Localizing groups with action

by

Georg Peschke

Publicacions Matemátiques 33 (1989) 227–234

Localization and genus in group theory and homotopy theory

Georg Peschke[1]

Abstract: When localizing the semidirect product of two groups, the effect on the factors is made explicit. As an application in Topology, we show that the loop space of a based connected CW-complex is a P-local group, up to homotopy, if and only if $\pi_1 X$ and the free homotopy groups $[\mathrm{S}^{k-1}, \Omega X]$, $k \geq 2$, are P-local.

Introduction

The study of groups G in which the functions $\rho_p : G \to G$, $\rho_p(g) = g^p$, are, for certain primes p, bijective, has a long history, see Malcev [9], Baumslag [1] and the references there. After Sullivan [16], Bousfield-Kan [3], Hilton [5] and Hilton-Mislin-Roitberg [8] this study appears now in the guise of localizing a group with respect to a given set of primes P. In a P-local group the functions ρ_n are bijective if n belongs to the multiplicative closure of the set of primes P', which is complementary to P.

According to Ribenboim [12, 13], there is a P-localizing functor from the category of groups to the category of P-local groups, $\mathcal{G} \to \mathcal{G}_P$. While the properties of this functor, when restricted to the category of nilpotent groups, are well understood (see [5] and [7]) its properties in general are not clear at all.

For example, on nilpotent groups the P-localizing functor is exact, but not in general. E.g., the exact sequence $\mathbb{Z} \rightarrowtail S_3 \twoheadrightarrow \mathbb{Z}/2$ for the symmetric group of 3 elements gets sent to $\mathbb{Z}/3 \to 0 \to 0$, when localizing at 3. S_3 is a semidirect product $\mathbb{Z}/3 \rtimes \mathbb{Z}/2$ and the purpose of this paper is to investigate the effect of localization on semidirect products $G = H \rtimes R$.

Since localization is functorial, G_P is again a semidirect product $G_P \cong K \rtimes R_P$. Therefore, it is desirable to understand the relation between H and K. We will discover that K is the P-localization of H with respect to the change of operator groups from R to R_P.

To explain this, we use the category ${}_R\mathcal{G}$ of R-groups (i.e. groups on which the group R acts on the left) and R-homomorphisms (i.e. group homomorphisms $f : H \to H'$ with $f(r.h) = r.f(h)$ for all $h \in H$ and $r \in R$). Further a group homomorphism $\gamma : R \to S$ induces the change-of-operator-groups functor $\gamma^* : {}_S\mathcal{G} \to {}_R\mathcal{G}$. For $H \in {}_R\mathcal{G}$, $K \in {}_S\mathcal{G}$, a group homomorphism $f : H \to K$ is a γ-homomorphism if $f : H \to \gamma^* K$ is an R-homomorphism. We then construct a left adjoint ${}_\gamma\mathrm{Ad}$ for γ^*; see (1.5).

Now ${}_S\mathcal{G}$ contains a subcategory ${}_S\mathcal{G}_P$ consisting of such groups on which S acts P-locally; see 1.2. Accordingly, we construct a left adjoint ${}_S L_P : {}_S\mathcal{G} \to {}_S\mathcal{G}_P$; see 1.6. The

[1]Supported by NSERC of Canada
Addr: Dept. of Mathem. Sciences, Univ. of Alberta, Edmonton, Canada T6G 2G1
E-mail: George.Peschke ualberta.ca

composite ${}_\gamma L_P := \ {}_S L_P \ {}_{\gamma|\mathrm{Ad}} : \ {}_R\mathcal{G} \to \ {}_S\mathcal{G}_P$ is left adjoint to the restriction of γ^* to ${}_S\mathcal{G}_P$. It then follows that $(H \rtimes R)_P \cong (\ {}_e L_P H) \rtimes R_P$, where $e : R \to R_P$ P-localizes.

Remarks

(1) The functor ${}_\gamma\mathrm{Ad}$ is of independent interest. For example, let ${}_S\mathrm{Ad}$ correspond to the unique homomorphism $\{1\} \to S$. Then ${}_S\mathrm{Ad}$ provides the foundation for a theory of S-groups by generators and relations.

