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# LOCALIZED INDEX AND $L^2$-LEFSCHETZ FIXED-POINT FORMULA FOR ORBIFOLDS

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## Abstract

We study a class of localized indices for the Dirac type operators on a complete Riemannian orbifold, where a discrete group acts properly, co-compactly, and isometrically. These localized indices, generalizing the $L^2$-index of Atiyah, are obtained by taking certain traces of the higher index for the Dirac type operators along conjugacy classes of the discrete group, subject to some trace assumption. Applying the local index technique, we also obtain an $L^2$-version of the Lefschetz fixed-point formulas for orbifolds. These cohomological formulas for the localized indices give rise to a class of refined topological invariants for the quotient orbifold.

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## 1. Introduction

The Lefschetz fixed-point formula calculates the supertrace of the action of a diffeomorphism on a closed Riemannian manifold with isolated fixed points. This was generalized by Atiyah and Bott in [**AB**] to an elliptic complex with a geometric endomorphism defined by a diffeomorphism with isolated fixed points. When the diffeomorphism $\gamma$ comes from a *compact* group $H$ of orientation-preserving isometries of a *compact* even-dimensional manifold $X$, the fixed-point formulas are a special case of the equivariant index formulas for the equivariant index

$$\operatorname{ind}_H(\gamma, \not{D}) = \operatorname{Tr}(\gamma|_{\ker \not{D}^+}) - \operatorname{Tr}(\gamma|_{\ker \not{D}^+}) \tag{1.1}$$

assocaited to an $H$-invariant Dirac operator $\not{D}$ on an equivariant Clifford module $\mathcal{E}$. The local index formula, also called the Lefschetz fixed-point formula,

$$\operatorname{ind}_H(\gamma, \not{D}) = \int_{X^\gamma} \hat{A}_\gamma(X) \operatorname{ch}_\gamma(\mathcal{E}/\mathcal{S}), \tag{1.2}$$

is obtained by an asymptotic expansion of the equivariant heat kernel to the operator $\gamma e^{-t\not{D}^2}$. Here, $X^\gamma$ is the fixed-point set of the $\gamma$-action, consisting of closed submanifolds of $X$, and $\hat{A}_\gamma(X) \operatorname{ch}_\gamma(\mathcal{E}/\mathcal{S})$ represents the local index density for the equivariant index. See [**BGV**, Chapter 6] for a detailed account of these developments. When $X$ is not compact, the equivariant index is not defined and there does not exist any Lefschetz fixed-point formula. We remark that the Lefschetz fixed-point formulas for compact orbifolds have been established in [**Dui**, **FT**, **EEK**].

To motivate our study, let us consider a compact even-dimensional Riemannian manifold $M$ with an isometric action of a *finite* group $H$. Let $\not{D}$ be an $H$-invariant Dirac operator on an equivariant Clifford module $\mathcal{E}$. Then $\not{D}$ defines a Dirac operator $\not{D}_{\mathfrak{X}}$ on the quotient orbifold $\mathfrak{X} = H\backslash M$. An orbifold locally looks like Euclidean space equipped with a finite group action and is a basic geometric model used in many areas such as mathematical physics, algebraic geometry, representation theory, and number theory. Index theory of Dirac type operators on orbifolds is an operator algebraic approach in detecting the topological, geometrical, and arithmetic information concerning the orbifolds.

The Kawasaki orbifold index for $\not{D}_{\mathfrak{X}}$ in [**K**] for the finite quotient orbifold $\mathfrak{X} = H\backslash M$ can be obtained by applying the Lefschetz fixed-point formula (the local equivariant index formula) for $\operatorname{ind}_H(\gamma, \not{D})$ to the following identity:

$$\operatorname{ind} \not{D}_{\mathfrak{X}} = \frac{1}{|H|} \sum_{\gamma \in H} \operatorname{ind}_H(\gamma, \not{D}) = \sum_{(\gamma) \in \langle H \rangle} \frac{1}{|Z_H(\gamma)|} \operatorname{ind}_H(\gamma, \not{D}). \tag{1.3}$$

Here, the second summand is taken over representatives for the conjugacy classes $\langle H \rangle$ in $H$. Applying the Lefschetz fixed-point formula (1.2), this is exactly the Kawasaki orbifold index for the Dirac operator $\not{D}_{\mathfrak{X}}$ on the orbifold $\mathfrak{X}$. From the view point of topology, the formula (1.3) for $\not{D}_{\mathfrak{X}}$ follows from its analogue of calculating the equivariant $K$-theory of the compact proper $H$-manifold $M$ using the extended quotient $\sqcup_{(\gamma)} Z_H(\gamma)\backslash M^\gamma$ indexed by the conjugacy classes of the finite group $H$ (see [**BC1**, **EE**]):

$$K_H^*(M) \otimes \mathbb{C} \cong \oplus_{(\gamma)} K^*(Z_H(\gamma)\backslash M^\gamma) \otimes \mathbb{C}. \tag{1.4}$$

In this paper, we are interested in the index problems for a Dirac type operator on an orbifold that is not necessarily compact. Namely, we consider a complete even-dimensional Riemannian orbifold $\mathfrak{X}$ where a discrete group $G$ acts properly, co-compactly, and isometrically. Let $\not{D}^{\mathcal{E}}$ be a $G$-invariant Dirac type operator on $\mathfrak{X}$ acting on the $L^2$-sections of a Dirac bundle $\mathcal{E}$ (cf. Definition 2.9), and let $\not{D}^{G\backslash\mathcal{E}}$ be the Dirac operator $\not{D}^{\mathcal{E}}$ passed from $\mathfrak{X}$ to the compact quotient orbifold $G\backslash\mathfrak{X}$. We study the index theory of $\not{D}^{\mathcal{E}}$ as well as its relation to $\not{D}^{G\backslash\mathcal{E}}$, based on the following facts when $\mathfrak{X}$ is a manifold.

1) The $G$-invariant elliptic operator $\not{D}^{\mathcal{E}}$ defines a cycle in the $K$-homology group of $C_0(\mathfrak{X})$ and is "Fredholm" with respect to the $C^*$-algebra $C^*(G)$ of the discrete group $G$. In other words, $\not{D}^{\mathcal{E}}$ has a higher index in the $K$-theory of the maximal group $C^*$-algebra of $G$ (see, for example, [**K1**]):

$$\mu : K_G^0(C_0(\mathfrak{X})) \longrightarrow K_0(C^*(G)) \qquad [\not{D}^{\mathcal{E}}] \mapsto \mu[\not{D}^{\mathcal{E}}]. \tag{1.5}$$

2) The Dirac operator $\not{D}^{G\backslash\mathcal{E}}$ on the quotient orbifold $G\backslash\mathfrak{X}$ has a Fredholm index according to the work of Kawasaki [**K**] and is related to the higher index $\mu[\not{D}^{\mathcal{E}}]$ of $\not{D}^{\mathcal{E}}$ by composing the homomorphism $\rho : C^*(G) \to \mathbb{C}$, given by the trivial representation of $G$, on the $K$-theory level (see [**Bu**]):

$$\operatorname{ind} \not{D}^{G\backslash\mathcal{E}} = \rho_*(\mu[\not{D}^{\mathcal{E}}]) \in \mathbb{Z}.$$

3) The $L^2$-index is an important topological invariant for a proper co-compact $G$-manifold. It was initially defined by Atiyah in [**At**] for a manifold admitting a free co-compact action of a discrete group. The $L^2$-index for $\not{D}^{\mathcal{E}}$ is a numerical index measuring the size of the space of $L^2$-solutions for $\not{D}^{\mathcal{E}}$. In [**W**], we showed that the $L^2$-index of $\not{D}^{\mathcal{E}}$ follows from taking the von Neumann trace of

the higher index (1.5):

$$L^2\text{- ind}\, \not{D}^{\mathcal{E}} = \tau_*(\mu[\not{D}^{\mathcal{E}}]) \in \mathbb{R}, \tag{1.6}$$

where $\tau_* : K_0(C^*(G)) \to \mathbb{R}$ is induced by the trace map

$$\tau : C^*(G) \longrightarrow \mathbb{C} \qquad \sum_{g \in G} \alpha_g g \mapsto \alpha_e.$$

Moreover, by the $L^2$-index formula derived in [**W**], the $L^2$-index of $\not{D}^{\mathcal{E}}$ is the top stratum of the Kawasaki index formula for $\not{D}^{G\backslash\mathcal{E}}$.

The paper is devoted to answering the following concrete questions for the index theory for $\not{D}^{\mathcal{E}}$:

1) How do we formulate the higher index and $L^2$-index in the setting when $\mathfrak{X}$ is a non-compact orbifold?
2) How is the orbifold index ind $\not{D}^{G\backslash\mathcal{E}}$ related to the $L^2$-index for $\not{D}^{\mathcal{E}}$?
3) Is there any Lefschetz fixed-point formula for a *non-compact* orbifold with a transformation preserving all the geometric data for the Dirac type operator?

The main results of the paper are then summarized as follows.

First of all, we have a positive answer to the first question, that is, the indices $\mu[\not{D}^{\mathcal{E}}]$, ind $\not{D}^{G\backslash\mathcal{E}}$, and $L^2$-ind $\not{D}^{\mathcal{E}}$ can be formulated when $\mathfrak{X}$ is a complete Riemannian orbifold admitting a proper, co-compact and isometric action of a discrete group $G$. Namely, we construct in (4.7) a higher index map

$$\mu : K^0_G(C^*_{red}(\mathfrak{X})) \longrightarrow K_0(C^*(G)) \qquad [\not{D}^{\mathcal{E}}] \mapsto \mu[\not{D}^{\mathcal{E}}],$$

where the Dirac operator $\not{D}^{\mathcal{E}}$ on the $G$-orbifold $\mathfrak{X}$ gives rise to an equivariant $K$-homology class of the reduced $C^*$-algebra $C^*_{red}(\mathfrak{X})$; see Lemma 4.7. The orbifold index ind $\not{D}^{G\backslash\mathcal{E}}$ of the quotient $G\backslash\mathfrak{X}$ is related to the higher index by the trivial representation of $G$; see Theorem 4.13.

Second, in searching for a relationship between the orbifold index of $\not{D}^{G\backslash\mathcal{E}}$ and the $L^2$-index of ind $\not{D}^{\mathcal{E}}$, we find it is necessary to introduce the notion of localized indices for $\not{D}^{\mathcal{E}}$, which is a major novelty of our paper. See Definition 5.7. For each conjugacy class $(g)$ of $g$ in $G$, the localized $(g)$-index of $\not{D}^{\mathcal{E}}$, denoted by $\mathrm{ind}_{(g)}(\not{D}^{\mathcal{E}})$, is defined as the pairing between the Banach algebra version higher index $\mu_{\mathcal{S}(G)}[\not{D}^{\mathcal{E}}]$ (cf. Definition 4.5) and a localized $(g)$-trace $\tau^{(g)}$, which is a cyclic cocycle of degree 0 given by the sum of the coefficients over elements in the conjugacy class $(g)$ for an element in the group algebra $\mathbb{C}G$ (cf. Definition 5.2). It then follows from Theorem 5.12 that the higher index $\mu_{\mathcal{S}(G)}[\not{D}^{\mathcal{E}}]$ is linked to the numerical indices ind $\not{D}^{G\backslash\mathcal{E}}$ and $L^2$-ind $\not{D}^{\mathcal{E}}$ by the localized indices as follows:

1) The localized indices factor through the Banach algebra version higher index by definition:

$$\operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}) = \tau_*^{(g)}(\mu_{\mathcal{S}(G)}[\not{D}^{\mathcal{E}}]). \tag{1.7}$$

In particular, the nonvanishing of the localized $(g)$-index implies the existence of $L^2$-solutions invariant under the action of $g \in G$.

2) The localized $(e)$-index generalizes the $L^2$-index and factors through the *usual* higher index $\mu[\not{D}^{\mathcal{E}}]$:

$$L^2\text{- ind}\,\not{D}^{\mathcal{E}} = \operatorname{ind}_{(e)}(\not{D}^{\mathcal{E}}) = \tau_*^{(e)}(\mu[\not{D}^{\mathcal{E}}]).$$

3) When the group $G$ satisfies some trace properties, the integer-valued orbifold index for $\not{D}^{G\backslash\mathcal{E}}$ on $G\backslash\mathfrak{X}$ is the sum of localized indices for $\not{D}^{\mathcal{E}}$ over all conjugacy classes (cf. Theorem 5.10):

$$\operatorname{ind}\not{D}^{G\backslash\mathcal{E}} = \sum_{(g)\in G} \operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}). \tag{1.8}$$

On the one hand, localized indices (1.7) give rise to a $K$-theoretic interpretation of each component of the orbifold index for $\not{D}^{G\backslash\mathfrak{X}}$ on the quotient orbifold $G\backslash\mathfrak{X}$. Hence, as an analogue of (1.4) in our more general setting, localized indices produce finer topological invariants on $G\backslash\mathfrak{X}$. On the other hand, comparing (1.8) with (1.3), the localized index should be thought of as a replacement of the equivariant index (1.1) for the Lefschetz fixed-point formula in the compact situation. This motivated us to find a cohomological formula for $\operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}})$; see Theorem 5.10.

Third, combining the techniques of $KK$-theory and the heat kernel method, we explicitly compute the localized $(g)$-index as the orbifold integration of the local index density over $\mathfrak{X}_{(g)}$, which is an orbifold of type $Z_G(g)\backslash\mathfrak{X}^g$ (the fixed-point sub-orbfiold). The formulas are presented in Theorems 3.23 and 5.10 and they are regarded as an analogue of the "Lefschetz fixed-point formulas" for the isometries given by the action of the discrete group $G$ on the complete Riemannian orbifold $\mathfrak{X}$. There are two nontrivial technical points in deriving the cohomological formulas for the localized indices. The first one is Theorem 3.4, which provides an asymptotic expansion of the heat kernel $K_t(x, y)$ of $\not{D}^{\mathcal{E}}$ and the uniform convergence of $\sum_{g\in G} K_t(x, gx)g$ over a relative compact suborbifold of $\mathfrak{X}$ that have nontrivial intersection with each orbit for the $G$-action. The second one is Lemma 5.6. The lemma relates two notions of traces in topological and analytic settings—namely, the continuous trace defined on a certain completion of the group algebra $\mathbb{C}G$ and some supertrace of the heat kernel. See Definition 5.2 and Definition 3.13 for the precise descriptions.

Finally, we provide some observations and applications for the localized indices and their index formulas:

1) When $\mathfrak{X}$ is replaced by a manifold $M$, we derive the $L^2$-version of the Lefschtez fixed-point formula for a complete Riemannian manifold $M$ with a proper co-compact action of a discrete group $G$, by introducing an average function, called a cut-off function, with respect to some group action on the fixed-point sub-manifold (see Theorem 6.1). This result extends the $L^2$-index formulas in [**At**, **W**] from the top stratum of $G\backslash M$ to the lower strata.
2) Let $\mathfrak{X} = G/K$ be a Riemannian symmetric manifold of noncompact type, where $G$ is a real semi-simple Lie group and $K$ is a maximal compact subgroup. Let $\Gamma$ be a discrete co-compact subgroup of $G$. The cohomological formulas for the localized indices for $\not{D}^{\mathcal{E}}$ give rise to the orbital integrals in the Selberg trace formula. See Theorem 6.2. This means that our localized indices in this special case are related to the multiplicity of unitary representations of the Lie group $G$.
3) The nonvanishing of the higher index for $\not{D}^{\mathcal{E}}$ is useful in solving topological and geometric problems for the orbifold $\mathfrak{X}$. Explicit formulas for the localized indices, which are derived from the higher index, provide a tool to tackle these problems for $\mathfrak{X}$. Theorem 6.4 provides an example of how the nonvanishing of the localized indices for groups satisfying some trace property derive the nonexistance of positive scalar curvature for complete spin orbifolds.

The paper is organized as follows: In Section 2, we review basic definitions concerning orbifolds and the Kawasaki orbifold index theorem. In Section 3, we give the definition of invariant elliptic operators on a proper co-compact $G$-orbifold and use the heat kernel method to calculate the localized supertraces of the heat kernel $e^{-t(\not{D}^{\mathcal{E}})^2}$ of the Dirac operator $\not{D}^{\mathcal{E}}$. In Section 4, we formulate the higher index map for $\not{D}^{\mathcal{E}}$ and show that the Fredholm index for $\not{D}^{G\backslash\mathcal{E}}$ factors through this higher index map. In Section 5, we introduce the localized indices and present their various connections to the other indices. We also use the analytical results from Section 3 to derive the cohomological index formulas for the localized indices that correspond to the $L^2$-version of the Lefschtez fixed-point formulas for a noncompact orbifold. In Section 6, we discuss several applications and remarks of the localized indices.

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## 2. A review of orbifolds and the orbifold index theorem

In this section, we give a preliminary review of orbifolds in terms of orbifold atlases and orbifold groupoids, and Kawasaki's orbifold index theorem. Some of basic references are [**Th**, **K**, **ALR**]. Chapter 2 of [**KL**] contains a nice account of differential and Riemannian geometries of orbifolds. Due to the subtle nature of orbifolds and the fact that there is not any universal agreement in describing orbifolds, we try to give a sufficiently self-contained review as best as we can.

**2.1. Orbifolds and discrete group actions.** An $n$-dimensional orbifold $\mathfrak{X}$ is a paracompact Hausdorff space $|\mathfrak{X}|$ equipped with an equivalence class of orbifold atlases. Let us recall the definition of an orbifold atlas.

**Definition 2.1. (Orbifolds and local groups)** *Let $|\mathfrak{X}|$ be a paracompact Hausdorff space. An orbifold atlas on $|\mathfrak{X}|$ is a coherent system of orbifold charts $\mathcal{O} = \{(\tilde{U}_i, H_i, \pi_i)\}$ such that:*

1) *$\{U_i\}$ is an open cover of $|\mathfrak{X}|$ that is closed under finite intersections.*
2) *For each $U_i$, there is an obifold chart $(\tilde{U}_i, H_i, \pi_i)$ where $\tilde{U}_i$ is a connected open neighborhood in some Euclidean space $\mathbb{R}^n$ with a right action of a finite group $H_i$ with the quotient map $\pi_i : \tilde{U}_i \to U_i$.*
3) *For any inclusion $\iota_{ij} : U_i \to U_j$, there is an embedding of orbifold charts*

$$(\phi_{ij}, \lambda_{ij}) : (\tilde{U}_i, H_i, \pi_i) \hookrightarrow (\tilde{U}_j, H_j, \pi_j), \tag{2.1}$$

*which is given by an injective group homomorphism $\lambda_{ij} : H_i \to H_j$ and an embedding $\phi_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ covering the inclusion $\iota_{ij}$ such that $\phi_{ij}$ is $H_i$-equivariant with respect to $\phi_{ij}$, that is,*

$$\phi_{ij}(x \cdot g) = \phi_{ij}(x) \cdot \lambda_{ij}(g) \qquad x \in \tilde{U}_i, g \in H_i. \tag{2.2}$$

*When the action of $H_i$ is not effective, the subgroup of $H_i$ acting trivially on $U_i$ is isomorphically mapped to the subgroup of $H_j$ acting trivially on $U_j$.*

*Here, the coherent condition for $\mathcal{O}$ is described as follows: given $U_i \subset U_j \subset U_k$, there exists an element $g \in H_k$ such that*

$$g \circ \phi_{ik} = \phi_{jk} \circ \phi_{ij}, \qquad g\lambda_{ik}(h)g^{-1} = \lambda_{jk} \circ \lambda_{ij}(h) \tag{2.3}$$

*for any $h \in H_i$. Two orbifold atlases are called equivalent if they are included in a third orbifold atlas (called a common refinement).*

*Given an orbifold $\mathfrak{X}$ and a point $x \in |\mathfrak{X}|$, let $(\tilde{U}, G, \pi)$ be an orbifold chart around $x$. Then the* **local group** *at $x$ is defined to be the stabilizer of $\tilde{x} \in \pi^{-1}(x)$, which is uniquely defined up to conjugation.*

As proved in Corollary 1.2.5 of [**MP1**], any orbifold atlas admits a refinement $\mathcal{O}$ such that for each orbifold atlas $(\tilde{U}, H, \pi, U)$ in $\mathcal{O}$, both $\tilde{U}$ and $U$ are contractible. Such an orbifold atlas is called a **good** atlas. For convenience, we may choose each orbifold chart $(\tilde{U}, H, \pi)$ in $\mathcal{O}$ such that $\tilde{U}$ is an open ball centred at the origin in $\mathbb{R}^n$ and $H$ is a finite group of linear transformations.

An orbifold $\mathfrak{X}$ is called **compact** (resp. **connected**) if $|\mathfrak{X}|$ is compact (resp. connected). An orbifold $\mathfrak{X}$ is **oriented** if there exists an orbifold atlas $\{(\tilde{U}_i, H_i, \pi_i)\}$ such that $\tilde{U}_i$ is an open set of an oriented Euclidean space $\mathbb{R}^n$, the $H_i$-action preserves the orientation, and moreover, all the embeddings $\{\phi_{ij}\}$ in (2.1) are orientation preserving.

A smooth map $f : \mathfrak{X} \to \mathfrak{Z}$ between two orbifolds is a continuous map $|f| : |\mathfrak{X}| \to |\mathfrak{Z}|$ with the following property: there exist orbifold atlases $\mathcal{O}_{\mathfrak{X}} = \{(\tilde{U}_i, H_i, \pi_i)\}$ and $\mathcal{O}_{\mathfrak{Z}} = \{(\tilde{V}_\alpha, G_\alpha, \pi_\alpha)\}$ for $\mathfrak{X} \to \mathfrak{Z}$, respectively, such that:

1) For each $(\tilde{U}_i, H_i, \pi_i) \in \mathcal{O}_{\mathfrak{X}}$, there is an orbifold chart $(\tilde{V}_{\alpha_i}, G_{\alpha_i}, \pi_{\alpha_i}) \in \mathcal{O}_{\mathfrak{Z}}$ with a local smooth map $f_i : (\tilde{U}_i, H_i) \to (\tilde{V}_{\alpha_i}, G_{\alpha_i})$ making the diagram

$$\begin{tikzcd} \tilde{U}_i \arrow[r, "f_i"] \arrow[d, "\pi_i"] & \tilde{V}_{\alpha_i} \arrow[d, "\pi_{\alpha_i}"] \\ U_i \arrow[r, "|f|_i"] & V_{\alpha_i} \end{tikzcd}$$

commute, where $f_i$ is $G_{\alpha_i}$-equivariant.

2) For any embedding of orbifold charts $(\tilde{U}_i, H_i, \pi_i) \hookrightarrow (\tilde{U}_j, H_j, \pi_j)$ in $\mathcal{O}_{\mathfrak{X}}$, there is an corresponding embedding of orbifold charts

$$(\tilde{V}_{\alpha_i}, G_{\alpha_i}, \pi_{\alpha_i}) \hookrightarrow (\tilde{V}_{\alpha_j}, G_{\alpha_j}, \pi_{\alpha_j})$$

in $\mathcal{O}_{\mathfrak{Z}}$ such that the following diagram commutes in the obvious equivariant sense:

$$\begin{tikzcd} (\tilde{U}_i, H_i, \pi_i) \arrow[r, "f_i"] \arrow[d] & (\tilde{V}_{\alpha_i}, G_{\alpha_i}, \pi_{\alpha_i}) \arrow[d] \\ (\tilde{U}_j, H_j, \pi_j) \arrow[r, "f_j"] & (\tilde{V}_{\alpha_j}, G_{\alpha_j}, \pi_{\alpha_j}). \end{tikzcd}$$

3) The coherent condition (2.3) is preserved under $f_i$'s.

A diffeomorphism $f : \mathfrak{X} \to \mathfrak{Z}$ is a smooth map with a smooth inverse. The set of all diffeomorphisms from $\mathfrak{X}$ to itself is denoted by $\mathrm{Diff}(\mathfrak{X})$.

A Riemannian metric on an orbifold $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O})$ is a collection of Riemannian metrics on $\tilde{U}_i$'s such that for each orbifold chart $(\tilde{U}_i, H_i, \pi_i)$ in $\mathcal{O}$, $H_i$ acts isometrically on $\tilde{U}_i$ and all the embeddings $\{\phi_{ij}\}$ in (2.1) are isometric. We have already assumed that the orbifold atlas for

$\mathfrak{X}$ is a good atlas. Note that when $\mathfrak{X}$ is an effective Riemannian $n$-dimensional orbifold, the orthonormal frame of $\mathfrak{X}$ is a smooth manifold with a locally free $O(n)$-action whose quotient space can be equipped a natural orbifold structure isomorphic to $\mathfrak{X}$.

REMARK 2.2. An orbifold is called **presentable** if it arises from a locally free action of a compact Lie group on a smooth manifold. It is obvious to see that any effective orbifold is presentable. Conjecturally, every orbifold is presentable (cf. Conjecture 1.55 in [**ALR**]). An orbifold is called a **good orbifold** if it is a quotient of a smooth manifold by a proper action of a discrete group. Equivalently, an orbifold is good if its orbifold universal cover is smooth; see [**Th**, **ALR**] for detailed discussions.

In this paper, we are mainly interested in a noncompact oriented Riemannian orbifold $\mathfrak{X}$ with a discrete group $G$-action that is proper, co-compact, and isometric. Given a discrete group $G$ and an orbifold $\mathfrak{X}$, a smooth action of $G$ on $\mathfrak{X}$ is a group homomorphism

$$G \longrightarrow \mathrm{Diff}(\mathfrak{X}).$$

An action of $G$ on $\mathfrak{X}$ is called proper if the induced action on $|\mathfrak{X}|$ is proper, that is, the map

$$G \times |\mathfrak{X}| \longrightarrow |\mathfrak{X}| \times |\mathfrak{X}| \qquad (g, x) \mapsto (x, gx) \tag{2.4}$$

is proper. It is easy to check that the quotient space $G\backslash|\mathfrak{X}|$ can be equipped with an orbifold structure as given by the following lemma.

**Lemma 2.3.** *Let $\mathfrak{X}$ be a noncompact Riemannian orbifold, and $G$ a discrete group acting properly, co-compactly, and isometrically on $\mathfrak{X}$.*

1) *Then $|\mathfrak{X}|$ is covered by finite number of $G$-slices of the form $G \times_{G_i} U_i$ for $i = 1, \cdots, N$, where $G_i$ is a finite subgroup of $G$ and $U_i$ is the quotient space of some Euclidean ball $\tilde{U}_i$ by a finite group $H_i$, where $\tilde{U}_i$ admits a left $G_i$-action and a right $H_i$-action.*
2) *Let $|\mathfrak{X}| = \cup_{i=1}^N G \times_{G_i} U_i$ be covered by $G$-slices as in (1). Then $\mathfrak{X}$ has the orbifold atlas generated by $\{G \times_{G_i} \tilde{U}_i, H_i\}_{i=1}^N$.*
3) *The orbit space $G\backslash\mathfrak{X}$, being a compact orbifold as a result of the properness of the action, admits an orbifold atlas generated by $\{\tilde{U}_i, G_i \times H_i\}_{i=1}^N$.*

*Proof.* As the action is proper, given an open set $U_i$ of $|\mathfrak{X}|$, then the pre-image of the closure of $U_i \times U_i$ under the map (2.4) is compact. This implies that there is a finite subgroup $G_i$ such that $U_i$ is $G_i$-invariant. By choosing $U_i$ small enough, we can assume that

$$\{g \in G | g(U_i) \cap U_i \neq \emptyset\} = G_i.$$

For each $U_i$, let $(\tilde{U}_i, H_i, \pi_i)$ be an orbifold chart over $U_i$. Then taking a smaller $U_i$, by the definition of a diffeomorphism of $\mathfrak{X}$, we know that $G_i$-action on $U_i$ can be lifted to an action on $\tilde{U}_i$ that is $H_i$-equivariant. This means that $G_i$-action and $H_i$-action commute. We can choose $\tilde{U}_i$ to be a Euclidean ball with a linear action of $H_i$. As $G$ acts co-compactly on $\mathfrak{X}$, there are finitely many $G$-slices of the form

$$G \times_{G_i} U_i.$$

This completes the proof of the Claim (1). Claims (2) and (3) are obvious. q.e.d.

