(S_{\rho}^{+}(f)) \chi_{Q_{\tau}^{\ell, \nu}} \right\|_{L^1(\mathcal{X})} \\ &\quad + \int_{\mathcal{X}} \int_{\mathcal{X}} \frac{|f(z)|}{V_{\rho(z)}(z) + V(x, z)} \left( \frac{\rho(z)}{\rho(z) + d(x, z)} \right)^{\gamma/(1+k_0)} d\mu(z) d\mu(x) \\ &\lesssim \|S_{\rho}^{+}(f)\|_{L^1(\mathcal{X})} + \|f\|_{L^1(\mathcal{X})} \lesssim \|S_{\rho}^{+}(f)\|_{L^1(\mathcal{X})}, \end{aligned}$$

which establishes the second inequality of (4.1) in Theorem 4.1.

To prove the claim (4.4), by Lemma 4.2 with the same notation as there, we write

$$\begin{aligned} \langle f, \varphi \rangle &= \sum_{\tau \in I_{\ell}} \sum_{\nu=1}^{N(\ell, \tau)} \int_{Q_{\tau}^{k, \nu}} \tilde{D}_{\ell}^{*}(\varphi)(y) d\mu(y) m_{Q_{\tau}^{\ell, \nu}}(S_{\ell}(f)) \\ &\quad + \sum_{k=\ell+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) \tilde{D}_k^{*}(\varphi)(y_{\tau}^{k, \nu}) D_k(f)(y_{\tau}^{k, \nu}) \equiv I_1 + I_2, \end{aligned}$$

where  $\tilde{D}_k^{*}$  denotes the integral operator with kernel  $\tilde{D}_k^{*}(x, y) \equiv \tilde{D}_k(y, x)$  for all  $x, y \in \mathcal{X}$ .

Observe that for all  $k \geq k'$ , by using  $\int_{\mathcal{X}} D_k(x, y) d\mu(y) = 0$ , (i) and (ii) of  $S_k$  in Definition 2.3 and the size condition of  $\varphi$ , we obtain that for all  $y \in \mathcal{X}$ ,

$$(4.5) \quad |\tilde{D}_k^{*}(\varphi)(y)| \lesssim 2^{-(k-k')\beta} \frac{1}{V_{2^{-k'}}(x) + V(x, y)} \left( \frac{2^{-k'}}{2^{-k'} + d(x, y)} \right)^{\gamma},$$

and that for all  $k < k'$ , by using  $\int_{\mathcal{X}} \varphi(y) d\mu(y) = 0$ , the size condition and the regularity of  $\varphi$ , and (i) for  $D_k$  in Definition 2.3, we have that for all  $y \in \mathcal{X}$ ,

$$(4.6) \quad |\tilde{D}_k^{*}(\varphi)(y)| \lesssim 2^{-(k'-k)\gamma'} \frac{1}{V_{2^{-k}}(x) + V(x, y)} \left( \frac{2^{-k}}{2^{-k} + d(x, y)} \right)^{\gamma},$$

where  $\gamma' \in (0, \gamma)$ . See the proof of Proposition 5.7 in [27] for details.

Moreover, in what follows, set  $\tilde{Q}_{\tau}^{k, \nu} \equiv \{y \in Q_{\tau}^{k, \nu} : 2^{-k} < \rho(y)/2\}$  and  $\bar{Q}_{\tau}^{k, \nu} \equiv (Q_{\tau}^{k, \nu} \setminus \tilde{Q}_{\tau}^{k, \nu})$ . With the subtle split of  $Q_{\tau}^{k, \nu}$ , together with the arbitrariness of  $y_{\tau}^{k, \nu} \in Q_{\tau}^{k, \nu}$ , we have

$$\inf_{y \in Q_{\tau}^{k, \nu}} |D_k(f)(y)| \lesssim \inf_{y \in \tilde{Q}_{\tau}^{k, \nu}} |S_{\rho}^{+}(f)(y)|$$

and

$$\begin{aligned} \inf_{y \in Q_{\tau}^{k, \nu}} |D_k(f)(y)| &\leq \inf_{y \in \bar{Q}_{\tau}^{k, \nu}} |D_k(f)(y)| \\ &\lesssim \inf_{y \in \bar{Q}_{\tau}^{k, \nu}} \int_{\mathcal{X}} \frac{|f(z)|}{V_{2^{-k}}(y) + V(y, z)} \left( \frac{2^{-k}}{2^{-k} + d(y, z)} \right)^{\gamma} d\mu(z). \end{aligned}$$

This, together with (4.5) and (4.6), implies that

$$\begin{aligned}
 |I_2| &\lesssim \sum_{k=\ell+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} 2^{-|k'-k|(\beta \wedge \gamma')} \frac{\mu(\tilde{Q}_\tau^{k, \nu})}{V_{2^{-(k' \wedge k)}}(x) + V(x, y)} \left( \frac{2^{-(k' \wedge k)}}{2^{-(k' \wedge k)} + d(x, y)} \right)^\gamma \\
 &\quad \times \inf_{y \in \tilde{Q}_\tau^{k, \nu}} |S_\rho^+(f)(y)| \\
 &\quad + \sum_{k=\ell+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} 2^{-|k'-k|(\beta \wedge \gamma')} \int_{\tilde{Q}_\tau^{k, \nu}} \frac{1}{V_{2^{-(k' \wedge k)}}(x) + V(x, y)} \\
 &\quad \times \left( \frac{2^{-(k' \wedge k)}}{2^{-(k' \wedge k)} + d(x, y)} \right)^\gamma \\
 &\quad \times \left\{ \int_{\mathcal{X}} \frac{|f(z)|}{V_{2^{-k}}(y) + V(y, z)} \left( \frac{2^{-k}}{2^{-k} + d(y, z)} \right)^\gamma d\mu(z) \right\} d\mu(y) \equiv I_{2,1} + I_{2,2}.
 \end{aligned}$$

We first estimate  $I_{2,2}$  by writing

$$\begin{aligned}
 I_{2,2} &\lesssim \int_{\mathcal{X}} |f(z)| \left\{ \sum_{k=\ell+1}^{\infty} 2^{-|k'-k|(\beta \wedge \gamma')} \int_{\rho(y) \leq 2^{-k+1}} \frac{1}{V_{2^{-(k' \wedge k)}}(x) + V(x, y)} \right. \\
 &\quad \times \left( \frac{2^{-(k' \wedge k)}}{2^{-(k' \wedge k)} + d(x, y)} \right)^\gamma \frac{1}{V_{2^{-k}}(y) + V(y, z)} \\
 &\quad \times \left. \left( \frac{2^{-k}}{2^{-k} + d(y, z)} \right)^\gamma d\mu(y) \right\} d\mu(z) \\
 &\equiv \int_{\mathcal{X}} |f(z)| I_{2,2}(x, z) d\mu(z).
 \end{aligned}$$

If we can show that for all  $x, z \in \mathcal{X}$ ,

$$(4.7) \quad I_{2,2}(x, z) \lesssim \frac{1}{V_{\rho(x)}(x) + V(x, z)} \left( \frac{\rho(x)}{\rho(x) + d(x, z)} \right)^\gamma,$$

then by Lemma 2.1 (ii) and (2.2) together with  $V_{\rho(x)} + V(x, z) \sim V_{\rho(z)}(z) + V(x, z)$  for all  $x, z \in \mathcal{X}$ , we have

$$(4.8) \quad I_{2,2} \lesssim \int_{\mathcal{X}} \frac{|f(z)|}{V_{\rho(z)}(z) + V(x, z)} \left( \frac{\rho(z)}{\rho(z) + d(x, z)} \right)^{\gamma/(1+k_0)} d\mu(z),$$

which is a desired estimate.

To see (4.7), notice that by Lemma 2.1 (i), if  $d(x, y) < \rho(x)$ , then there exists a positive constant  $\tilde{C}$  such that  $(\tilde{C})^{-1}\rho(y) < \rho(x) < \tilde{C}\rho(y)$ . Thus if  $\tilde{C}2^{-(k' \wedge k)+1} \leq \rho(x)$  and  $\rho(y) \leq 2^{-k+1}$ , then we have  $d(x, y) \geq \rho(x)$ . From this it follows that  $(2^{-(k' \wedge k)} + d(x, y)) \gtrsim \rho(x)$ . Therefore, if  $d(x, z) < 2\rho(x)$ , then we have

$$\begin{aligned}
 I_{2,2}(x, z) &\lesssim \frac{1}{V_{\rho(x)}(x)} \sum_{k=\ell+1}^{\infty} 2^{-|k'-k|(\beta \wedge \gamma')} \int_{\mathcal{X}} \frac{1}{V_{2^{-k}}(y) + V(y, z)} \\
 &\quad \times \left( \frac{2^{-k}}{2^{-k} + d(y, z)} \right)^\gamma d\mu(y) \lesssim \frac{1}{V_{\rho(x)}(x)}.
 \end{aligned}$$

If  $d(x, z) \geq 2\rho(x)$  and  $d(x, y) > d(x, z)/2$ , similarly we have

$$\begin{aligned} I_{2,2}(x, z) &\lesssim \frac{1}{V_{\rho(x)}(x)} \left( \frac{\rho(x)}{\rho(x) + d(x, z)} \right)^\gamma \sum_{k=\ell+1}^{\infty} 2^{-|k'-k|(\beta \wedge \gamma')} \\ &\quad \times \int_{\mathcal{X}} \frac{1}{V_{2^{-(k' \wedge k)}}(y) + V(y, z)} \left( \frac{2^{-k}}{2^{-k} + d(y, z)} \right)^\gamma d\mu(y) \\ &\lesssim \frac{1}{V_{\rho(x)}(x)} \left( \frac{\rho(x)}{\rho(x) + d(x, z)} \right)^\gamma. \end{aligned}$$

If  $d(x, z) \geq 2\rho(x)$  and  $d(y, z) > d(x, z)/2$ , then by  $2^{-(k' \wedge k)} \leq \rho(x)$ ,

$$\begin{aligned} I_{2,2}(x, z) &\lesssim \frac{1}{V_{\rho(x)}(x)} \left( \frac{\rho(x)}{\rho(x) + d(x, z)} \right)^\gamma \sum_{k=\ell+1}^{\infty} 2^{-|k'-k|(\beta \wedge \gamma')} \\ &\quad \times \int_{\mathcal{X}} \frac{1}{V_{2^{-(k' \wedge k)}}(x) + V(x, y)} \left( \frac{2^{-(k' \wedge k)}}{2^{-(k' \wedge k)} + d(x, y)} \right)^\gamma d\mu(y) \\ &\lesssim \frac{1}{V_{\rho(x)}(x)} \left( \frac{\rho(x)}{\rho(x) + d(x, z)} \right)^\gamma. \end{aligned}$$

Combining these estimates implies (4.7).