(2) The problem of localizing semidirect products has also been studied by Casacuberta [4] in the case where the normal subgroup H is abelian, and by A. Reynol when H is finite abelian [11].

(3) Our study is also of interest in Topology; see (1.7) and (1.8).

It is a pleasure to acknowledge several useful conversations with K. Varadarajan. Also I owe insight into the matter to correspondence with P. Hilton and C. Casacuberta.

1. We now take up the announced investigation. So let R be a group acting on another group H via a homomorphism $\phi : R \to \mathrm{Aut}(H)$. The corresponding semidirect product is denoted by $H \rtimes_\phi R$ or $H \rtimes R$ if there is no risk of confusion.

1.1 Lemma $G = H \rtimes R$ is P-local if and only if the following two conditions hold:

(i) R is P-local;

(ii) For all $r \in R$ and $n \in P'$, the function

$$\rho_{r,n} : H \longrightarrow H, \qquad h \longmapsto h\phi_r(h)\phi_{r^2}(h) \cdot \cdots \cdot \phi_{r^{n-1}}(h)$$

is a bijection, where ϕ_r denotes the automorphism $\phi(r)$ of H.

Proof This follows from $(h, r)^n = (\rho_{r,n}(h), r^n)$. □

The functions $\rho_{r,n}$ have been used already by Baumslag in a setting involving wreath products; see [2].

1.2 Definition R acts P-locally on H if, for all $r \in R$ and $n \in P'$, the function $\rho_{r,n}$ of (1.1) is a bijection.

The notion of a P-local action has independently been introduced by Rodicio [14]. Since $\rho_{1,n}(h) = h^n$, if R acts P-locally on H, then H is P-local. We write ${}_R\mathcal{G}_P$ for the category of R-groups on which R acts P-locally.

It is straight forward to prove

1.3 Lemma Let

$$\begin{array}{ccccc} R & \rightarrowtail & H \rtimes_\phi R & \twoheadrightarrow & R \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ K & \rightarrowtail & K \rtimes_\psi S & \twoheadrightarrow & S \end{array}$$

be a commuting diagram of split exact sequences of groups. Then β P-localizes in $\mathcal{G}$ if and only if the following three conditions hold:

(i) γ P-localizes in $\mathcal{G}$;

(ii) S acts P-locally on K;

(iii) For all $L \in {}_S\mathcal{G}$ on which S acts P-locally and every γ-homomorphism $\nu : H \to L$, there is a unique S-homomorphism $\nu' : K \to L$, with $\nu = \nu'\alpha$.

This suggests

1.4 Definition Let $H \in {}_R\mathcal{G}$, $K \in {}_S\mathcal{G}$ and let $\gamma : R \to S$ be a homomorphism. Then $\alpha : H \to K$ P-localizes with respect to γ if and only if the following three conditions hold:

(i) S acts P-locally on K;

(ii) α is a γ-homomorphism;

(iii) α satisfies the universal property (1.3.iii) above.

Thus, Lemma 1.3 can be restated as

1.3' Lemma β P-localizes in $\mathcal{G}$ if and only if γ P-localizes in $\mathcal{G}$ and α P-localizes with respect to γ. □

Now let $\gamma\colon R \to S$ be given. The construction of a left adjoint functor ${}_\gamma L_P\colon {}_R\mathcal{G} \to {}_S\mathcal{G}_P$ to the composite functor ${}_S\mathcal{G}_P \xrightarrow{\text{inclusion}} {}_S\mathcal{G} \xrightarrow{\gamma^*} {}_R\mathcal{G}$ is done in two steps.

1.5 Theorem $\gamma^* : {}_S\mathcal{G} \to {}_R\mathcal{G}$ has a left adjoint ${}_\gamma\mathrm{Ad} : {}_R\mathcal{G} \to {}_S\mathcal{G}$.

1.6 Theorem The inclusion functor ${}_S\mathcal{G}_P \to {}_S\mathcal{G}$ has a left adjoint left inverse ${}_SL_P : {}_S\mathcal{G} \to {}_S\mathcal{G}_P$.

It then follows from 1.3' that $(H \rtimes R)_P \cong ({}_eL_PH) \rtimes R_P$, where $e : R \to R_P$ P-localizes.

Here is an interesting application of P-local actions in Topology.