It becomes a majority view that the language of groupoids provides a convenient and economical way to describe orbifolds. In this paper, the groupoid viewpoint is also essential to getting a correct $C^*$-algebra associated to an orbifold in order to define a correct version of $K$-homology for orbifolds. We briefly recall the definition of an orbifold groupoid that is just a proper étale groupoid constructed from its orbifold atlas.

**Definition 2.4. (Proper étale groupoids)** *A Lie groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ consists of two smooth manifolds $\mathcal{G}_1$ and $\mathcal{G}_0$, together with five smooth maps $(s, t, m, u, i)$ satisfying the following properties.*

1) *The source map and the target map $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are submersions.*
2) *The composition map*
$$m : \mathcal{G}_1^{[2]} := \{(g_1, g_2) \in \mathcal{G}_1 \times \mathcal{G}_1 : t(g_1) = s(g_2)\} \longrightarrow \mathcal{G}_1,$$
*written as $m(g_1, g_2) = g_1 \cdot g_2$ for composable elements $g_1$ and $g_2$, satisfies the obvious associative property.*
3) *The unit map $u : \mathcal{G}_0 \to \mathcal{G}_1$ is a two-sided unit for the composition.*
4) *The inverse map $i : \mathcal{G}_1 \to \mathcal{G}_1$, $i(g) = g^{-1}$, is a two-sided inverse for the composition $m$.*

*A Lie groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ is* **proper** *if $(s, t) : \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is proper, and called* **étale** *if $s$ and $t$ are local diffeomorphisms.*

In the category of proper étale groupoids, a very important notion of morphism is the so-called generalized morphisms developed by Hilsum and Skandalis (for the category of general Lie groupoids in [**HS**]). Let $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ and $\mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ be two proper étale Lie groupoids. A **generalized morphism** between $\mathcal{G}$ and $\mathcal{H}$, denoted by $f : \mathcal{G} \dashrightarrow \mathcal{H}$, is a right principal $\mathcal{H}$-bundle $P_f$ over $\mathcal{G}_0$ that is also a left $\mathcal{G}$-bundle over $\mathcal{H}_0$ such that the left $\mathcal{G}$-action and the right $\mathcal{H}$-action commute, formally denoted by

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & P_f \arrow[dl] \arrow[dr] & \mathcal{H}_1 \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G}_0 & & \mathcal{H}_0. \end{tikzcd} \tag{2.5}$$

See [**CaW**] for more details. Note that a generalized morphism $f$ between $\mathcal{G}$ and $\mathcal{H}$ is invertible if $P_f$ in (2.5) is also a principal $\mathcal{G}$-bundle over $\mathcal{H}_0$. Then $\mathcal{G}$ and $\mathcal{H}$ are called Morita-equivalent.

REMARK 2.5. Given a proper étale Lie groupoid $\mathcal{G}$, there is a canonical orbifold structure on its orbit space $|\mathcal{G}|$ [**ALR**, Prop. 1.44]. Two Morita-equivalent proper étale Lie groupoids define two diffeomorphic orbifolds [**ALR**, Theorem 1.45]. Conversely, given an orbifold $\mathfrak{X}$ with an orbifold atlas $\mathcal{O} = \{(\tilde{U}_i, H_i, \pi_i)\}$, there is a canonical proper étale Lie groupoid $\mathcal{G}_{\mathfrak{X}}$, locally given by the action groupoid $\tilde{U}_i \rtimes H_i \rightrightarrows \tilde{U}_i$; see [**MP2**, **LU**]. Two equivalent orbifold atlases define two Morita-equivalent proper étale Lie groupoids. A proper étale Lie groupoid arising from an orbifold atlas will be called an **orbifold groupoid** for simplicity in this paper. Our notion of orbifold groupiods is slightly different from that in literature, where any proper étale Lie groupoid is called an orbifold groupoid. One can check that a smooth map between two orbifolds corresponds to a generalized morphism between the associated orbifold groupoids.

Using the language of orbifold groupoids, it is simpler to describe de Rham cohomology and orbifold $K$-theory for a compact orbifold $\mathfrak{X}$. Let $\mathcal{G} = \mathcal{G}_{\mathfrak{X}}$ be the associated orbifold groupoid. Then the de Rham cohomology of $\mathfrak{X}$, denoted by $H^*_{orb}(\mathfrak{X}, \mathbb{R})$, is just the cohomology of the $\mathcal{G}$-invariant de Rham complex $(\Omega^*(\mathfrak{X}), d)$, where

$$\Omega^*(\mathfrak{X}) = \{\omega \in \Omega^*(\mathcal{G}_0) | s^*\omega = t^*\omega\}.$$

The Satake–de Rham theorem says that there is a natural isomorphism

$$H^*_{orb}(\mathfrak{X}, \mathbb{R}) \cong H^*(|\mathfrak{X}|, \mathbb{R}).$$

Hence, the orbifold de Rham cohomology of $\mathfrak{X}$ does not provide any orbifold information about $\mathfrak{X}$. In terms of orbifold atlas $\mathcal{O} = \{(\tilde{U}_i, H_i, \pi_i)\}$ of $\mathfrak{X}$, a differential form $\omega \in \Omega^*(\mathfrak{X})$ can be written as a family of local equivariant differential forms on $\tilde{U}_i$'s that respect the equivariant condition (2.2) and the coherent condition (2.3). Let $\omega$ be a compactly supported $n$-form on $\mathfrak{X}$; we define

$$\int^{orb}_{\mathfrak{X}} \omega = \sum_i \frac{1}{|H_i|} \int_{\tilde{U}_i} \rho_i \omega, \tag{2.6}$$

where $\rho_i$ is a smooth partition of unity subordinate to $\{\tilde{U}_i\}$ in an obvious sense. Then this is a well-defined integration map for an $n$-dimensional orbifold $\mathfrak{X}$.

An orbifold vector bundle $\mathcal{E}$ over an orbifold $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O})$ is a coherent family of equivariant vector bundles

$$\{(\tilde{E}_i \to \tilde{U}_i, H_i)\}$$

such that for any embedding of orbifold charts $\phi_{ij} : (\tilde{U}_i, H_i) \hookrightarrow (\tilde{U}_j, H_j)$, there is an $H_i$-equivariant bundle map $\tilde{\phi}_{ij} : \tilde{E}_i \to \tilde{E}_j$ covering $\phi_{ij} : \tilde{U}_i \to \tilde{U}_j$. The coherent condition for $\tilde{\phi}_{ij}$ is given by (2.3) with $\phi_{ij}$'s replaced by $\tilde{\phi}_{ij}$'s. Then the total space $E = \bigcup(\tilde{E}_i/H_i)$ of an orbifold vector bundle $\mathcal{E} \to \mathfrak{X}$ has a canonical orbifold structure given by $\{(\tilde{E}_i, H_i)\}$. We remark that orbifold vector bundles in this paper are those called proper orbifold vector bundles in [**K**, **ALR**]. An orbifold vector bundle is called real (resp. complex) if each $E_i$ is real (resp. complex). For example, for an orbifold $\mathfrak{X}$, the local tangent and cotangent bundles form real orbifold vector bundles over $\mathfrak{X}$, denoted by $T\mathfrak{X}$ and $T^*\mathfrak{X}$ respectively.

A smooth section of an orbifold vector bundle is a family of invariant smooth sections of $\tilde{E}_i$'s that behave well under the equivariant condition (2.2) and the coherent condition (2.3). The space of smooth sections of $\mathcal{E}$ is denoted by $\Gamma(\mathfrak{X}, \mathcal{E})$. A connection $\nabla$ on a complex orbifold vector bundle $\mathcal{E} \to \mathfrak{X}$ is a family of invariant connections $\{\nabla_i\}$ on $\{(\tilde{E}_i \to \tilde{U}_i, H_i)\}$ that are compatible with the bundle maps $\{\tilde{\phi}_{ij}\}$. A connection $\nabla$ on $\mathcal{E}$ defines a covariant derivative, which is a differential operator

$$\nabla : \Gamma(\mathfrak{X}, \mathcal{E}) \longrightarrow \Gamma(\mathfrak{X}, T^*\mathfrak{X} \otimes \mathcal{E})$$

satisfying Leibniz's rule in the usual sense.

Recall that a vector bundle over a Lie groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ is a $\mathcal{G}$-vector bundle $E$ over $\mathcal{G}_0$, that is, a vector bundle $E$ with a fiberwise linear action of $\mathcal{G}_1$ covering the canonical action of $\mathcal{G}_1$ on $\mathcal{G}_0$. The corresponding principal bundle can be described in terms of a Hilsum–Skandalis morphism

$$\mathcal{G} \dashrightarrow GL(k) \ ,$$

where $k$ is the rank of $E$ and the general linear group $GL(k)$, over $\mathbb{R}$ or $\mathbb{C}$, is treated as a Lie groupoid $GL(k) \rightrightarrows \{e\}$. See [**CaW**] for details.

One can check that an orbifold vector bundle over $\mathfrak{X}$ defines a canonical vector bundle over the orbifold groupoid $\mathcal{G}$. The orbifold $K$-theory of $\mathfrak{X}$, denoted by $K^0_{orb}(\mathfrak{X})$, is defined to be the Grothendieck ring of stable isomorphism classes of complex oribifold vector bundles over $\mathfrak{X}$. Let $K^0(\mathcal{G})$ be the Grothendieck ring of stable isomorphism classes of complex vector bundles over $\mathcal{G}$. Then we have an obvious isomorphism

$$K^0_{orb}(\mathfrak{X}) \cong K^0(\mathcal{G}).$$

Here, $\mathcal{G}$ is the canonical orbifold groupiod associated to $\mathfrak{X}$, so there is no ambiguity in the above isomorphism caused by another Morita-equivalent proper étale Lie groupoid.

Given an orbifold groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ for an orbifold $\mathfrak{X}$, there are two canonical convolution $C^*$-algebras: the reduced and maximal $C^*$-algebras, denoted by $C^*_{red}(\mathfrak{X})$ and $C^*_{max}(\mathfrak{X})$. For readers' benefit, we recall the definition from Chapter 2.5 in [**Co2**, **Ren**].

**Definition 2.6. (Reduced and maximal $C^*$-algebras)** *The reduced $C^*$-algebra $C^*_{red}(\mathfrak{X})$ is the completion of the convolution algebra $C^\infty_c(\mathcal{G}_1)$ of smooth compactly supported functions on $\mathcal{G}_1$ for the norm*

$$\|f\| = \sup_{x \in \mathcal{G}_0} \|\pi_x(f)\|$$

*for $f \in C_c^\infty(\mathcal{G}_1)$. Here, $\pi_x$ is the involution representation of $C_c^\infty(\mathcal{G}_1)$ in the Hilbert space $l^2(s^{-1}(x))$. The maximal $C^*$-algebra $C^*_{max}(\mathfrak{X})$ is the completion of the convolution algebra $C_c^\infty(\mathcal{G}_1)$ for the norm*

$$\|f\|_{max} = \sup_\pi\{\|\pi(f)\|\},$$

*where the supremum is taken over all possible Hilbert space representation of $C_c^\infty(\mathcal{G}_1)$.*

REMARK 2.7. For an effective or a presentable orbifold, it is straightforward to see that there is an isomorphism

$$K^*_{orb}(\mathfrak{X}) \cong K_*(C^*_{red}(\mathfrak{X})),$$

where $K_*(C^*_{red}(\mathfrak{X}))$ is the $K$-theory of the reduced $C^*$-algebra $C^*_{red}(\mathfrak{X})$. For a noneffective orbifold, this becomes a very subtle issue. Nevertheless, the above isomorphism still holds for general orbifolds as established in [**TTW**].

**2.2. Kawasaki's index theorem for compact orbifolds.** The Atiyah–Singer type index theorem for elliptic operators over a compact orbifold was established by Kawasaki in [**K**]. When the underlying compact orbifold is good, a higher index was studied in [**F1**, **Bu**], and a twisted $L^2$-index with trivial Dixmier–Douady invariant was developed in [**MM**]. In this subsection, we will review the original Kawasaki's orbifold index theorem using modern language of twisted sectors and delocalized characteristic classes.

Let $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O})$ be an orbifold. Then the set of pairs

$$\{(x, (g)_{H_x}) | x \in |v|, g \in H_x\},$$

where $(g)_{H_x}$ is the conjugacy class of $g$ in the local group $H_x$, has a natural orbifold structure given by

$$\{(\tilde{U}^g, Z(g), \tilde{\pi}) | g \in H_x\}. \tag{2.7}$$

Here, for each orbifold chart $(\tilde{U}, H, \pi) \in \mathcal{O}$ around $x$, $Z(g)$ is the centralizer of $g$ in $H_x$ and $\tilde{U}^g$ is the fixed-point set of $g$ in $\tilde{U}$. This orbifold, denoted by $I\mathfrak{X}$, is called the **inertia orbifold** of $\mathfrak{X}$. The inertia orbifold $I\mathfrak{X}$ consists of a disjoint union of sub-orbifolds of $\mathfrak{X}$. There is a canonical orbifold immersion

$$ev : I\mathfrak{X} \longrightarrow \mathfrak{X}$$

given by the obvious inclusions $\{(\tilde{U}^g, Z(g), \tilde{\pi}) \to (\tilde{U}, H, \pi)\}$.

To describe the connected components of $I\mathfrak{X}$, we need to introduce an equivalence relation on the set of conjugacy classes in local groups as in [**ALR**]. For each $x \in \mathfrak{X}$, let $(\tilde{U}_x, H_x, \pi_x, U_x)$ be a local orbifold chart at $x$. If $y \in U_x$, then up to conjugation there is an injective homomorphism of local groups $H_y \to H_x$. Under this injective map, the conjugacy class $(g)_{H_x}$ in $H_x$ is well defined for $g \in H_y$. We define the equivalence to be generated by the relation $(g)_{H_y} \sim (g)_{H_x}$. Let $\mathcal{T}_{\mathfrak{X}}$ be the set of equivalence classes. Then the inertia orbifold is given by

$$I\mathfrak{X} = \bigsqcup_{(g)\in\mathcal{T}_{\mathfrak{X}}} \mathfrak{X}_{(g)},$$

where $\mathfrak{X}_{(g)} = \{(x, (g')_{H_x}) | g' \in H_x, (g')_{H_x} \sim (g)\}$. Note that $\mathfrak{X}_{(e)} = \mathfrak{X}$ is called the nontwisted sector and $\mathfrak{X}_{(g)}$ for $g \neq e$ is called a **twisted sector** of $\mathfrak{X}$.

Given a complex orbifold vector bundle $E$ over $\mathfrak{X}$, the pull-back bundle $ev^*E$ over $I\mathfrak{X}$ has a canonical automorphism $\Phi$. With respect to a Hermitian metric on $E$, there is an eigen-bundle decomposition of $ev^*E$,

$$ev^*E = \bigoplus_{\theta\in\mathbb{Q}\cap[0,1)} E_\theta,$$

where $E_\theta$ is a complex orbifold vector bundle over $I\mathfrak{X}$, on which $\Phi$ acts by multiplying $e^{2\pi\sqrt{-1}\theta}$. Define the **delocalized Chern character** of $E$ by

$$\mathrm{ch}_{deloc}(E) = \sum_\theta e^{2\pi\sqrt{-1}\theta}\,\mathrm{ch}(E_\theta) \in H^{ev}_{orb}(I\mathfrak{X}, \mathbb{C}),$$

where $\mathrm{ch}(E_\theta) \in H^{ev}(I\mathfrak{X}, \mathbb{C})$ is the ordinary Chern character of $E_\theta$. The odd delocalized Chern character

$$\mathrm{ch}_{deloc} : K^1_{orb}(\mathfrak{X}) \longrightarrow H^{odd}(I\mathfrak{X}, \mathbb{C})$$

can be defined in the usual way. The following proposition is established in [**HW**] for compact presentable orbifolds, and in [**TTW**] for general compact orbfiolds.

**Proposition 2.8.** [**HW**, Proposition 2.7] *For any compact orbifold $\mathfrak{X}$, the delocalized Chern character gives a ring isomorphism*

$$\mathrm{ch}_{deloc} : K^*_{orb}(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^*(I\mathfrak{X}, \mathbb{C})$$

*over* $\mathbb{C}$.

Let $\mathfrak{X}$ be a compact Riemannian orbifold with an orbifold atlas $\mathcal{O} = \{(\tilde{U}_i, H_i, \pi)\}$. Then the bundles of Clifford algebras over $\tilde{U}_i$'s,

$$\{(\mathrm{Cliff}(T\tilde{U}_i) \to \tilde{U}_i, H_i)\},$$

whose fiber at $x \in \tilde{U}_i$ is the real Clifford algebra $\mathrm{Cliff}(T_x\tilde{U}_i)$, define an **orbifold Clifford bundle**, denoted by $\mathrm{Cliff}(T\mathfrak{X})$. As in the manifold case (cf. Chapter 3.3 in [**BGV**]), there is a connection on $\mathrm{Cliff}(T\mathfrak{X})$, induced from the Levi–Civita connection $\nabla^{T\mathfrak{X}}$, which is compatible with the Clifford multiplication on $\Gamma(\mathfrak{X}, \mathrm{Cliff}(T\mathfrak{X}))$.

**Definition 2.9.** *Let $\mathfrak{X}$ be a compact Riemannian even-dimensional orbifold with an orbifold atlas $\mathcal{O} = \{\tilde{U}_i, H_i, \pi)\}$.*

1) *A* **Dirac** *bundle over $\mathfrak{X}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian orbifold vector bundle $\mathcal{E}$ with a graded self-adjoint smooth action of Cliff$(T\mathfrak{X})$ in the sense that, for each orbifold chart $(\tilde{U}_i, H_i, \pi)$, the action of $v \in$ Cliff$(T_x\tilde{U}_i)$ is of degree 1 and skew-adjoint.*
2) *A connection $\nabla^{\mathcal{E}}$ on a Dirac bundle is called a Clifford connection if for any $v \in \Gamma(\mathfrak{X}, \mathit{Cliff}(T\mathfrak{X}))$ and $\xi \in \Gamma(\mathfrak{X}, T\mathfrak{X})$,*

$$[\nabla^{\mathcal{E}}_{\xi}, c(v)] = c(\nabla_{\xi} v).$$

*Here, $c$ is the Clifford action Cliff$(T\mathfrak{X}) \otimes \mathcal{E} \to \mathcal{E}$.*

REMARK 2.10. 1) By means of a smooth partition of unity on $\mathfrak{X}$, any Dirac bundle over $\mathfrak{X}$ admits a Clifford connection. The space of all Clifford connections is an affine space modeled on $\Gamma(\mathfrak{X}, T^*\mathfrak{X} \otimes \mathrm{End}_{\mathrm{Cliff}(T\mathfrak{X})}(\mathcal{E}))$. Here, $\mathrm{End}_{\mathrm{Cliff}(T\mathfrak{X})}(\mathcal{E})$ is the bundle of degree 0 endomorphisms that commute with the Clifford action.
2) For an even-dimensional compact Riemannian orbifold $\mathfrak{X}$, $\mathfrak{X}$ is spin$^c$ if and only if there exists an orbifold complex vector bundle $\mathcal{S}$, called a complex spinor bundle, such that the complexified version of Cliff$(T\mathfrak{X})$, denoted by $\mathrm{Cliff}_{\mathbb{C}}(T\mathfrak{X})$, is isomorphic to $\mathrm{End}_{\mathbb{C}}(\mathcal{S})$. A choice of such an $\mathcal{S}$ is called a Spin$^c$ structure on $\mathfrak{X}$. Then any Dirac bundle $\mathcal{E}$ can be written as $\mathcal{S} \otimes \mathcal{W}$ for an orbifold complex vector bundle $\mathcal{W}$ over $\mathfrak{X}$.
3) The Grothendieck group of Dirac bundles over $\mathfrak{X}$ forms an abelian group, denoted by $K^0(\mathfrak{X}, \mathrm{Cliff}_{\mathbb{C}}(T\mathfrak{X}))$. The relative delocalized Chern character

$$\mathrm{ch}^{\mathcal{S}}_{deloc} : K^0_{orb}(\mathfrak{X}, \mathrm{Cliff}_{\mathbb{C}}(T\mathfrak{X})) \longrightarrow H^{ev}(I\mathfrak{X}, \mathbb{R})$$

can be defined by the same construction as in Section 4.1 of [**BGV**]. When $\mathfrak{X}$ is equipped with a Spin$^c$ structure $\mathcal{S}$, then

$$K^0_{orb}(\mathfrak{X}, \mathrm{Cliff}_{\mathbb{C}}(T\mathfrak{X})) \cong K^0_{orb}(\mathfrak{X}),$$

and for any Dirac bundle $\mathcal{E}$ written as $\mathcal{S} \otimes \mathcal{W}$, we have

$$\mathrm{ch}^{\mathcal{S}}_{deloc}(\mathcal{S} \otimes \mathcal{W}) = \mathrm{ch}_{deloc}(\mathcal{W}).$$

Given a Dirac bundle $\mathcal{E}$ with a Clifford connection $\nabla^{\mathcal{E}}$ over a compact Riemannian even-dimensional orbifold $\mathfrak{X}$, the Dirac operator on $\mathcal{E}$ is defined to be the following composition:

$$D\!\!\!/^{\mathcal{E}} : \Gamma(\mathfrak{X}, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \Gamma(\mathfrak{X}, T^*\mathfrak{X} \otimes \mathcal{E}) \cong \Gamma(\mathfrak{X}, T\mathfrak{X} \otimes \mathcal{E}) \xrightarrow{c} \Gamma(\mathfrak{X}, \mathcal{E}).$$

When $\mathfrak{X}$ is a spin$^c$ orbifold with its spinor bundle $\mathcal{S}$, then $\mathcal{E} = \mathcal{S} \otimes \mathcal{W}$ for a complex orbifold vector bundle $\mathcal{W}$, and $D\!\!\!/^{\mathcal{E}}$ will be written as $D\!\!\!/^{\mathcal{W}}$. With respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$,

$$D\!\!\!/^{\mathcal{E}} = \begin{bmatrix} 0 & D\!\!\!/^{\mathcal{E}}_- \\ D\!\!\!/^{\mathcal{E}}_+ & 0 \end{bmatrix}$$

where $\not{D}^{\mathcal{E}}_-$ is the formal adjoint of $\not{D}^{\mathcal{E}}_+$. Let $L^2(\mathfrak{X}, \mathcal{E})$ and $L^2_1(\mathfrak{X}, \mathcal{E})$ be the completions of $\Gamma(\mathfrak{X}, \mathcal{E})$ with respect to $L^2$-norm and $L^2_1$-norm respectively. Then

$$\not{D}^{\mathcal{E}} : L^2_1(\mathfrak{X}, \mathcal{E}) \longrightarrow L^2(\mathfrak{X}, \mathcal{E})$$

is a Fredholm operator by showing that the unbounded operator

$$\not{D}^{\mathcal{E}} : L^2(\mathfrak{X}, \mathcal{E}) \longrightarrow L^2(\mathfrak{X}, \mathcal{E})$$

has compact resolvent. One could also establish the finite dimensionality of $\ker \not{D}^{\mathcal{E}}$ by showing that the heat operator $e^{-t(\not{D}^{\mathcal{E}})^2}$ is a trace-class operator for any compact Riemannian orbifold $\mathfrak{X}$. This latter claim follows from the fact that $e^{-t(\not{D}^{\mathcal{E}})^2}$ has a smooth kernel. In next section we will prove the smooth heat kernel for any complete Riemannian orbifold, so we omit the details here.

The Fredholm index of $\not{D}^{\mathcal{E}}$ is given by

$$\operatorname{ind} \not{D}^{\mathcal{E}} = \dim_{\mathbb{C}} \ker \not{D}^{\mathcal{E}}_+ - \dim_{\mathbb{C}} \ker \not{D}^{\mathcal{E}}_-.$$

To rephrase the orbifold index theorem in [**K**] (cf. [**Bu**]), we introduce the notations for the delocalized A-hat class and the delocalized Todd class of $\mathfrak{X}$ as follows. For simplicity, we assume that each twisted sector $\mathfrak{X}_{(g)}$ is connected; otherwise, these characteristic classes are defined on each connected component. Considering the orbifold immersion

$$ev : I\mathfrak{X} \longrightarrow \mathfrak{X},$$

we can decompose the pull-back of the orbifold tangent bundle along each twisted sector $\mathfrak{X}_{(g)}$ (using the Riemannian metric)

$$T\mathfrak{X}|_{\mathfrak{X}_{(g)}} \cong T\mathfrak{X}_{(g)} \oplus \mathcal{N}_{(g)}$$

such that the pull-back of the Riemannian curvature of $T\mathfrak{X}$ is decomposed as

$$ev^* R^{T\mathfrak{X}} = R^{T\mathfrak{X}_{(g)}} \oplus R^{\mathcal{N}_{(g)}},$$

where $R^{T\mathfrak{X}_{(g)}}$ is the Riemannian curvature of $\mathfrak{X}_{(g)}$ and $R^{\mathcal{N}_{(g)}}$ is the curvature of the induced connection on $\mathcal{N}_{(g)}$. Then the delocalized A-hat class of $\mathfrak{X}$ is a cohomology class on $I\mathfrak{X}$. When restricted to each twisted sector $\mathfrak{X}_{(g)}$, it is defined by

$$\hat{A}_{deloc}(\mathfrak{X}) = \frac{\hat{A}(\mathfrak{X}_{(g)})}{[\det\left(1 - \Phi_{(g)} e^{R^{\mathcal{N}_{(g)}}/2\pi i}\right)]^{1/2}}.$$

Here, $\hat{A}(\mathfrak{X}_{(g)})$ is the A-hat form of $\mathfrak{X}_{(g)}$ defined by

$$\left[\det\left(\frac{R^{T\mathfrak{X}_{(g)}}/4\pi i}{\sinh(R^{T\mathfrak{X}_{(g)}}/4\pi i)}\right)\right]^{1/2}$$

and $\Phi_{(g)}$ is the automorphism of $\mathcal{N}_{(g)}$ defined by the inertia orbifold structure. If $I\mathfrak{X}$ is a spin$^c$ orbifold, then the delocalized Todd class of

$\mathfrak{X}$ is a cohomology class on $I\mathfrak{X}$. When restricted to each twisted sector $\mathfrak{X}_{(g)}$, it is defined by the closed differential form

$$\mathrm{Td}_{deloc}(\mathfrak{X}) = \frac{\mathrm{Td}(\mathfrak{X}_{(g)})}{\det\left(1 - \Phi_{(g)}^{-1} e^{R^{\mathcal{N}_{(g)}}/2\pi i}\right)},$$

where $\mathrm{Td}(\mathfrak{X}_{(g)}) = \det\left(\dfrac{iR^{\mathfrak{X}_{(g)}}/2\pi}{1 - e^{-iR^{\mathfrak{X}_{(g)}}/2\pi}}\right)$ is the usual Todd form of the spin$^c$ orbifold $\mathfrak{X}_{(g)}$.