On the other hand, if we choose  $r \in (n/(n + \epsilon), 1)$  such that  $1/r > 1 - (\beta \wedge \gamma')$ , then by Lemma 4.3 we have

$$\begin{aligned} I_{2,1} &\lesssim \sum_{k=\ell+1}^{\infty} 2^{-|k'-k|(\beta \wedge \gamma')} 2^{[(k' \wedge k) - k]n(1-1/r)} \\ &\quad \times \left[ M \left( \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \left| \inf_{y \in \tilde{Q}_\tau^{k, \nu}} S_\rho^+(f)(y) \right|^r \chi_{\tilde{Q}_\tau^{k, \nu}}(x) \right) \right]^{1/r} \\ &\lesssim \sum_{k=\ell+1}^{\infty} 2^{-|k'-k|(\beta \wedge \gamma')} 2^{[(k' \wedge k) - k]n(1-1/r)} [M([S_\rho^+(f)]^r)(x)]^{1/r} \\ &\lesssim [M([S_\rho^+(f)]^r)(x)]^{1/r}. \end{aligned}$$

Combining the estimates for  $I_{2,1}$  and  $I_{2,2}$  yields the fact that

$$\begin{aligned} |I_2| &\lesssim [M([S_\rho^+(f)]^r)(x)]^{1/r} \\ &\quad + \int_{\mathcal{X}} \frac{|f(z)|}{V_{\rho(z)}(z) + V(x, z)} \left( \frac{\rho(z)}{\rho(z) + \rho(x, z)} \right)^{\gamma/(1+k_0)} d\mu(z). \end{aligned}$$

To estimate  $I_1$ , set

$$\tilde{m}_{\tilde{Q}_\tau^{k, \nu}}(g) \equiv \frac{1}{\mu(Q_\tau^{k, \nu})} \int_{\tilde{Q}_\tau^{k, \nu}} g(z) d\mu(z).$$

Then by  $k' \geq \ell$ , Lemma 4.4 and (4.6), we have

$$\begin{aligned}
 |I_1| &\lesssim \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \frac{\mu(\tilde{Q}_\tau^{\ell, \nu})}{V_{2^{-\ell}}(x) + V(x, y)} \left( \frac{2^{-\ell}}{2^{-\ell} + d(x, y)} \right)^\gamma \tilde{m}_{\tilde{Q}_\tau^{\ell, \nu}}(|S_\rho^+(f)(y)|) \\
 &+ \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \int_{\tilde{Q}_\tau^{\ell, \nu}} \frac{1}{V_{2^{-\ell}}(x) + V(x, y)} \left( \frac{2^{-\ell}}{2^{-\ell} + d(x, y)} \right)^\gamma \\
 &\times \left\{ \int_{\mathcal{X}} \frac{|f(z)|}{V_{2^{-\ell}}(y) + V(y, z)} \left( \frac{2^{-\ell}}{2^{-\ell} + d(y, z)} \right)^\gamma d\mu(z) \right\} d\mu(y) \equiv I_{1,1} + I_{1,2}.
 \end{aligned}$$

By an argument similar to that used in (4.8), we have that (4.8) still holds by replacing  $I_{2,2}$  with  $I_{1,2}$ . For  $I_{1,1}$ , similar to the estimate for  $I_{2,1}$ , by Lemma 4.3 we have

$$I_{1,1} \lesssim \left[ M \left( \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \left[ \tilde{m}_{\tilde{Q}_\tau^{\ell, \nu}}(S_\rho^+(f)) \right]^r \chi_{\tilde{Q}_\tau^{\ell, \nu}} \right) (x) \right]^{1/r},$$

which together with the obvious inequality

$$\tilde{m}_{\tilde{Q}_\tau^{\ell, \nu}}(S_\rho^+(f)) \chi_{\tilde{Q}_\tau^{\ell, \nu}} \leq m_{Q_\tau^{\ell, \nu}}(S_\rho^+(f)) \chi_{Q_\tau^{\ell, \nu}}$$

further implies the desired estimate. Thus, we have

$$\begin{aligned}
 |I_1| &\lesssim \left[ M \left( \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \left[ \tilde{m}_{\tilde{Q}_\tau^{\ell, \nu}}(S_\rho^+(f)) \right]^r \chi_{\tilde{Q}_\tau^{\ell, \nu}} \right) \right]^{1/r} \\
 &+ \int_{\mathcal{X}} \frac{|f(z)|}{V_{\rho(z)}(z) + V(x, z)} \left( \frac{\rho(z)}{\rho(z) + d(x, z)} \right)^{\gamma/(1+k_0)} d\mu(z).
 \end{aligned}$$

Combining the estimates for  $I_1$  and  $I_2$  yields (4.4) and hence completes the proof of Theorem 4.1.

*Remark 4.2.* Theorem 4.1 still holds with the  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI replaced by the continuous  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI. In fact, if  $\{\tilde{S}_t\}_{t>0}$  is a continuous  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI, letting  $S_k \equiv \tilde{S}_{2^{-k}}$  for  $k \in \mathbb{Z}$ , then  $\{S_k\}_{k \in \mathbb{Z}}$  is an  $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI and, by Theorem 4.1,

$$\|\tilde{S}_\rho^+(f)\|_{L^1(\mathcal{X})} \lesssim \|G_\rho(f)\|_{L^1(\mathcal{X})} \lesssim \|S_\rho^+(f)\| \lesssim \|\tilde{S}_\rho^+(f)\|_{L^1(\mathcal{X})},$$

where  $\tilde{S}_\rho^+(f)(x) \equiv \sup_{0 < t < \rho(x)} |\tilde{S}_t(f)(x)|$  for all  $x \in \mathcal{X}$ . The above claim is true.

To prove Theorem 4.2, we need the following estimate. Let  $T_t$  and  $\tilde{T}_t$  be as in Theorem 4.2. For  $x, y \in \mathcal{X}$ , set  $E_t(x, y) \equiv T_t(x, y) - \tilde{T}_t(x, y)$  and

$$E_\rho^+(f)(x) \equiv \sup_{0 < t < \rho(x)} |E_t(f)(x)|.$$

**Lemma 4.4.** *Under the same assumptions as in Theorem 4.2, there exists a positive constant  $C$  such that for all  $f \in L^1(\mathcal{X})$ ,  $\|E_\rho^+(f)\|_{L^1(\mathcal{X})} \leq C\|f\|_{L^1(\mathcal{X})}$ .*

*Proof.* By Lemmas 2.3 and 2.4, it suffices to prove that for all  $\alpha$ ,

$$(4.9) \quad \|E_\rho^+(\chi_{B_\alpha^*} f)\|_{L^1(\mathcal{X})} \lesssim \|\chi_{B_\alpha^*} f\|_{L^1(\mathcal{X})}.$$

To this end, notice that for all  $x, y \in \mathcal{X}$  with  $x \neq y$ , by (2.1),

$$\left[ \frac{t + d(x, y)}{d(x, y)} \right]^\kappa V(x, y) \lesssim V_{t+d(x, y)}(x) \sim V_t(x) + V(x, y).$$

Thus for any  $x \in B_\alpha^{**}$  and  $y \in B_\alpha^*$ , since  $\rho(y) \sim \rho(x_\alpha) \sim \rho(x)$  via Lemma 2.1 (i), by assumption (ii) of Theorem 4.2 we have

$$\begin{aligned} |E_t(x, y)| &\lesssim \frac{1}{V_t(x) + V(x, y)} \left[ \frac{t}{t + d(x, y)} \right]^{\delta_2} \left[ \frac{t}{t + \rho(x)} \right]^{\delta_1} \\ &\lesssim \frac{1}{V(x, y)} \left[ \frac{d(x, y)}{t + d(x, y)} \right]^\kappa \left[ \frac{t}{t + \rho(x_\alpha)} \right]^{\delta_1} \lesssim \frac{1}{V(x, y)} \left[ \frac{d(x, y)}{\rho(x_\alpha)} \right]^{\kappa \wedge \delta_1}, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{B_\alpha^{**}} \sup_{0 < t < \rho(x)} |E_t(\chi_{B_\alpha^*} f)(x)| d\mu(x) \\ &\lesssim \int_{B_\alpha^{**}} \int_{B_\alpha^*} \frac{1}{V(x, y)} \left[ \frac{d(x, y)}{\rho(x_\alpha)} \right]^{\kappa \wedge \delta_1} |(\chi_{B_\alpha^*} f)(y)| d\mu(x) d\mu(y) \lesssim \|\chi_{B_\alpha^*} f\|_{L^1(\mathcal{X})}. \end{aligned}$$

For any  $x \notin B_\alpha^{**}$  and  $t < \rho(x)$ , it is easy to see that  $\rho(x_\alpha) \lesssim d(x, x_\alpha) \sim d(x, y)$  for all  $y \in B_\alpha^*$ , and by (2.2),  $t < \rho(x) \lesssim [d(x, x_\alpha)]^{k_0/(1+k_0)} [\rho(x_\alpha)]^{1/(1+k_0)}$ , from which it follows that

$$\begin{aligned} |E_t(f)(x)| &\lesssim \int_{B_\alpha^*} \frac{1}{V_t(x) + V(x, y)} \left[ \frac{t}{t + d(x, y)} \right]^{\delta_2} |f(y)| d\mu(y) \\ &\lesssim \frac{1}{V(x, x_\alpha)} \left[ \frac{\rho(x_\alpha)}{d(x, x_\alpha)} \right]^{\delta_2/(1+k_0)} \|\chi_{B_\alpha^*} f\|_{L^1(\mathcal{X})}. \end{aligned}$$

By this we have

$$\begin{aligned} &\int_{(B_\alpha^{**})^c} \sup_{0 < t < \rho(x)} |E_t(\chi_{B_\alpha^*} f)(x)| d\mu(x) \\ &\lesssim \|\chi_{B_\alpha^*} f\|_{L^1(\mathcal{X})} \int_{(B_\alpha^{**})^c} \frac{1}{V(x, x_\alpha)} \left[ \frac{\rho(x_\alpha)}{d(x, x_\alpha)} \right]^{\delta_2/(1+k_0)} d\mu(x) \lesssim \|\chi_{B_\alpha^*} f\|_{L^1(\mathcal{X})}, \end{aligned}$$

which completes the proof of (4.9) and hence the proof of Lemma 4.4.