1.7 Theorem Let X be a based connected CW-complex. Then the two conditions below are equivalent.

(i) π_1X and the free homotopy groups $[\mathrm{S}^{k-1}, \Omega X]$, $k \geq 2$, are P-local;

(ii) ΩX is P-local group up to homotopy; i.e. for each $n \in P'$, the map $\overline{\rho}_n : \Omega X \to \Omega X$, $\overline{\rho}_n(x) = x^n$, is a homotopy equivalence.

Proof “(ii) $\implies$ (i)’ Recall that $\overline{\rho}_n$ induces ρ_n on $\pi_0 \Omega X$ and on all free homotopy groups $[\mathrm{S}^{k-1}, \Omega X]$, $k \geq 2$. If $\overline{\rho}_n$ is a homotopy equivalence, then ρ_n is a bijection. Thus $\pi_0 \Omega X \cong \pi_1 X$ and $[\mathrm{S}^{k-1}, \Omega X]$ are P-local.
(i) $\implies$ (ii) Recall from [10] that ωX is an H-semidirect product: $\Omega X \simeq (\Omega X)_0 \rtimes \pi_1 X$ and, as a consequence, that $[\mathrm{S}^{k-1}, \Omega X] \cong \pi_k X \rtimes \pi_1 X$, for all $k \geq 2$. Since $\pi_1 X$ is P-local, $\overline{\rho}_n$ determines a bijection of the connected components of ΩX. Since $(\Omega X)_0$ is a simple space, the restriction of ρ_n to $(\Omega X)_0 \times \{r\}$, $r \in \Pi_1 X$, induces the homomorphism

$$\pi_{k-1}(\Omega X)_0 \times \{r\} \cong [\mathrm{S}^{k-1}, (\Omega X)_0 \times \{r\}] \longrightarrow [\mathrm{S}^{k-1}, (\Omega X)_0 \times \{r^n\}] \cong \pi_{k-1}(\Omega X)_0 \times \{r^n\}.$$

By hypothesis, this is a bijection. Thus, $\overline{\rho}_n$ is a homotopy equivalence. □

1.8 Corollary The loop space of a P-local nilpotent CW-complex is a P-local group up to homotopy.

Proof If X is a P-local nilpotent space, then $\pi_1 X$ is P-local. Furthermore, the groups $[\mathrm{S}^{k-1}, \Omega X]$, $k \geq 2$, are semidirect products of the P-local groups $\pi_k X$ and $\pi_1 X$ with respect to a nilpotent action of $\pi_1 X$ on $\pi_k X$. By a result of Hilton [6], the groups $[\mathrm{S}^{k-1}, X]$ are P-local, for $k \geq 2$; compare also Roitberg [15]. Now apply (1.7). □

2 Proof of Theorem 1.5

We need the following lemma whose proof is a little tedious but straightforward.

2.1 Lemma Let R act on H via $\phi : R \to \mathrm{Aut}(H)$. Let

$$D \;:=\; \{rhr^{-1}\phi_r(h^{-1}) : r \in R, \quad h \in H\} \;\subset\; H * R.$$

Let $\overline{H}$, $\overline{D}$ denote the normal closure of H, D in $H * R$. Then $\overline{D}$ is normal in $\overline{H}$ and $\overline{H}/\overline{D}$ is isomorphic to H. □

Step 1 for the proof of (1.5): Construction of ${}_\gamma\mathrm{Ad}$
Let R act on H via $\phi : R \to \mathrm{Aut}(H)$ and consider the diagram

$$\begin{array}{ccccc}
\overline{H} & \xrightarrow{i} & H * R & \xrightarrow{\pi} & R \\
\downarrow \eta & 1 & \downarrow \mathrm{Id}*\gamma & 2 & \downarrow \gamma \\
\widehat{H} & \xrightarrow[i']{} & H * S & \xrightarrow[\pi']{} & S
\end{array}$$