**Theorem 2.11.** *Let $\mathfrak{X}$ be a compact Riemannian even-dimensional orbifold. Let $\not{D}^{\mathcal{E}}$ be a Dirac operator on a Dirac bundle $\mathcal{E}$ over $\mathfrak{X}$. Then the index of $\not{D}^{\mathcal{E}}$ is given by*

$$\mathrm{ind}\, \not{D}^{\mathcal{E}} = \int_{I\mathfrak{X}}^{orb} \hat{A}_{deloc}(\mathfrak{X})\, \mathrm{ch}^{\mathcal{S}}_{deloc}(\mathcal{E}).$$

*In particular, if $I\mathfrak{X}$ is a spin$^c$ orbifold and $\mathcal{W}$ is an orbifold Hermitian vector bundle with a Hermitian connection, then the index of the spin$^c$ Dirac operator $\not{D}^{\mathcal{W}}_{+}$ is given by the formula*

$$\mathrm{ind}\, \not{D}^{\mathcal{W}} = \int_{I\mathfrak{X}}^{orb} \mathrm{Td}_{deloc}(\mathfrak{X})\, \mathrm{ch}_{deloc}(\mathcal{W}).$$

Remark 2.12. One can adapt the proof of Theorem 4.1 in [**BGV**] to compact orbifolds to establish the local index version of Theorem 2.11. It amounts to studying the heat kernel asymptotics for $e^{-t(\not{D}^{\mathcal{E}})^2}$ on the compact orbifold. We will apply the same heat kernel approach to establish a more refined index formula for a discrete group action on a complete Riemannian orbifold, so we will not reproduce a proof here.

## 3. Invariant elliptic operators on complete orbifolds

In this section, after a review of invariant elliptic pseudo-differential operators in Section 3.1, we prepare some analytical tools to study indices of Dirac type operators on a complete even-dimensional Riemannian orbifold $\mathfrak{X}$, where a discrete group $G$ acts properly, co-compactly, and isometrically. When the orbifold is compact, the index of the Dirac operator $\not{D}^{\mathcal{E}}$ is known to be calculated by the supertrace of the corresponding heat operator. However, when $\mathfrak{X}$ is not compact, the convergence of the heat kernel asymptotic expansion as $t \to 0^+$ has to be modified accordingly. This is worked out in Section 3.2. Also, the operator trace of $e^{-t(\not{D}^{\mathcal{E}})^2}$ does not make sense. In Section 3.3 we introduce $(g)$-trace class operators, and in Section 3.4 we compute the $(g)$-trace for the heat operator. On the one hand, these traces are related to the orbifold index of $\not{D}^{G\backslash\mathcal{E}}$ on the quotient. On the other hand, as we shall see in Section 5, they are topological invariants for $\mathfrak{X}$ coming from the higher index for $\not{D}^{\mathcal{E}}$.

**3.1. Invariant elliptic pseudo-differential operators.** Let $\mathfrak{X}$ be a complete $n$-dimensional oriented Riemannian orbifold, and let $\mathcal{E}$ be a Hermitian orbifold vector bundle with a Hermitian connection $\nabla$. Denote by $\Gamma_c(\mathfrak{X}, \mathcal{E}) \subset \Gamma(\mathfrak{X}, \mathcal{E})$ the subspace of smooth sections of $\mathcal{E}$ with compact support. A differential operator

$$D : \Gamma_c(\mathfrak{X}, \mathcal{E}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{E}) \tag{3.1}$$

of order $m$ is a linear map such that for any orbifold chart $(\tilde{U}_i, H_i, \pi_i, U_i)$, the operator $D$ is locally represented by

$$\tilde{D}_i : \Gamma(\tilde{U}_i, \tilde{\mathcal{E}}|_{\tilde{U}_i})^{H_i} \longrightarrow \Gamma(\tilde{U}_i, \tilde{\mathcal{E}}|_{\tilde{U}_i})^{H_i},$$

an $H_i$-invariant differential operator of order $m$. The completion of $\Gamma_c(\mathfrak{X}, \mathcal{E})$ in the Sobolev $k$-norm

$$\|\psi\|_k^2 = \sum_{i=0}^{k} \int_{\mathfrak{X}}^{orb} |\nabla^i \psi|^2 d\,\mathrm{vol}_{\mathfrak{X}} \tag{3.2}$$

is the Sobolev space denoted by $L_k^2(\mathfrak{X}, \mathcal{E})$. Here, $d\,\mathrm{vol}_{\mathfrak{X}}$ is the Riemannian volume element (the unique $n$-form of unit length) on $\mathfrak{X}$ defined by the metric. Then the differential operator (3.1) extends to a bounded linear map $D : L_k^2(\mathfrak{X}, \mathcal{E}) \to L_{k-m}^2(\mathfrak{X}, \mathcal{E})$ for all $k \geq m$. A linear map $A : \Gamma_c(\mathfrak{X}, \mathcal{E}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{E})$ is called a **smoothing operator** if $A$ extends to a bounded linear map $A : L_k^2(\mathfrak{X}, \mathcal{E}) \to L_{k+m}^2(\mathfrak{X}, \mathcal{E})$ for all $k$ and $m \geq 0$. The Sobolev embedding theorem implies that, for a smoothing operator $A$, we have

$$A(L_k^2(\mathfrak{X}, \mathcal{E})) \subset \Gamma(\mathfrak{X}, \mathcal{E})$$

for all $k$.

**Definition 3.1** (**Pseudo-differential operators on an orbifold**). *A linear map*

$$D : \Gamma_c(\mathfrak{X}, \mathcal{E}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{E})$$

*is a* **pseudo-differential operator** *of order $m$ if modulo smoothing operators, for any orbifold chart $\{(\tilde{U}_i, H_i, \pi_i, U_i)\}$, the operator $D$ is represented by*

$$\tilde{D}_i : \Gamma_c(\tilde{U}_i, \tilde{\mathcal{E}}|_{\tilde{U}_i}) \longrightarrow \Gamma(\tilde{U}_i, \tilde{\mathcal{E}}|_{\tilde{U}_i}), \tag{3.3}$$

*which is an $H_i$-invariant pseudo-differential operator of order $m$. A pseudo-differential operator $D$ is* **elliptic** *if $\tilde{D}_i$ is elliptic for each orbifold chart $(\tilde{U}_i, H_i)$. Denote by $\Psi_{orb}^m(\mathfrak{X}, \mathcal{E})$ the linear space of all pseudo-differential operators of order $m$ on $(\mathfrak{X}, \mathcal{E})$.*

Due to the local nature of pseudo-differential operators, an operator $D \in \Psi_{orb}^m(\mathfrak{X}, \mathcal{E})$ has a well-defined principal symbol of order $m$ that is an element in

$$Sym_{orb}^m(\mathfrak{X}, \mathcal{E}) / Sym_{orb}^{m-1}(\mathfrak{X}, \mathcal{E})$$

where $Sym^m_{orb}(\mathfrak{X}, \mathcal{E})$ is the space of all sections of order $m$ of the orbifold vector bundle $\mathrm{End}(p^*\mathcal{E})$ over $T^*\mathfrak{X}$; here, $p : T^*\mathfrak{X} \to \mathfrak{X}$ is the obvious projection. Associated to a pseudo-differential operator $D$ of order $m$, the Schwartz kernel

$$k \in \mathcal{D}'(\mathfrak{X} \times \mathfrak{X}, \mathcal{E} \boxtimes \mathcal{E}^*)$$

as a distributional-valued section, is smooth off the diagonal. Here,

$$\mathcal{E} \boxtimes \mathcal{E}^* = \pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^*,$$

with the projections $\mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}$ to the first factor and the second factor, respectively. As explained in [**Ma**], in terms of a local orbifold chart $(\tilde{U}_i, H_i, \pi_i, U)_i$ there is a distributional-valued section

$$\tilde{k}_D(\tilde{x}, \tilde{y}) \in \Gamma(\tilde{U}_i \times \tilde{U}_i, \tilde{\mathcal{E}}_i \boxtimes \tilde{\mathcal{E}}_i^*),$$

which is the kernel for the representing operator $\tilde{D}_i$ in (3.3), such that

$$k_D(x, y) = \sum_{h \in H_i} h\tilde{k}_D(\tilde{x} \cdot h^{-1}, \tilde{y}), \tag{3.4}$$

where $\tilde{x} \in \pi_i^{-1}(x)$ and $\tilde{y} \in \pi_i^{-1}(y)$. Here, we assume that $H_i$ acts on $\tilde{U}_i$ from the right.

A smoothing pseudo-differential operator $A$ has a smooth kernel, which is a smooth section $k(x, y)$ of the bundle $\mathcal{E} \boxtimes \mathcal{E}^* = \pi_1^*(\mathcal{E}) \otimes \pi_2^*(\mathcal{E}^*)$ over $\mathfrak{X} \times \mathfrak{X}$ such that

$$A\psi(x) = \int_{\mathfrak{X}}^{orb} k(x, y)\psi(y) d\,\mathrm{vol}_{\mathfrak{X}}(y).$$

When $\mathfrak{X}$ is equipped with a proper, co-compact and isometric action of a discrete group $G$, we can define $G$-invariant pseudo-differential operators on $\mathcal{E}$ in the usual sense. Let $\mathcal{E}$ be an orbifold Hermitian vector bundle with a $G$-invariant Hermitian connection $\nabla$. Consider a collection of orbifold charts of $\mathfrak{X}$

$$\{(G \times_{G_i} \tilde{U}_i, H_i, G \times_{G_i} U_i)\}$$

as provided by Lemma 2.3. Here, $\{(G \times_{G_i} \tilde{U}_i, H_i, G \times_{G_i} U_i)\}$ is a collection of orbifold charts indexed by the coset space $G/G_i$. We further assume $\{U_i\}$ to be an open covering of a relatively compact open set $C$ in $|\mathfrak{X}|$ such that $\mathfrak{X} = G \cdot C$. A $G$-invariant pseudo-differential operator

$$D_{\mathfrak{X}} : \Gamma_c(\mathfrak{X}, \mathcal{E}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{E})$$

is locally represented by a $G$-invariant pseudo-differential operator on the orbifold $G \times_{G_i} U_i$. Note that the orbifold structure on the quotient space $G\backslash|\mathfrak{X}|$ is defined by the collection of finitely many orbifold charts

$$\{(\tilde{U}_i, G_i \times H_i, U_i)\}.$$

Then a $G$-invariant pseudo-differential operator $D_{\mathfrak{X}}$ of order $m$ on $\mathfrak{X}$ defines a pseudo-differential operator of order $m$ on the compact orbifold $G\backslash\mathfrak{X}$.

REMARK 3.2. Let $k(x,y) \in \mathcal{E}_x \otimes \mathcal{E}_y^*$ be the distributional kernel of a $G$-invariant pseudo-differential operator $D_{\mathfrak{X}}$. Then we have

$$gk(g^{-1}x, y) = k(x, gy)g \qquad \forall x, y \in \mathfrak{X}, \forall g \in G. \tag{3.5}$$

Here, $g$ stands for the action of $g \in G$ on the smooth sections in $\Gamma(\mathfrak{X}, \mathcal{E})$.

An important class of pseudo-differential operators being studied have proper support. Recall that an operator with the distributional kernel $k(x,y)$ is **properly supported** if for all compact sets $C \subset \mathfrak{X}$, the following set is compact in $\mathfrak{X} \times \mathfrak{X}$ :

$$\{(x,y) \in \mathfrak{X} \times \mathfrak{X} | k(x,y) \neq 0, x \text{ or } y \in C\}. \tag{3.6}$$

If $S$ is a $G$-invariant properly supported operator with Schwartz kernel $K_S$, then by Remark 3.2 and the co-compactness of the action, (3.6) implies the existence of $R > 0$ such that

$$K_S(x,y) = 0 \qquad \forall d(x,y) > R.$$

In view of Definition 3.1, any pseudo-differential operator is properly supported up to a smoothing operator.

The following proposition is a key property for elliptic operators. When $\mathfrak{X}$ is compact, it implies that an elliptic operator has the Fredholm index. In general, this proposition leads to an elliptic operator model for $K$-homology and higher index. As a pseudo-differential operator on an orbifold is locally defined on the orbifold charts, the proof is similar to the manifold case ([**W**, Proposition 2.7]), where we assumed $D_{\mathfrak{X}}$ to be properly supported and have order 0. The proof of the proposition is then a trivial generalization. In fact, any pseudo-differential operator of positive order can be normalized to an order 0 operator and is properly supported up to a difference of a smoothing operator.

**Proposition 3.3.** *Let $(D_{\mathfrak{X}})_+ : L^2_k(\mathfrak{X}, \mathcal{E}_+) \to L^2_{k-m}(\mathfrak{X}, \mathcal{E}_-)$, where $k > m$, be a $G$-invariant elliptic operator with nonnegative order $m$. Then there exists a $G$-invariant parametrix $Q_{\mathfrak{X}} : L^2_{k-m}(\mathfrak{X}, \mathcal{E}_-) \to L^2_k(\mathfrak{X}, \mathcal{E}_+)$ of $(D_{\mathfrak{X}})_+$ so that $1 - Q_{\mathfrak{X}}(D_{\mathfrak{X}})_+ = S_0$ and $1 - (D_{\mathfrak{X}})_+ Q_{\mathfrak{X}} = S_1$ are smoothing operators.*

**3.2. Dirac operator and heat kernel asymptotics.** In this paper, we focus on Dirac operators, which are first-order elliptic differential operators. Let $\mathfrak{X}$ be a complete even-dimensional Riemannian orbifold. Let $\nabla^{T\mathfrak{X}}$ be the Levi–Civita connection on $T\mathfrak{X}$. Given a Dirac bundle $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ with a Clifford connection $\nabla^{\mathcal{E}}$, the Dirac operator

$$\not{D}^{\mathcal{E}} = \begin{bmatrix} 0 & \not{D}^{\mathcal{E}}_- \\ \not{D}^{\mathcal{E}}_+ & 0 \end{bmatrix} : L^2(\mathfrak{X}, \mathcal{E}) \longrightarrow L^2(\mathfrak{X}, \mathcal{E}), \tag{3.7}$$

with domain $\Gamma_c(\mathfrak{X}, \mathcal{E})$ is an essentially self-adjoint elliptic differential operator. Moreover, we have the Lichnerowicz formula for $(\not{D}^{\mathcal{E}})^2$ as follows:

$$(\not{D}^{\mathcal{E}})^2 = \Delta^{\mathcal{E}} + c(F^{\mathcal{E}/\mathcal{S}}) + \frac{1}{4} r_{\mathfrak{X}}, \tag{3.8}$$

where $\Delta^{\mathcal{E}}$ is the Laplace operator on $\mathcal{E}$, $c(F^{\mathcal{E}/\mathcal{S}})$ is the Clifford action of the twisted curvature of the Clifford connection $\nabla^{\mathcal{E}}$, and $r_{\mathfrak{X}}$ is the scalar curvature of $\mathfrak{X}$. We remark that the proof of the essential self-adjointness of $\not{D}^{\mathcal{E}}$ and the Lichnerowicz formula for $(\not{D}^{\mathcal{E}})^2$ follow from the proof in the case of smooth manifolds without significant changes. Then the heat operator

$$e^{-t(\not{D}^{\mathcal{E}})^2} : L^2(\mathfrak{X}, \mathcal{E}) \longrightarrow L^2(\mathfrak{X}, \mathcal{E})$$

is a one-parameter semi-group of operators consisting of positive, self-ajoint operators of norm $\leq 1$, satisfying the following properties, for $\psi \in L^2(\mathfrak{X}, \mathcal{E})$, $t \geq 0$:

1) $\left(\dfrac{d}{dt} + (\not{D}^{\mathcal{E}})^2\right) e^{-t(\not{D}^{\mathcal{E}})^2}\psi = 0.$

2) $\lim_{t\to 0} e^{-t(\not{D}^{\mathcal{E}})^2}\psi = \psi$ in $L^2(\mathfrak{X}, \mathcal{E})$.

Moreover, $e^{-t(\not{D}^{\mathcal{E}})^2}$ is a smoothing operator. By the Schwartz kernel theorem, there is a kernel

$$K_t(x, y) \in \Gamma(\mathfrak{X} \times \mathfrak{X}, \mathcal{E} \boxtimes \mathcal{E}^*),$$

called the heat kernel of $\not{D}^{\mathcal{E}}$ with respect to $d\,\mathrm{vol}_{\mathfrak{X}}$. For $\psi \in L^2(\mathfrak{X}, \mathcal{E})$,

$$(e^{-t(\not{D}^{\mathcal{E}})^2}\psi)(x) = \int_{\mathfrak{X}}^{orb} K_t(x, y)(\psi(y))d\,\mathrm{vol}_{\mathfrak{X}}(y).$$

When $\mathfrak{X}$ is compact, that is, $|\mathfrak{X}|$ is compact, we have the following asymptotic expansion: for each Riemannian orbifold chart $(\tilde{U}_i, H_i, U_i)$, using the notations in (3.4), there exists a smooth section $u_j$ of $\mathcal{E} \boxtimes \mathcal{E}^*$ over $\tilde{U}_i \times \tilde{U}_i$ such that for every $l > n = \dim \mathfrak{X}$ and $x, y \in U_i$, as $t \to 0^+$,

$$\begin{aligned} K_t(x, y) &= (4\pi t)^{-n/2} \frac{1}{|H_i|} \textstyle\sum_{h \in H_i} e^{-\frac{\tilde{d}(\tilde{x}, \tilde{y})^2}{4t}} \textstyle\sum_{j=0}^{l} t^j h u_j(\tilde{x} \cdot h^{-1}, \tilde{y}) \\ &\quad + O(t^{l - n/2}). \end{aligned}$$

Here, $\tilde{d}(\tilde{x}, \tilde{y})$ is the distance function on $\tilde{U}_i$ defined by the Riemannian metric on $\tilde{U}_i$. Moreover, the off-diagonal estimate is given by

$$K_t(x, y) = O(e^{-a^2/4t})$$

as $t \to 0$, for $(x, y) \in |\mathfrak{X}| \times |\mathfrak{X}|$ with $d(x, y) > a > 0$. Here, $d(x, y)$ is the distance function defined by the Riemannian metric on $\mathfrak{X}$. See [**Ma**] and Chapter 5.4 in [**MaMa**] for relevant discussions. The local index technique in [**BGV**, **MaMa**] gives rise to a local index formula for the Kawasaki orbifold index theorem (Theorem 2.11).

Let $\mathfrak{X}$ be a complete even-dimensional Riemannian orbifold equipped with a proper, co-compact, and isometric action of a discrete group $G$. Let $\nabla^{T\mathfrak{X}}$ be the $G$-invariant Levi–Civita connection on $T\mathfrak{X}$. Then the scalar curvature $r_{\mathfrak{X}}$ is a $G$-invariant function on $\mathfrak{X}$. Given a $G$-equivariant Dirac bundle $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ with a $G$-invariant Clifford connection $\nabla^{\mathcal{E}}$, the Dirac operator

$$\not{D}^{\mathcal{E}} : L^2(\mathfrak{X}, \mathcal{E}) \longrightarrow L^2(\mathfrak{X}, \mathcal{E})$$

is then a $G$-invariant elliptic differential operator. The corresponding Dirac operator on $G\backslash\mathfrak{X}$ is denoted by

$$\not{D}^{G\backslash\mathcal{E}} : L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E}) \longrightarrow L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E}).$$

**Theorem 3.4.** *Let $\mathfrak{X}$ be a complete even-dimensional Riemannian orbifold with a proper, co-compact, and isometric action of a discrete group $G$. Let $K_t(x, y)$ be the heat kernel of a $G$-invariant elliptic differential operator $\not{D}^{\mathcal{E}}$, that is, the Schwartz kernel of the semigroup $e^{-t(\not{D}^{\mathcal{E}})^2}$. Then $K_t(x, y)$ is a $G$-invariant smooth kernel with an asymptotic expansion, as $t \to 0^+$,*

$$K_t(x, y) \sim (4\pi t)^{-n/2} e^{-\frac{d(x,y)^2}{4t}} \sum_{j=0}^{\infty} t^i u_j(x, y) \tag{3.9}$$

*on a sufficiently small neighborhood of the diagonal in $\mathfrak{X} \times \mathfrak{X}$. Here $u_j(x, y)$ is a smooth section of $\mathcal{E} \boxtimes \mathcal{E}^*$ in the sense of (3.4). Moreover, choose a relative compact sub-orbifold $C$ in $\mathfrak{X}$ with $\mathfrak{X} = G \cdot C$. Let $\pi : C \to G\backslash\mathfrak{X}$ be the natural map defined by the quotient map $\mathfrak{X} \to G\backslash\mathfrak{X}$. Then the series*

$$\sum_{g\in G} gK_t(g^{-1} \cdot x, y) = \sum_{g\in G} K_t(x, g \cdot y)g \in \mathcal{E}_x \otimes \mathcal{E}^*_y$$

*converges uniformly on $[t_1, t_2]\times C\times C$ to the kernel $\overline{K}_t(\bar{x}, \bar{y})$ of $e^{-t(\not{D}^{G\backslash\mathcal{E}})^2}$ for $\bar{x} = \pi(x)$ and $\bar{y} = \pi(y)$. Here, $g$ acts on a section $\psi \in L^2(\mathfrak{X}, \mathcal{E})$ by $(g \cdot \psi)(y) = g\psi(g^{-1} \cdot y)$.*

*Proof.* In the case of smooth manifolds, this theorem is due to Donnelly [**Do**]. For convenience, we use the same notation $(\not{D}^{\mathcal{E}})^2$ to denote its unique self-adjoint extension. By the Lichnerowicz formula, $(\not{D}^{\mathcal{E}})^2$ is a generalized Laplacian operator

$$\Delta^{\mathcal{E}} + c(F^{\mathcal{E}/\mathcal{S}}) + \frac{1}{4} r_{\mathfrak{X}}$$

where $c(F^{\mathcal{E}/\mathcal{S}}) + \frac{1}{4} r_{\mathfrak{X}}$ is bounded from below due to the isometric co-compact action of $G$ on $\mathfrak{X}$. Then by Theorem D.1.2 in [**MaMa**] and the standard functional calculus,

$$e^{-t(\not{D}^{\mathcal{E}})^2} : L^2(\mathfrak{X}, \mathcal{E}) \longrightarrow L^2(\mathfrak{X}, \mathcal{E})$$

is a smoothing operator and the heat kernel $K_t(x,y)$ is a smooth section of $\mathcal{E} \boxtimes \mathcal{E}^*$ in $x, y \in \mathfrak{X}$ and $t \in (0,\infty)$.

The asymptotic expansion follows from the uniqueness of the fundamental solution to the heat equation on $(0,\infty) \times \mathfrak{X} \times \mathfrak{X}$,

$$\left(\frac{\partial}{\partial t} + (\not{D}_y^{\mathcal{E}})^2\right) K_t(x,y) = 0,$$

where $(\not{D}_y^{\mathcal{E}})^2$ acts in the second variable, with the following initial boundary condition at $t = 0$: if $\psi$ is a smooth section of $\mathcal{E}$, then

$$\lim_{t\to 0} \int_{\mathfrak{X}}^{orb} K_t(x,y)\psi(y) d\operatorname{vol}_{\mathfrak{X}}(y) = \psi(x)$$

in the $L^2$-norm or the uniform norm on any compact set in $\mathfrak{X}$. It is done by building an approximate heat kernel in normal orbifold coordinate charts as the asymptotic expansion in question holds on a sufficiently small neighborhood of the diagonal in $\mathfrak{X} \times \mathfrak{X}$. Then the asymptotic expansion (3.9) follows from the construction in [**BGV**, Chapter 2] and [**CGT**] carried over to the case of orbifolds. Moreover, for any $T > 0$ and $0 < t \leq T$, we have the following fiberwise norm estimate,

$$\|(\frac{\partial}{\partial t})^i (\nabla_x^{\mathcal{E}})^j (\nabla_y^{\mathcal{E}})^k K_t(x,y)\| \leq c_1 t^{-n/2-i-j/2-k/2} e^{-\frac{d(x,y)^2}{4t}}.$$

To check that the series $\sum_{g\in G} K_t(x, g\cdot y)g$ converges uniformly on $[t_1,t_2] \times C \times C$, we need to estimate the number

$$n(i) = \#\{g \in G | (i-1)r \leq d(x, g\cdot y) < ir\}$$

for $i \in \mathbb{N}$, $x, y \in C$ with $r > diam(C)$. As the $G$-action is proper, let

$$m = \#\{g \in G | (g\cdot C) \cap C \neq \emptyset\}.$$

Suppose that $B(x,r)$ contains $p_r$ $G$-translates of $y$, that is,

$$p(r) = \#\{g | d(x, g\cdot y) < r\}.$$

Then $B(x,2r)$ contains $p(r)$ $G$-translates of $C$ as $r > diam(C)$. Any point of $\mathfrak{X}$ is contained in at most $m$ translates of $C$, so we have

$$p(r)\cdot \operatorname{vol}(C) \leq m\cdot \operatorname{vol}(B(x,2r)).$$

Note that since $G$ acts on $\mathfrak{X}$ isometrically and co-compactly, then the sectional curvatures of $\mathfrak{X}$ are bounded from below. By comparing with a space of constant curvature (cf. [**Bo**, Proposition 20] and [**BiCr**, Corollary 4, p.256]), we have

$$\operatorname{vol}(B(x,r)) \leq c_3 e^{c_4 r}.$$

This implies that $n(i) < p(ir) < \dfrac{mc_3 e^{2c_4 ir}}{\operatorname{vol}(C)}$ for any $i \in \mathbb{N}$. Then by partitioning $G$ into subsets

$$G(i) = \{g \in G | (i-1)r \leq d(x, g\cdot y) < ir\},$$

for $i \in \mathbb{N}$, we have the operator norm estimate

$$\begin{aligned}
& \sum_{g\in G} \|K_t(x, g\cdot y)g\| \\
\leq \; & c_1 t^{-n/2} \sum_{g\in G} e^{-\frac{d^2(x,g\cdot y)}{4t}} \\
= \; & c_1 t^{-n/2} \sum_{i=1}^{\infty} \sum_{g\in G(i)} e^{-\frac{-(i-1)^2r^2}{4t}} \\
\leq \; & c_1 t^{-n/2} \sum_{i=1}^{\infty} n_i e^{-\frac{-(i-1)^2r^2}{4t}} \\
\leq \; & \frac{mc_1c_3}{\mathrm{vol}(C)} t^{-n/2} \sum_{i=1}^{\infty} e^{2c_4 ir} e^{-\frac{-(i-1)^2r^2}{4t}} \\
= \; & \frac{mc_1c_3}{\mathrm{vol}(C)} t^{-n/2} \sum_{i=1}^{\infty} e^{-\frac{-(i-1)^2r^2+2c_4 ir}{4t}},
\end{aligned}$$

which converges uniformly on $[t_1, t_2] \times C \times C$. By a similar argument, we have uniform convergence for all the derivatives of the series. One can check that

$$\overline{K}_t(\bar{x}, \bar{y}) = \sum_{g\in G} K_t(x, g\cdot y)g$$

is a fundamental solution to the heat equation for $(\not{D}^{G\backslash\mathcal{E}})^2$ on $G\backslash\mathfrak{X}$. By the uniqueness of the heat kernel, we know that the series

$$\sum_{g\in G} K_t(x, g\cdot y)g$$

converges uniformly on $[t_1, t_2]\times C\times C$ to the heat kernel of $\not{D}^{G\backslash\mathcal{E}}$. q.e.d.

As $K_t$ is $G$-invariant and $C$ has nonempty intersection with each orbit, the proof of Theorem 3.4 gives rise to an estimate on the heat kernel $K_t(x, y)$, which will be used later.

**Corollary 3.5.** *There exists $L > 0$ such that, for all $x \in \mathfrak{X}$, we have*

$$\sum_{g\in G} \|K_t(x, g\cdot x)g\|_{\mathcal{E}_x} \leq L. \tag{3.10}$$

**3.3. Cut-off functions and $(g)$-trace class operators.** We aim to define some meaningful traces for the heat kernel operator $e^{-t(\not{D}^{\mathcal{E}})^2}$. We shall first introduce a class of operators on $\mathfrak{X}$ "approximating" the heat kernel operator.

**Definition 3.6.** *Let $\mathbf{S}$ be the algebra of bounded operators on $L^2(\mathfrak{X}, \mathcal{E})$ having the following properties:*

1) *The Schwartz kernel $K_S$ of $S \in \mathbf{S}$ is smooth and $G$-invariant—in particular,*

$$K_S(gx, gy) = g[K_S(x, y)]g^{-1} \qquad \forall x, y \in \mathfrak{X}, \forall g \in G.$$

2) *The kernel $K_S$ for $S \in \mathbf{S}$ is properly supported, in the sense of (*3.6*).*

Because of condition (1) in Definition 3.6, the usual trace of $S \in \mathbf{S}$ might be infinite. To eliminate the repeated summations caused by the $G$-invariance, we will use of the following "weight" function over $\mathfrak{X}$.

