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Research Article

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Locally adequate semigroup algebras

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Abstract: We build up a multiplicative basis for a locally adequate concordant semigroup algebra by constructing Rukolaĭne idempotents. This allows us to decompose the locally adequate concordant semigroup algebra into a direct product of primitive abundant 0-$\mathcal{J}^*$-simple semigroup algebras. We also deduce a direct sum decomposition of this semigroup algebra in terms of the $\mathcal{R}^*$-classes of the semigroup obtained from the above multiplicative basis. Finally, for some special cases, we provide a description of the projective indecomposable modules and determine the representation type.

Keywords: Contracted semigroup algebras, Rukolaĭne idempotents, Multiplicative basis, Direct product decomposition, Representation type

MSC: 16G30, 16G60

1 Introduction

Munn [1, 2] gave a direct product decomposition of finite inverse semigroup algebras into matrix algebras over group algebras using principal ideal series. In [3], this result was independently obtained by Rukolaĭne. His approach was to construct a multiplicative basis by defining the so-called Rukolaĭne idempotents. Munn later showed that the technique developed by Rukolaĭne worked for inverse semigroups with idempotents sets locally finite, see [4].

Recent interest in Möbius functions arose due to the work of Solomon on decomposing the semigroup algebra of a finite semilattice into a direct product of fields [5], and the work of Brown on studying random walks on bands by using representation theory of their semigroup algebras [6]. By using the Möbius functions on the natural partial orders on inverse semigroups, Steinberg extended the results of Solomon and Munn on direct product decomposition of finite inverse semigroups to inverse semigroups with idempotents sets finite, and he explicitly computed the corresponding orthogonal central idempotents [7]. Guo generalized the results described above to finite locally inverse semigroups and finite ample semigroups, again using Möbius functions, see [8, 9].

Decomposing an algebra with an identity into direct sum of projective indecomposable modules is an important problem in representation theory because it provides a complete set of primitive orthogonal idempotents. It also allows for an explicit computation of the Gabriel quiver, the Auslander-Reiten quiver and the representation type of an algebra. It has shown that the semigroup algebras of finite $\mathcal{R}$-trivial monoids are basic; furthermore the projective indecomposable modules have been described, see [10–12]. However, there is no much description of the projective indecomposable modules of the semigroup algebras which are not basic.

The first part of this paper is primarily concerned with carrying over certain results of inverse semigroup algebras to locally adequate concordant semigroup algebras. In general, the contracted semigroup algebras of locally adequate concordant semigroups are not basic. The remainder of the paper is devoted to exploring a description of the

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projective indecomposable modules and to determining whether or not these semigroup algebras are representation-finite.

The paper is organized as follows. In Section 2, we provide some background on semigroups and algebras. If $R_0[S]$ is the contracted semigroup algebra of a locally adequate concordant semigroup S with idempotents set $E(S)$ pseudofinite, in Section 3, we generalize the concepts and results of Rukolaĭne idempotents of inverse semigroup algebras to $R_0[S]$. Section 4 involves constructing a multiplicative basis $\overline{\mathcal{B}}$ of $R_0[S]$, see Theorem 4.8, and developing basic properties of the semigroup $\overline{S} = \overline{\mathcal{B}} \cup \{0\}$. In Section 5, $R_0[S]$ is decomposed into a direct product of primitive abundant 0-$\mathcal{J}^*$-simple semigroup algebras, see Theorem 5.1. In Section 6, if $R_0[S]$ contains an identity, the multiplicative basis $\overline{\mathcal{B}}$ also allows for a direct sum decomposition of $R_0[S]$. Theorem 6.5 translates the problem involving the projective indecomposable modules of $R_0[S]$ into cancellative monoids theory terms. Furthermore, we determine the representation type of these semigroup algebras.

2 Preliminaries

In this section, we recall some basic definitions and results on semigroups and representation theory of algebras. Throughout this paper, let R denote a commutative ring with identity, and denote the zero element of a R-algebra by the symbol 0.

We first recall some definitions and results on semigroups which can be found in [13, 14].

Without loss of generality, we always assume a semigroup S is with a zero element (denoted by θ). Denote the set of all nonzero elements of S and $E(S)$ (the idempotents set of S) by S^* and $E(S)^*$, respectively.