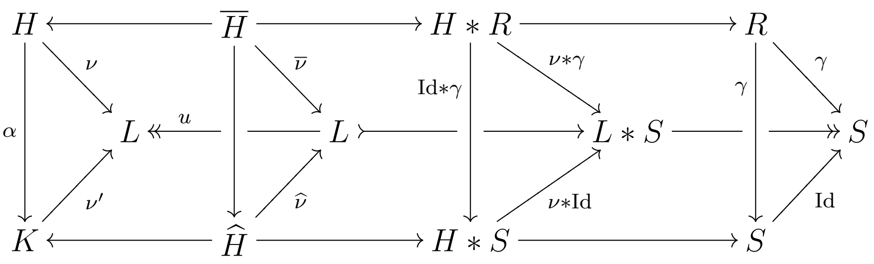
where π, π' are the canonical epimorphisms making the right hand square commute. $\overline{H} := \ker(\pi)$ and $\widehat{H} := \ker(\pi')$. Note that $(\mathrm{Id} * \gamma)(\overline{H}) \subset \widehat{H}$ and let η be the restriction of $\mathrm{Id} * \gamma$ to $\overline{H}$. Then the left hand square also commutes.

By design, R acts on $\overline{H}$ by conjugation and S acts on $\widehat{H}$ by conjugation and η is a γ-homomorphism. Using (2.1), we relate these actions to the given action of R on H. We have refined the method of HNN-extensions.

By (2.1), $\overline{D} \subset \overline{H}$. Let $\widehat{D}$ be the normal closure of $\eta(D)$ in $H * S$. Since $\eta(D) \subset \widehat{H} \lhd (H * S)$, $\widehat{D} \subset \widehat{H}$. Take $K := {}_\gamma\mathrm{Ad}(H) := \widehat{H}/\widehat{D}$. Then η defines $\alpha\colon H \cong \overline{H}/\overline{D} \to \widehat{H}/\widehat{D} = K$.

The action of S on $\widehat{H}$ by conjugation passes down to an action $\psi : S \to \mathrm{Aut}(K)$. Explicitly, $\psi_S(\widehat{h}\widehat{D}) = s\widehat{h}s^{-1}\widehat{D}$. The action of R on $\overline{H}$ by conjugation passes down to the original action ϕ, by (2.1). It is clear that α is a γ-homomorphism.

Step 2: Verification of the universal property of $\alpha : H \to K$. Let S act on L via $\theta : S \to \mathrm{Aut}(L)$. Let $\nu : H \to L$ be a γ-homomorphism. Consider the diagram



The right hand prism commutes and induces homomorphisms $\overline{\nu}$, $\widehat{\nu}$ by restriction. Further $\overline{\nu}(D) \subset \ker(u)$. Consequently, $\widehat{\nu}(\widehat{D}) \subset \ker(u)$, showing that $\widehat{\nu}$ factors through K with $\nu' : K \to L$. Since $\widehat{\nu}$ is an S-homomorphism, so is ν'.

It is straightforward to check uniqueness of ν' on the generators xhx^{-1} of K, where $x \in H * S$, $h \in H$. That ${}_\gamma\mathrm{Ad}$ is a functor is immediate. This completes the proof of (1.5). $\square$

It follows directly from the construction of ${}_\gamma\mathrm{Ad}$ that

2.2 Proposition ${}_\gamma\mathrm{Ad}$ preserves epimorphisms. $\square$

3 Proof of Theorem 1.6

Let

$$\begin{aligned} {}_S\mathcal{U}_P &:= \{K \in {}_S\mathcal{G} : \rho_{s,n} \text{ is 1-1 for all } s \in S,\ n \in P'\} \\ {}_S\mathcal{E}_P &:= \{K \in {}_S\mathcal{G} : \rho_{s,n} \text{ is onto for all } s \in S,\ n \in P'\}. \end{aligned}$$

Then ${}_S\mathcal{G}_P := {}_S\mathcal{U}_P \cap {}_S\mathcal{E}_P$ is the category of S-groups on which S acts P-locally.

We construct functors ${}_S\sqrt{}_P : {}_S\mathcal{G} \to {}_S\mathcal{G}$, which create preimages for the functions $\rho_{s,n}$ as well as ${}_SU_P : {}_S\mathcal{G} \to {}_S\mathcal{U}_P$, which make preimages of the functions $\rho_{s,n}$ unique.