Now we turn to the proof of Theorem 4.2.

*Proof of Theorem 4.2.* Assume that  $f \in L^1(\mathcal{X})$  and  $\|T_\rho^+(f)\|_{L^1(\mathcal{X})} < \infty$ . Then by Remark 4.1,  $\|f\|_{L^1(\mathcal{X})} \leq \|T_\rho^+(f)\|_{L^1(\mathcal{X})}$ . From this, Remark 4.2 and Lemma 4.4, it follows that  $f \in H_\rho^1(\mathcal{X})$  and

$$\begin{aligned} \|f\|_{H_\rho^1(\mathcal{X})} &\lesssim \|\tilde{T}_\rho^+(f)\|_{L^1(\mathcal{X})} \leq \|T_\rho^+(f)\|_{L^1(\mathcal{X})} + \|E_\rho^+(f)\|_{L^1(\mathcal{X})} \\ &\lesssim \|T_\rho^+(f)\|_{L^1(\mathcal{X})} + \|f\|_{L^1(\mathcal{X})} \lesssim \|T_\rho^+(f)\|_{L^1(\mathcal{X})} \lesssim \|T^+(f)\|_{L^1(\mathcal{X})}. \end{aligned}$$

Conversely, we need to prove that  $T_\rho^+$  and  $T^+$  are bounded from  $H_\rho^1(\mathcal{X})$  to  $L^1(\mathcal{X})$ . To this end, by Proposition 3.2 it suffices to prove that for all  $(1, 2)_\rho$ -atoms  $a$ ,

$$\|T_\rho^+(a)\|_{L^1(\mathcal{X})} + \|T^+(a)\|_{L^1(\mathcal{X})} \lesssim 1.$$

Assume that  $a$  is a  $(1, 2)_\rho$ -atom supported in  $B(y_0, r)$  with  $r < \rho(y_0)$ . By Theorem 4.1 and Remark 4.2, we have  $\|\tilde{T}_\rho^+(a)\|_{L^1(\mathcal{X})} \lesssim 1$ . By Lemma 4.4, we

further obtain  $\|E_\rho^+(a)\|_{L^1(\mathcal{X})} \lesssim \|a\|_{L^1(\mathcal{X})} \lesssim 1$ , which yields  $\|T_\rho^+(a)\|_{L^1(\mathcal{X})} \lesssim 1$ . This also implies that, to show  $\|T^+(a)\|_{L^1(\mathcal{X})} \lesssim 1$ , it suffices to prove that

$$\left\| \sup_{t \geq \rho(\cdot)} |T_t(a)(\cdot)| \right\|_{L^1(\mathcal{X})} \lesssim 1.$$

To see this, by the Hölder inequality and the  $L^2(\mathcal{X})$ -boundedness of  $T^+$  (see Remark 4.1 (i)), we have

$$\int_{B(y_0, 2r)} \sup_{t \geq \rho(x)} |T_t(a)(x)| d\mu(x) \lesssim \|a\|_{L^2(\mathcal{X})} [V_{2r}(y_0)]^{1/2} \lesssim 1.$$

Since for any  $x \in (B(y_0, 4\rho(y_0)) \setminus B(y_0, 2r))$ ,  $\rho(x) \sim \rho(y_0)$  via Lemma 2.1 (i), by assumption (i) of Theorem 4.2 we have that for all  $x \in (B(y_0, 4\rho(y_0)) \setminus B(y_0, 2r))$  and  $t \geq \rho(x)$ ,

$$|T_t(a)(x)| \leq \int_{\mathcal{X}} |T_t(x, y)a(y)| d\mu(y) \lesssim \frac{1}{V_t(x)} \lesssim \frac{1}{V_{\rho(y_0)}(y_0)}.$$

This implies that

$$\int_{B(y_0, 4\rho(y_0)) \setminus B(y_0, 2r)} \sup_{t \geq \rho(x)} |T_t(a)(x)| d\mu(x) \lesssim 1.$$

For any  $x \notin B(y_0, 4\rho(y_0))$ , since (2.2) implies that

$$\rho(x) \lesssim [d(x, y_0)]^{k_0/(1+k_0)} [\rho(y_0)]^{1/(1+k_0)},$$

by assumption (i) of Theorem 4.2 we have that

$$|T_t(a)(x)| \lesssim \frac{1}{V(x, y_0)} \left[ \frac{\rho(x)}{d(x, y_0)} \right]^{\delta_2 \wedge \delta_3} \lesssim \frac{1}{V(x, y_0)} \left[ \frac{\rho(y_0)}{d(x, y_0)} \right]^{(\delta_2 \wedge \delta_3)/(1+k_0)},$$

which implies that

$$\begin{aligned} & \int_{B(y_0, 4\rho(y_0))^c} \sup_{t \geq \rho(x)} |T_t(a)(x)| d\mu(x) \\ & \lesssim \int_{B(y_0, 4\rho(y_0))^c} \frac{1}{V(x, y_0)} \left[ \frac{\rho(y_0)}{d(x, y_0)} \right]^{(\delta_2 \wedge \delta_3)/(1+k_0)} d\mu(x) \lesssim 1. \end{aligned}$$

This shows that  $\|\sup_{t \geq \rho(\cdot)} |T_t(a)(\cdot)|\|_{L^1(\mathcal{X})} \lesssim 1$  and hence finishes the proof of Theorem 4.2.

### 5. SOME APPLICATIONS

In this section, we present several applications of results in Sections 3 and 4.

**5.1. Schrödinger operators on  $\mathbb{R}^n$ .** Let  $n \geq 3$  and  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space endowed with the Euclidean norm  $|\cdot|$  and the Lebesgue measure  $dx$ . Denote the Laplace operator  $-\sum_{j=1}^n (\frac{\partial}{\partial x_j})^2$  on  $\mathbb{R}^n$  by  $\Delta$  and the corresponding heat semigroup  $\{e^{-t\Delta}\}_{t>0}$  by  $\{\tilde{T}_t\}_{t>0}$ . By the Gaussian estimates for the heat kernel and the Markov property for  $\{\tilde{T}_t\}_{t>0}$ , we know that  $\{\tilde{T}_{t^2}\}_{t>0}$  forms a continuous  $(1, N, N)$ -AOTI as in Remark 2.2 (ii) for any  $N > 0$ .

Let  $U$  be a nonnegative locally integrable function on  $\mathbb{R}^n$ ,  $\mathcal{L} \equiv \Delta + U$  the Schrödinger operator and  $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$  the corresponding heat semigroup. Define

$$H_{\mathcal{L}}^1(\mathbb{R}^n) \equiv \{f \in L^1(\mathbb{R}^n) : \|f\|_{H_{\mathcal{L}}^1(\mathbb{R}^n)} \equiv \|T^+(f)\|_{L^1(\mathbb{R}^n)} < \infty\},$$

where  $T^+(f)(x) \equiv \sup_{t>0} |T_t(f)(x)|$  for all  $x \in \mathbb{R}^n$ .

If  $q > n/2$  and  $U \in \mathcal{B}_q(\mathbb{R}^n, |\cdot|, dx)$ , where  $\mathcal{B}_q(\mathbb{R}^n, |\cdot|, dx)$  is the reverse Hölder class as in Subsection 2.1, then Dziubański and Zienkiewicz [11] first established the atomic decomposition characterizations of  $H_{\mathcal{L}}^1(\mathbb{R}^n)$  via the auxiliary function  $\rho$  defined as in (2.3). In fact, Dziubański and Zienkiewicz in [11] proved that  $H_{\mathcal{L}}^1(\mathbb{R}^n) = H_{\rho}^{1,\infty}(\mathbb{R}^n)$  with equivalent norms. Moreover, in [12], for  $f \in L^1(\mathbb{R}^n)$  with compact support, Dziubański and Zienkiewicz also proved that  $\|f\|_{H_{\mathcal{L}}^1(\mathcal{X})} \sim \|\tilde{T}_{\rho}^+(f)\|_{L^1(\mathcal{X})}$ , where  $\tilde{T}_{\rho}^+$  is defined as in Remark 4.2 with  $\tilde{S}_t$  replaced by  $\tilde{T}_{t^2}$ .

On the other hand, Proposition 2.1 implies that  $\rho$  defined in (2.3) is admissible,  $\{\tilde{T}_{t^2}\}_{t>0}$  is a continuous  $(1, N, N)_{\rho}$ -AOTI for any  $N > 0$ , and  $\{T_{t^2}\}_{t>0}$  and  $\{\tilde{T}_{t^2}\}_{t>0}$  satisfy the assumptions (i) and (ii) of Theorem 4.2; see [12]. Thus, by Theorem 4.2,  $H_{\mathcal{L}}^1(\mathbb{R}^n) = H_{\rho}^1(\mathbb{R}^n)$  with equivalent norms. Moreover, all of the results obtained in Sections 3 and 4 are valid for  $H_{\mathcal{L}}^1(\mathbb{R}^n)$ . In particular, the results established in Section 3 are new compared to the results in [11, 12].

**5.2. Degenerate Schrödinger operators on  $\mathbb{R}^n$ .** Let  $n \geq 3$  and  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space endowed with the Euclidean norm  $|\cdot|$  and the Lebesgue measure  $dx$ . Recall that a nonnegative locally integrable function  $w$  is said to be an  $\mathcal{A}_2(\mathbb{R}^n)$  weight in the sense of Muckenhoupt if

$$\sup_B \left\{ \frac{1}{|B|} \int_B w(x) dx \right\}^{1/2} \left\{ \frac{1}{|B|} \int_B [w(x)]^{-1} dx \right\}^{1/2} < \infty,$$

where the supremum is taken over all the balls in  $\mathbb{R}^n$ . See [39] and also [47] for the definition of  $\mathcal{A}_2(\mathbb{R}^n)$  weights and their properties. Observe that if we set  $w(E) \equiv \int_E w(x) dx$  for any measurable set  $E$ , then there exist positive constants  $C, Q$  and  $\kappa$  such that for all  $x \in \mathbb{R}^n, \lambda > 1$  and  $r > 0$ ,

$$C^{-1}\lambda^{\kappa}w(B(x, r)) \leq w(B(x, \lambda r)) \leq C\lambda^Qw(B(x, r));$$

namely, the measure  $w(x) dx$  satisfies (2.1). Thus  $(\mathbb{R}^n, |\cdot|, w(x) dx)$  is an RD-space.