Using Lemma 2.3 on the local structure of the $G$-orbifold $\mathfrak{X}$, let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ be the corresponding proper étale grouppoid that locally looks like (cf. Remark 2.5)

$$(G \times_{G_i} \tilde{U}_i) \rtimes H_i \rightrightarrows G \times_{G_i} \tilde{U}_i$$

and naturally admits a $G$-action on the left. Recall that a smooth function $f$ on $\mathfrak{X}$ corresponds to a smooth $\mathcal{G}$-invariant function on $\mathcal{G}_0$, that is, $s^*f = t^*f$. A smooth function $f$ on $\mathfrak{X}$ defines a unique continuous function of $|\mathfrak{X}|$ that will also be denoted by $f$. A function on $\mathfrak{X}$ is called compactly supported if its support in $|\mathfrak{X}|$ is compact.

**Definition 3.7** (**Cut-off function**). *A nonnegative function $c \in C_c^\infty(\mathfrak{X}) = C_c^\infty(\mathcal{G})$ is called a* **cut-off function** *of $\mathfrak{X}$ associated to the $G$ action if the values over the orbits of $G$-action add up to be* 1*, that is,*

$$\sum_{g \in G} c(g^{-1}x) = 1, \quad \forall x \in |\mathfrak{X}|. \tag{3.11}$$

Notice that the action of the cut-off function $c \in C_c^\infty(\mathfrak{X})$ on $f \in \Gamma(\mathfrak{X}, \mathcal{E})$ is given by point-wise multiplication:

$$[c \cdot f](x) = c(x)f(x). \tag{3.12}$$

Remark 3.8. A cut-off function always exists for a proper co-compact $G$-action on $\mathfrak{X}$. In fact, let $h \in C_c^\infty(\mathfrak{X})$ be a nonnegative function whose support has nonempty intersection with each $G$-orbit. Then

$$c(x) = \frac{h(x)}{\sum_{g \in G} h(g^{-1}x)}$$

is a cut-off function.

We shall construct one particular cut-off function $c$ on $\mathfrak{X}$, where we suppose

$$|\mathfrak{X}| = \bigcup_{i=1}^N G \times_{G_i} U_i$$

and each $U_i$ has an orbifold chart $(\tilde{U}_i, H_i, \pi_i, U_i)$.

Let $\{\bar{\phi}_i\}$ be a partition of unity of $G \backslash \mathfrak{X}$ subordinate to the open cover

$$\{V_i = G_i \backslash U_i = G_i \backslash \tilde{U}_i / H_i\}_{i=1}^N$$

such that the lift of $\bar{\phi}_i$ to $\tilde{U}_i$ is a smooth $(G_i \times H_i)$-invariant function, denoted by $\tilde{\phi}_i$.

We use $[x]$ to denote the equivalence class of an element $x$. For example, if $g \in G$ and $u \in U_i$, then $[g, u] \in G \times_{G_i} U_i$ and $[u] \in G_i \backslash U_i \subset |G \backslash \mathfrak{X}|$. Let $\varphi_i$ be a function on $G \times_{G_i} U_i \subset \mathfrak{X}$ given by

$$\varphi_i([g, u]) = \begin{cases} \bar{\phi}_i([u]) & g \in G_i \\ 0 & g \notin G_i. \end{cases}$$

As $\bar{\phi}_i$ is $G_i$-invariant, $\varphi_i$ is well defined. Also, $\varphi_i$ extends to a smooth $G_i$-invariant function on $\mathfrak{X}$ when setting $\varphi_i(x) = 0$ for $x \notin G \times_{G_i} U_i$.

Define a smooth nonnegative function on $\mathfrak{X}$ by

$$c(x) := \sum_{i=1}^{N} \frac{1}{|G_i|} \varphi_i(x). \tag{3.13}$$

It is compactly supported because each summand is compactly supported.

**Lemma 3.9.** *The nonnegative function $c \in C_c^\infty(\mathfrak{X})$ given by (3.13) is a cut-off function of $\mathfrak{X}$.*

*Proof.* Fix $x \in \mathfrak{X}$. For each $j \in J := \{i | x \in G \times_{G_i} U_i\}$, there exists $h_j \in G$ and $u_j \in U_j$ so that $x = [h_j, u_j]$. Note also that if $j \notin J$, $\varphi_j(x) = 0$. Hence, by (3.13)

$$\sum_{g \in G} c(g^{-1}x) = \sum_{g \in G} \sum_{j \in J} \frac{1}{|G_j|} \varphi_j[g^{-1}h_j, u_j].$$

As $\varphi_j$ vanishes unless $g^{-1}h_j = k \in G_j$, then by the $G_j$-invariance of $\varphi_j$ we obtain

$$\sum_{g \in G} c(g^{-1}x) = \sum_{k \in G_j} \sum_{j \in J} \frac{1}{|G_j|} \varphi_j([k, u_j]) = \sum_{j \in J} \bar{\phi}_j([u_j]).$$

Let $\pi : \mathfrak{X} \to G \backslash \mathfrak{X}$ be the quotient map then by definition $\pi(x) = [u_j]$ for all $j \in J$ and $\bar{\phi}_j(\pi(x)) = 0$ if $j \notin J$. As $\{\bar{\phi}_i\}$ is a partition of unity on $G \backslash \mathfrak{X}$, we have

$$\sum_{j \in J} \bar{\phi}_j([u_j]) = \sum_{j=1}^{N} \bar{\phi}_j(\pi(x)) = 1.$$

The lemma is then proved. q.e.d.

The cut-off function is designed to deal with $G$-invariant sections and $G$-invariant operators. Denote by $\Gamma(\mathfrak{X}, \mathcal{E})^G$ the subset of $G$-invariant sections in $\Gamma(\mathfrak{X}, \mathcal{E})$. We shall present some properties of the cut-off function to be used later.

**Lemma 3.10.** *Let $c \in C_c^\infty(\mathcal{G}_0)$ be the cut-off function given by (3.13). Then:*

1) *For a smooth function $f$ on $G\backslash\mathfrak{X}$ and its lift, the $G$-invariant function $\tilde{f} \in C^\infty(\mathfrak{X})$, we have*

$$\int_{\mathfrak{X}}^{orb} c(x)\tilde{f}(x) d\,\mathrm{vol}_{\mathfrak{X}}(x) = \int_{G\backslash\mathfrak{X}}^{orb} f(x) d\,\mathrm{vol}_{G\backslash\mathfrak{X}}(x).$$

2) *If a continuous function $h$ on $\mathfrak{X}\times\mathfrak{X}$ satisfies that*

$$h(gx, gy) = h(x,y), \qquad \forall g \in G, \forall x, y \in \mathfrak{X} \tag{3.14}$$

*and that $c(x)h(x,y), c(y)h(x,y)$ are integrable on $\mathfrak{X}\times\mathfrak{X}$, then we have*

$$\int_{\mathfrak{X}\times\mathfrak{X}}^{orb} c(x)h(x,y) d\,\mathrm{vol}_{\mathfrak{X}}(x) d\,\mathrm{vol}_{\mathfrak{X}}(y) = \int_{\mathfrak{X}\times\mathfrak{X}}^{orb} c(y)h(x,y) d\,\mathrm{vol}_{\mathfrak{X}}(x) d\,\mathrm{vol}_{\mathfrak{X}}(y).$$

*Proof.* Let $\{\bar{\phi}_i\}$ be a partition of unity of $G\backslash\mathfrak{X}$ subordinate to the open cover

$$\{V_i = G_i\backslash U_i = G_i\backslash \tilde{U}_i/H_i\}_{i=1}^N$$

such that the lift of $\bar{\phi}_i$ to $\tilde{U}_i$ is a smooth $(G_i \times H_i)$-invariant function, denoted by $\tilde{\phi}_i$. Let $\{\phi_i\} \subset C^\infty(\mathfrak{X})$ be the $G$-invariant partition of unity of the cover $\{G \times_{G_i} U_i\}_{i=1}^N$ given by $\phi_i[g,u] = \bar{\phi}_i[u]$ for $g \in G$ and $u \in U_i$ locally.

(1) Using the cut-off function as in (3.13) , we have

$$\begin{aligned}
& \int_{\mathfrak{X}}^{orb} c(x)\tilde{f}(x) d\,\mathrm{vol}_{\mathfrak{X}}(x) \\
= & \int_{\mathfrak{X}}^{orb} \sum_{i=1}^N \frac{1}{|G_i|}\varphi_i(x)\tilde{f}(x) d\,\mathrm{vol}_{\mathfrak{X}}(x) \\
= & \sum_{i=1}^N \int_{U_i}^{orb} \frac{1}{|G_i|}\varphi_i(x)\tilde{f}(x) d\,\mathrm{vol}_{\mathfrak{X}}(x) \\
= & \sum_{i=1}^N \frac{1}{|H_i \times G_i|} \int_{\tilde{U}_i} \tilde{\phi}_i(x)\tilde{f}(x) dx \\
= & \int_{G\backslash\mathfrak{X}}^{orb} f(x) d\,\mathrm{vol}_{G\backslash\mathfrak{X}}(x).
\end{aligned}$$

(2) Using (3.13) for the cut-off function and the partition of unity $\{\phi_j\}$ for $\mathfrak{X}$ subordinated to $\{G \times_{G_j} U_j\}_{j=1}^N$, we get

(3.15)
$$\begin{aligned}
&\int^{orb}_{\mathfrak{X}\times\mathfrak{X}} c(x)h(x,y)d\operatorname{vol}_{\mathfrak{X}}(x)d\operatorname{vol}_{\mathfrak{X}}(y)\\
= &\sum_{i=1}^N \int^{orb}_{U_i}\int^{orb}_{\mathfrak{X}} \frac{1}{|G_i|}\varphi_i(x)h(x,y)d\operatorname{vol}_{\mathfrak{X}}(x)d\operatorname{vol}_{\mathfrak{X}}(y)\\
= &\sum_{i=1}^N \frac{1}{|G_i\times H_i|}\int_{\tilde{U}_i}\int^{orb}_{\mathfrak{X}} \tilde{\phi}_i(x)\ h(x,y)dxd\operatorname{vol}_{\mathfrak{X}}(y)\\
= &\sum_{i,j=1}^N \frac{1}{|G_i||H_i|}\int_{\tilde{U}_i}\int^{orb}_{G\times_{G_j}U_j} \tilde{\phi}_i(x)\phi_j(y)h(x,y)dxd\operatorname{vol}_{\mathfrak{X}}(y)\\
= &\sum_{k\in G}\sum_{i,j=1}^N \frac{1}{|G_i||G_j||H_i||H_j|}\int_{\tilde{U}_i\times\tilde{U}_j} \tilde{\phi}_i(x)\tilde{\phi}_j(y)h(x,k\cdot y)dxdy.
\end{aligned}$$

Here, the last equality follows from

$$\int^{orb}_{G\times_{G_j}U_j}\phi_j(y)h(x,y)d\operatorname{vol}_{\mathfrak{X}}(y) = \sum_{k\in G}\frac{1}{|G_j|}\int^{orb}_{U_j}\tilde{\phi}_j(y)h(x,k\cdot y)dy.$$

Similarly, we have

(3.16)
$$\begin{aligned}
&\int^{orb}_{\mathfrak{X}\times\mathfrak{X}} c(y)h(x,y)d\operatorname{vol}_{\mathfrak{X}}(x)d\operatorname{vol}_{\mathfrak{X}}(y)\\
= &\sum_{k\in G}\sum_{i,j=1}^N \frac{1}{|G_i||G_j||H_i||H_j|}\int_{\tilde{U}_i\times\tilde{U}_j} \tilde{\phi}_i(x)\tilde{\phi}_j(y)h(k\cdot x,y)dxdy.
\end{aligned}$$

By (3.14), we have $h(x, k\cdot y) = h(k^{-1}\cdot x, y)$ for $k \in G$. Thus, the $k$-term in (3.15) is the $k^{-1}$-term in (3.16). Then the integrals (3.15) and (3.16) coincide. The lemma is then proved. q.e.d.

Remark 3.11. Condition (2) in Definition 3.6 is to reduce the complexity in calculating the new traces. However, $e^{-t(\not{D}^{\mathcal{E}})^2}$ does not have proper support. In view of Theorem 3.4, when $t > 0$ is small, the heat kernel decays exponentially off the diagonal. Therefore, we shall use the following lemma to replace the heat kernel by an operator in the class **S** having the same value on the diagonal.

**Lemma 3.12.** *For any $G$-invariant smoothing operator $S$, the operator*

$$S_0 := \sum_{g\in G} g\cdot(c^{\frac{1}{2}}Sc^{\frac{1}{2}}) \tag{3.17}$$

*belongs to* **S** *(cf. Definition 3.6) and has the same Schwartz kernel as that of $S$ along the diagonal.*

*Proof.* Let $K_S$ and $K_{S_0}$ be the heat kernels for the operators $S$ and $S_0$, respectively. Then by definition (3.17) we have

$$K_{S_0}(x,y) = \sum_{g\in G} c(g^{-1}x)^{\frac{1}{2}} K_S(x,y) c(g^{-1}y)^{\frac{1}{2}} \tag{3.18}$$

and conclude that $S_0$ is a properly supported smoothing operator. Moreover, if $x = y$, then

$$K_{S_0}(x,x) = \sum_{g\in G} c(g^{-1}x) K_S(x,x) = K_S(x,x).$$

The lemma is then proved. q.e.d.

In the following, we shall define a family of traces, indexed by the conjugacy classes of elements in $G$, for the set of operators slightly larger than the set $\mathbf{S}$ defined in Definition 3.6. These traces are closely related to the localized indices in Section 5 (see Lemma 5.6).

For any $g \in G$, denote by $(g)_G \subset G$ (or $(g)$ when there is no ambiguity) the conjugacy class of $g \in G$. Also, *whenever* traces of an operator involved in this paper, if the space is $\mathbb{Z}/2\mathbb{Z}$-graded, we shall mean a graded traces or a supertrace:

$$\mathrm{tr}_s \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \mathrm{tr}(A) - \mathrm{tr}(B).$$

**Definition 3.13** (**($g$)-trace class operator).** *A bounded properly supported $G$-invariant operator $S : L^2(\mathfrak{X}, \mathcal{E}) \to L^2(\mathfrak{X}, \mathcal{E})$ is of* **($g$)-trace class** *if the operator*

$$\sum_{h\in(g)} h^{-1}\phi|S|\psi$$

*is of trace class for all positive functions $\phi, \psi \in C_c^\infty(\mathcal{G}_0)$. Here, $|S| = (SS^*)^{\frac{1}{2}}$ and $h^{-1}$ stands for the unitary operator on $L^2(\mathfrak{X}, \mathcal{E})$ given by*

$$(h^{-1}\cdot u)(x) := h^{-1}u(hx) \qquad u \in L^2(\mathfrak{X}, \mathcal{E}).$$

*If $S$ is a $(g)$-trace class operator, the $(g)$-supertrace (or simply the $(g)$-trace) is defined by the formula*

$$\mathrm{tr}_s^{(g)}(S) := \sum_{h\in(g)} \mathrm{tr}_s[h^{-1}cS], \tag{3.19}$$

*where $c$ is a cut-off function on $\mathfrak{X}$.*

Remark 3.14. In case of a $G$-manifold, where $g$ is the group identity, the $(e)$-trace is the same notion as the $G$-trace of bounded $G$-invariant operators on $L^2(\mathfrak{X}, \mathcal{E})$ introduced in [**W**].

Denote by $K$ a chosen subset of $G$ having the following properties:

$$\begin{aligned} &\{kgk^{-1} | k \in K\} = (g); \\ &\text{if } k_1 \neq k_2, \text{ for all } k_j \in K, \text{ then } k_1gk_1^{-1} \neq k_2gk_2^{-1}. \end{aligned} \tag{3.20}$$

We shall sometimes use $\{kgk^{-1}\}_{k\in K}$ to denote the conjugacy class $(g)$ of $g \in G$. In view of the following lemma, we shall identify $K$ with the quotient $G/Z_G(g)$, where $Z_G(g) = \{h \in G | hg = gh\}$ is the centralizer of $g$ in $G$. Note that, if finite, $K$ has the same cardinality as that of $(g)$.

**Lemma 3.15.** *Let $K$ be a subset of $G$ having the property (3.20). Then*

$$K \cdot Z_G(g) = G.$$

*Proof.* If $k_1, k_2 \in K$ and $h_1, h_2 \in Z_G(g)$ satisfy $k_1h_1 = k_2h_2$, then

$$k_1h_1gh_1^{-1}k_1^{-1} = k_2h_2gh_2^{-1}k_2^{-1}$$

and it is the same as $k_1gk_1^{-1} = k_2gk_2^{-1}$. By definition of $K$, we have $k_1 = k_2$, and hence $h_1 = h_2$. Then the map

$$m : K \times Z_G(g) \longrightarrow G \qquad (k, h) \mapsto kh \tag{3.21}$$

is injective. For each $l \in G$, then by definition there is a $k \in K$ so that $lgl^{-1} = kgk^{-1}$. Thus, $k^{-1}l \in Z_G(g)$ and (3.21) is surjective. The lemma is then proved. q.e.d.

**Proposition 3.16.** *Let $S$ be a $(g)$-trace class operator having a smoothing Schwartz kernel $K_S$, and let $c$ be a cut-off function on $\mathfrak{X}$. Then*

$$\operatorname{tr}_s^{(g)} S = \sum_{h\in(g)} \int_{\mathfrak{X}}^{orb} c(x) \operatorname{Tr}_s[h^{-1}K_S(hx, x)] d\operatorname{vol}_{\mathfrak{X}}(x). \tag{3.22}$$

*Here,* $\operatorname{Tr}_s$ *is the matrix supertrace of* $\operatorname{End}(\mathcal{E}_x, \mathcal{E}_x)$*. Alternatively,*

$$\operatorname{tr}_s^{(g)} S = \sum_{k\in G/Z_G(g)} \int_{\mathfrak{X}}^{orb} c(kx) \operatorname{Tr}_s[g^{-1}K_S(gx, x)] d\operatorname{vol}_{\mathfrak{X}}(x), \tag{3.23}$$

*where $G/Z_G(g)$ is identified as a subset $K$ of $G$ having property (3.20). In particular, operators from* $\mathbf{S}$ *given by Definition 3.6 are of $(g)$-trace classes.*

*Proof.* As $(h^{-1} \cdot u)(x) = h^{-1}u(hx)$ for $u \in L^2(\mathfrak{X}, \mathcal{E})$, we have

$$[h^{-1}cS]u(x) = c(hx)h^{-1}Su(hx) = \int_{\mathfrak{X}}^{orb} c(hx)h^{-1}[K_S(hx, y)]u(y) d\operatorname{vol}_{\mathfrak{X}}(y).$$

Then by the change of variable $x \to h^{-1}x$ and the invariance of the measure $d\operatorname{vol}_{\mathfrak{X}}(h^{-1}x) = d\operatorname{vol}_{\mathfrak{X}}(x)$, we obtain

$$\operatorname{tr}_s[h^{-1}cS] = \int_{\mathfrak{X}}^{orb} c(x) \operatorname{Tr}_s[h^{-1}K_S(hx, x)] d\operatorname{vol}_{\mathfrak{X}}(x).$$

If $h = kgk^{-1}$, then as $S$ is $G$-invariant, by (3.5) we have $h^{-1}[K_S(hx', x')] = g^{-1}[K_S(gx, x)]$, where $x' = kx$. Therefore,

$$\begin{aligned}\operatorname{tr}_s^{(g)} S &= \sum_{h\in(g)} \operatorname{tr}_s[h^{-1}cS]\\ &= \sum_{h\in(g)} \int_{\mathfrak{X}}^{orb} c(x)\operatorname{Tr}_s[h^{-1}K_S(hx,x)]d\operatorname{vol}_{\mathfrak{X}}(x)\\ &= \sum_{k\in G/Z_G(g)} \int_{\mathfrak{X}}^{orb} c(kx)\operatorname{Tr}_s[g^{-1}K_S(gx,x)]d\operatorname{vol}_{\mathfrak{X}}(x).\end{aligned}$$

If $S \in \mathbf{S}$, then as the action of $G$ on $\mathfrak{X}$ is proper, there are finitely many nonvanishing terms in the sum in (3.22). Hence operators from $\mathbf{S}$ are of $(g)$-trace class. The proposition is proved. q.e.d.

Remark 3.17. The trace $\operatorname{tr}_s^{(g)}$ given by Definition 3.13 does not depend on the choice of cut-off function $c$. In fact, set

$$m(x) := \sum_{h\in(g)} \operatorname{Tr}_s[h^{-1}K_S(hx,x)]$$

and observe that for any $k \in G$, we have

$$\begin{aligned} m(kx) &= \textstyle\sum_{h\in(g)} \operatorname{Tr}_s h^{-1}[K_S(hkx,kx)]\\ &= \textstyle\sum_{h\in(g)} \operatorname{Tr}_s[(k^{-1}hk)^{-1}K_S(k^{-1}hkx,x)] = m(x).\end{aligned}$$

Then by Lemma 3.10 (1) the integral over the orbifold $\mathfrak{X}$ does not depend on choice of a cut-off function $c$.

The $(g)$-trace defined in (3.19) has the tracial property as is shown in the following proposition. As we shall see later in the proof of Lemma 5.6, this trace is the composition of a matrix trace and the localized $(g)$-trace given by Definition 5.2.

**Proposition 3.18.** *Let $S, T$ be $G$-invariant operators and let $ST$ and $TS$ be of $(g)$-trace class. Then*

$$\operatorname{tr}_s^{(g)}(ST) = \operatorname{tr}_s^{(g)}(TS).$$

*Proof.* By Proposition 3.16, $\operatorname{tr}_s^{(g)}(ST)$ is equal to

$$\sum_{h\in(g)} \int_{\mathfrak{X}\times\mathfrak{X}}^{orb} c(x)\operatorname{Tr}_s[h^{-1}(K_S(hx,y)K_T(y,x))]d\operatorname{vol}_{\mathfrak{X}}(x)d\operatorname{vol}_{\mathfrak{X}}(y). \tag{3.24}$$

Similarly, we arrive at the $(g)$-trace for $TS$:

(3.25)
$$\begin{aligned}\operatorname{tr}_s^{(g)}(TS) &= \sum_{h\in(g)} \int^{orb}_{\mathfrak{X}\times\mathfrak{X}} c(y)\operatorname{Tr}_s[h^{-1}(K_T(hy,x)K_S(x,y))]dxdy\\ &= \sum_{h\in(g)} \int^{orb}_{\mathfrak{X}\times\mathfrak{X}} c(y)\operatorname{Tr}_s[h^{-1}(K_T(y,x)K_S(x,h^{-1}y))]dxdy,\end{aligned}$$

where the latter equality comes from the change of variables $y \mapsto h^{-1}y$. Set

$$m(x,y) = \sum_{h\in(g)} \operatorname{Tr}_s[h^{-1}(K_S(hx,y)K_T(y,x))].$$

Then the $G$-invariance property of $S$ and $T$ implies that $m$ satisfies (3.14):

$$\begin{aligned} m(kx,ky) &= \sum_{h\in(g)} \operatorname{Tr}_s[h^{-1}K_S(hkx,ky)K_T(ky,kx)]\\ &= \sum_{h\in(g)} \operatorname{Tr}_s[(k^{-1}hk)^{-1}K_S(k^{-1}hkx,y)K_T(y,x)]\\ &= m(x,y), \qquad \forall k\in G.\end{aligned}$$

Notice that

$$\begin{aligned}\operatorname{Tr}_s[h^{-1}(K_S(hx,y)K_T(y,x))] &= \operatorname{Tr}_s[K_T(y,x)h^{-1}K_S(hx,y)]\\ &= \operatorname{Tr}_s[h^{-1}K_T(y,x)K_S(x,h^{-1}y)].\end{aligned}$$

Then we have

$$\begin{aligned}\operatorname{tr}_s^{(g)}(ST) &= \int^{orb}_{\mathfrak{X}\times\mathfrak{X}} c(x)m(x,y)dxdy\\ &= \int^{orb}_{\mathfrak{X}\times\mathfrak{X}} c(y)m(x,y)dxdy = \operatorname{tr}_s^{(g)}(TS),\end{aligned}$$

by Lemma 3.10(2). The proposition is then proved. q.e.d.

Finally, we look at the $G$-invariant heat operator $S_t := e^{-t(\not{D}^{\mathcal{E}})^2}$ for $t>0$, where $\not{D}^{\mathcal{E}}$ is the $G$-invariant Dirac operator on $\mathfrak{X}$ given by (3.7). Denote by $K_t(x,y)$ the Schwartz kernel of $e^{-t(\not{D}^{\mathcal{E}})^2}$, which can be "approximated" by elements of $\mathbf{S}$, that is, the norm of $K_t(x,y)$ decays rapidly when $d(x,y)\to\infty$ for a fixed positive number $t$ (cf. Theorem 3.4).

Following from Theorem 3.4, when $g\in G$ does not have a fixed point on $\mathfrak{X}$, we have

$$\lim_{t\to 0^+} g^{-1}K_t(gx,x) = 0. \tag{3.26}$$

If $g\in G$ has a fixed point $(h,x)\in\mathfrak{X}$ where $h\in G$ and $x\in U_i$, then

$$g(h,x) = (gh,x) = (h(h^{-1}gh),x)$$

implies that $h^{-1}gh \in G_i$ and fixes $x \in U_i$. Then the following lemma implies that we may assume $g \in G_i$ and $g$ fixes the point $(e, x)$.

**Lemma 3.19.** *If $g \in G$ has a fixed point on $\mathfrak{X}$, then there exists an $i \in \{1, \ldots, N\}$, so that $(g) \cap G_i$ is not empty.*

**Proposition 3.20.** *Let $S_t$ be the heat operator $e^{-t(\not{D}^{\mathcal{E}})^2}$, and let $K_t$ be its Schwartz kernel. Then it is of $(g)$-trace class and its $(g)$-trace is independent of $t$ given by the finite sum*

$$\operatorname{tr}_s^{(g)} S_t = \sum_{h\in(g)\cap(\cup_{i=1}^N G_i)} \lim_{t\to 0^+} \int_{\mathfrak{X}}^{orb} c(x) \operatorname{Tr}_s[h^{-1}K_t(hx, x)] d\operatorname{vol}_{\mathfrak{X}}(x). \tag{3.27}$$

*Proof.* Let us estimate the sum $\sum_{h\in(g)} |\operatorname{Tr}_s[h^{-1}\phi S_t \psi]|$, where $\phi, \psi \in C_c^\infty(\mathfrak{X})$, following from Definition 3.13, as follows:

$$\begin{aligned} &\sum_{h\in(g)} |\operatorname{Tr}_s[h^{-1}\phi S_t\psi]| \\ &\le \sum_{h\in(g)} \int_{\mathfrak{X}}^{orb} \phi(hx)\psi(x)\dim(\mathcal{E})\|h^{-1}K_t(hx,x)\|_{\mathcal{E}_x} d\operatorname{vol}_{\mathfrak{X}}(x) \\ &\le \dim\mathcal{E}\|\phi\|_{L^\infty} \int_{\mathfrak{X}}^{orb} \psi(x) \sum_{h\in G} \|h^{-1}K_t(hx,x)\|_{\mathcal{E}_x} d\operatorname{vol}_{\mathfrak{X}}(x) \\ &\le L\dim\mathcal{E}\|\phi\|_{L^\infty}\|\psi\|_{L^1} < \infty. \end{aligned} \tag{3.28}$$

Here, in the last inequality, we have used the uniform upper estimate in Corollary 3.5. Therefore, the heat operator $S_t = e^{-t(\not{D}^{\mathcal{E}})^2}$ on $\mathfrak{X}$ is of $(g)$-trace class for all $g \in G$ and $t > 0$.