Let S act on K via $\psi : S \to \mathrm{Aut}(K)$. Let FK denote the free group with basis $\{k_{s,n} : k \in K,\ s \in S,\ n \in P'\}$ and let $\xi K := {}_S\mathrm{Ad}(FK)$ denote the free S-group with that basis. If $\theta : S \to \mathrm{Aut}(\xi K)$ denotes the corresponding S-action, then S acts on $K * \xi K$ by $S \ni s \mapsto \psi_s * \theta_s \in \mathrm{Aut}(K * \xi K)$. Let N denote the S-invariant normal closure of the set $\{\rho_{s,n}(k_{s,n}k^{-1} : k \in K,\ s \in S,\ n \in P'\}$ in $K * \xi K$.

3.1 Definition ${}_S\sqrt{}_P K := K * \xi K / N$.

There is a canonical homomorphism $t\colon K \to {}_S\sqrt{}_P K$. By design, $\mathrm{im}(t) \subset \mathrm{im}(\rho_{s,n})$, for all $s \in S$ and $n \in P'$. Further, an S-homomorphism $f : K \to K'$ induces the S-homomorphism $\xi f : \xi K \to \xi K'$ via the function $k_{s,n} \mapsto [f(k)]_{s,n}$ on bases. Hence, the S-homomorphism $(f * \xi f) : K * \xi K \to K' * \xi K'$ is defined. Passing to quotients, it yields the S-homomorphism ${}_S\sqrt{}_P f : {}_S\sqrt{}_P K \to {}_S\sqrt{}_P K'$.

3.2 Lemma The following hold.

(i) ${}_S\sqrt{}_P : {}_S\mathcal{G} \to {}_S\mathcal{G}$ is a covariant functor.

(ii) ${}_S\sqrt{}_P$ preserves epimorphisms.

(iii) The homomorphism $t : K \to {}_S\sqrt{}_P K$ defines a natural transformation of the identity functor on ${}_S\mathcal{G}$ to ${}_S\sqrt{}_P$.

(iv) If $f : K \to L$ is an S-homomorphism such that $\rho_{l,n}$ is (1–1) and onto $\mathrm{im}(f)$ for all $l \in L$ and $n \in P'$, then there is a unique S-homomorphism $f' : {}_S\sqrt{}_P K \to L$ with $f = f't$.

Proof (i), (ii) and (iii) are straightforward from the construction.
(iv) The universal property of ${}_S\mathrm{Ad}$ yields a unique S-homomorphism $d : \xi K \to L$ corresponding to the homomorphism $FK \to L$, $k_{s,n} \mapsto \rho_{s,n}^{-1} f(k)$. Observe that $\ker(K * \xi K \twoheadrightarrow {}_S\sqrt{}_P K) \subset \ker(f * d)$. Hence f' exists. Uniqueness of f' follows from $f''\rho_{s,n} = \rho_{s,n} f''$, for any $f'' : {}_S\sqrt{}_P K \to L$ with $f = f''t$. □

3.3 Definition Let K be any S-group.

$$ {}_SE_P K \ :=\ \lim\{K \to {}_S\sqrt{}_P K \to ({}_S\sqrt{}_P)^2 K \to \cdots\}. $$

By induction, using lemma (3.2), we get

3.4 Proposition The following hold:

(i) ${}_SE_P : {}_S\mathcal{G} \to {}_S\mathcal{E}_P$ is a covariant functor.

(ii) ${}_SE_P$ preserves epimorphisms.

(iii) The canonical homomorphism $\tau : K \to {}_SE_PK$ defines a natural transformation of the identity functor on ${}_S\mathcal{G}$ to ${}_SE_P$.

(iv) If $f : K \to L$ is an S-homomorphism and S acts P-locally on L, then there is a unique S-homomorphism $f' : {}_SE_PK \to L$ with $f = f'\tau$.

To make the functions $\rho_{s,n}$ of an S-group K (1–1), we factor out a suitable subgroup. Let

$${}_Sa_PK := \cap\{\ker(f : K \to U) : U \in {}_S\mathfrak{U}_P, \quad f \text{ any } S\text{-homomorphism}\}.$$

3.5 Definition ${}_SU_PK := K/\, {}_Sa_PK$.

It follows that ${}_SU_PK \in {}_S\mathfrak{U}_P$. Further, if $f : K \to K'$ is an S-homomorphism, then $f({}_Sa_PK) \subset {}_Sa_PK'$. So f induces ${}_SU_Pf : {}_SU_PK \to {}_SU_PK'$. The lemma below is a direct consequence of this definition.