Let  $w \in \mathcal{A}_2(\mathbb{R}^n)$  and  $\{a_{i,j}\}_{1 \leq i,j \leq n}$  be a real symmetric matrix function satisfying that for all  $x, \xi \in \mathbb{R}^n$ ,

$$C^{-1}|\xi|^2 \leq \sum_{1 \leq i,j \leq n} a_{i,j}(x)\xi_i\bar{\xi}_j \leq C|\xi|^2.$$

Then the degenerate elliptic operator  $\mathcal{L}_0$  is defined by

$$\mathcal{L}_0f(x) \equiv -\frac{1}{w(x)} \sum_{1 \leq i,j \leq n} \partial_i(a_{i,j}(\cdot)\partial_jf)(x),$$

where  $x \in \mathbb{R}^n$ . Denote by  $\{\tilde{T}_t\}_{t>0} \equiv \{e^{-t\mathcal{L}_0}\}_{t>0}$  the semigroup generated by  $\mathcal{L}_0$ . We also denote the kernel of  $\tilde{T}_t$  by  $\tilde{T}_t(x, y)$  for all  $x, y \in \mathbb{R}^n$  and  $t > 0$ . Then it is known that there exist positive constants  $C, C_7, \tilde{C}_7$  and  $\alpha \in (0, 1]$  such that for all



$t > 0$  and  $x, y \in \mathbb{R}^n$ ,

$$(5.1) \quad \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{|x-y|^2}{\tilde{C}_7 t} \right\} \leq \tilde{T}_t(x, y) \leq \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{|x-y|^2}{C_7 t} \right\};$$

that for all  $t > 0$  and  $x, y, y' \in \mathbb{R}^n$  with  $|y - y'| < |x - y|/4$ ,

$$(5.2) \quad |\tilde{T}_t(x, y) - \tilde{T}_t(x, y')| \leq \frac{1}{V_{\sqrt{t}}(x)} \left( \frac{|y - y'|}{\sqrt{t}} \right)^\alpha \exp \left\{ -\frac{|x-y|^2}{C_7 t} \right\};$$

and, moreover, that for all  $t > 0$  and  $x, y \in \mathbb{R}^n$ ,

$$(5.3) \quad \int_{\mathbb{R}^n} \tilde{T}_t(x, z) w(z) dz = 1 = \int_{\mathbb{R}^n} \tilde{T}_t(z, y) w(z) dz.$$

See, for example, Theorems 2.1, 2.7, 2.3 and 2.4, and Corollary 3.4 of [28].

Let  $U$  be a nonnegative locally integrable function on  $w(x) dx$ . Define the degenerate Schrödinger operator by  $\mathcal{L} \equiv \mathcal{L}_0 + U$ . Then  $\mathcal{L}$  generates a semigroup  $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$  with kernels  $\{T_t(x, y)\}_{t>0}$  for all  $x, y \in \mathbb{R}^n$ . By Kato-Trotter's product formula (see [29]),  $0 \leq T_t(x, y) \leq \tilde{T}_t(x, y)$  for all  $x, y \in \mathbb{R}^n$  and  $t > 0$ . Define the radial maximal operator  $T^+$  by  $T^+(f)(x) \equiv \sup_{t>0} |e^{-t\mathcal{L}}(f)(x)|$  for all  $x \in \mathbb{R}^n$ . Then  $T^+$  is bounded on  $L^p(w(x) dx)$  for  $p \in (1, \infty]$  and from  $L^1(w(x) dx)$  to weak- $L^1(w(x) dx)$ . The Hardy space associated to  $\mathcal{L}$  is defined by

$$H_{\mathcal{L}}^1(w(x) dx) \equiv \{f \in L^1(w(x) dx) : \|f\|_{H_{\mathcal{L}}^1(w(x), dx)} \equiv \|T^+(f)\|_{L^1(w(x) dx)} < \infty\}.$$

If  $q > Q/2$  and  $U \in \mathcal{B}_q(\mathbb{R}^n, |\cdot|, w(x) dx)$ , letting  $\rho$  be as in (2.3), then Dziubański [15] proved that there exists a positive constant  $C'_7$  such that for all  $x, y \in \mathbb{R}^n$ ,

$$(5.4) \quad 0 \leq T_t(x, y) \leq \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\rho(x) + |x - y|} \right]^N \exp \left\{ -\frac{|x - y|^2}{C'_7 t} \right\}$$

and

$$(5.5) \quad 0 \leq \tilde{T}_t(x, y) - T_t(x, y) \leq C \left[ \frac{\sqrt{t}}{\sqrt{t} + \rho(x)} \right]^{q-Q/2} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{|x - y|^2}{C'_7 t} \right\}.$$

By this, Dziubański [15] proved that  $H_{\mathcal{L}}^1(w(x) dx) = H_{\rho}^{1, \infty}(w(x) dx)$  with equivalent norms via using a different theory of Hardy spaces on spaces of homogeneous type from here.

On the other hand, by Proposition 2.1,  $\rho$  is an admissible function. From (5.1) through (5.3), Remark 2.2 (iii) and the semigroup property, it follows that  $\{\tilde{T}_t\}_{t>0}$  is a continuous  $(1, N, N)_{\rho}$ -AOTI for any  $N > 0$ . Observe that (5.4) and (5.5) implies that  $\{T_t\}_{t>0}$  and  $\{\tilde{T}_t\}_{t>0}$  satisfy the assumptions of Theorem 4.2. Thus by Theorem 4.2 in Section 4,  $H_{\mathcal{L}}^1(w(x) dx) = H_{\rho}^1(w(x) dx)$ . Moreover, all of the results in Sections 3 and 4 are valid for  $H_{\mathcal{L}}^1(w(x) dx)$ . In particular, all the results in Section 3 and Theorem 4.1 are new compared to the known results in [15].

**5.3. Sub-Laplace Schrödinger operators on Heisenberg groups.** The  $(2n + 1)$ -dimensional Heisenberg group  $\mathbb{H}^n$  is a connected and simply connected Lie group with underlying manifold  $\mathbb{R}^{2n} \times \mathbb{R}$  and the multiplication

$$(x, t)(y, s) = \left( x + y, t + s + 2 \sum_{j=1}^n [x_{n+j}y_j - x_jy_{n+j}] \right).$$

The homogeneous norm on  $\mathbb{H}^n$  is defined by  $|(x, t)| = (|x|^4 + |t|^2)^{1/4}$  for all  $(x, t) \in \mathbb{H}^n$ , which induces a left-invariant metric  $d((x, t), (y, s)) = |(-x, -t)(y, s)|$ . Moreover, there exists a positive constant  $C$  such that  $|B((x, t), r)| = Cr^Q$ , where  $Q \equiv (2n + 2)$  is the homogeneous dimension of  $\mathbb{H}^n$  and  $|B((x, t), r)|$  is the Lebesgue measure of the ball  $B((x, t), r)$ . The triplet  $(\mathbb{H}^n, d, dx)$  is an RD-space.

A basis for the Lie algebra of left-invariant vector fields on  $\mathbb{H}^n$  is given by

$$X_{2n+1} = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j} \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

All nontrivial commutators are  $[X_j, X_{n+j}] = 4X_{2n+1}$ ,  $j = 1, \dots, n$ . The sub-Laplacian has the form  $\Delta_{\mathbb{H}^n} = -\sum_{j=1}^{2n} X_j^2$ . See [19, 47] for the theory of the Hardy spaces associated to the sub-Laplacian  $\Delta_{\mathbb{H}^n}$ .

Let  $U$  be a nonnegative locally integrable function on  $\mathbb{H}^n$ . Define the sub-Laplacian Schrödinger operator by  $\mathcal{L} \equiv \Delta_{\mathbb{H}^n} + U$ . Denote by  $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$  the semigroup generated by  $\mathcal{L}$ . Define the Hardy space associated to  $\mathcal{L}$  by

$$H_{\mathcal{L}}^1(\mathbb{H}^n) \equiv \{f \in L^1(\mathbb{H}^n) : \|f\|_{H_{\mathcal{L}}^1(\mathbb{H}^n)} \equiv \|T^+(f)\|_{L^1(\mathbb{H}^n)} < \infty\}.$$

If  $q > Q/2$  and  $U \in \mathcal{B}_q(\mathbb{H}^n)$ , then C. Lin, H. Liu and Y. Liu [34] proved that  $H_{\mathcal{L}}^1(\mathbb{H}^n) = H_{\rho}^{1,q}(\mathbb{H}^n)$  with equivalent norms for all  $q \in (1, \infty]$ , where  $\rho$  is as in (2.3).

On the other hand, Proposition 2.1 implies that  $\rho$  is an admissible function. It is easy to check that  $\{\tilde{T}_{t^2}\}_{t>0} \equiv \{e^{-t^2\Delta_{\mathbb{H}^n}}\}_{t>0}$  is a continuous  $(1, N, N)_{\rho}$ -AOTI for any  $N$ ; see, for example, [19]. C. Lin, H. Liu and Y. Liu [34] proved that  $\{T_{t^2}\}_{t>0}$  and  $\{\tilde{T}_{t^2}\}_{t>0}$  satisfy the assumptions of Theorem 4.2. Thus applying Theorem 4.2, we obtain  $H_{\mathcal{L}}^1(\mathbb{H}^n) = H_{\rho}^1(\mathbb{H}^n)$  with equivalent norms. Moreover, all the results in Sections 3 and 4 are valid for  $H_{\mathcal{L}}^1(\mathbb{H}^n)$ . In this case, compared to the results obtained in [34], Theorem 3.1, Theorem 3.2 (ii), Proposition 3.3 and Theorem 4.1 are new.