Let $a(t) = \operatorname{tr}_s^{(g)} S_t$. Then

$$\begin{aligned} \frac{da(t)}{dt} &= -\operatorname{tr}_s^{(g)}((\not{D}^{\mathcal{E}})^2 e^{-t(\not{D}^{\mathcal{E}})^2}) \\ &= -\tfrac{1}{2}\operatorname{tr}_s^{(g)}([\not{D}^{\mathcal{E}}, \not{D}^{\mathcal{E}} e^{-t(\not{D}^{\mathcal{E}})^2}]), \end{aligned}$$

which vanishes by Proposition 3.18. So the function $a(t) = \operatorname{tr}_s^{(g)} S_t$ is constant in $t$. Then, by Proposition 3.16, we have

$$\operatorname{tr}_s^{(g)} S_t = \lim_{t\to 0^+} \sum_{h\in(g)} \int_{\mathfrak{X}}^{orb} c(x)\operatorname{Tr}_s[h^{-1}K_t(hx,x)] d\operatorname{vol}_{\mathfrak{X}}(x). \tag{3.29}$$

Using the same argument as (3.28), the sum in (3.29) is absolutely convergent. Hence, when we take the limit of the sum, the limit commutes with the infinite sum as well as the integral $\int_{\mathfrak{X}}^{orb}$. Further, the limit commutes with the integral. As $t \to 0^+$ then from (3.26) a summand in (3.29) tends to 0 if $h$ does not fix any point in $\mathfrak{X}$. The nonzero ones are the ones where $hx = x$ for some $x \in \mathfrak{X}$. By Lemma 3.19, we see it happens only when $h \in (g) \cap G_i$ for some $i$. Thus, there are only finitely many summands and

$$\operatorname{tr}_s^{(g)} S_t = \sum_{h \in (g) \cap (\cup_{i=1}^N G_i)} \lim_{t \to 0^+} \int_{\mathfrak{X}}^{orb} c(x) \operatorname{Tr}_s[h^{-1} K_{S_t}(hx, x)] d \operatorname{vol}_{\mathfrak{X}}(x).$$

The proposition is then proved. q.e.d.

**Corollary 3.21.** *Let $S_{0,t} := \sum_{g \in G} g \cdot (c^{\frac{1}{2}} S_t c^{\frac{1}{2}})$, where $S_t = e^{-t(\not{D}^{\mathcal{E}})^2}$. Then*

$$\operatorname{tr}_s^{(g)}(S_t) = \lim_{t \to 0^+} \operatorname{tr}_s^{(g)}(S_{0,t}).$$

*(The LHS is independent of $t$.)*

*Proof.* From Proposition 3.16 and Lemma 3.12, we see that $S_{0,t}$ is of $(g)$-trace class for all $g \in G$. As the sum $\sum_{h \in (g)} K_{S_{0,t}}(hx, x)$ has finitely many nonvanishing terms as $t \to 0^+$, we have

(3.30)
$$\lim_{t \to 0^+} \operatorname{tr}_s^{(g)} S_{0,t} = \sum_{h \in (g)} \int_{\mathfrak{X}}^{orb} c(x) \operatorname{Tr}_s[\lim_{t \to 0^+} h^{-1} K_{S_{0,t}}(hx, x)] d \operatorname{vol}_{\mathfrak{X}}(x).$$

Observe that the kernels of $S_{0,t}$ and $S_t$ have the same diagonal value by Lemma 3.12. In view of Proposition 3.20, we have the same formula for $\operatorname{tr}_s^{(g)} S_t$ in (3.27) as (3.30). Hence, the corollary is proved. q.e.d.

**3.4. Calculation of $(g)$-trace for the heat operator.** We shall derive the formula for the $(g)$-trace of the heat operator $e^{-t(\not{D}^{\mathcal{E}})^2}$ in terms of the local data on the quotient orbifold $G \backslash \mathfrak{X}$.

To calculate the $(g)$-trace of the heat operator $S_t = e^{-t(\not{D}^{\mathcal{E}})^2}$, we need to describe the twisted sectors for the quotient orbfiold $G \backslash \mathfrak{X}$. Let $\mathcal{T}_{\mathfrak{X}}$ be the set for the twisted sectors of $\mathfrak{X}$ consisting of equivalence classes of conjugacy classes in the local isotropy groups of $\mathfrak{X}$. Then the set for the twisted sectors of $G \backslash \mathfrak{X}$ consists of pairs

$$((g), (h)),$$

where $(g)$ is the conjugacy class of $g$ in $G$ such that $g$ has nonempty fixed points and $(h) \in \mathcal{T}_{\mathfrak{X}}$. We now describe the twisted sector $(G \backslash \mathfrak{X})_{((g),(h))}$. Let

$$\{(G \times_{G_i} \tilde{U}_i, H_i, G \times_{G_i} \tilde{U}_i)\}$$

be orbifold charts as in Lemma 2.3 such that $G \times_{G_i} U_i$ consists of a disjoint union of $G$-translate of $U_i$. Suppose that $g \in G$ has nonempty fixed points in $U_i$; then

$$(g) \cap G_i = (g_i)_{G_i}$$

for a conjugacy class of $g_i$ in $G_i$. Let $(h) \in \mathcal{T}_{\mathfrak{X}}$ have a representative $(h_i)$, a conjugacy class in $H_i$. Then

$$\{(\tilde{U}_i^{g_i, h_i}, Z_{G_i}(g_i) \times Z_{H_i}(h_i))\} \tag{3.31}$$

are orbifold charts for the twisted sector $(G\backslash\mathfrak{X})_{((g),(h))}$.

**Definition 3.22.** *The* $(g)$**-twisted sector** *of* $G\backslash\mathfrak{X}$ *is given by*

$$(G\backslash\mathfrak{X})_{(g)} = \bigsqcup_{(h)\in\mathcal{T}_{\mathfrak{X}}} (G\backslash\mathfrak{X})_{((g),(h))}, \tag{3.32}$$

*which is a sub-orbifold of the inertia orbifold* $I(G\backslash\mathfrak{X})$. *Let*

$$\hat{A}_{(g)}(\mathfrak{X}) \text{ and } \mathrm{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E}) \tag{3.33}$$

*be the restrictions of* $\hat{A}_{deloc}(G\backslash\mathfrak{X})$ *and* $\mathrm{ch}^{\mathcal{S}}_{deloc}(G\backslash\mathcal{E})$ *to the* $(g)$*-twisted sector* $(G\backslash\mathfrak{X})_{(g)}$.

**Theorem 3.23.** *For all* $g\in G$, *the heat operator* $S_t = e^{-t(\not{D}^{\mathcal{E}})^2}$ *is a* $(g)$*-trace class operator, and the* $(g)$*-trace is given by*

$$\begin{aligned} \mathrm{tr}^{(g)}_s S_t &= \lim_{t\to 0^+} \sum_{h\in(g)} \int^{orb}_{\mathfrak{X}} c(x)\,\mathrm{Tr}_s[h^{-1}K_t(hx,x)]d\,\mathrm{vol}_{\mathfrak{X}}(x) \\ &= \int^{orb}_{(G\backslash\mathfrak{X})_{(g)}} \hat{A}_{(g)}(\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E}). \end{aligned} \tag{3.34}$$

*Proof.* If $g\in G$ does not fix any point in $\mathfrak{X}$, then by (3.26) we have $\mathrm{tr}^{(g)}_s S_t = 0$. On the other hand, $(G\backslash\mathfrak{X})_{(g)}$ is an empty set. So the second integral also vanishes.

If $g\in G$ has a fixed point in $\mathfrak{X}$, then by Lemma 3.19 we may assume $g\in G_i$ for some $i$. Let $c$ be the cut-off function of $\mathfrak{X}$ given by (3.13) (see also Lemma 3.9). Let $K_t$ (resp. $\bar{K}_t$) be the heat kernel of $\not{D}^{\mathcal{E}}$ (resp. $\not{D}^{G\backslash\mathcal{E}}$), and let $\tilde{K}_t$ be the lift of $\bar{K}_t$ to $\tilde{U}_i\times\tilde{U}_i$. Then we have

$$\begin{aligned} \mathrm{tr}^{(g)}_s S_t &= \sum_{k\in(g)} \int^{orb}_{\mathfrak{X}} c(x)\,\mathrm{Tr}_s[k^{-1}K_t(kx,x)]d\,\mathrm{vol}_{\mathfrak{X}}(x) \\ &= \sum_{k\in(g)}\sum_{i=1}^{N} \int^{orb}_{G_i\times_{G_i}U_i} \frac{1}{|G_i|}\varphi_i(x)\,\mathrm{Tr}_s[k^{-1}K_t(kx,x)]d\,\mathrm{vol}_{\mathfrak{X}}(x). \end{aligned} \tag{3.35}$$

By Corollary 3.5, the sum is absolute convergent. Hence, when we take the limit of the sum, the limit commutes with the infinite sum. Then, notice that by (3.26), as $t\to 0^+$, $\int_{G_i\times_{G_i}U_i}\varphi(x)\,\mathrm{Tr}_s[k^{-1}K_t(kx,x)]\to 0$ unless there exists a point $x=[m,u]\in G_i\times_{G_i}U_i$ fixed by $k\in G$, that is,

$$[m,u] = k[m,u] = [m(m^{-1}km),u],$$

which means that $k' := m^{-1}km\in G_i$. In this situation, we have

$$k^{-1}K_t(kx,x) = K_t([m,u],[m,m^{-1}kmu])k = \bar{K}_t([u],k'[u])k'.$$

To simplify the notation, let us still use $k$ for $k' \in (g)_{G_i}$. Hence, as we take the limit of each summand of (3.35), it vanishes except for the terms $k \in (g)_{G_i}$. Thus, $\lim_{t\to 0^+} \mathrm{tr}_s^{(g)} S_t$ is equal to

$$\sum_{i=1}^N \frac{1}{|G_i|}\frac{1}{|H_i|}\lim_{t\to 0^+}\int_{\tilde U_i}\tilde\phi_i(u)\sum_{k\in(g)_{G_i}}\sum_{(h)\in\mathcal{T}_{\mathfrak{X}}\cap H_i}\sum_{l\in(h)}\mathrm{Tr}_s[\tilde K_t(u,kul)kl]du.$$

As $\tilde K_t$ is $G_i$ and $H_i$ invariant, $\tilde K_t$ remains constant on the conjugacy classes in $G_i$ and in $H_i$. Also, by Lemma 3.15 we have $\frac{|(g)_{G_i}|}{|G_i|} = \frac{1}{|Z_{G_i}(g)|}$ and $\frac{|(h)_{H_i}|}{|H_i|} = \frac{1}{|Z_{H_i}(h)|}$. Therefore, we conclude that

$$\begin{aligned}&\lim_{t\to 0^+}\mathrm{tr}_s^{(g)} S_t\\ &\quad = \sum_{i=1}^N \frac{1}{|Z_{G_i}(g)|}\lim_{t\to 0^+}\int_{\tilde U_i}\tilde\phi_i(u)\sum_{(h)\in\mathcal{T}_{\mathfrak{X}}\cap H_i}\frac{1}{|Z_{H_i}(h)|}\mathrm{Tr}_s[\tilde K_t(u,guh)gh]du.\end{aligned}$$

Applying the standard local index techniques as in [**BGV**, **Bis**, **LYZ**, **Ma**], we get

$$\begin{aligned}&\sum_{(h)\in\mathcal{T}_{\mathfrak{X}}\cap H_i}\frac{1}{|Z_{G_i}(g)|}\frac{1}{|Z_{H_i}(h)|}\lim_{t\to 0^+}\int_{\tilde U_i}\tilde\phi_i(u)\,\mathrm{Tr}_s[K_t(u,guh)gh]du\\ = &\sum_{(h)\in\mathcal{T}_{\mathfrak{X}}\cap H_i}\frac{1}{|Z_{G_i}(g)|}\frac{1}{|Z_{H_i}(h)|}\int_{\tilde U_i^{g,h}}\tilde\phi_i(u)\hat A_{deloc}(\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{deloc}(G\backslash\mathcal{E})\\ = &\int^{orb}_{(G\backslash\mathfrak{X})_{(g)}}\bar\phi_i\hat A_{(g)}(\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E}),\end{aligned}$$

for $g \in G_i$. Therefore,

$$\begin{aligned}\mathrm{tr}_s^{(g)} S_t &= \lim_{t\to 0^+}\mathrm{tr}_s^{(g)} S_t\\ &= \sum_{i=1}^N\int^{orb}_{(G\backslash\mathfrak{X})_{(g)}}\bar\phi_i\hat A_{(g)}(\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E})\\ &= \int^{orb}_{(G\backslash\mathfrak{X})_{(g)}}\hat A_{(g)}(\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E}).\end{aligned}$$

This completes the proof of the theorem. q.e.d.

Remark 3.24. Together with Kawasaki's orbifold index theorem, Theorem 3.23 implies that

$$\begin{aligned}\sum_{(g)}\mathrm{tr}_s^{(g)} S_t &= \sum_{(g)}\int^{orb}_{(G\backslash\mathfrak{X})_{(g)}}\hat A_{(g)}(\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E})\\ &= \int^{orb}_{I(G\backslash\mathfrak{X})}\hat A_{deloc}(G\backslash\mathfrak{X})\,\mathrm{ch}^{\mathcal{S}}_{deloc}(G\backslash\mathcal{E}) = \mathrm{ind}(\not{D}^{G\backslash\mathcal{E}}).\end{aligned}\tag{3.36}$$

In Sections 4 and 5, we shall give a $K$-theoretic interpretation of $\mathrm{tr}_s^{(g)}\, e^{-t(\not{D}^{\mathcal{E}})^2}$, which is called the localized index of $\not{D}^{\mathcal{E}}$ at the conjugacy class $(g)$ of $g$ in $G$.

## 4. Higher index theory for orbifolds

In this section, we propose a higher index of a $G$-invaraint Dirac operator

$$\not{D}^{\mathcal{E}} : L^2(\mathfrak{X}, \mathcal{E}) \to L^2(\mathfrak{X}, \mathcal{E})$$

on a complete Riemannian even-dimensional orbifold $\mathfrak{X}$ with a proper, co-compact, and isometric action of a discrete group $G$. This higher index generalizes the notion of higher index for smooth noncompact manifolds with a proper and free co-compact action of a discrete group $G$ as in [**K1**, **K2**]. This higher index for smooth noncompact manifolds plays an important role in the study of the Novikov conjecture and the Baum–Connes conjecture. See [**BC2**, **BCH**, **Yu**].

When $X$ is a proper co-compact Riemannian $G$-manifold with a $G$-equivaraint Clifford module $\mathcal{E}$, the Dirac operator $\not{D}^{\mathcal{E}}_X$ gives rise to a $K$-homology class

$$[(L^2(X, \mathcal{E}), F = \not{D}^{\mathcal{E}}_X \left[(\not{D}^{\mathcal{E}}_X)^2 + 1\right]^{-\frac{1}{2}})] \in K^0_G(C_0(X)). \tag{4.1}$$

The higher index of $\not{D}^{\mathcal{E}}_X$ is defined to be the image of the homology class (4.1) under the higher index map (see [**K1**])

$$\mu : K^0_G(C_0(X)) \longrightarrow K_0(C^*(G)). \tag{4.2}$$

We refer to [**K2**, **Bl**] for the definition and basic properties of $KK$-group and its relation to $K$-theory and $K$-homology. In Section 4.1, we construct the higher index map for a complete Riemannian orbifold $\mathfrak{X}$ with a proper, co-compact, and isometric action of a discrete group $G$. Then we calculate the higher index of $\not{D}^{\mathcal{E}}$ as a $KK$-cycle in Section 4.2. After that, we show that the higher index of $\not{D}^{\mathcal{E}}$ is related to the orbifold index of $\not{D}^{G\backslash\mathcal{E}}$ via the trivial representation of $G$ (cf. Theorem 4.13).

**4.1. Higher analytic index map.** We formulate the higher index in the context of a proper co-compact $G$-orbifold following the philosophy of [**K1**]. In view of Lemma 2.3 on the local structure of the $G$-orbifold $\mathfrak{X}$, choose the corresponding proper étale grouppoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$, that is, $|\mathfrak{X}| \cong \mathcal{G}_0/\mathcal{G}_1$, which locally looks like (cf. Remark 2.5)

$$(G \times_{G_i} \tilde{U}_i) \rtimes H_i \rightrightarrows G \times_{G_i} \tilde{U}_i$$

and naturally admits a $G$-action on the left. As $G$ acts properly and co-compactly on $\mathfrak{X}$, $G$ acts properly and co-compactly on $\mathcal{G}$. Let $C^*_{red}(\mathfrak{X})$ be the reduced $C^*$-algebra given by Definition 2.6. As the reduced $C^*$-norm is preserved under the left action of $G$ on $C^\infty_c(\mathcal{G}_1)$,

$$(h \cdot f)(g) = f(h^{-1} \cdot g) \qquad \forall h \in G, \forall f \in C^\infty_c(\mathcal{G}_1),$$

the $G$-action extends to $C^*_{red}(\mathfrak{X})$. The convolution algebra $C_c(G, C^*_{red}(\mathfrak{X}))$ is represented as a set of bounded operators on $L^2(G, C^*_{red}(\mathfrak{X}))$ given by the integration of the left regular representation of $G$ on $L^2(G, C^*_{red}(\mathfrak{X}))$

$$C_c(G, C^*_{red}(\mathfrak{X})) \longrightarrow \mathcal{B}(L^2(G, C^*_{red}(\mathfrak{X}))).$$

Denote by $C^*_{red}(\mathfrak{X}) \rtimes_r G$ the closure of $C_c(G, C^*_{red}(\mathfrak{X}))$ under the operator norm of $\mathcal{B}(L^2(G, C^*_{red}(\mathfrak{X})))$. Denote by $C^*_{red}(\mathfrak{X}) \rtimes G$ the closure of the maximal operator norm of all the covariant representations of the convolution algebra $C_c(G, C^*_{red}(\mathfrak{X}))$.

The first ingredient is a projection in $C^*_{red}(\mathfrak{X}) \rtimes G$ constructed from a cut-off function $c$. Let $\mathcal{P}$ be the closure of $C^\infty_c(\mathcal{G}_1)$ under the norm of $C^*_{red}(\mathfrak{X}) \rtimes G$. Then $\mathcal{P}$ is a Hilbert $C^*_{red}(\mathfrak{X}) \rtimes G$-module. Let $c \in C^\infty_c(\mathcal{G})$ be a cut-off function of $\mathfrak{X}$ associated to the $G$ action (cf. Definition 3.7), and let $s : \mathcal{G}_1 \to \mathcal{G}_0$ be the source map. Then the pullback function

$$s^*c(\gamma) := c(s(\gamma)), \quad \forall \gamma \in \mathcal{G}_1$$

gives rise to a "cut-off" function in $C^\infty_c(\mathcal{G}_1)$, which satisfies

$$\sum_{g\in G} s^*c(g^{-1}\gamma) = \sum_{g\in G} c(s(g^{-1}\gamma)) = \sum_{g\in G} c(g \cdot s(\gamma)) = 1.$$

Define a projection $p \in C_c(G, C^\infty_c(\mathcal{G}_1)) \subset C^*_{red}(\mathfrak{X}) \rtimes G$ by

$$p(g)(x) = [s^*c(g^{-1}x)s^*c(x)]^{\frac{1}{2}}, \quad \forall g \in G, \forall x \in \mathcal{G}_1. \tag{4.3}$$

The Hilbert module $\mathcal{P}$ is related to $p$. In fact, the image of the convoluting operation of $p$ on $C^*_{red}(\mathfrak{X}) \rtimes G$ is $\mathcal{P}$, that is,

$$p \cdot [C^*_{red}(\mathfrak{X}) \rtimes G] = \mathcal{P}. \tag{4.4}$$

They represent the same element in the following identifications:

$$\begin{gathered} KK(\mathbb{C}, C^*_{red}(\mathfrak{X}) \rtimes G) \cong K_0(C^*_{red}(\mathfrak{X}) \rtimes G) \cong K_0(C^*_{red}(G\backslash\mathfrak{X})) \\ [(\mathcal{P}, 1_{\mathbb{C}}, 0)] \mapsto [p] \mapsto [1]. \end{gathered} \tag{4.5}$$

The second ingredient is the analytic $K$-homology of $C^*_{red}(\mathfrak{X})$. Let $H$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space equipped with a unitary representation $\pi$ of the group $G$ and with a $*$-homomorphism $\phi : C^*_{red}(\mathfrak{X}) \to \mathcal{B}(H)$, where the two representations respect the $\mathbb{Z}/2\mathbb{Z}$-grading and are compatible with the action of $G$ on $C^*_{red}(\mathfrak{X})$:

$$\phi(g \cdot a) = \pi(g)\phi(a)\pi(g)^{-1} \qquad \forall g \in G, \forall a \in C^*_{red}(\mathfrak{X}).$$

Let $F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}$, where $F_+^* = F_-$, be a bounded operator on $H$ such that

$$\phi(a)(F^2 - 1), [\phi(a), F], [\pi(g), F] \in \mathcal{K}(H) \qquad \forall a \in C^*_{red}(\mathfrak{X}), \forall g \in G. \tag{4.6}$$

Here, $\mathcal{K}(H)$ is the set of compact operators on $H$ and $[\cdot,\cdot]$ is the graded commutator. Then the triple $(H,\pi,F)$ gives rise to a $K$-homology element in $K^0_G(C^*_{red}(\mathfrak{X}))$. This $K$-homology cycle is the abstract model for $G$-invariant elliptic pseudo-differential operators of order 0 on $\mathfrak{X}$. Denote by $[F]$ the equivalence class $[(H,\pi,F)]\in K^0_G(C^*_{red}(\mathfrak{X}))$.

**Definition 4.1** (**Analytic index** [**K3**]). *The* **analytic $K$-theoretic index map** *is the homomorphism*

$$\mu : K^0_G(C^*_{red}(\mathfrak{X})) \longrightarrow K_0(C^*(G)) \tag{4.7}$$

*from a $K$-homology element $[F]\in K^0_G(C^*_{red}(\mathfrak{X}))$ to the $KK$-product of the following elements:*

$$[\mathcal{P}]\in KK(\mathbb{C}, C^*_{red}(\mathfrak{X})\rtimes G) \quad \text{and} \quad j^G([F])\in KK_0(C^*_{red}(\mathfrak{X})\rtimes G, C^*(G)).$$

*Here, $j^G$ is given by*

$$j^G : KK^G(C^*_{red}(\mathfrak{X}),\mathbb{C}) \longrightarrow KK_0(C^*_{red}(\mathfrak{X})\rtimes G, C^*(G)), \tag{4.8}$$

*the descent homomorphism (cf.* [**K2**, 3.11]*).*

In order to accommodate the localized indices to be introduced in the next section, we introduce some variations of the analytic index map taking values in the $K$-theory of some completions of $\mathbb{C}G$ in some other norms.

First of all, recall that the left regular representation of $L^1(G)$ on $L^2(G)$ extends to a natural surjective $*$-homomorphism $r : C^*(G)\to C^*_r(G)$, which induces a $K$-theory homomorphism

$$r_* : K_0(C^*(G)) \longrightarrow K_0(C^*_r(G)).$$

Composing $r_*$ with (4.7) gives rise to the reduced version of the analytic index map.

**Definition 4.2** (**Analytic index (reduced version)**). *The* **reduced** *analytic $K$-theoretic index map is the homomorphism $\mu_{red} := r_*\circ\mu$*

$$\mu_{red} : K^0_G(C^*_{red}(\mathfrak{X})) \longrightarrow K_0(C^*_r(G)) \tag{4.9}$$

*from a $K$-homology element $[F]\in K^0_G(C^*_{red}(\mathfrak{X}))$ to the $KK$-product of $r_*[\mathcal{P}]\in KK(\mathbb{C}, C^*_{red}(\mathfrak{X})\rtimes_r G)$ with*

$$j^G_r([F]) := r_*\circ j^G[F]\in KK_0(C^*_{red}(\mathfrak{X})\rtimes_r G, C^*_r(G)).$$

Remark 4.3. Observe that the $K$-theory element $r_*[\mathcal{P}]=[r\circ\mathcal{P}]$ is represented by $\mathcal{P}_{red}:=r(\mathcal{P})$, which is the $C^*_{red}(\mathfrak{X})\rtimes_r G$-module given by $r(\mathcal{P})=p[C^*_{red}(\mathfrak{X})\rtimes_r G]$ in view of (4.4).

Moreover, assume that $\mathcal{S}(G)$ is a Banach algebra containing $\mathbb{C}G$ as a dense subalgebra. Then there is a condition for $\mathcal{S}(G)$ due to Lafforgue [**L1**], which gives rise to the Banach algebra version of the analytic index map.

**Definition 4.4** (**Unconditional completion** [**L1**])**.** *Let $\mathcal{S}(G)$ be a Banach algebra equipped with a norm $\|\cdot\|_{\mathcal{S}(G)}$ and containing $\mathbb{C}G$ as a dense subalgebra. Then $\mathcal{S}(G)$ is called an* **unconditional completion** *of $\mathbb{C}G$ if for any $f_1, f_2 \in \mathbb{C}G$ satisfying $|f_1(g)| \leq |f_2(g)|, \forall g \in G$, we have $\|f_1\|_{\mathcal{S}(G)} \leq \|f_2\|_{\mathcal{S}(G)}$.*

Let $B$ be a $G$-Banach-algebra with norm $\|\cdot\|_B$. Denote by $\mathcal{S}(G, B)$ the completion of $C_c(G, B)$ with respect to the norm

$$\|\sum_{g\in G} a_g g\| := \|\sum_{g\in G} \|a_g\|_B g\|_{\mathcal{S}(G)}, \quad \sum_{g\in G} a_g g \in C_c(G, B).$$

If $\mathcal{S}(G)$ is an unconditional completion, then by [**L1**] the descent map formulated using Banach $KK$-theory is well defined:

$$j^G_{\mathcal{S}(G)} : KK(C^*_{red}(\mathfrak{X}), \mathbb{C}) \to KK^{ban}(\mathcal{S}(G, C^*_{red}(\mathfrak{X})), \mathcal{S}(G)). \tag{4.10}$$

In (4.10) we used the $KK$-group $KK^{ban}_G(A, B)$ associated to two $G$-Banach algebras $A$ and $B$. The group is defined by generalized Kasparov cycles of the form $(E, \phi, F)$ modulo suitable equivalence relations [**L1**]. Here, $E$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Banach $B$-module and $F \in \mathcal{B}(E)$ is an odd self-adjoint operator. In addition, $G$ represents in $\mathcal{B}(E)$ as a grading-preserving unitary representation and $\phi : A \to \mathcal{B}(E)$ is a grading-preserving homomorphism.

Denote by $\mathcal{K}(E)$ the set of compact operators over the Banach $B$-module $E$. Then the triple $(E, \phi, F)$ is a **generalized Kasparov cycle** if

$$[\phi(a), F], \phi(a)(F^2 - 1), [\pi(g), F] \in \mathcal{K}(E) \qquad \forall a \in A, \forall g \in G.$$

Analogously, the Banach algebra version of analytic index map can be defined as follows.