3.6 Lemma The following hold

(i) ${}_SU_P : {}_S\mathcal{G} \to {}_S\mathfrak{U}_P$ is a covariant functor.

(ii) The canonical epimorphism $\sigma : K \twoheadrightarrow {}_SU_PK$ defines a natural transformation of the identity on ${}_S\mathcal{G}$ to ${}_SU_P$.

(iii) ${}_SU_P$ preserves epimorphisms.

(iv) If $f : K \to L$ is an S-homomorphism and $L \in {}_S\mathfrak{U}_P$, then there is a unique homomorphism $f' : {}_SU_PK \to L$ with $f = f'\sigma$. ${}_SU_P$ is left adjoint left inverse to the inclusion functor ${}_S\mathfrak{U}_P \to {}_S\mathcal{G}$. □

3.7 Definition Let $\gamma : R \to S$ be a group homomorphism. Let ${}_\gamma L_P := {}_SU_P\,{}_SE_P\,{}_\gamma\mathrm{Ad} : \mathcal{G}_R \to {}_S\mathcal{G}_P$ be the composite of the three functors.

Note that the natural transformations associated with ${}_SU_P$, ${}_SE_P$, ${}_\gamma\mathrm{Ad}$ define a natural transformation e $(= {}_\gamma e_P)$ of the identity functor on ${}_R\mathcal{G}$ to ${}_SL_P$.

3.8 Proposition Let $\gamma : R \to S$ be a group homomorphism.

(i) ${}_SL_P : {}_R\mathcal{G} \to {}_S\mathcal{G}_P$ is a covariant functor which is left adjoint to the change-of-operator-groups functor $\gamma^* : {}_S\mathcal{G}_P \to {}_R\mathcal{G}$.

(ii) ${}_SL_P$ preserves epimorphisms.

Proof Combine (1.5),(2.2),(3.4),(3.6). □

This completes the proof of (1.6). □

The author is grateful to the referee for pointing out the connection of [2], [6], [14] with the present paper.

References

[1] G. Baumslag. Some Aspects of Groups with Unique Roots. *Acta Math.* **104** (1960) 217–303.

[2] G. Baumslag Roots and Wreath Products. *Proc. Cam. Phil. Soc.* **56** (1960) 109–117.

[3] A.K. Bousfield, D.M. Kan *Homotopy Limits, Completions and Localizations.* Springer-Verlag, Berlin/New York 1987 (second corrected printing)

[4] C. Casacuberta Espais topoloógics amb accions P-locals. Ph.D. Thesis, Universitat Autònoma de Barcelona 1988

[5] P.J. Hilton. Localization and Cohomology of Nilpotent Groups. *Math. Z.* **132** (1973) 265–286.

[6] P.J. Hilton Nilpotent actions on nilpotent groups. Springer LNM 450, Springer Verlag Berlin/New York 1975, 174–196.

[7] P.J. Hilton. *Nilpotente Gruppen und nilpotente Räume.* Springer LNM 1053, Springer Verlag Berlin/New York 1984.

[8] P.J. Hilton, G. Mislin, J. Roitberg. *Localization of Nilpotent Groups and Spaces.* North-Holland, Amsterdam 1975.

[9] A.I. Malcev Nilpotent torsion free groups. *Izv. Akad. Nauk. SSSR Ser. Mat.* (Russian) **13** (1949) 202–212.

[10] G. Peschke H-semidirect Products. *Can. Math. Bull.* **30** (1987) 402–511.

[11] A. Reynol. P-localization of some classes of groups. preprint.

[12] P. Ribenboim. Equations in groups, with special emphasis on localization and torsion I. *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez.* Ia(8) **19** (1987) 23–60.

[13] P. Ribenboim. Equations in groups, with special emphasis on localization and torsion II. *Portugal. Math.* **44** (1987) 417–445.

[14] A.G. Rodicio. Métodos homológicos en grupos P-locales. Ph.D. thesis 1986.

[15] J. Roitberg. Note on nilpotent spaces and localization. *Math. Z.* **137** (1974) 67–74.

[16] D. Sullivan. Genetics of homotopy theory and the Adams conjecture. *Ann. of Math.* **100** (1974) 1–79.