**5.4. Sub-Laplace Schrödinger operators on connected and simply connected nilpotent Lie groups.**

Let  $\mathbb{G}$  be a connected and simply connected nilpotent Lie group. Let  $X \equiv \{X_1, \dots, X_k\}$  be left invariant vector fields on  $\mathbb{G}$  satisfying the Hörmander condition. Namely,  $X$ , together with their commutators of order  $\leq m$ , generates the tangent space of  $\mathbb{G}$  at each point of  $\mathbb{G}$ . Let  $d$  be the Carnot-Carathéodory (control) distance on  $\mathbb{G}$  associated to  $X$ . Fix a left invariant Haar measure  $\mu$  on  $\mathbb{G}$ . Then for all  $x \in \mathbb{G}$ ,  $V_r(x) = V_r(e)$ , and moreover, there exist  $0 < \kappa \leq D < \infty$  such that for all  $x \in \mathbb{G}$ ,  $C^{-1}r^{\kappa} \leq V_r(x) \leq Cr^{\kappa}$  when  $0 < r \leq 1$ , and  $C^{-1}r^D \leq V_r(x) \leq Cr^D$  when  $r > 1$ . See [42], [52] and [53] for the details. Thus  $(\mathbb{G}, d, \mu)$  is an RD-space.

The sub-Laplacian is given by  $\Delta_{\mathbb{G}} \equiv -\sum_{j=1}^k X_j^2$ . Denote by  $\{\tilde{T}_t\}_{t>0} \equiv \{e^{-t\Delta_{\mathbb{G}}}\}_{t>0}$  the semigroup generated by  $\Delta_{\mathbb{G}}$ . Then there exist positive constants  $C, C_8$  and  $\tilde{C}_8$  such that for all  $t > 0$  and  $x, y \in \mathbb{G}$ ,

$$(5.6) \quad C^{-1} \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{\tilde{C}_8 t}\right\} \leq \tilde{T}_t(x, y) \leq C \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{C_8 t}\right\},$$

that for all  $t > 0$  and  $x, y, y' \in \mathcal{X}$  with  $d(y, y') \leq d(x, y)/4$ ,

$$(5.7) \quad |\tilde{T}_t(x, y) - \tilde{T}_t(x, y')| \leq C \frac{d(y, y')}{\sqrt{t}} \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{C_8 t}\right\},$$

and moreover, that for all  $t > 0$  and  $x, y \in \mathbb{G}$ ,

$$(5.8) \quad \int_{\mathbb{G}} \tilde{T}_t(x, z) d\mu(z) = 1 = \int_{\mathbb{G}} \tilde{T}_t(z, y) d\mu(z).$$

See, for example, [52] and [53] for the details.

Define the radial maximal operator  $\tilde{T}^+$  by  $\tilde{T}^+(f)(x) \equiv \sup_{t>0} |\tilde{T}_t(f)(x)|$  for all  $x \in \mathbb{G}$ . Then  $\tilde{T}^+$  is bounded on  $L^p(\mathbb{G})$  for  $p \in (1, \infty]$  and from  $L^1(\mathbb{G})$  to weak- $L^1(\mathbb{G})$ . The Hardy space associated to  $\Delta_{\mathbb{G}}$  is defined by

$$H^1(\mathbb{G}) \equiv \left\{ f \in L^1(\mathbb{G}) : \|f\|_{H^1(\mathbb{G})} \equiv \|\tilde{T}^+(f)\|_{L^1(\mathbb{G})} < \infty \right\}.$$

See, for example, [43, 44, 45, 27] for the theory of Hardy spaces associated with the sub-Laplace operator  $\Delta_{\mathbb{G}}$ .

Let  $U$  be a nonnegative locally integrable function on  $\mathbb{G}$ . Then the sub-Laplace Schrödinger operator is defined by  $\mathcal{L} \equiv \Delta_{\mathbb{G}} + U$ . The operator  $\mathcal{L}$  generates a semigroup  $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$  whose kernels are denoted by  $\{T_t(x, y)\}_{t>0}$  for all  $x, y \in \mathbb{G}$ . By Kato-Trotter’s product formula (see [29]),  $0 \leq T_t(x, y) \leq \tilde{T}_t(x, y)$  for all  $t > 0$  and  $x, y \in \mathbb{G}$ . Define the radial maximal operator  $T^+$  by  $T^+(f)(x) = \sup_{t>0} |e^{-t\mathcal{L}}(f)(x)|$  for all  $x \in \mathbb{G}$ . Then  $T^+$  is bounded on  $L^p(\mathbb{G})$  for  $p \in (1, \infty]$  and from  $L^1(\mathbb{G})$  to weak- $L^1(\mathbb{G})$ . The Hardy space associated to  $\mathcal{L}$  is defined by

$$H_{\mathcal{L}}^1(\mathbb{G}) \equiv \{f \in L^1(\mathbb{G}) : \|f\|_{H_{\mathcal{L}}^1(\mathbb{G})} \equiv \|T^+(f)\|_{L^1(\mathbb{G})} < \infty\}.$$

Let  $q > D/2$ ,  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$  and  $\rho(x)$  for all  $x \in \mathbb{G}$  be as in (2.3). Then Li [33] established some basic results concerning  $\mathcal{L}$ , which include estimates for fundamental solutions of  $\mathcal{L}$  and the boundedness on Lebesgue spaces of some operators associated to  $\mathcal{L}$ . To apply the results obtained in Sections 3 and 4 to  $\mathcal{L}$ , we need the following estimates.

**Proposition 5.1.** *Let  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ . Then for each  $N \in \mathbb{N}$  there exists a positive constant  $C$  such that for all  $f \in L^2(\mathbb{G})$  and  $x \in \mathbb{G}$ ,*

$$(5.9) \quad |[\rho(x)]^{-2N} \mathcal{L}^{-N} f(x)| \leq CM^N(f)(x),$$

where  $M$  denotes the Hardy-Littlewood maximal operator on  $\mathbb{G}$  and  $M^N \equiv M \circ \dots \circ M$ .

*Proof.* Let  $G_0$  be the kernel of  $\mathcal{L}^{-1}$ . For any fixed  $N > 0$ , by Theorem 3.6 of [33] there exists a positive constant  $C$  such that for all  $x, y \in \mathbb{G}$ ,

$$(5.10) \quad 0 \leq G_0(x, y) \leq C \frac{[d(x, y)]^2}{[1 + d(x, y)/\rho(x)]^N} \frac{1}{V(x, y)}.$$

By this, for all  $x \in \mathbb{G}$  we have

$$\begin{aligned} & |[\rho(x)]^{-2} \mathcal{L}^{-1} f(x)| \\ & \lesssim [\rho(x)]^{-2} \int_{\mathbb{G}} G_0(x, y) |f(y)| d\mu(y) \\ & \lesssim \int_{\mathbb{G}} \frac{1}{V(x, y)} \frac{[d(x, y)/\rho(x)]^2}{[1 + d(x, y)/\rho(x)]^N} |f(y)| d\mu(y) \\ & \lesssim \left( \sum_{j=1}^{\infty} 2^{-j(N-2)} + \sum_{j=-\infty}^0 2^{2j} \right) \frac{1}{V_{2^{j+1}\rho(x)}(x)} \int_{d(x, y) \leq 2^{j+1}\rho(x)} |f(y)| d\mu(y) \\ & \lesssim M(f)(x), \end{aligned}$$

where  $N > 2$ . Since (2.2) and Lemma 2.1 imply that

$$\frac{1}{\rho(y)} \gtrsim \frac{1}{\rho(x)} \left[ 1 + \frac{d(x, y)}{\rho(x)} \right]^{-k_0/(1+k_0)},$$

thus, by (5.10) we have

$$\begin{aligned} |[\rho(x)]^{-4} \mathcal{L}^{-2} f(x)| &\lesssim [\rho(x)]^{-4} \int_{\mathbb{G}} G_0(x, y) |\mathcal{L}^{-1} f(y)| d\mu(y) \\ &\lesssim \int_{\mathbb{G}} \frac{1}{V(x, y)} \frac{[d(x, y)/\rho(x)]^2}{[1 + d(x, y)/\rho(x)]^N} |M(f)(y)| d\mu(y) \lesssim M^2(f)(x). \end{aligned}$$

Repeating the above arguments then completes the proof of Proposition 5.1.

**Proposition 5.2.** *If  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ , then for any  $N > 0$  there exist positive constants  $C$  and  $C'_s$  such that for all  $t > 0$  and  $x, y \in \mathbb{G}$ ,*

$$0 \leq T_t(x, y) \leq C \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\rho(x) + d(x, y)} \right]^N \exp \left\{ -\frac{[d(x, y)]^2}{C'_s t} \right\}.$$

*Proof.* By (5.6) and  $0 \leq T_t(x, y) \leq \tilde{T}_t(x, y)$  for all  $t > 0$  and  $x, y \in \mathbb{G}$ , we have

$$(5.11) \quad 0 \leq T_t(x, y) \leq \tilde{T}_t(x, y) \lesssim \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{C_s t} \right\}.$$

To prove Proposition 5.2, it suffices to prove that for  $t \geq C[\rho(x)]^2$ ,

$$(5.12) \quad T_t(x, y) \lesssim \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\sqrt{t}} \right]^N.$$

In fact, if this holds, then for  $t \geq C[\rho(x)]^2$ ,

$$T_t(x, y) \lesssim \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\sqrt{t}} \right]^N \lesssim \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\rho(x) + d(x, y)} \right]^N \left[ \frac{\sqrt{t} + d(x, y)}{\sqrt{t}} \right]^N,$$

which together with (5.11) via the geometric mean yields the desired conclusion. For  $t \leq C[\rho(x)]^2$ , since the function  $f(t) = \frac{t}{t+a}$  is increasing in  $t$ , by (5.11) we have

$$T_t(x, y) \lesssim \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\rho(x) + d(x, y)} \right]^N \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\}.$$

To prove (5.12), observe that  $\mathcal{L}$  is self-adjoint. For any  $f \in L^2(\mathbb{G})$ , by the well-known spectral theorem we have

$$(5.13) \quad \|\partial_t^N T_t(f)\|_{L^2(\mathbb{G})} = t^{-N} \|(t\mathcal{L})^N e^{-t\mathcal{L}} f\|_{L^2(\mathbb{G})} \leq C(N)t^{-N} \|f\|_{L^2(\mathbb{G})}.$$