**Definition 4.5** (**Analytic index (Banach algebra version)** [**L1**])**.** *Let $\mathcal{S}(G)$ be an unconditional completion of $\mathbb{C}G$. The* **Banach algebra version** *of the analytic $K$-theoretic index map is the homomorphism*

$$\mu_{\mathcal{S}(G)} : K^0_G(C^*_{red}(\mathfrak{X})) \longrightarrow K_0(\mathcal{S}(G)) \tag{4.11}$$

*from a $K$-homology element $[F] \in K^0_G(C^*_{red}(\mathfrak{X}))$ to the $KK$-product of the element*

$$[\mathcal{P}_{\mathcal{S}(G)}] \in KK^{ban}(\mathbb{C}, \mathcal{S}(G, C^*_{red}(\mathfrak{X}))),$$

*represented by the Banach $\mathcal{S}(G, C^*_{red}(\mathfrak{X}))$-module $\mathcal{P}_{\mathcal{S}(G)} := p[\mathcal{S}(G, C^*_{red}(\mathfrak{X}))]$, with the element*

$$j^G_{\mathcal{S}(G)}([F]) \in KK^{ban}(\mathcal{S}(G, C^*_{red}(\mathfrak{X})), \mathcal{S}(G))$$

*given by (4.10).*

REMARK 4.6. The Banach algebra $L^1(G) \subset C_r^*(G)$ is an unconditional completion of $\mathbb{C}G$. In general, if we have an inclusion $i^{\mathcal{S},C_r^*}$ : $\mathcal{S}(G) \to C_r^*(G)$ for $\mathcal{S}(G)$, then the (reduced) higher index and the Banach algebra version higher index are related as follows:

$$r_*\mu[F] = \mu_{red}[F] = i_*^{\mathcal{S},C_r^*}\mu_{\mathcal{S}(G)}[F], \qquad \forall [F] \in K_G^0(C_{red}^*(\mathfrak{X})).$$

**4.2. Higher index for a discrete group action on a noncompact orbifold.** Given the abstract setting of the higher index maps in Section 4.1, we shall describe the $K$-homological cycles in $K_G^0(C_{red}^*(\mathfrak{X}))$ of the Dirac operator $\not{D}^{\mathcal{E}}$ and its analytic index in $K_0(C^*(G))$ and in $K_0(\mathcal{S}(G))$ in terms of (generalized) Kasparov cycles.

Let $G$ be a discrete group acting properly, co-compactly, and isometrically on a complete Riemannian orbifold $\mathfrak{X}$. Let $\mathcal{E}$ be a Hermitian orbifold vector bundle over $\mathfrak{X}$. The Hilbert space $L^2(\mathfrak{X},\mathcal{E})$ is the completion of the compactly supported smooth sections $\Gamma_c(\mathfrak{X},\mathcal{E})$ with respect to the following inner product:

$$\langle f,g\rangle_{L^2} = \int_{\mathfrak{X}}^{orb} \langle f(x),g(x)\rangle_{\mathcal{E}_x}\, d\,\mathrm{vol}_{\mathfrak{X}}(x) \qquad \forall f,g \in \Gamma_c(\mathfrak{X},\mathcal{E}). \tag{4.12}$$

It is a $G$-algebra with the action given by

$$[g\cdot f](x) := gf(g^{-1}x), \qquad \forall g \in G, \forall f \in L^2(\mathfrak{X},\mathcal{E}).$$

A natural representation $\phi : C_{red}^*(\mathfrak{X}) \to \mathcal{B}(L^2(\mathfrak{X},\mathcal{E}))$ is given by the following action: for every $m \in C_c^\infty(\mathcal{G}_1), x \in \mathcal{G}_0$ and $f \in L^2(\mathfrak{X},\mathcal{E})$,

$$[m\cdot f](x) = \sum_{g\in s^{-1}(x)} m(g)(g\cdot f)(x) = \sum_{g\in s^{-1}(x)} m(g)[g(f(g^{-1}x))] \in \tilde{\mathcal{E}}_x. \tag{4.13}$$

As discussed in Section 3, a properly supported $G$-invariant pseudo-differential operator $D_{\mathfrak{X}}$ on $\Gamma_c(\mathfrak{X},\mathcal{E}) \to \Gamma_c(\mathfrak{X},\mathcal{E})$ of order $m$ extends to a bounded linear operator between Sobolev spaces

$$D_{\mathfrak{X}} : L_k^2(\mathfrak{X},\mathcal{E}) \to L_{k-m}^2(\mathfrak{X},\mathcal{E}),$$

for all $k \geq m$. By the compact embedding theorem, for any $f \in C_c(\mathcal{G}_1)$ and for any pseudo-differntial operator $D_{\mathfrak{X}}$ of negative order, the operator $\phi(f)D_{\mathfrak{X}}$ is compact.

**Lemma 4.7.** *Let $\not{D}^{\mathcal{E}}$ be a Dirac operator on $\mathfrak{X}$. Then the Hilbert space $L^2(\mathfrak{X},\mathcal{E})$, the representation (*4.13*) and the operator $F = \frac{\not{D}^{\mathcal{E}}}{\sqrt{1+(\not{D}^{\mathcal{E}})^2}}$ form a cycle in the $K$-homology group*

$$[\not{D}^{\mathcal{E}}] := [(L^2(\mathfrak{X},\mathcal{E}),\phi,F)] \in K_G^0(C_{red}^*(\mathfrak{X})).$$

*Proof.* Note that $\not{D}^{\mathcal{E}} = \begin{pmatrix} 0 & \not{D}_-^{\mathcal{E}} \\ \not{D}_+^{\mathcal{E}} & 0 \end{pmatrix}$ is a $G$-invariant Dirac operator on $L^2(\mathfrak{X},\mathcal{E})$. Then $F = \frac{\not{D}^{\mathcal{E}}}{\sqrt{1+(\not{D}^{\mathcal{E}})^2}}$ is an order 0 pseudo-differential operator on $L^2(\mathfrak{X},\mathcal{E})$. $F$ can be extended to a bounded operator on $L^2(\mathfrak{X},\mathcal{E})$.

Observe that $1 - F^2 = (1 + (\not{D}^{\mathcal{E}})^2)^{-1}$ is a pseudo-differential operator of order $-2$. Thus,

$$\phi(f)(1 - F^2) \in \mathcal{K}(L^2(\mathfrak{X}, \mathcal{E})) \qquad \forall f \in C_0(\mathfrak{X}) = C_0(\mathcal{G}).$$

Note that by the $G$-invariance of $F$, we have $[F, \pi(g)] = 0, \forall g \in G$. It is straightforward to check $[F, \phi(f)] \in \mathcal{K}(L^2(\mathfrak{X}, \mathcal{E})), \forall f \in C_c(\mathcal{G}_1)$. For the conditions in (4.6), we only need to check for $f \in C_c(\mathcal{G}_1)$ the dense subalgebra of $C^*_{red}(\mathfrak{X})$, which can be done as in [**Bl**] for the manifold cases. q.e.d.

**Definition 4.8.** *The* **higher index** *of $\not{D}^{\mathcal{E}}$, denoted by* $\mathrm{Ind}\,\not{D}^{\mathcal{E}}$*, is defined to be*

$$\mathrm{Ind}\,\not{D}^{\mathcal{E}} = \mu([\not{D}^{\mathcal{E}}]) \in K_0(C^*(G)),$$

*the image in $K_0(C^*(G))$ of $[\not{D}^{\mathcal{E}}]$ under the analytic index map (*4.7*).*

We now represent the higher index of $\not{D}^{\mathcal{E}}$ as a $KK$-cycle in

$$K_0(C^*(G)) \cong KK(\mathbb{C}, C^*(G)),$$

following from the work of Kasparov [**K3**].

Recall that in (4.12) the algebra $\Gamma_c(\mathfrak{X}, \mathcal{E})$ is equipped with a pre-Hilbert space inner product. However, this inner product is not sufficient for us to derive an analogue of the Fredholm property for $\not{D}^{\mathcal{E}}$. Instead, we equip $\Gamma_c(\mathfrak{X}, \mathcal{E})$ with a $\mathbb{C}G$-valued inner product given by

$$(4.14) \qquad \begin{aligned} \langle f_1, f_2 \rangle_{\mathbb{C}G}(g) &:= \int_{\mathfrak{X}}^{orb} (f_1(x), g(f_2(g^{-1}x)))_{\mathcal{E}_x} \, d\,\mathrm{vol}_{\mathfrak{X}}(x) \\ (f \cdot b)(x) &:= \sum_{g \in G} g(f(g^{-1}x)) b(g^{-1}) \end{aligned}$$

for all $f, f_1, f_2 \in \Gamma_c(\mathfrak{X}, \mathcal{E}), g \in G$, and $b \in \mathbb{C}G, x \in \mathcal{G}_0$. It is routine to check that (4.14) gives rise to a pre-Hilbert $\mathbb{C}G$-module. For example, by (4.14) we have

$$\langle f_1, f_2 b \rangle_{\mathbb{C}G} = \langle f_1, f_2 \rangle_{\mathbb{C}G} b, \qquad \forall f_1, f_2 \in \Gamma_c(\mathfrak{X}, \mathcal{E}), b \in \mathbb{C}G.$$

Denote by $\mathcal{A}$ the completion of $\Gamma_c(\mathfrak{X}, \mathcal{E})$ in the Hilbert $C^*(G)$-norm given by the inner product (4.14). Let $c \in C_c^\infty(\mathcal{G})$ be a cut-off function of $\mathfrak{X}$ with respect to the $G$-action, where the pullback $s^* c^{\frac{1}{2}} \in C_c^\infty(\mathcal{G}_1)$ acts on $g \cdot e \in L^2(\mathfrak{X}, \mathcal{E})$ in the sense of (4.13). Then there is an inclusion $\iota : \Gamma_c(\mathfrak{X}, \mathcal{E}) \hookrightarrow C_c(G, L^2(\mathfrak{X}, \mathcal{E}))$ given by

$$(4.15) \qquad \iota(e)(g) = s^* c^{\frac{1}{2}} \cdot [g \cdot e] \qquad \forall e \in \Gamma_c(\mathfrak{X}, \mathcal{E}), \forall g \in G.$$

Denote by $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$ the maximal crossed product as the completion of $C_c(G, L^2(\mathfrak{X}, \mathcal{E}))$ under the norm $\| \sum_{g \in G} (\|a_g\|_{L^2}) g \|_{max}$ for $f = \sum_{g \in G} a_g g \in C_c(G, L^2(\mathfrak{X}, \mathcal{E}))$, where $\| \cdot \|_{max}$ is the norm of $C^*(G)$. It is a Hilbert $C^*(G)$-module.

**Proposition 4.9** ([**K3**]). *The inclusion map $\iota$ given by (4.15) extends to a injective homomorphism between Hilbert $C^*(G)$-modules*

$$\iota : \mathcal{A} \hookrightarrow L^2(\mathfrak{X}, \mathcal{E}) \rtimes G.$$

*Moreover, $\mathcal{A}$ is a direct summand of $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$ as a Hilbert $C^*(G)$-submodule.*

*Proof.* Notice that $C_c^\infty(G, L^2(\mathfrak{X}, \mathcal{E}))$ carries a natural $\mathbb{C}G$-module structure given by convolution. It is easy to check that the inclusion $\iota$ is compatible with the pre-Hilbert $\mathbb{C}G$-module structures. By taking the completion, $\mathcal{A}$ is regarded as a Hilbert $C^*(G)$-submodule for $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$. To show the second claim, note that $\iota$ admits an adjoint $q := \iota^*$ with respect to the $\mathbb{C}G$-valued inner product on $C_c^\infty(G, L^2(\mathfrak{X}, \mathcal{E}))$ given by

$$q : C_c^\infty(G, L^2(\mathfrak{X}, \mathcal{E})) \to \Gamma_c(\mathfrak{X}, \mathcal{E}) \qquad f \mapsto \sum_{g \in G} c(g^{-1}\cdot)^{\frac{1}{2}} \cdot \{g \cdot [f(g^{-1})]\}. \tag{4.16}$$

As a consequence, $q \circ \iota$ is an identity on $\Gamma_c(\mathfrak{X}, \mathcal{E})$ and $\iota \circ q$ is the projection from $C_c^\infty(G, L^2(\mathfrak{X}, \mathcal{E}))$ to $\Gamma_c(\mathfrak{X}, \mathcal{E})$. The projection $\iota \circ q$ extends to $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$ and has $\mathcal{A}$ as its image. The proposition is then proved. q.e.d.

Remark 4.10. The projection map $\iota \circ q$ applied to $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$ agrees with the left convolution of the projection $p$ (defined in (4.3)) with $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$. In other words, we have

$$p[L^2(\mathfrak{X}, \mathcal{E}) \rtimes G] = \mathcal{A}. \tag{4.17}$$

**Proposition 4.11** ([**K3**]). *Let $\not{D}^{\mathcal{E}}$ be a Dirac operator on $\mathfrak{X}$, and let $[\not{D}^{\mathcal{E}}]$ be the $K$-homology element in Lemma 4.7. Then the $KK$-cycle of the higher index $\operatorname{Ind} \not{D}^{\mathcal{E}} \in K_0(C^*(G))$ is given by*

$$[(\mathcal{A}, 1_{\mathbb{C}}, p\tilde{F}p)] = [(q(L^2(\mathfrak{X}, \mathcal{E}) \rtimes G), 1_{\mathbb{C}}, q \circ \tilde{F} \circ i)] \in KK(\mathbb{C}, C^*(G)). \tag{4.18}$$

*Here, $\tilde{F}$ is the lift of the operator $F = \not{D}^{\mathcal{E}}[1 + (\not{D}^{\mathcal{E}})^2]^{-\frac{1}{2}}$ to $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$ given by*

$$[\tilde{F}(h)]g = F(h(g)), \qquad \forall h \in C_c(G, L^2(\mathfrak{X}, \mathcal{E})), \forall g \in G. \tag{4.19}$$

*Proof.* By definition of the higher index, we have

$$\operatorname{Ind} \not{D}^{\mathcal{E}} = \mu[F] = [\mathcal{P}] \otimes_{C^*_{red}(\mathfrak{X}) \rtimes G} j^G([L^2(\mathfrak{X}, \mathcal{E}), \phi, F]),$$

where $F = \frac{\not{D}^{\mathcal{E}}}{\sqrt{1+(\not{D}^{\mathcal{E}})^2}}$ is the bounded operator on $\Gamma_c(\mathfrak{X}, \mathcal{E})$. Denote by $\tilde{F}$ the lift of the operator $F$ to $L^2(\mathfrak{X}, \mathcal{E}) \rtimes G$ given by (4.19). Then the image of $[F]$ under the descent map $j^G$ is given by

$$j^G([L^2(\mathfrak{X}, \mathcal{E}), \phi, F]) = [L^2(\mathfrak{X}, \mathcal{E}) \rtimes G, \tilde{\phi}, \tilde{F}].$$

Thus, we have

$$\begin{aligned}\operatorname{Ind} D\!\!\!/^{\mathcal{E}} =& [\mathcal{P}, 1_{\mathbb{C}}, 0] \otimes_{C^*_{red}(\mathfrak{X})\rtimes G} [L^2(\mathfrak{X},\mathcal{E}) \rtimes G, \tilde{\phi}, \tilde{F}] \\ =& [\mathcal{P} \otimes_{C^*_{red}(\mathfrak{X})\rtimes G} [L^2(\mathfrak{X},\mathcal{E}) \rtimes G], 1_{\mathbb{C}}, \mathcal{P}\tilde{F}\mathcal{P}].\end{aligned}$$

The statement is proved by noting that $\mathcal{A}$ is obtained from the Hilbert $C^*_{red}(\mathfrak{X}) \rtimes G$-module $\mathcal{P}$ and the Hilbert $C^*(G)$-module $L^2(\mathfrak{X},\mathcal{E}) \rtimes G$. In fact, by Remark 4.10 we have

$$\begin{aligned}\mathcal{P} \otimes_{C^*_{red}(\mathfrak{X})\rtimes G} [L^2(\mathfrak{X},\mathcal{E}) \rtimes G] =& p \cdot [C^*_{red}(\mathfrak{X}) \rtimes G] \otimes_{C^*_{red}(\mathfrak{X})\rtimes G} [L^2(\mathfrak{X},\mathcal{E}) \rtimes G] \\ =& p \cdot [L^2(\mathfrak{X},\mathcal{E}) \rtimes G] = \mathcal{A}.\end{aligned}$$

In addition, the compression $\mathcal{P}\tilde{F}\mathcal{P}$ of $\tilde{F}$ and $\mathcal{P}$ on $p\cdot[L^2(\mathfrak{X},\mathcal{E})\rtimes G] = \mathcal{A}$ is alternatively written as $q \circ \tilde{F} \circ \iota$ on $q(L^2(\mathfrak{X},\mathcal{E}) \rtimes G) \cong \mathcal{A}$. This follows from $\iota \circ q = p\cdot$ on $L^2(\mathfrak{X},\mathcal{E}) \rtimes G$. Therefore, the higher index of $D\!\!\!/^{\mathcal{E}}$ is represented by the following $KK$-cycle:

(4.20)
$$[(\mathcal{A}, 1_{\mathbb{C}}, p\tilde{F}p)] = [(q(L^2(\mathfrak{X},\mathcal{E}) \rtimes G), 1_{\mathbb{C}}, q \circ \tilde{F} \circ i)] \in KK(\mathbb{C}, C^*(G)).$$

The proposition is then proved. q.e.d.

Let $\mathcal{S}(G)$ be an unconditional completion of $\mathbb{C}G$, under a Banach norm $\|\cdot\|_{\mathcal{S}(G)}$. Similarly, we have the following norm-preserving inclusion between Banach $\mathcal{S}(G)$-modules extending the map (4.15)

$$\iota_{\mathcal{S}(G)} : \mathcal{A}_{\mathcal{S}(G)} \longrightarrow \mathcal{S}(G, L^2(\mathfrak{X},\mathcal{E})).$$

Here, $\mathcal{A}_{\mathcal{S}(G)}$ is the Banach $\mathcal{S}(G)$-module in the same fashion as (4.17) given by

$$\mathcal{A}_{\mathcal{S}(G)} := p[\mathcal{S}(G, L^2(\mathfrak{X},\mathcal{E}))]. \tag{4.21}$$

It is a direct summand of $\mathcal{S}(G, L^2(\mathfrak{X},\mathcal{E}))$. Denote by $\tilde{F}$ the lift of $F$ from $\Gamma_c(\mathfrak{X},\mathcal{E})$ to $\mathcal{S}(G, L^2(\mathfrak{X},\mathcal{E}))$ given by (4.19). Then the Banach algebra version of the higher index is then stated as follows.

**Proposition 4.12.** *Let $D\!\!\!/^{\mathcal{E}}$ be a Dirac operator on $\mathfrak{X}$, and let $[D\!\!\!/^{\mathcal{E}}]$ be the $K$-homology element in Lemma 4.7. Then the $KK$-cycle of the higher index $\mu_{\mathcal{S}(G)}[D\!\!\!/^{\mathcal{E}}] \in K_0(\mathcal{S}(G))$ is given by the following $KK$-cycle:*

$$[(\mathcal{A}_{\mathcal{S}(G)}, 1_{\mathbb{C}}, p\tilde{F}p)] = [(q(\mathcal{S}(G, L^2(\mathfrak{X},\mathcal{E}))), 1_{\mathbb{C}}, q \circ \tilde{F} \circ i)]. \tag{4.22}$$

*Here, $\tilde{F}$ is the lift of the operator $F = D\!\!\!/^{\mathcal{E}}[1+(D\!\!\!/^{\mathcal{E}})^2]^{-\frac{1}{2}}$ to $\mathcal{S}(G, L^2(\mathfrak{X},\mathcal{E}))$ given by (4.19).*

**4.3. Orbifold index and the higher index.** We relate the higher index of $D\!\!\!/^{\mathcal{E}}$ to the Kawasaki index of $D\!\!\!/^{G\backslash\mathcal{E}}$ on the quotient $G\backslash\mathfrak{X}$. This is essentially a result of Theorem 3.4 and Theorem 5.15 (see also Remark 3.24). Here, we present an alternative proof only using $KK$-theory.

**Theorem 4.13.** *The Kawasaki's index for closed orbifold $G\backslash\mathfrak{X}$ is equal to the trivial representation of $G$ induced on the higher index of $[\not{D}^{\mathcal{E}}]$:*

$$\operatorname{ind} \not{D}^{G\backslash\mathcal{E}} = \rho_*(\mu[\not{D}^{\mathcal{E}}]) \tag{4.23}$$

*where $\rho_* : K_0(C^*(G)) \to \mathbb{Z}$ is induced by*

$$\rho : C^*(G) \to \mathbb{C} : \sum \alpha_g g \mapsto \sum \alpha_g \tag{4.24}$$

*and $\mu$ is the higher index map* (4.7)*.*

*Proof.* Let $F = \frac{\not{D}^{\mathcal{E}}}{\sqrt{1+(\not{D}^{\mathcal{E}})^2}}$ and $F_{G\backslash\mathfrak{X}} = \frac{\not{D}^{G\backslash\mathcal{E}}}{\sqrt{1+(\not{D}^{G\backslash\mathcal{E}})^2}}$. Then the action of $F_{G\backslash\mathfrak{X}}$ on $\Gamma(G\backslash\mathfrak{X}, G\backslash\mathcal{E})$ can be identified to that of $F$ on the $G$-equivariant sections $\Gamma(\mathfrak{X},\mathcal{E})^G$. The functorial map $\rho_*$ applied to the higher index $\mu[\not{D}^{\mathcal{E}}]$ is essentially the $KK$-product of $[(\mathbb{C},\rho,0)]$ with $\mu[F]$ over $C^*(G)$ in the following map (see [**Bl**]):

$$KK(\mathbb{C}, C^*(G)) \times KK(C^*(G),\mathbb{C}) \to KK(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}.$$

Then the right-hand side of (4.23) is

$$\begin{aligned} \rho_*(\mu[F]) =& [(\mathcal{A}, 1_{\mathbb{C}}, q\circ\tilde{F}\circ\iota)] \otimes_{C^*(G)} [(\mathbb{C},\rho,0)] \\ =& [(\mathcal{A}\otimes_{C^*(G)}\mathbb{C}, 1_{\mathbb{C}}, (q\circ\tilde{F}\circ\iota)\otimes 1)], \end{aligned} \tag{4.25}$$

where $\mathcal{A} = q(L^2(\mathfrak{X},\mathcal{E}) \rtimes G)$, and by (4.15), (4.16), and (4.19) we have

$$q\circ\tilde{F}\circ\iota = \sum_{g\in G} g(c^{\frac{1}{2}} F c^{\frac{1}{2}}). \tag{4.26}$$

In the following, we shall identify $L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E})$ and $\mathcal{A}\otimes_{C^*(G)}\mathbb{C}$. For a fixed cut-off function $c$, we have the map

$$j : \Gamma(G\backslash\mathfrak{X}, G\backslash\mathcal{E}) \longrightarrow \Gamma_c(\mathfrak{X},\mathcal{E})\otimes_{\mathbb{C}G}\mathbb{C} \qquad f\mapsto c\cdot\tilde{f}\otimes 1.$$

Here, $\Gamma_c(\mathfrak{X},\mathcal{E})$ is a pre-Hilbert $\mathbb{C}G$-module and $\tilde{f}$ is the image of $f$ under the natural map $\Gamma(G\backslash\mathfrak{X}, G\backslash\mathcal{E}) \cong \Gamma(\mathfrak{X},\mathcal{E})^G$. We show that $j(f)$ does not depend on the choice of $c$. In fact, for another cut-off function $d$ and any $h\otimes 1 \in \Gamma_c(\mathfrak{X},\mathcal{E})\otimes_{\mathbb{C}G}\mathbb{C}$, as $\tilde{f}$ is $G$-invariant, we have

$$\langle (d-c)\tilde{f}\otimes 1, h\otimes 1\rangle = \sum_{g\in G}\langle [(d-c)\tilde{f}](gx), h(x)\rangle = 0.$$

Hence, the nondegeneracy of the inner product implies that $d\tilde{f}\otimes 1 = c\tilde{f}\otimes 1$.

We claim that $j$ preserves the inner products. In fact, the claim follows from

$$\begin{aligned}\langle f, h\rangle &= \int_{G\backslash\mathfrak{X}}^{orb} (f(x), h(x))_{(G\backslash\mathcal{E})_x}\, d\,\mathrm{vol}_{G\backslash\mathfrak{X}}(x)\\ &= \int_{\mathfrak{X}}^{orb} c(x)(f(x), h(x))_{\tilde{\mathcal{E}}_x}\, d\,\mathrm{vol}_{\mathfrak{X}}(x) \quad \text{and}\end{aligned}$$

$$\begin{aligned}\langle j(f), j(h)\rangle &= \sum_{g\in G}\int_{\mathfrak{X}}^{orb} (c(x)\tilde{f}(x), c(g^{-1}x)\tilde{h}(gx))_{\tilde{\mathcal{E}}_x}\, d\,\mathrm{vol}_{\mathfrak{X}}(x)\\ &= \int_{\mathfrak{X}}^{orb} (c(x)\tilde{f}(x), \sum_{g\in G} c(g^{-1}x)\tilde{h}(x))_{\tilde{\mathcal{E}}_x}\, d\,\mathrm{vol}_{\mathfrak{X}}(x)\end{aligned}$$

for all $f, h \in C(G\backslash\mathfrak{X}, G\backslash\mathcal{E})$. Hence, the map $j$ extends to an isomorphism of two Hilbert spaces

$$j : L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E}) \to \mathcal{A}\otimes_{\mathbb{C}G}\mathbb{C}.$$

It is straightforward to check that the inverse of $j$ is given by

$$j^{-1} : \mathcal{A}\otimes_{C^*(G)}\mathbb{C} \to L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E}) \quad h\otimes 1 \mapsto \sum_{g\in G} h(g^{-1}\cdot).$$

Then, together with (4.25) and (4.26), we have

$$\rho_*(\mu[F]) = [(\mathcal{A}\otimes_{C^*(G)}\mathbb{C}, 1_{\mathbb{C}}, \sum_{g\in G} g(c^{\frac{1}{2}}Fc^{\frac{1}{2}})\otimes 1)] = [(L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E}), F_0)],$$

where

$$F_0 = j^{-1}\circ[\sum_{g\in G} g(c^{\frac{1}{2}}Fc^{\frac{1}{2}})\otimes 1]\circ j = \sum_{l,g\in G} c(g^{-1}l^{-1}x)^{\frac{1}{2}}Fc(g^{-1}l^{-1}x)^{\frac{1}{2}}c(l^{-1}x).$$

Finally, observe that the left-hand side of (4.23) is

$$\mathrm{ind}\, \not{D}^{G\backslash\mathcal{E}} = [(L^2(G\backslash\mathfrak{X}, G\backslash\mathcal{E}), 1_{\mathbb{C}}, F_{G\backslash\mathfrak{X}})].$$

One need only to show that $F_0$ and $F$ coincide up to compact operators on $\Gamma(\mathfrak{X}, \mathcal{E})^G$ (denoted $F_0 \equiv F$), that is, they have the same Fredholm index. As we have $\sum_{g\in G} g(c^{\frac{1}{2}}Fc^{\frac{1}{2}}) \equiv \sum_{g\in G} g(cF) = F$, then

$$F_0 = \sum_{l\in G} l[\sum_{g\in G} g(c^{\frac{1}{2}}Fc^{\frac{1}{2}})c] \equiv \sum_{l\in G} l(Fc) = F = F_{G\backslash\mathfrak{X}}.$$

The theorem is proved. q.e.d.

Remark 4.14. It is important to emphasis $G$ being discrete to ensure

$$\Gamma(\mathfrak{X}, \mathcal{E})^G \cong \Gamma(G\backslash\mathfrak{X}, G\backslash\mathcal{E})$$

and so that $\not{D}^{G\backslash\mathcal{E}}$ is a restriction of $\not{D}^{\mathcal{E}}$ to the invariant sections. We used this identification in the proof of Theorem 4.13. If $G$ is a locally compact group acting on $\mathfrak{X}$ properly, co-compactly, and isometrically, then an elliptic operator $\not{D}^{G\backslash\mathcal{E}}$ on $G\backslash\mathfrak{X}$ lifts to a transversally elliptic operator on $\mathfrak{X}$, which is elliptic when $G$ is discrete. When a group $G$ is continuous, the restriction $\not{D}^{G\backslash\mathcal{E}}$ loses informations on the longitudinal part of a $G$-invariant operator $\not{D}^{\mathcal{E}}$ on $\mathfrak{X}$. However, a result similar to Theorem 4.13, when a locally compact group acts on a manifold properly and co-compactly, can be found in [**MZ**].

## 5. Localized indices

In this section, we introduce the *localized trace* associated to each conjugacy class $(g)$ of $g \in G$ and define the corresponding localized index. We show that localized indices are well-defined topological invariants for the $G$-invariant Dirac operator $\not{D}^{\mathcal{E}}$. In particular, the localized index at the group identity is just the $L^2$-index, obtained from taking the canonical von Neumann trace on the group von Neumann algebra $\mathcal{N}G$ of the higher index. We also show that the localized index can be computed from the heat kernel of the Dirac operator when $G$ satisfies some trace property.

**5.1. Localized traces.** Denote by $(g)$ the conjugacy class of $g$ in $G$. Define a map $\tau^{(g)} : \mathbb{C}G \to \mathbb{C}$ given by

$$\sum_{h\in G} \alpha_h h = \sum_{h\in(g)} \alpha_h. \tag{5.1}$$

**Lemma 5.1.** *The linear map $\tau^{(g)}$ in (5.1) is a trace for $\mathbb{C}G$, that is, $\tau^{(g)}(ab) = \tau^{(g)}(ba)$ for any $a, b \in \mathbb{C}G$.*

*Proof.* Let $a = \sum_{g\in G} a_g g$ and $b = \sum_{g\in G} b_g g$, where all but finite coefficients are 0, that is, $a, b \in \mathbb{C}G$. Let $c_g, d_g$ be the coefficients of the products

$$ab = \sum_{g\in G} c_g g \quad \text{and} \quad ba = \sum_{g\in G} d_g g.$$

Thus, $c_k = \sum_{h\in G} a_{kh^{-1}} b_h$ and $d_k = \sum_{h\in G} b_{kh^{-1}} a_h$. Let $K$ be defined as in (3.20). Then

$$\tau^{(g)}\left(\sum_{h\in G} \alpha_h h\right) = \sum_{k\in K} \alpha_{kgk^{-1}}.$$

By Definition 5.1 and Lemma 3.15, we have

$$\tau^{(g)}(ba) = \sum_{k\in K, h\in G} a_h b_{kgk^{-1}h^{-1}}$$

and

$$\begin{aligned}\tau^{(g)}(ab) &= \sum_{k\in K, h\in G} a_{kgk^{-1}h^{-1}} b_h = \sum_{k\in K}\left(\sum_{h\in G} a_{kgk^{-1}h} b_{h^{-1}}\right)\\ &= \sum_{k\in K}\left(\sum_{h\in G} a_h b_{h^{-1}kgk^{-1}}\right) = \sum_{h\in G}\left(\sum_{k\in K} a_h b_{h^{-1}kgk^{-1}}\right)\\ &= \sum_{h\in G}\left(\sum_{k\in hK} a_h b_{kgk^{-1}h^{-1}}\right).\end{aligned}$$

It is easy to verify that $hK$ also satisfies (3.20) for each $h \in G$. Then $\tau^{(g)}(ab) = \tau^{(g)}(ba)$. The lemma is proved. q.e.d.

**Definition 5.2** (**Localized** $(g)$**-trace).** *Let $\mathcal{S}(G)$ be a Banach algebra being an unconditional completion of $\mathbb{C}G$ satisfying*

$$L^1(G) \subset \mathcal{S}(G) \subset C_r^*(G). \tag{5.2}$$

*A* **localized** $(g)$**-trace** *on $\mathcal{S}(G)$ is a continuous trace map*

$$\tau^{(g)} : \mathcal{S}(G) \longrightarrow \mathbb{C}, \tag{5.3}$$

*which extends the map (*5.1*).*

Remark 5.3. The localized $(g)$-trace map always exists. We can choose $\mathcal{S}(G)$ to be $L^1(G)$. Note that $L^1(G)$ is an unconditional completion of $\mathbb{C}G$. The continuity of $\tau^{(g)} : L^1(G) \to \mathbb{C}$ can be proved as follows. For any $\epsilon > 0$, choose $\delta = \varepsilon$, for all $\|a\|_{L^1} < \delta$ where $a = \sum_{g\in G} a_g g$, then we have

$$|\tau^{(g)}(a)| := \left|\sum_{h\in(g)} a_h\right| \leq \sum_{h\in G} |a_h| = \|a\|_{L^1} < \varepsilon.$$

Remark 5.4. Let $e$ be the group identity of $G$. The localized $(e)$-trace is "global" in the sense that it is given by the canonical continuous trace on $C_r^*(G)$:

$$\tau^{(e)} : C_r^*(G) \longrightarrow \mathbb{C} \qquad \sum_{h\in G} \alpha_h h \mapsto \alpha_e. \tag{5.4}$$

In fact, $\tau^{(e)}$ can be further extended to a continuous normalized trace on the group von Neumann algebra $\mathcal{N}G$, the weak closure of $C_r^*(G)$. Note that we have the $*$-homomorphisms

$$C^*(G) \twoheadrightarrow C_r^*(G) \hookrightarrow \mathcal{N}G,$$

which induce the following homomorphisms on the level of $K$-theory:

$$K_*(C^*(G)) \longrightarrow K_*(C_r^*(G)) \longrightarrow K_*(\mathcal{N}G). \tag{5.5}$$

Recall that a trace $\tau$ on a $C^*$-algebra $A$ is **normalized** if it is a state, that is,

$$\tau(a^*a) \geq 0, \quad \forall a \in A \qquad \text{and} \qquad \tau(e) = 1.$$

The trace $\tau^{(e)}$ is normalized. For a general conjugacy class $(g)$ consisting of infinite elements, $\tau^{(g)}$ is *not* normalized, hence it may not be a continuous trace on $\mathcal{N}G$.

REMARK 5.5. If $G$ is abelian or if $(g)$ is a finite set, then $\tau^{(g)}$, for $g \neq e$, can be extended to a continuous trace $C_r^*(G) \to \mathbb{C}$. In general, however, a trace map $C_r^*(G) \to \mathbb{C}$ can fail to be continuous. Thus, for some $G$, $C_r^*(G)$ may be too large to be an unconditional completion of $G$.

The localized $(g)$-trace $\tau^{(g)}$ in Definition 5.2 and the $(g)$-trace $\mathrm{tr}^{(g)}$ introduced in Section 3 are closely related. Let $S : \Gamma_c(\mathfrak{X}, \mathcal{E}) \to \Gamma_c(\mathfrak{X}, \mathcal{E})$ be a $G$-invariant operator that extends to a bounded operator on $L^2(\mathfrak{X}, \mathcal{E})$. Denote by $\tilde{S} : C_c(G, L^2(\mathfrak{X}, \mathcal{E})) \to C_c(G, L^2(\mathfrak{X}, \mathcal{E}))$ the lift of $S$ given by

$$(\tilde{S}u)(g) = S(u(g)), \qquad \forall u \in C_c(G, L^2(\mathfrak{X}, \mathcal{E})), \forall g \in G.$$

Let $\mathcal{S}(G)$ be an unconditional completion of $\mathbb{C}G$ such that the localized $(g)$-trace on $\mathcal{S}(G)$ is continuous. Then the properly supported operator

$$S_{\mathcal{A}} := q\tilde{S}\iota = \sum_{g \in G} g \cdot (c^{\frac{1}{2}} S c^{\frac{1}{2}}) \tag{5.6}$$

can be extended to a bounded operator on the $\mathcal{S}(G)$-module $\mathcal{A}_{\mathcal{S}(G)}$ (cf. (4.21)).

**Lemma 5.6.** *Let $S : L^2(\mathfrak{X}, \mathcal{E}) \to L^2(\mathfrak{X}, \mathcal{E})$ be a bounded self-adjoint $G$-invariant smoothing operator. Let $S_{\mathcal{A}} : \mathcal{A}_{\mathcal{S}(G)} \to \mathcal{A}_{\mathcal{S}(G)}$ be the operator given by (*5.6*). Then:*

1) *$S_{\mathcal{A}}$ on $L^2(\mathfrak{X}, \mathcal{E})$ is of $(g)$-trace class in the sense of Definition* 3.13 *for all $g \in G$.*
2) $\mathrm{Tr}_s\, S_{\mathcal{A}} \in \mathcal{S}(G)$ *(cf. Definition* 5.2*) and its localized $(g)$-trace coincides with the $(g)$-trace of $\mathcal{S}_{\mathcal{A}}$, that is,*

$$\mathrm{tr}_s^{(g)}(S_{\mathcal{A}}) = \tau^{(g)}\left(\mathrm{Tr}_s\, S_{\mathcal{A}}\right) \in \mathbb{R}. \tag{5.7}$$

*Proof.* (1) As the cut-off function $c$ is compactly supported, the kernel of the operator $c^{\frac{1}{2}} S c^{\frac{1}{2}}$ is also compactly supported. Then $S_{\mathcal{A}}$ given by $\sum_{g \in G} g \cdot (c^{\frac{1}{2}} S c^{\frac{1}{2}})$ as in (5.6) is properly supported and is of $(g)$-trace class (cf. Lemma 3.12 and Proposition 3.16).

(2) Recall from Section 4.2 that $\Gamma_c(\mathfrak{X}, \mathcal{E})$ has both the pre-Hilbert space structure (4.12) and the $\mathbb{C}G$-module structure (4.14). There is an injection $\iota$ from $\Gamma_c(\mathfrak{X}, \mathcal{E})$ to $C_c(G, L^2(\mathfrak{X}, \mathcal{E}))$ given by (4.15), and it

preserves the $\mathbb{C}G$-inner product. Now, $\mathcal{A}_{\mathcal{S}(G)}$ is the closure of $\iota(\Gamma_c(\mathfrak{X}, \mathcal{E}))$ under the Banach norm.

Let $\{u_i\}_{i\in\mathbb{N}} \in \Gamma_c(\mathfrak{X}, \mathcal{E})$ so that $\{\iota(u_i)\}_{\iota\in\mathbb{N}}$ forms an orthonormal basis for the $\mathcal{S}(G)$-module $\mathcal{A}_{\mathcal{S}(G)}$. Without loss of generality, let us ignore the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{E}$ and work on trace instead of supertrace. Then

$$(\operatorname{Tr} S_{\mathcal{A}})(k) = \sum_i \langle S_{\mathcal{A}} u_i, u_i\rangle_{\mathbb{C}G}(k) \qquad \forall k \in G.$$

As $S_{\mathcal{A}}$ is properly supported, $u_i$ is compactly supported, and the action of $G$ on $\mathfrak{X}$ is proper, $\operatorname{Tr} S_{\mathcal{A}}(k)$ vanishes for all but finite $k \in G$. Therefore, $\operatorname{Tr} S_{\mathcal{A}} \in \mathbb{C}G \subset \mathcal{S}(G)$.

To see (5.7), we calculate the localized $(g)$-trace of $\operatorname{Tr} S_{\mathcal{A}}$ as follows:

$$\tau^{(g)}((\operatorname{Tr} S_{\mathcal{A}})(\cdot)) = \sum_{k\in(g)} \sum_i \langle S_{\mathcal{A}} u_i, u_i\rangle_{\mathbb{C}G}(k).$$

By definition of the $C^*(G)$-inner product (4.14), we have

$$\begin{aligned} \langle u_i, u_j\rangle_{L^2} =& \langle u_i, u_j\rangle_{\mathbb{C}G}(e) = \delta_{ij} \quad \forall i, j \in \mathbb{N} \\ \langle h\cdot u_i, g\cdot u_j\rangle_{L^2} =& \langle u_i, u_j\rangle_{\mathbb{C}G}(h^{-1}g) = 0 \quad \forall g \neq h, \forall i, j \in \mathbb{N}. \end{aligned}$$

Then

$$\{g\cdot u_i\}_{g\in G, i\in\mathbb{N}} \in \Gamma_c(\mathfrak{X}, \mathcal{E}) \tag{5.8}$$

forms an orthonormal subset of $L^2(\mathfrak{X}, \mathcal{E})$.

We claim that (5.8) forms a basis for $L^2(\mathfrak{X}, \mathcal{E})$. If not, let $v \in L^2(\mathfrak{X}, \mathcal{E})$ be a vector perpendicular to all elements in (5.8). Then

$$\langle v, u_i\rangle_{\mathbb{C}G}(g) = \langle v, g\cdot u_i\rangle_{L^2} = 0 \quad \forall g \in G, \forall i \in \mathbb{N}.$$

This implies that $\iota(v)$ is perpendicular to the basis $\{\iota(u_j)\}_{j\in\mathbb{N}}$ in $\mathcal{A}_{\mathcal{S}(G)}$. So $\iota(v) = 0$. As $\iota$ is injective, we conclude that $v = 0$. Thus, elements of the set (5.8) form an orthonormal basis for $L^2(\mathfrak{X}, \mathcal{E})$.

To compute the trace of $S_{\mathcal{A}}$, we consider

$$\langle S_{\mathcal{A}} u_i, u_i\rangle_{\mathbb{C}G}(k) = \int_{\mathfrak{X}}^{orb} \langle S_{\mathcal{A}} u_i(x), k\cdot u_i(x)\rangle_{\mathcal{E}_x} d\operatorname{vol}_{\mathfrak{X}}(x), \tag{5.9}$$

obtained by (4.14). Note that $S_{\mathcal{A}}$ is $G$-invariant by (5.6). In particular, we may replace $S_{\mathcal{A}}$ on the right-hand side of (5.9) by

$$S_{\mathcal{A}} = \sum_{h\in G} h\cdot (cS_{\mathcal{A}}) = \sum_{h\in G} h(cS_{\mathcal{A}})h^{-1}.$$

Therefore, for all $k \in G$, we have

$$\begin{aligned}
&\langle S_{\mathcal{A}} u_i, u_i \rangle_{\mathbb{C}G}(k) \\
&= \sum_{h \in G} \int_{\mathfrak{X}}^{orb} \langle h(cS_{\mathcal{A}})h^{-1} u_i(x), k u_i(x) \rangle_{\mathcal{E}_x} \, d\,\mathrm{vol}_{\mathfrak{X}}(x) \\
&= \sum_{h \in G} \int_{\mathfrak{X}}^{orb} \langle (cS_{\mathcal{A}})h^{-1} u_i(x), (h^{-1}kh)h^{-1} u_i(x) \rangle_{\mathcal{E}_x} \, d\,\mathrm{vol}_{\mathfrak{X}}(x).
\end{aligned}$$

As we sum all $k \in (g)$, we obtain

$$\begin{aligned}
&\sum_{k \in (g)} \langle S_{\mathcal{A}} u_i, u_i \rangle_{\mathbb{C}G}(k) \\
&= \sum_{h \in G} \sum_{k \in (g)} \int_{\mathfrak{X}}^{orb} \langle (cS_{\mathcal{A}})h^{-1} u_i(x), k h^{-1} u_i(x) \rangle_{\mathcal{E}_x} \, d\,\mathrm{vol}_{\mathfrak{X}}(x) \\
&= \sum_{k \in (g)} \sum_{h \in G} \int_{\mathfrak{X}}^{orb} \langle k^{-1}(cS_{\mathcal{A}})h^{-1} u_i(x), h^{-1} u_i(x) \rangle_{\mathcal{E}_x} \, d\,\mathrm{vol}_{\mathfrak{X}}(x).
\end{aligned}$$

Observe that $\{\iota(u_i)\}_{i \in \mathbb{N}}$ forms an orthonormal basis for $\mathcal{A}_{\mathcal{S}(G)}$ and the set $\{g \cdot u_i\}_{g \in G, i \in \mathbb{N}}$ forms an orthonormal basis for $L^2(\mathfrak{X}, \mathcal{E})$. Then by summing the equality for all $i \in \mathbb{N}$, we conclude that

$$\sum_{k \in (g)} (\mathrm{Tr}\, S_{\mathcal{A}})(k) = \sum_{k \in (g)} \mathrm{Tr}(k^{-1} c S_{\mathcal{A}}).$$

By Definition 3.13 and Definition 5.2 this is equivalent to saying that

$$\tau^{(g)}(\mathrm{Tr}\, S_{\mathcal{A}}(\cdot)) = \mathrm{tr}^{(g)}(S_{\mathcal{A}}).$$

The lemma is proved. q.e.d.

**5.2. Localized indices.** The localized trace in the sense of Definition 5.2 induces a group homomorphism on the level of $K$-theory,

$$\tau_*^{(g)} : K_0(\mathcal{S}(G)) \longrightarrow \mathbb{R}, \tag{5.10}$$

as follows. Let $P$ be a projection in $M_n(\mathcal{S}(G))$, the algebra of $n \times n$ matrices with entries in $\mathcal{S}(G)$. Define

$$\tau_*^{(g)}(P) = \mathrm{Tr}(\tau^{(g)}(P)).$$

An element of $K_0(\mathcal{S}(G))$ is represented by $[P_1] - [P_2]$, where $P_1$ and $P_2$ are projections in the matrix algebra with entries in $\mathcal{S}(G)$. Then the map (5.10) is well defined because $\tau^{(g)}$ is a continuous trace map. Also, it is real-valued, as any projection can be written as the difference of two positive operators on which $\tau_*^{(g)}$ takes real values.

Let $\mu_{\mathcal{S}(G)}[\not{D}^{\mathcal{E}}] \in K_0(\mathcal{S}(G))$ be the Banach algebra version higher index of $\not{D}^{\mathcal{E}}$ given by (4.11). Recall that in Theorem 4.13 we applied the homomorphism $\rho$ in (4.24) to get the orbifold index on $G \backslash \mathfrak{X}$. Also, combining (5.4) with (5.5), we have the **$L^2$-index** $\tau_*^{(e)}\mu[\not{D}^{\mathcal{E}}]$ of the Dirac operator $\not{D}^{\mathcal{E}}$.

Note that $\rho$ is a trace and $\rho = \sum_{(g)} \tau^{(g)}$ when adding up $\tau^{(g)}$ over all conjugacy classes of $G$. This means that the higher index of $\not{D}^{\mathcal{E}}$ can be localized to each conjugacy class.

**Definition 5.7** (**Localized index**). *The real number*

$$\operatorname{ind}_{(g)} \not{D}^{\mathcal{E}} := \tau_*^{(g)}(\mu_{\mathcal{S}(G)}[\not{D}^{\mathcal{E}}])$$

*is called the* **localized** $(g)$**-index** *of $\not{D}^{\mathcal{E}}$. In general, we call them* **localized indices**.

Remark 5.8. When the conjugacy class $(g)$ of $g \in G$ has finite elements, $\tau^{(g)}$ extends to a continuous trace on $C^*(G)$. Thus, $\operatorname{ind}_{(g)} \not{D}^{\mathfrak{X}} = \tau_*^{(g)}\mu[\not{D}^{\mathfrak{X}}]$. In particular, $L^2$-index factors through the higher index in the $K$-theory for the reduced group $C^*$-algebra. Another case worth mentioning is that when $G$ has the RD property, we have an isomorphism of $K$-theory $K_0(\mathcal{S}(G)) \cong K_0(C_r^*(G))$. Then all localized $(g)$-indices are images of an element in $K_0(C_r^*(G))$. For example, in [**Pu**] Puschnigg showed that every reduced $C^*$-algebra of a hyperbolic group $G$ contains a subalgebra that is an unconconditional completion $\mathcal{S}(G)$ of $G$. Thus, the two algebras have the same $K$-theory. These provide a large and important class of nontrivial examples where our localized $(g)$-indices factor through $K_0(C_r^*(G))$. In general, it is not known if localized $(g)$-indices always come from the higher index in the $K$-theory of the group $C^*$-algebra. Fortunately, this does not affect the results of our paper.

The following analogue of Mckean–Singer formula provides an explicit calculation of the localized $(g)$-index in terms of localized $(g)$-supertrace $\operatorname{tr}_s^{(g)}$ of the heat kernel operator of $\not{D}^{\mathcal{E}}$ (cf. Definition 3.13).

**Proposition 5.9.** *Suppose the localized $(g)$-trace $\tau^{(g)}$ extends to a trace on $C_r^*(G)$. The localized $(g)$-index of $\not{D}^{\mathcal{E}}$ is calculated by*

$$\operatorname{ind}_{(g)} \not{D}^{\mathcal{E}} = \operatorname{tr}_s^{(g)}\left(e^{-t(\not{D}^{\mathcal{E}})^2}\right).$$

*Proof.* Let $Q$ be a parametrix of $\not{D}_+^{\mathcal{E}}$ in the sense of Proposition 3.3. Then there are $G$-invariant smoothing operators $S_0$ and $S_1$, where

$$1 - Q\not{D}_+^{\mathcal{E}} = S_0 \text{ and } 1 - \not{D}_+^{\mathcal{E}}Q = S_1. \tag{5.11}$$

In particular, if we choose

$$Q = \not{D}_-^{\mathcal{E}}(1 - e^{-t\not{D}_+^{\mathcal{E}}\not{D}_-^{\mathcal{E}}/2})(\not{D}_+^{\mathcal{E}}\not{D}_-^{\mathcal{E}})^{-1} = (1 - e^{-t\not{D}_-^{\mathcal{E}}\not{D}_+^{\mathcal{E}}/2})(\not{D}_+^{\mathcal{E}})^{-1},$$

then we have $S_0 = e^{-t\not{D}_-^{\mathcal{E}}\not{D}_+^{\mathcal{E}}/2}$ and $S_1 = e^{-t\not{D}_+^{\mathcal{E}}\not{D}_-^{\mathcal{E}}/2}$.

Consider the normalization $F = \not{D}^{\mathcal{E}}[1+(\not{D}^{\mathcal{E}})^2]^{-\frac{1}{2}}$ of $\not{D}^{\mathcal{E}}$. The positive part of this operator is

$$F_+ = \not{D}^{\mathcal{E}}_+(1+\not{D}^{\mathcal{E}}_-\not{D}^{\mathcal{E}}_+)^{-\frac{1}{2}} = (1+\not{D}^{\mathcal{E}}_+\not{D}^{\mathcal{E}}_-)^{-\frac{1}{2}}\not{D}^{\mathcal{E}}_+.$$

If we choose the parametrix of $F_+$ as $R := (1+\not{D}^{\mathcal{E}}_-\not{D}^{\mathcal{E}}_+)^{\frac{1}{2}}Q$, then a direct calculation shows that

$$(5.12)\quad 1 - RF_+ = S_0 = e^{-t\not{D}^{\mathcal{E}}_-\not{D}^{\mathcal{E}}_+/2} \qquad 1 - F_+R = S_1 = e^{-t\not{D}^{\mathcal{E}}_+\not{D}^{\mathcal{E}}_-/2}.$$

As we assumed the localized $(g)$-trace can be extended to $C^*_r(G)$, we can apply $\tau^{(g)}_*$ to $\mu_{red}(\not{D}^{\mathcal{E}})$ to obtain $\mathrm{ind}_{(g)}\,\not{D}^{\mathcal{E}}$.

Let $\tilde{F}$ be the lift of $F$ from $\Gamma_c(\mathfrak{X},\mathcal{E})$ to $L^2(\mathfrak{X},\mathcal{E})\rtimes_r G$ defined by (4.19). Recall that the higher index for $\not{D}^{\mathcal{E}}$ is given by

$$\mu_{red}[F] = [(\mathcal{A}_r, 1_{\mathbb{C}}, q\circ\tilde{F}\circ\iota)] \in KK(\mathbb{C}, C^*_r(G)),$$

where $\mathcal{A}_r := q[L^2(\mathfrak{X},\mathcal{E})\rtimes_r G]$ is the $C^*_r(G)$-module as $\mathcal{A}$ in (4.17) adapted to the reduced case.

Denote by $\mathcal{B}(\mathcal{A}_r)$ the set of bounded operators on $\mathcal{A}_r$. Let $\mathcal{K}(\mathcal{A}_r)$ be the closed ideal of compact operators over $\mathcal{A}_r$. The algebra $\mathcal{K}(\mathcal{A}_r)$ is the closure in the norm for $\mathcal{B}(\mathcal{A}_r)$ of the set of integral operators with $G$-invariant continuous kernel and with proper support. Let $\mathcal{S}(\mathcal{A}_r)\subset\mathcal{K}(\mathcal{A}_r)$ be an ideal in $\mathcal{B}(\mathcal{A}_r)$ where

1) $\mathcal{S}(\mathcal{A}_r)$ is closed under holomorphic functional calculus and
2) $\mathcal{S}(\mathcal{A}_r)$ contains the algebra of $G$-invariant properly supported smoothing operators.

In view of [**Co1**, Section 2], the algebra $\mathcal{S}(\mathcal{A}_r)$ exists and the densely defined localized traces, viewed as degree 0 cyclic cocycles, can be extended to this algebra.

As $q\circ\tilde{F}\circ\iota = \sum_{g\in G} g(c^{-\frac{1}{2}}Fc^{\frac{1}{2}})$ and it differs from $F$ by a compact operator,

$$(5.13)\quad q\circ\tilde{F}\circ\iota - F = \sum_{g\in G} g(c^{-\frac{1}{2}}Fc^{\frac{1}{2}}) - \sum_{g\in G} g(cF) = \sum_{g\in G} g(c^{-\frac{1}{2}}[F,c^{\frac{1}{2}})],$$

we can choose a different representative $F$ in place of $q\circ\tilde{F}\circ\iota$ in the same equivalence class of $KK(\mathbb{C}, C^*_r(G))$ and the parametrix $R$ so that

$$1 - RF_+ = S_0 \qquad 1 - F_+R = S_1.$$

By the Fredholm picture of $KK(\mathbb{C},\mathcal{S}(G))$, the operator $F$ is invertible in $\mathcal{B}(\mathcal{A}_r)$ up to an operator in $\mathcal{S}(\mathcal{A}_r)$. In fact, $S_0, S_1\in\mathcal{S}(\mathcal{A}_r)$. Thus, $F_+$ and $R$ give rise to elements of $K_1(\mathcal{B}(\mathcal{A}_r)/\mathcal{S}(\mathcal{A}_r))$. Moreover, the set of compact operators $\mathcal{K}(\mathcal{A}_r)$ over the $C^*_r(G)$-module $\mathcal{A}_r$ is Morita equivalent to $C^*_r(G)$. Thus, we have

$$K_0(\mathcal{S}(\mathcal{A}_r)) \to K_0(\mathcal{K}(\mathcal{A}_r)) \cong K_0(C^*_r(G)).$$

Therefore, the identification between $KK(\mathbb{C}, C_r^*(G))$ and $K_0(C_r^*(G))$ are given by the boundary map

$$KK(\mathbb{C}, C_r^*(G)) \to K_1(\mathcal{B}(\mathcal{A}_r)/\mathcal{S}(\mathcal{A}_r)) \to K_0(\mathcal{S}(\mathcal{A}_r)) \to K_0(C_r^*(G))$$
$$[(\mathcal{A}, 1_{\mathbb{C}}, q\widetilde{F}\iota)] \mapsto [F] \mapsto \left[\begin{pmatrix} S_0^2 & S_0(1+S_0)R \\ F_+S_1 & 1-S_1^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Here, the entries of the $2 \times 2$ matrix belong to $\mathcal{S}(\mathcal{A}_r)^+$, the unitalization of $\mathcal{S}(\mathcal{A}_r)$. By convention, the trace of this identity element in $\mathcal{S}(\mathcal{A}_r)^+ \backslash \mathcal{S}(\mathcal{A}_r)$ is always assumed to be 0 (see [**Co1**]). Thus, we have

$$\operatorname{ind}_{(g)} \not{D}^{\mathcal{E}} = \tau_*^{(g)}(\mu[\not{D}^{\mathcal{E}}]) = \tau_*^{(g)}([{S_0}^2]) - \tau_*^{(g)}([{S_1}^2]). \tag{5.14}$$

Note that $q\tilde{S}_i^2\iota = \sum_{g\in G} g(c^{\frac{1}{2}} S_i^2 c^{\frac{1}{2}})$, that by Lemma 3.12 it is a $G$-invariant properly supported smoothing operator for $i = 0, 1$ and that the operator $S_0^2 = e^{-t\not{D}^{\mathcal{E}}_- \not{D}^{\mathcal{E}}_+}$ and $S_1^2 = e^{-t\not{D}^{\mathcal{E}}_+ \not{D}^{\mathcal{E}}_-}$ in $\mathcal{S}(\mathcal{A}_r)$ can be approximated by the properly supported operators $q\tilde{S}_0^2\iota, q\tilde{S}_1^2\iota$, respectively, as $t \to 0^+$. Hence, when $t$ is sufficiently close to 0, they are in the same equivalence class of $K$-theory $K_0(\mathcal{S}(\mathcal{A}_r))$. So we are reduced to find $\tau_*^{(g)}([\sum_{g\in G} g(c^{\frac{1}{2}} e^{-t(\not{D}^{\mathcal{E}})^2} c^{\frac{1}{2}})])$. We shall regard $\sum_{g\in G} g(c^{\frac{1}{2}} e^{-t(\not{D}^{\mathcal{E}})^2} c^{\frac{1}{2}})$ as a matrix with coefficient in $C_r^*(G)$. Therefore, by (5.10), which defines $\tau_*^{(g)}$, and Lemma 5.6, we obtain

$$\operatorname{ind}_{(g)} \not{D}^{\mathcal{E}} = \operatorname{tr}^{(g)}(\sum_{g\in G} g(c^{\frac{1}{2}} e^{-t\not{D}^{\mathcal{E}}_- \not{D}^{\mathcal{E}}_+} c^{\frac{1}{2}})) - \operatorname{tr}^{(g)}(\sum_{g\in G} g(c^{\frac{1}{2}} e^{-t\not{D}^{\mathcal{E}}_+ \not{D}^{\mathcal{E}}_-} c^{\frac{1}{2}}))$$

when $t$ is small and the number is independent of $t$. So letting $t \to 0$ and applying Corollary 3.21, we obtain

$$\operatorname{ind}_{(g)} \not{D}^{\mathcal{E}} = \operatorname{tr}^{(g)}(e^{-t\not{D}^{\mathcal{E}}_- \not{D}^{\mathcal{E}}_+}) - \operatorname{tr}^{(g)}(e^{-t\not{D}^{\mathcal{E}}_+ \not{D}^{\mathcal{E}}_-}).$$

The proposition is then proved. q.e.d.