Set  $T_t^{(N)}(x, y) \equiv \partial_s^N T_s(x, y)|_{s=t}$  for all  $x, y \in \mathbb{G}$ . Notice that

$$\begin{aligned} T_{2t}^{(N)}(x, y) &= \partial_s^N T_s(x, y)|_{s=2t} = \partial_s^N T_{t+s}(x, y)|_{s=t} = \partial_s T_s(T_t(x, \cdot))(y)|_{s=t} \\ &= \left( \partial_s^N T_s|_{s=t} \right) (T_t(x, \cdot))(y) = (\partial_t^N T_t)(T_t(x, \cdot))(y), \end{aligned}$$

which together with (5.13) and (5.11) implies that

$$\begin{aligned} \|T_{2t}^{(N)}(x, \cdot)\|_{L^2(\mathbb{G})} &\lesssim \frac{1}{t^N} \|T_t(x, \cdot)\|_{L^2(\mathbb{G})} \\ &\lesssim \frac{1}{t^N} \left\{ \int_{\mathbb{G}} \frac{1}{[V_{\sqrt{t}}(x)]^2} \exp\left\{-\frac{[d(x, y)]^2}{ct}\right\} d\mu(y) \right\}^{1/2} \\ &\lesssim \frac{1}{t^N} \frac{1}{[V_{\sqrt{t}}(x)]^{1/2}} \left\{ \sum_{j=0}^{\infty} 2^{jD/2} \exp\{-2^j\} \right\}^{1/2} \lesssim \frac{1}{t^N} \frac{1}{[V_{\sqrt{t}}(x)]^{1/2}}. \end{aligned}$$

This, together with the Hölder inequality, (5.13) and (5.11), again further yields that

$$|T_{2t}^{(N)}(x, y)| \lesssim \|T_t^{(N)}(x, \cdot)\|_{L^2(\mathbb{G})} \|T_t(x, \cdot)\|_{L^2(\mathbb{G})} \lesssim \frac{1}{t^N} \frac{1}{V_{\sqrt{t}}(x)}.$$

Thus, by Proposition 5.1 we have

$$\begin{aligned} T_t(x, y) &= [\rho(x)]^{2N} |[\rho(x)]^{-2N} \mathcal{L}_y^{-N} \mathcal{L}_y^N(T_t(x, \cdot))(y)| \\ &= [\rho(x)]^{2N} |[\rho(x)]^{-2N} \mathcal{L}_y^{-N} \partial_t^{(N)}(T_t(x, \cdot))(y)| \\ &\lesssim [\rho(x)]^{2N} M^N \left(\partial_t^{(N)} T_t(x, \cdot)\right)(y) \\ &\lesssim [\rho(x)]^{2N} \left\| \partial_t^{(N)} T_t(x, \cdot) \right\|_{L^\infty(\mathbb{G})} \lesssim \frac{1}{V_{\sqrt{t}}(x)} \left[ \frac{\rho(x)}{\sqrt{t}} \right]^{2N}, \end{aligned}$$

which implies (5.12) and hence completes the proof of Proposition 5.2.

For  $t \geq 0$ , set  $E_t \equiv \tilde{T}_t - T_t$ . Also denote by  $E_t$  the kernel of  $E_t$ . Then for all  $x, y \in \mathbb{G}$ ,

$$(5.14) \quad E_t(x, y) = \tilde{T}_t(x, y) - T_t(x, y) = \int_0^t \int_{\mathbb{G}} \tilde{T}_s(x, z) U(z) T_{t-s}(z, y) d\mu(z) ds;$$

see, for example, [15]. To estimate  $E_t$ , we need the following estimate.

**Lemma 5.1.** *If  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ , then for any positive constants  $\tilde{C}$  and  $C'$  there exists a positive constant  $C$  such that for all  $x \in \mathbb{G}$  and  $t > 0$  with  $\sqrt{t} < \tilde{C}\rho(x)$ ,*

$$\int_{\mathbb{G}} \frac{U(z)}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, z)]^2}{C't}\right\} d\mu(z) \leq C \frac{1}{t} \left[ \frac{\sqrt{t}}{\rho(x)} \right]^{2-D/q}.$$

*Proof.* We first recall that Li in [33, Lemma 2.8] proved that there exists a positive constant  $\ell$  such that for all  $x \in \mathbb{G}$  and  $R \geq \rho(x)$ ,

$$\frac{R^2}{V_R(x)} \int_{B(x, R)} U(z) d\mu(z) \lesssim \left[ \frac{R}{\rho(x)} \right]^\ell,$$

which is also easy to be deduced from (2.5) and (2.6). By this, (2.4) and (2.6), letting  $j_0 \in \mathbb{N}$  such that  $2^{j_0-1} \leq \tilde{C}\rho(x)/\sqrt{t} < 2^{j_0}$ , we then have

$$\begin{aligned} & \int_{\mathbb{G}} \frac{U(z)}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, z)]^2}{C't} \right\} d\mu(z) \\ & \lesssim \sum_{j=0}^{\infty} \frac{1}{V_{\sqrt{t}}(x)} e^{-2^j} \int_{d(x, z) < \sqrt{2C'2^j t}} U(z) d\mu(z) \\ & \lesssim \sum_{j=1}^{j_0} \frac{1}{2^j t} e^{-2^j} \left[ \frac{\sqrt{2^j t}}{\rho(x)} \right]^{2-D/q} + \sum_{j=j_0+1}^{\infty} \frac{1}{2^j t} e^{-2^j} \left[ \frac{\sqrt{2^j t}}{\rho(x)} \right]^\ell \lesssim \frac{1}{t} \left[ \frac{\sqrt{t}}{\rho(x)} \right]^{2-D/q}, \end{aligned}$$

which completes the proof of Lemma 5.1.

**Proposition 5.3.** *If  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ , then for each  $N > 0$  there exist positive constants  $C$  and  $C_9$  such that for all  $t > 0$  and  $x, y \in \mathbb{G}$ ,*

$$(5.15) \quad 0 \leq E_t(x, y) \leq C \left[ \frac{\sqrt{t}}{\sqrt{t} + \rho(x)} \right]^{2-D/q} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{C_9 t} \right\}.$$

*Proof.* By (5.11) we have

$$0 \leq E_t(x, y) \lesssim \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\}.$$

Thus, if  $\sqrt{t} \geq C\rho(x)$ , then (5.15) follows from this estimate. If  $\sqrt{t} \geq C\rho(y)$  and  $\sqrt{t} \leq C\rho(x)$ , then by (2.2) and Lemma 2.1 (ii) we obtain

$$1 \lesssim \frac{\sqrt{t}}{\rho(y)} \lesssim \frac{\sqrt{t}}{\rho(x)} \left[ \frac{\rho(x) + d(x, y)}{\rho(x)} \right]^{k_0} \lesssim \frac{\sqrt{t}}{\rho(x)} \left[ \frac{\sqrt{t} + d(x, y)}{\sqrt{t}} \right]^{k_0},$$

which implies that

$$E_t(x, y) \lesssim \left[ \frac{\sqrt{t}}{\rho(x)} \right]^{2-D/q} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{[d(x, y)]^2}{Ct} \right\}.$$

If  $t \leq C[\rho(x) \wedge \rho(y)]$ , then set  $W_1 \equiv \{z \in \mathbb{G} : d(z, x) \geq d(x, y)/2\}$  and  $W_2 \equiv \mathbb{G} \setminus W_1$ . By (5.14) and (5.11) we have

$$\begin{aligned} E_t(x, y) & \leq \left\{ \int_0^{t/2} \int_{W_1} + \int_0^{t/2} \int_{W_2} + \int_{t/2}^t \int_{W_1} + \int_{t/2}^t \int_{W_2} \right\} \frac{1}{V_{\sqrt{s}}(x)} \\ & \quad \times \exp \left\{ -\frac{[d(x, z)]^2}{Cs} \right\} \frac{U(z)}{V_{\sqrt{t-s}}(y)} \exp \left\{ -\frac{[d(z, y)]^2}{C(t-s)} \right\} d\mu(z) ds \\ & \equiv Z_1 + Z_2 + Z_3 + Z_4. \end{aligned}$$

Notice that if  $0 < s < t/2$ , then  $t - s \sim t$ . Then, by Lemma 5.1 we obtain

$$\begin{aligned} Z_1 &\lesssim \frac{1}{V_{\sqrt{t}}(y)} \exp\left\{-\frac{[d(x, y)]^2}{Ct}\right\} \int_0^{t/2} \int_{W_1} \frac{U(z)}{V_{\sqrt{s}}(x)} \exp\left\{-\frac{[d(x, z)]^2}{Cs}\right\} d\mu(z) ds \\ &\lesssim \frac{1}{V_{\sqrt{t}}(y)} \exp\left\{-\frac{[d(x, y)]^2}{Ct}\right\} \int_0^{t/2} \frac{1}{s} \left[\frac{\sqrt{s}}{\rho(x)}\right]^{2-D/q} ds \\ &\lesssim \left[\frac{\sqrt{t}}{\rho(x)}\right]^{2-D/q} \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{Ct}\right\}. \end{aligned}$$

If  $0 < s < t/2$  and  $z \in W_2$ , then  $t \sim t - s$  and  $d(y, z) \geq d(x, y) - d(x, z) \geq d(x, y)/2$ , which, together with Lemma 5.1, implies that

$$\begin{aligned} Z_2 &\lesssim \frac{1}{V_{\sqrt{t}}(y)} \exp\left\{-\frac{[d(x, y)]^2}{Ct}\right\} \int_0^{t/2} \int_{W_2} \frac{U(z)}{V_{\sqrt{s}}(x)} \exp\left\{-\frac{[d(x, z)]^2}{Cs}\right\} d\mu(z) ds \\ &\lesssim \left[\frac{\sqrt{t}}{\rho(x)}\right]^{2-D/q} \frac{1}{V_{\sqrt{t}}(x)} \exp\left\{-\frac{[d(x, y)]^2}{Ct}\right\}. \end{aligned}$$

The estimates for  $Z_3$  and  $Z_4$  are similar, and we omit the details. This completes the proof of Proposition 5.3.

By Proposition 2.1,  $\rho$  as in (2.3) is an admissible function. From (5.6), (5.7), (5.8) and Remark 2.2 (iii), together with the semigroup property of  $\{\tilde{T}_t\}_{t>0}$ , it follows that, for any  $N > 0$ ,  $\{\tilde{T}_{t^2}\}_{t>0}$  is a  $(1, N, N)$ -AOTI. Moreover, Propositions 5.2 and 5.3 imply that assumptions (i) and (ii) of Theorem 4.2 hold for  $\{\tilde{T}_{t^2}\}_{t>0}$  and  $\{T_{t^2}\}_{t>0}$ . Applying the results obtained in Sections 3 and 4 directly to  $\mathcal{L}$ , we have the following conclusions.

**Theorem 5.1.** *Let  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ . If  $f \in H^1_{\mathcal{L}}(\mathbb{G})$ , then  $f \in L^1(\mathbb{G})$  and  $K_{\rho}(f) - f \in H^1(\mathbb{G})$ . Moreover, there exists a positive constant  $C$  such that for all  $f \in H^1_{\mathcal{L}}(\mathbb{G})$ ,  $\|K_{\rho}(f) - f\|_{H^1(\mathbb{G})} \leq C\|f\|_{H^1_{\rho}(\mathbb{G})}$ .*

**Theorem 5.2.** *If  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ , then the following are equivalent:*

- (i)  $f \in H^1_{\mathcal{L}}(\mathbb{G})$ ;
- (ii)  $f, T^+_{\rho}(f) \in L^1(\mathbb{G})$ , where  $T^+_{\rho}$  is defined as in Remark 4.2 with  $\tilde{S}_t$  replaced by  $T_{t^2}$ ;
- (iii)  $f \in H^1_{\rho}(\mathbb{G})$ ;
- (iv) there exists  $r \in (1, \infty]$  such that  $f \in H^{1,r}_{\rho}(\mathbb{G})$ ;
- (v) there exist  $\epsilon \in (0, 1)$  and  $\beta, \gamma \in (0, \epsilon)$  such that  $f \in (\mathcal{G}^{\epsilon}_0(\beta, \gamma))'$  and  $\tilde{T}^+_{\rho}(f) \in L^1(\mathbb{G})$ , where  $\tilde{T}^+_{\rho}$  is defined as in Remark 4.2 with  $\tilde{S}_t$  replaced by  $\tilde{T}_{t^2}$ .

Moreover, if  $r \in (1, \infty]$ , then for all  $f \in L^1(\mathbb{G})$ ,

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{G})} \sim \|T^+_{\rho}(f)\|_{L^1(\mathbb{G})} \sim \|\tilde{T}^+_{\rho}(f)\|_{L^1(\mathbb{G})} \sim \|G_{\rho}(f)\|_{L^1(\mathbb{G})} \sim \|f\|_{H^{1,r}_{\rho}(\mathbb{G})}.$$

**Theorem 5.3.** *Let  $q > D/2$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ . If  $r \in (1, \infty]$ , then  $\|\cdot\|_{H^{1,r}_{\rho, \text{fin}}(\mathbb{G})}$  and  $\|\cdot\|_{H^1_{\mathcal{L}}(\mathbb{G})}$  are equivalent on  $H^{1,r}_{\rho, \text{fin}}(\mathbb{G})$ .*

Moreover, applying Proposition 3.2 to the Riesz transforms  $\nabla\mathcal{L}^{-1}$ , we have the following conclusion.

**Theorem 5.4.** *If  $q \in (D/2, D)$  and  $U \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ , then Riesz transforms  $\nabla\mathcal{L}^{-1/2}$  are bounded from  $H^1_{\mathcal{L}}(\mathbb{G})$  to  $L^1(\mathbb{G})$ .*

*Proof.* It has been proved in [33, Theorem C] that  $\nabla\mathcal{L}^{-1/2}$  is bounded on  $L^{p_1}(\mathbb{G})$  for any  $p_1 \in (1, p)$ , where  $1/p = 1/q - 1/D$ . By Proposition 3.2, it suffices to prove that for all  $(1, 2)_\rho$ -atoms  $a$ ,  $\|\nabla\mathcal{L}^{-1/2}(a)\|_{L^1(\mathbb{G})} \lesssim 1$ .

Let  $K$  and  $\tilde{K}$  be the integral kernels of  $\nabla\mathcal{L}^{-1/2}$  and  $\nabla\Delta_{\mathbb{G}}^{-1/2}$ , respectively. Let  $1/p_1 + 1/p'_1 = 1$ . Then Li [33] proved that for all  $f \in L^{p'_1}_{loc}(\mathbb{G})$  and  $x \in \mathbb{G}$ ,

$$(5.16) \quad \int_{d(x,y) > \rho(x)} |K(y, x)||f(y)| d\mu(y) \lesssim [M(|f|^{p'_1})(x)]^{1/p'_1}$$

(see Lemma 6.1, Corollary 6.2 and their proofs therein) and that

$$(5.17) \quad \int_{d(x,y) \leq \rho(x)} |K(y, x) - \tilde{K}(y, x)||f(y)| d\mu(y) \lesssim [M(|f|^{p'_1})(x)]^{1/p'_1}$$

(see Lemma 6.4 and its proofs therein). Let  $C$  be a positive constant such that  $1/2 \leq C_3C^{-1/(1+k_0)}(1 + 1/C)^{k_0/(1+k_0)} < 1$ . If  $d(x, y) > C\rho(x)$ , then by (2.2) we have

$$(5.18) \quad \rho(y) \leq C_3C^{-1/(1+k_0)}(1 + 1/C)^{k_0/(1+k_0)}d(x, y) < d(x, y).$$

Let  $\eta$  be as in (2.8), and for all  $x \in \mathbb{G}$  set

$$A_1(f)(x) \equiv \int_{\mathbb{G}} K(x, y) \left[1 - \eta\left(\frac{d(x, y)}{C\rho(x)}\right)\right] f(y) d\mu(y),$$

$$A_2(f)(x) \equiv \int_{\mathbb{G}} [K(x, y) - \tilde{K}(x, y)]\eta\left(\frac{d(x, y)}{C\rho(x)}\right) f(y) d\mu(y),$$

and

$$A_3(f)(x) \equiv T(f) - A_1(f)(x) - A_2(f)(x) \equiv \int_{\mathbb{G}} \tilde{K}(x, y)\eta\left(\frac{d(x, y)}{C\rho(x)}\right) f(y) d\mu(y).$$

Then by (5.18),

$$|A_1(f)(x)| \leq \int_{d(x,y) > C\rho(x)} |K(x, y)||f(y)| d\mu(y) \leq \int_{d(x,y) > \rho(y)} |K(x, y)||f(y)| d\mu(y),$$

which, together with the duality, (5.16) and the boundedness of the Hardy-Littlewood maximal operator  $M$ , implies that  $\|A_1(f)\|_{L^{p_1}(\mathbb{G})} \lesssim \|f\|_{L^{p_1}(\mathcal{X})}$  for any  $p_1 \in (1, p)$ . Moreover, for all  $(1, 2)_\rho$ -atoms  $a$ , by (5.16)

$$\|A_1(a)\|_{L^1(\mathcal{X})} \lesssim \int_{\mathbb{G}} |a(y)| \int_{d(x,y) > \rho(y)} |K(x, y)| d\mu(x) d\mu(y) \lesssim \|a\|_{L^1(\mathbb{G})} \lesssim 1,$$

which, together with Proposition 3.2, implies that  $A_1$  is bounded from  $H^1_\rho(\mathbb{G})$  to  $L^1(\mathbb{G})$ .

Similarly, we have

$$|A_2(f)(x)| \leq \int_{d(x,y) \leq \rho(y)} |K(x, y) - \tilde{K}(x, y)||f(y)| d\mu(y)$$

$$+ \int_{\rho(y) \leq d(x,y) < 2C\rho(x)} |K(x, y)||f(y)| d\mu(y)$$

$$+ \int_{\rho(y) \leq d(x,y) < 2C\rho(x)} \frac{1}{V(x, y)}|f(y)| d\mu(y).$$



By duality, the boundedness of  $M$ , and  $\rho(x) \sim \rho(y)$  when  $d(x, y) < \rho(x)$ , we have  $\|A_2(f)\|_{L^{p_1}(\mathbb{G})} \lesssim \|f\|_{L^{p_1}(\mathcal{X})}$  for any  $p_1 \in (1, p)$ . Moreover, for all  $(1, 2)_\rho$ -atoms  $a$ , by Lemma 2.1 (i), (5.15) and (5.17), we have

$$\begin{aligned} \|A_2(a)\|_{L^1(\mathcal{X})} &\lesssim \int_{\mathbb{G}} |a(y)| \int_{d(x, y) < \rho(y)} |K(x, y) - \tilde{K}(x, y)| d\mu(x) d\mu(y) \\ &\quad + \int_{\mathbb{G}} |a(y)| \int_{\rho(y) \leq d(x, y) < 2C\rho(x)} |K(x, y)| d\mu(x) d\mu(y) \\ &\quad + \int_{\mathbb{G}} |a(y)| \int_{\rho(y) \leq d(x, y) < 2C\rho(x)} \frac{1}{V(x, y)} d\mu(x) d\mu(y) \lesssim \|a\|_{L^1(\mathbb{G})} \lesssim 1, \end{aligned}$$

which, together with Proposition 3.2, implies that  $A_2$  is bounded from  $H_\rho^1(\mathbb{G})$  to  $L^1(\mathbb{G})$ . Here we used the fact that  $\tilde{K}$  satisfies (K1); see [32, 1].

Obviously, for any  $p_1 \in (1, p)$ ,  $A_3$  is bounded on  $L^{p_1}(\mathcal{X})$ . Moreover, for any  $f \in L_b^\infty(\mathbb{G})$  and  $x \notin \text{supp } f$ ,

$$A_3(f)(x) = \int_{\mathbb{G}} \tilde{K}(x, y) \eta\left(\frac{d(x, y)}{C\rho(x)}\right) f(y) d\mu(y).$$

Since  $\tilde{K}$  satisfies conditions (K1) and (K2) (see [32, 1] again), by Proposition 3.3  $A_3$  is also bounded from  $H_\rho^1(\mathbb{G})$  to  $L^1(\mathbb{G})$ . Thus, by Theorem 5.2,  $\nabla \mathcal{L}^{-1/2}$  are bounded from  $H_{\mathcal{L}}^1(\mathbb{G})$  to  $L^1(\mathbb{G})$ , which completes the proof of Theorem 5.4.

#### ACKNOWLEDGEMENTS

The first author would like to thank Professor Jacek Dziubański for some stimulating conversations on this subject and Professor Heping Liu for providing his preprint [34] and some useful discussions. Both authors would like to thank Professor Wengu Chen for some useful advice on this subject and also Doctor Liguang Liu for her improvement of Proposition 3.1. They would also like to thank the referee for many valuable remarks which improved the presentation of this article.