We are now ready to state the main theorem of this section. We combine the heat kernel asymptotics in Theorem 3.23 with Proposition 5.9 to calculate the localized indices in the following theorem.

**Theorem 5.10.** *Let $\mathfrak{X}$ be a complete Riemannian orbifold where a discrete group $G$ acts properly, co-compactly, and isometrically. Suppose the localized $(g)$-trace defined by (5.2) extends continuously to $C_r^*(G)$. Let $\not{D}^{\mathcal{E}}$ be the $G$-invariant Dirac operator on $\mathfrak{X}$. The localized $(g)$-index is calculated by*

$$\operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}) = \int^{orb}_{(G\backslash\mathfrak{X})_{(g)}} \hat{A}_{(g)}(\mathfrak{X}) ch^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E}), \tag{5.15}$$

*where $\hat{A}_{(g)}(\mathfrak{X})$ and $ch^{\mathcal{S}}_{(g)}$ are the delocalized characteristic classes of $G\backslash\mathfrak{X}$ restricted to the $(g)$-twisted sector $(G\backslash\mathfrak{X})_{(g)}$ given by (3.33).*

REMARK 5.11. The assumption that the localized $(g)$-trace extends continuously to the reduced group $C^*$-algebra is not essential. Assuming this will enable us to perturb $q \circ \tilde{F} \circ \iota$ by a compact operator in (5.13). Proceeding without this assumption produces further complications in analysis. For this reason, we shall investigate the index formula removing this condition in a future paper. The analytical result for this paper is presented in Section 3 without any trace assumption on $G$. But to show these results are topological in nature, it is convenient to add this assumption for groups. There are a lot of groups satisfying this condition such as hyperbolic groups [**Pu**] and property RD (rapid decay) groups with polynomial growth [**G**]. Geometric group theorists have studied many concrete examples of property RD groups, which provide the main examples of groups satisfying the Buam–Connes conjecture [**L2**].

These localized indices for $\not{D}^{\mathcal{E}}$ gives rise to refined topological invariants for the Dirac operator on the orbifold $G\backslash\mathfrak{X}$. In fact, in view of Theorem 5.10 and Theorem 3.23, the following theorem is immediate. We provide in addition a $K$-theory proof. Note that in the analytic result Theorem 3.23 we do not pose any trace assumption on $G$.

**Theorem 5.12.** *Suppose the localized $(g)$-trace defined by (5.2) extends continuously to $C^*_r(G)$. The orbifold index on $G\backslash\mathfrak{X}$ is the sum of localized $(g)$-indices over all conjugacy classes of $G$, that is,*

$$\operatorname{ind} \not{D}^{G\backslash\mathcal{E}} = \sum_{(g)} \operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}).$$

*Proof.* Note that $\operatorname{ind} \not{D}^{G\backslash\mathcal{E}} = \rho_*(\mu[\not{D}^{\mathcal{E}}])$ and $\operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}) = \tau_*^{(g)}(\mu[\not{D}^{\mathcal{E}}])$. Denote by $P_0$ and $P_1$ two $\mathcal{S}(G)$-valued projection matrices ($P_i^2 = P_i = P_i^*$ where $i = 0, 1$) such that $\mu[\not{D}^{\mathcal{E}}] = [P_0] - [P_1] \in K_0(\mathcal{S}(G))$. As the localized indices for Dirac operators are finite, $|\tau_*^{(g)}([P_i])| < \infty$. We need only to show that

$$\rho_*([P]) = \sum_{(g)} \tau_*^{(g)}([P]), \qquad P = P_i, \qquad i = 0 \text{ or } 1. \tag{5.16}$$

Denote by $\rho(P)$ (resp. $\tau^{(g)}(P)$) the $\mathbb{C}$-valued matrix whose $(i, j)$th entry is $\rho$ (resp. $\tau^{(g)}$) applied to the $(i, j)$th-entry of $P$. Then, the left-hand side of (5.16) is equal to the rank of $\rho(P)$, and the right-hand side of (5.16) is the sum of the trace of $\tau^{(g)}(P)$ over all conjugacy classes of $G$, which is also the trace of $\rho(P)$ observing that $\rho(P) = \sum_{(g)} \tau^{(g)}(P)$. Hence, it is sufficient to show that the rank of $\rho(P)$ equals its trace.

As $\rho$ is a homomorphism, the image $\rho(P)$ of the projection $P$ is still a projection, that is, $\rho(P)^2 = \rho(P) = \rho(P)^*$. The $\mathbb{C}$-valued projection $\rho(P)$ is then unitary equivalent to a diagonal matrix $Q$ whose entries are either 1 or 0. Note that the trace and rank are invariant under unitary equivalence. Thus, the rank of $\rho(P)$ is the same as its trace. This completes the proof of the theorem. q.e.d.

Remark 5.13. Theorem 5.10 gives rise to a local index formula for Dirac operators. Then the localized index can be defined for any $G$-invariant elliptic operator $D$ on $\mathfrak{X}$ by adapting the argument of [**W**] to the case of orbifold. In fact, $D$ gives rise to an element in $K^0_G(C^*_{red}(\mathfrak{X}))$ by Lemma 4.7. Then using a similar construction as in [**ABP**, Section 7] and in [**W**], we can find a $G$-invariant Dirac type operator $\not{D}$, representing the same $K$-homology class as $D$. Hence, $\not{D}$ and $D$ have the same higher index and localized indices. Thus, localized index formulas are well defined for $[D] \in K^0_G(C^*_{red}(\mathfrak{X}))$ and are calculated by local index formula (5.15) for the Dirac operator $\not{D}$ under the trace assumption on $G$.

## 6. Applications and further remarks

**6.1. $L^2$-Lefschtez fixed-point formula.** To illustrate that Theorem 5.10 is indeed an $L^2$-Lefschtez fixed-point formula for noncompact orbifolds, we restrict ourselves to the case of a complete Riemannian manifold where a discrete group $G$ acts properly, co-compactly, and isometrically.

Let $\mathfrak{X}$ be a good orbifold arising from a complete Riemannian manifold $M$ with a proper, co-compact, and isometric action of a discrete group $G$. In this situation, the twisted sector of the orbifold $\mathfrak{X} = G\backslash M$ is simply indexed by the conjugacy classes of the group $G$. Then over $M$ associated to each conjugacy class $(g) \subset G$, we have the localized $(g)$-index for the $G$-invariant Dirac operator $\not{D}^{\mathcal{E}}$. By Theorem 5.10, we know the following:

1) When $g$ is the group identity, the localized index of $\not{D}^{\mathcal{E}}$, also known as the $L^2$-index, gives rise to the top stratum of the Kawasaki index formula for the Dirac operator $\not{D}^{G\backslash\mathcal{E}}$ on $\mathfrak{X} = G\backslash M$.
2) When $g \in G$ is not the group identity, the localized indices of $\not{D}^{\mathcal{E}}$ characterize the lower strata of the orbifold index formula of $\not{D}^{G\backslash\mathcal{E}}$.

Therefore, we have related the higher index of $\not{D}^{\mathcal{E}}$ to the orbifold index restricted to each twisted sector $\mathfrak{X}_{(g)}$ by the localized $(g)$-trace (cf. Definition 5.2).

Denote by $M^g$ the fixed-point submanifold of $M$ by $g \in G$. Then the component for the inertia orbifold $I(G\backslash M)$ has the following structure indexed by the conjugacy class $(g)$ of $G$:

$$(G\backslash M)_{(g)} = G\backslash \cup_{h\in(g)} M^h = Z_G(g)\backslash M^g. \tag{6.1}$$

In the following, we shall derive from Theorem 5.10 a formula of $\mathrm{ind}_{(g)} \not{D}^{\mathcal{E}}$ as integration over fixed-point submanifolds by introducing a suitable cut-off function. For example, when $g = e$, the localized $(e)$-index of $\not{D}^{\mathcal{E}}$ is the $L^2$-index of $\not{D}^{\mathcal{E}}$ and is equal to the top stratum of the formula for $\operatorname{ind} \not{D}^{\mathcal{E}}_{G\backslash M}$:

(6.2)
$$L^2\text{-}\operatorname{ind}(\not{D}^{\mathcal{E}}) = \int_{G\backslash M}^{orb} \hat{A}(G\backslash M)\operatorname{ch}^{\mathcal{S}}(G\backslash\mathcal{E}) = \int_M c(x)\hat{A}(M)\operatorname{ch}^{\mathcal{S}}(\mathcal{E}),$$

where $c$ is a cut-off function on $M$ with respect to the $G$ action. We shall show in this subsection a localized index formula for all $g \in G$ in the fashion of (6.2).

Note that $Z_G(g)$ acts on $M^g$ isometrically. Given a cut-off function $c$ on $M$ with respect to the $G$ action, we construct a function $c^{(g)}$ on $M^g$ as follows:

$$c^{(g)}(y) = \sum_{k\in G/Z_G(g)} c(k^{-1}y) \quad y \in M^g, \tag{6.3}$$

where $G/Z_G(g)$ is identified as a subset $K$ of $G$ in view of Lemma 3.15 (see (3.20) for the definition of $K$). By Lemma 3.15, the function given by (6.3) is in fact a cut-off function on $M^g$ with respect to the $Z_G(g)$-action:

$$\sum_{l\in Z_G(g)} c^{(g)}(l^{-1}y) = \sum_{l\in Z_G(g)}\sum_{k\in K} c(l^{-1}k^{-1}y) = \sum_{g\in G} c(g^{-1}y) = 1.$$

The localized $(g)$-index of $\not{D}^{\mathcal{E}}$ is given by

$$\begin{aligned}\operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}) &= \int_{\mathfrak{X}_{(g)}}^{orb} \frac{\hat{A}(\mathfrak{X}_{(g)})\operatorname{ch}^{\mathcal{S}}_{(g)}(G\backslash\mathcal{E})}{\det\left(1-\Phi_{(g)}e^{R_{\mathcal{N}_{(g)}}/2\pi i}\right)^{\frac{1}{2}}} \\ &= \int_{M^g} c^{(g)}(x)\frac{\hat{A}(M^g)\operatorname{ch}^{\mathcal{S}}_g(\mathcal{E})}{\det(1-ge^{R_{\mathcal{N}^g}/2\pi i})^{\frac{1}{2}}}.\end{aligned}$$

We summarize these results as the following $L^2$-version of the Lefschetz fixed-point formula for a complete Riemannian manifold $M$ with a proper co-compact action of a discrete group $G$.

**Theorem 6.1.** *Let $M$ be a complete Riemannian manifold where a discrete group $G$ acts properly, co-compactly, and isometrically. Let $\not{D}^{\mathcal{E}}$ be the $G$-invariant Dirac operator on $M$, and let $\not{D}^{\mathcal{E}}_{G\backslash M}$ be the corresponding Dirac operator on the quotient orbifold $G\backslash M$. Then $L^2\text{-}\operatorname{ind}\not{D}^{\mathcal{E}} = \operatorname{ind}_{(e)}(\not{D}^{\mathcal{E}})$, the localized index of $\not{D}^{\mathcal{E}}$ at the identity conjugacy class $(e)$. For $g \neq e$, the localized $(g)$-index of $\not{D}^{\mathcal{E}}$ is given by*

$$\operatorname{ind}_{(g)}(\not{D}^{\mathcal{E}}) = \int_{M^g} c^{(g)}(x)\frac{\hat{A}(M^g)\operatorname{ch}^{\mathcal{S}}_g(\mathcal{E})}{\det(1-ge^{R_{\mathcal{N}^g}/2\pi i})^{\frac{1}{2}}},$$

*where $c^{(g)}$ is the cut-off function on $M^g$ with respect to the action of $Z_G(g)$ given by (6.3).*

**6.2. Selberg trace formula.** We shall present an interesting connection between the localized indices and the orbital integrals in the Selberg trace formula. To begin with, we recall the set up of the Selberg trace formula [**S**] (see also the survey article [**Ar**]).

Let $G$ be a real unimodular Lie group, and let $\Gamma$ be a discrete cocompact subgroup of $G$. Denote by $R$ the right regular unitary representation of $G$ on $L^2(G)$:

$$R(g)f(h) = f(g^{-1}h) \qquad \forall g, h \in G, \forall f \in L^2(G).$$

It extends to a representation of $L^1(G)$ on $L^2(G)$ as follows:

$$R(f) = \int_G f(g)R(g)dg \qquad \forall f \in L^1(G). \tag{6.4}$$

As $\Gamma$ acts on the left of $G$, $R$ is reduced to

$$R : L^1(G) \longrightarrow \mathrm{End}(L^2(\Gamma\backslash G)). \tag{6.5}$$

Let $f \in C^\infty(G) \cap L^1(G)$ be a test function, where $R(f)$ is a trace class operator on $L^2(\Gamma\backslash G)$. The Selberg trace formula is an equality relating two ways in calculating $\mathrm{Tr}\, R(f)$, where Tr is the operator trace on $L^2(\Gamma\backslash G)$. On the one hand, $\mathrm{Tr}\, R(f)$ has the spectral decomposition indexed by all irreducible unitary representations of $G$ (denoted by $\mathrm{Irr}(G)$):

$$\mathrm{Tr}\, R(f) = \sum_{\pi \in \mathrm{Irr}(G)} m(\pi)\, \mathrm{Tr}(\pi(f)), \tag{6.6}$$

where $m(\pi)$ is the multiplicity of $\pi$ in $R$. The equality (6.6) is called the **spectral side** of the Selberg trace formula. On the other hand, the Schwartz kernel $K(x, y)$ of $R(f)$, which has the expression

$$K(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y), \tag{6.7}$$

can be used to calculate $\mathrm{Tr}\, R(f)$ as follows:

$$\mathrm{Tr}\, R(f) = \int_{\Gamma\backslash G} K(x, x)dx = \sum_{(\gamma)\subset\Gamma} \mathrm{vol}(Z_G(\gamma)/Z_\Gamma(\gamma)) \int_{Z_G(\gamma)\backslash G} f(x^{-1}\gamma x)dx, \tag{6.8}$$

where the sum is over representatives of all conjugacy classes of $G$. This equality (6.8) is called the **geometric side** of the Selberg trace formula. We denote the $(\gamma)$-summand in (6.8) by $\mathcal{O}_\gamma$ and call it the **orbital integral**:

$$\mathcal{O}_\gamma := \mathrm{vol}(Z_G(\gamma)/Z_\Gamma(\gamma)) \int_{Z_G(\gamma)\backslash G} f(x^{-1}\gamma x)dx.$$

The orbital integral is easier to calculate and is an important tool in finding the multiplicity of a representation.

To relate Selberg trace formula to localized indices, we recall the similar setting that appeared in [**CM**, **BM**]. Let $H$ be a maximal compact subgroup of $G$ with volume 1. Let $M = G/H$ be the Riemannian symmetric manifold of noncompact type. Then the discrete co-compact subgroup $\Gamma$ of $G$ acts properly and co-compactly on $M$. Let $\mathcal{E}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded homogeneous vector bundle over $M$, that is, there exists a $\mathbb{Z}/2\mathbb{Z}$-graded $H$-representation $E$, where $H$ respects the grading of $E$, so that $\mathcal{E} = G \times_H E$. Denote by $\not{D}^{\mathcal{E}} : L^2(M, \mathcal{E}) \to L^2(M, \mathcal{E})$ the $G$-invariant Dirac operator. We then have the identification

$$L^2(M, \mathcal{E}) \cong (L^2(G) \otimes E)^H.$$

Therefore, the corresponding Dirac operator $\not{D}^{\Gamma\backslash\mathcal{E}}$ on the quotient orbifold $\Gamma\backslash M$ is regarded as an operator on $(L^2(\Gamma\backslash G) \otimes E)^H$. Let us denote by $K_t$ (resp. $\bar{K}_t$) the heat kernel for $\not{D}^{\mathcal{E}}$ (resp. $\not{D}^{\Gamma\backslash\mathcal{E}}$). As $M$ is homogeneous, we have:

$$K_t \in (C^\infty(G \times G) \otimes \mathrm{End}(E))^{H\times H} \quad \bar{K}_t \in (C^\infty(\Gamma\backslash G \times \Gamma\backslash G) \otimes \mathrm{End}(E))^{H\times H}.$$

As $D^{\mathcal{E}}$ is $G$-invariant, $K_t$ gives rise to a well-defined function $k_t \in (C^\infty(G) \otimes \mathrm{End}(E))^{H\times H}$ given by

$$k_t(x^{-1}y) = K_t(x, y), \quad \forall x, y \in G. \tag{6.9}$$

We shall still consider the right regular representation of a test function, but we replace the representation space $L^2(\Gamma\backslash G)$ by the $\mathbb{Z}/2\mathbb{Z}$-graded space $(L^2(\Gamma\backslash G) \otimes E)^H$ and the operator trace by the supertrace. Note that by (6.7) and Theorem 3.4, we observe that the Schwartz kernel of $R(k_t)$ is exactly $\bar{K}_t(x, y)$, the heat kernel for $\not{D}^{\Gamma\backslash\mathcal{E}}$. Applying Theorem 3.4 to (6.7), we see that the supertrace $\mathrm{Tr}_s\, R(k_t)$ is finite. Therefore, even though $k_t$ is not compactly supported on $G$, we can choose $k_t$ as a test function. Comparing the localized index formula and the Selberg trace formula in this situation, we obtain the following theorem, which states that the orbital integrals for $\mathrm{Tr}\, R(k_t)$ have a one-to-one correspondence with the localized indices for $\not{D}^{\mathcal{E}}$.

**Theorem 6.2.** *Let $k_t$ be the test function (6.9) determined by the heat operator for the Dirac operator $\not{D}^{\mathcal{E}}$ on the Riemannian symmetric manifold $M = G/H$ of noncompact type. Let $R(k_t)$ be the right regular representation of $L^1(G)$ on $(L^2(\Gamma\backslash G) \otimes E)^H$ in the sense of (6.5). Then the geometric side of the Selberg trace formula of $\mathrm{Tr}_s(R(k_t))$, which was expressed as a sum of orbital integrals $\mathcal{O}_\gamma$ in (6.8) over all conjugacy classes of $\Gamma$, corresponds exactly to the sum of localized indices for $\not{D}^{\mathcal{E}}$. Moreover,*

$$\mathrm{Tr}_s(R(k_t)) = \sum_{(\gamma)\subset\Gamma} \mathcal{O}_\gamma = \mathrm{ind}\, \not{D}^{\Gamma\backslash\mathcal{E}}, \qquad \mathcal{O}_\gamma = \mathrm{ind}_{(\gamma)}\, \not{D}^{\mathcal{E}}.$$

*Proof.* By the geometric side (6.8) of the Selberg trace formula and the definition of the localized indices (cf. Theorem 5.10), we shall only need to show that the orbital integral

$$\mathcal{O}_\gamma = \mathrm{vol}(Z_G(\gamma)/Z_\Gamma(\gamma)) \int_{Z_G(\gamma)\backslash G} \mathrm{Tr}_s[k_t(x^{-1}\gamma x)\gamma]dx, \tag{6.10}$$

where $\mathrm{Tr}_s$ is the supertrace of $\mathrm{End}\, E$, is equal to

$$\mathrm{tr}_s^{(\gamma)_\Gamma}\, e^{-t(\not{D}^{\mathcal{E}})^2} := \sum_{h\in(\gamma)} \int_G c(x)\,\mathrm{Tr}_s[K_t(x,hx)h]dx,$$

where $c$ is a cut-off function on $G$ with respect to the $\Gamma$-action.

Let us rewrite $\mathrm{tr}_s^{(\gamma)_\Gamma}\, e^{-t(\not{D}^{\mathcal{E}})^2}$ as

$$\begin{aligned} \mathrm{tr}_s^{(\gamma)_\Gamma}\, e^{-t(\not{D}^{\mathcal{E}})^2} &= \sum_{k\in K} \int_G c(x)\,\mathrm{Tr}_s[k_t(x^{-1}k\gamma k^{-1}x)k^{-1}\gamma k]dx \\ &= \int_G c^{(\gamma)}(x)\,\mathrm{Tr}_s[k_t(x^{-1}\gamma x)\gamma]dx, \end{aligned} \tag{6.11}$$

where $K$ is the set generating the conjugacy class $(\gamma)_\Gamma$ defined in (3.20) and

$$c^{(\gamma)}(x) := \sum_{k\in K} c(kx).$$

Identify the space of the right cosets $Z_G(\gamma)\backslash G$ of $Z_G(\gamma)$ in $G$ as a subset of $G$ consisting of representatives of the right cosets. Then any $x \in G$ can be decomposed uniquely into $x = ba$, where $b \in Z_G(\gamma)$ and $a \in Z_G(\gamma)\backslash G$. Notice that $k_t(a^{-1}b^{-1}\gamma ba)\gamma = k_t(a^{-1}\gamma a)\gamma$ for all $b \in Z_G(\gamma)$. Thus, (6.11) is equal to

$$\mathrm{tr}_s^{(\gamma)_\Gamma}\, e^{-t(\not{D}^{\mathcal{E}})^2} = \int_{Z_G(\gamma)\backslash G} \mathrm{Tr}_s[k_t(a^{-1}\gamma a)\gamma]\left[\int_{Z_G(\gamma)} c^{(\gamma)}(ba)db\right] da. \tag{6.12}$$

Lemma 3.15 implies that $K\cdot Z_\Gamma(\gamma) = \Gamma$. Then for any $l \in G$, we have

$$\sum_{b\in Z_\Gamma(\gamma)} c^{(\gamma)}(bla) = \sum_{k\in K, b\in Z_\Gamma(\gamma)} c(kbla) = \sum_{h\in\Gamma} c(hla) = 1.$$

The term $\int_{Z_G(\gamma)} c^{(\gamma)}(ba)db$ in (6.12) can be calculated similarly as the argument we used to derive (6.12):

$$\begin{aligned} \int_{Z_G(\gamma)} c^{(\gamma)}(ba)db = \int_{Z_G(\gamma)/Z_\Gamma(\gamma)} &\left[\sum_{l\in Z_\Gamma(\gamma)} c^{(\gamma)}(hla)dl\right] \\ dh &= \mathrm{vol}(Z_G(\gamma)/Z_\Gamma(\gamma)). \end{aligned}$$

Together with (6.12) and (6.10), we see that $\mathrm{tr}_s^{(\gamma)_\Gamma}\, e^{-t(\not{D}^{\mathcal{E}})^2} = \mathcal{O}_\gamma$. The theorem is then proved by comparing (6.8) and Theorem 5.12. q.e.d.

Remark 6.3. The theorems in Sections 6.1 and 6.2 only concern the heat kernel analysis discussed in Section 3. Thus, they did not need the assumption of the extension of $(g)$-traces in Theorem 5.10. In case we are not sure if $\operatorname{ind}_{(g)} \not{D}$ is equal to the $(g)$-supertrace of the heat kernel $\operatorname{tr}_s^{(g)} e^{-t\not{D}^2}$, we use the latter to replace the definition of the localized $(g)$-index.

**6.3. An application in positive scalar curvature.** Localized indices produce finer topological invariants for the $G$-orbifold $\mathfrak{X}$ and the quotient $G\backslash\mathfrak{X}$. They also reveal some geometric information of the orbifold. We present a result regarding positive scalar curvature. As we know, the higher index of an elliptic operator is the obstruction of the invertibility of the operator. However, the higher index is difficult to compute. Therefore, as a weaker condition but easier to compute, the nonvanishing of the localized indices is an alternative obstruction of invertibility of the operator. We then formulate the nonvanishing result as follows.

Let $\mathfrak{Y}$ be a compact spin orbifold obtained from the quotient of a complete orbifold $\mathfrak{X}$ by a discrete $G$ group acting properly, co-compactly, and isometrically. Let $\mathcal{S}$ be the spinor bundle over $\mathfrak{X}$. Denote by $\not{D}^{\mathcal{S}}$ (resp. $\not{D}^{G\backslash\mathcal{S}}$) the Dirac operator on $\mathfrak{X}$ (resp. $\mathfrak{Y}$).

**Theorem 6.4.** *Assume $G$ is a group such that localized $(g)$-traces can be extended to the reduced group $C^*$-algebra. If any of the localized indices of the $G$-invariant Dirac operator $\not{D}^{\mathcal{S}}$ on $\mathfrak{X}$ is nonzero, then the quotient orbifold $\mathfrak{Y}$ cannot have positive scalar curvature.*

*Proof.* If the quotient orbifold $\mathfrak{Y} = G\backslash\mathfrak{X}$ has a metric leading to a positive scalar curvature, then as $G$ acts by isometry, the covering orbifold $\mathfrak{X}$ also has a positive scalar curvature, denoted by $r_{\mathfrak{X}}$. Then by the Lichnerowicz formula (3.8), we have

$$(\not{D}^{\mathcal{S}})^2 = \Delta^{\mathcal{S}} + \frac{1}{4} r_{\mathfrak{X}},$$

which is a strictly positive operator. Hence, $\not{D}^{\mathcal{S}}$ is invertible. Choosing the parametric to be $(\not{D}^{\mathcal{S}})^{-1}$, we then observe that $T_0 = T_1 = 0$ in the proof of Proposition 5.9, and this implies that $\operatorname{ind}_{(g)} \not{D}^{\mathcal{S}} = 0$ in this situation. Thus, all the localized indices are 0, which contradicts the assumption. The theorem is then proved. q.e.d.

Remark 6.5. If the orbifold index of the Dirac operator $\not{D}^{G\backslash\mathcal{S}}$ on the quotient is 0, it does not prove the nonexistance of positive scalar curvature of $\mathfrak{Y} = G\backslash\mathfrak{X}$. But if the localized indices, which sum up to be 0 by Theorem 5.12, are not all 0, then Theorem 6.4 shows that $\mathfrak{Y}$ cannot admit a positive scalar curvature.

On the other hand, while the localized indices (for $g \neq e$) would vanish for spaces with positive scalar curvature, we expect the localized indices to be useful in the study of nonpositively curved space, for example, Riemannian symmetric manifolds of noncompact type with compact orbifold quotients.

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