#### REFERENCES

- [1] G. Alexopoulos, An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth, *Canad. J. Math.* 44 (1992), 691-727. MR1178564 (93j:22006)
- [2] P. Auscher and B. B. Ali, Maximal inequalities and Riesz transform estimates on  $L^p$  spaces for Schrödinger operators with nonnegative potentials, *Ann. Inst. Fourier (Grenoble)* 57 (2007), 1975-2013. MR2377893 (2009h:35067)
- [3] M. Bownik, Boundedness of operators on Hardy spaces via atomic decompositions, *Proc. Amer. Math. Soc.* 133 (2005), 3535-3542. MR2163588 (2006d:42028)
- [4] M. Christ, A  $T(b)$  theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* 60/61 (1990), 601-628. MR1096400 (92k:42020)
- [5] R. R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures Appl.* (9) 72 (1993), 247-286. MR1225511 (95d:46033)
- [6] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certain Espaces Homogènes*, Lecture Notes in Math. 242, Springer, Berlin, 1971. MR0499948 (58:17690)
- [7] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83 (1977), 569-645. MR0447954 (56:6264)
- [8] G. David, J. L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-acrétives et interpolation*, *Rev. Mat. Ibero.* 1 (1985) 1-56. MR850408 (88f:47024)
- [9] X. T. Duong and L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, *J. Amer. Math. Soc.* 18 (2005), 943-973. MR2163867 (2006d:42037)

- [10] J. Dziubański, Atomic decomposition of  $H^p$  spaces associated with some Schrödinger operators, *Indiana Univ. Math. J.* 47 (1998), 75-98. MR1631616 (99j:46030)
- [11] J. Dziubański and J. Zienkiewicz, Hardy space  $H^1$  associated to Schrödinger operator with potential satisfying reverse Hölder inequality, *Rev. Mat. Ibero.* 15 (1999), 279-296. MR1715409 (2000j:47085)
- [12] J. Dziubański and J. Zienkiewicz,  $H^p$  spaces for Schrödinger operators, *Fourier analysis and related topics (Bédlewo, 2000)*, 45-53, Banach Center Publ., 56, Polish Acad. Sci., Warsaw, 2002. MR1971563 (2004f:42035)
- [13] J. Dziubański and J. Zienkiewicz,  $H^p$  spaces associated with Schrödinger operators with potentials from reverse Hölder classes, *Colloq. Math.* 98 (2003), 5-38. MR2032068 (2004k:42038)
- [14] J. Dziubański and J. Zienkiewicz, Hardy spaces  $H^1$  for Schrödinger operators with certain potentials, *Studia Math.* 164 (2004), 39-53. MR2079769 (2005f:47097)
- [15] J. Dziubański, Note on  $H^1$  spaces related to degenerate Schrödinger operators, *Illinois J. Math.* 49 (2005), 1271-1297. MR2210363 (2006m:42037)
- [16] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea and J. Zienkiewicz,  $BMO$  spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality, *Math. Z.* 249 (2005), 329-356. MR2115447 (2005k:35064)
- [17] C. Fefferman, The uncertainty principle, *Bull. Amer. Math. Soc. (N. S.)* 9 (1983), 129-206. MR707957 (85f:35001)
- [18] C. Fefferman and E. M. Stein,  $H^p$  spaces of several variables, *Acta Math.* 129 (1972), 137-193. MR0447953 (56:6263)
- [19] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, Princeton, N. J., 1982. MR657581 (84h:43027)
- [20] L. Grafakos, *Modern Fourier Analysis*, Second Edition, Graduate Texts in Math., No. 250, Springer, New York, 2008. MR2463316
- [21] L. Grafakos, L. Liu and D. Yang, Maximal function characterizations of Hardy spaces on RD-spaces and their applications, *Sci. China Ser. A* 51 (2008), 2253-2284. MR2462027 (2009j:42017)
- [22] L. Grafakos, L. Liu and D. Yang, Radial maximal function characterizations for Hardy spaces on RD-spaces, *Bull. Soc. Math. France* 137 (2009), 225-251. MR2543475 (2010f:42034)
- [23] F. W. Gehring, The  $L^p$ -integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* 130 (1973), 265-277. MR0402038 (53:5861)
- [24] D. Goldberg, A local version of real Hardy spaces, *Duke Math. J.* 46 (1979), 27-42. MR523600 (80h:46052)
- [25] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001. MR1800917 (2002c:30028)
- [26] Y. Han, D. Müller and D. Yang, Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type, *Math. Nachr.* 279 (2006), 1505-1537. MR2269253 (2007g:42035)
- [27] Y. Han, D. Müller and D. Yang, A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces, *Abstr. Appl. Anal.* 2008, Art. ID 893409, 250 pp. MR2485404
- [28] W. Hebisch and L. Saloff-Coste, On the relation between elliptic and parabolic Harnack inequalities, *Ann. Inst. Fourier (Grenoble)* 51 (2001), 1437-1481. MR1860672 (2002g:58024)
- [29] T. Kato, Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups, *Topics in functional analysis (essays dedicated to M. G. Kreĭn on the occasion of his 70th birthday)*, pp. 185-195, *Adv. in Math. Suppl. Stud.*, 3, Academic Press, New York, 1978. MR538020 (80i:47040)
- [30] K. Kurata and S. Sugano, Fundamental solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials, *Studia Math.* 138 (2000), 101-119. MR1749075 (2001c:35172)
- [31] K. Kurata and S. Sugano, A remark on estimates for uniformly elliptic operators on weighted  $L^p$  spaces and Morrey spaces, *Math. Nachr.* 209 (2000), 137-150. MR1734362 (2000k:35064)
- [32] N. Lohoué and N. Th. Varopoulos, Remarques sur les transformées de Riesz sur les groupes de Lie nilpotents, *C. R. Acad. Sci. Paris Sér. I Math.* 301 (1985), 559-560. MR816628 (87b:43008)
- [33] H. Li, Estimations  $L^p$  des opérateurs de Schrödinger sur les groupes nilpotents, *J. Funct. Anal.* 161 (1999), 152-218. MR1670222 (2000e:58030)
- [34] C. Lin, H. Liu and Y. Liu, The Hardy space  $H_L^1$  associated with Schrödinger operators on the Heisenberg group, Submitted.

- [35] G. Lu, A Fefferman-Phong type inequality for degenerate vector fields and applications, *Panamer. Math. J.* 6 (1996), 37-57. MR1414658 (97h:35068)
- [36] R. A. Macías and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. in Math.* 33 (1979) 257-270. MR546295 (81c:32017a)
- [37] R. A. Macías and C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, *Adv. in Math.* 33 (1979) 271-309. MR546296 (81c:32017b)
- [38] S. Meda, P. Sjögren and M. Vallarino, on the  $H^1 - L^1$  boundedness of operators, *Proc. Amer. Math. Soc.* 136 (2008), 2921-2931. MR2399059 (2009b:42025)
- [39] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* 165 (1972), 207-226. MR0293384 (45:2461)
- [40] A. Nagel and E. M. Stein, On the product theory of singular integrals, *Rev. Mat. Ibero.* 20 (2004), 531-561; Corrigenda, *Rev. Mat. Ibero.* 21 (2005), 693-694. MR2073131 (2006i:42023)
- [41] A. Nagel and E. M. Stein, The  $\bar{\partial}_b$ -complex on decoupled boundaries in  $\mathbb{C}^n$ , *Ann. of Math.* (2) 164 (2006), 649-713. MR2247970 (2007d:32036)
- [42] A. Nagel, E. M. Stein and S. Wainger, Balls and metrics defined by vector fields I. Basic properties, *Acta Math.* 155 (1985), 103-147. MR793239 (86k:46049)
- [43] L. Saloff-Coste, Analyse sur les groupes de Lie nilpotents, *C. R. Acad. Sci. Paris Sér. I Math.* 302 (1986), 499-502. MR844151 (88j:22008)
- [44] L. Saloff-Coste, Fonctions maximales sur certains groupes de Lie, *C. R. Acad. Sci. Paris Sér. I Math.* 305 (1987), 457-459. MR916309 (89a:46080)
- [45] L. Saloff-Coste, Analyse sur les groupes de Lie à croissance polynômiale, *Ark. Mat.* 28 (1990), 315-331. MR1084020 (92d:22014)
- [46] Z. Shen,  $L^p$  estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier (Grenoble)* 45 (1995), 513-546. MR1343560 (96h:35037)
- [47] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, N. J., 1993. MR1232192 (95c:42002)
- [48] E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of  $H^p$ -spaces, *Acta Math.* 103 (1960), 25-62. MR0121579 (22:12315)
- [49] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics, 1381, Springer-Verlag, Berlin, 1989. MR1011673 (90j:42053)
- [50] H. Triebel, *Theory of Function Spaces III*, Birkhäuser Verlag, Basel, 2006. MR2250142 (2007k:46058)
- [51] A. Uchiyama, A maximal function characterization of  $H^p$  on the space of homogeneous type, *Trans. Amer. Math. Soc.* 262 (1980), 579-592. MR586737 (81j:46085)
- [52] N. Th. Varopoulos, Analysis on Lie groups, *J. Funct. Anal.* 76 (1988), 346-410. MR924464 (89i:22018)
- [53] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Univ. Press, New York, 1992. MR1218884 (95f:43008)
- [54] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge University Press, Cambridge, 1991. MR1144277 (93d:46001)
- [55] D. Yang and Y. Zhou, A boundedness criterion via atoms for linear operators in Hardy spaces, *Constr. Approx.* 29 (2009), 207-218. MR2481589
- [56] D. Yang and Y. Zhou, New properties of Besov and Triebel-Lizorkin spaces on RD-spaces, *Manuscripta Math.* (to appear), DOI: 10.1007/s00229-010-0384-y.
- [57] J. Zhong, The Sobolev estimates for some Schrödinger type operators, *Math. Sci. Res. Hot-Line* 3:8 (1999), 1-48. MR1717791 (2000k:35065)

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS, MINISTRY OF EDUCATION, BEIJING 100875, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* dcyang@bnu.edu.cn

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS, MINISTRY OF EDUCATION, BEIJING 100875, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* yuanzhou@mail.bnu.edu.cn