Let S be a semigroup and $\mathcal{K}$ be an equivalence relation on S. The $\mathcal{K}$-class containing an element a of the semigroup S will be denoted by K_a or $K_a(S)$ in case of ambiguity. We denote the set of nonzero $\mathcal{K}$-classes of S by $(S/\mathcal{K})^*$.

Denote by S^1 the semigroup obtained from a semigroup S by adding an identity if S has no identity, otherwise, let $S^1 = S$. It is well known that Green's relations play an important role in the theory of semigroups. They were introduced by Green in 1951: for $a, b \in S$,

$$\begin{aligned} a\,\mathcal{L}\,b &\Leftrightarrow S^1a = S^1b,\\ a\,\mathcal{R}\,b &\Leftrightarrow aS^1 = bS^1,\\ a\,\mathcal{J}\,b &\Leftrightarrow S^1aS^1 = S^1bS^1,\\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R},\\ \mathcal{D} &= \mathcal{L} \vee \mathcal{R}. \end{aligned}$$

It is clear that $\mathcal{L}$ (resp., $\mathcal{R}$) is a right (resp., left) congruence on S and $\mathcal{D} \subseteq \mathcal{J}$. A semigroup S is called *regular* if every $\mathcal{L}$-class and every $\mathcal{R}$-class contain idempotents. The regularity of a semigroup S can be characterized by the property that the set $V(a) = \{a' \in S \mid aa'a = a,\ a'aa' = a'\}$ is nonempty for each $a \in S$.

Pastijn first extended the Green's relations to the so called "Green's $*$-relations" on a semigroup S [15]: for $a, b \in S$,

$$\begin{aligned} a\,\mathcal{L}^*\,b &\Leftrightarrow (\forall x, y \in S^1)(ax = ay \leftrightarrow bx = by),\\ a\,\mathcal{R}^*\,b &\Leftrightarrow (\forall x, y \in S^1)(xa = ya \leftrightarrow xb = yb),\\ a\,\mathcal{J}^*\,b &\Leftrightarrow J^*(a) = J^*(b),\\ \mathcal{H}^* = \mathcal{L}^* \wedge \mathcal{R}^* &\text{ and } \mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*, \end{aligned}$$

where $J^*(a)$ is the smallest ideal containing a which is saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$.

Clearly, $\mathcal{L}^*$ (resp., $\mathcal{R}^*$) is a right (resp., left) congruence on S. It is easy to see that $\mathcal{L} \subseteq \mathcal{L}^*$ (resp., $\mathcal{R} \subseteq \mathcal{R}^*$), and for $a, b \in \mathrm{Reg}(S)$, $a\,\mathcal{L}\,b$ (resp., $a\,\mathcal{R}\,b$) if and only if $a\,\mathcal{L}^*\,b$ (resp., $a\,\mathcal{R}^*\,b$). So $\mathcal{L} = \mathcal{L}^*$, $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{J} = \mathcal{J}^*$ on regular semigroups.

We say a semigroup is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of it contains an idempotent. It is clear that regular semigroups are abundant semigroups.

Let S be an abundant semigroup and $a \in S^*$. We use $a^\dagger$ (resp., a^*) to denote a typical idempotent related to a by $\mathcal{R}^*$ (resp., $\mathcal{L}^*$).

Define two partial orders $\leq_r$ and $\leq_l$ on S [16] by

$$a \leq_r b \Longleftrightarrow R_a^* \leq R_b^* \text{ and } a = a^\dagger b \text{ for some } a^\dagger,$$
$$a \leq_l b \Longleftrightarrow L_a^* \leq L_b^* \text{ and } a = ba^* \text{ for some } a^*.$$

The *natural partial order* $\leq$ on S is defined to be $\leq = \leq_r \cap \leq_l$. We have an alternative characterisation of $\leq$: for $x, y \in S$, $x \leq y$ if and only if there exist $e, f \in E(S)$ such that $x = ey = yf$.

Let S be an abundant semigroup and $e \in E(S)^*$. Define $\omega(e) = \{f \in E(S) \mid f \leq e\}$. Clearly, $\omega(e) = E(eSe)$. For convenience, denote the subsemigroup of S generated by $\omega(e)$ by $\langle e \rangle$.

An abundant semigroup S is called *idempotent connected (IC)* [17], if for all $a \in S^*$, $a^\dagger \in R_a^*(S) \cap E(S)$ and $a^* \in L_a^*(S) \cap E(S)$, there is an isomorphism

$$\alpha_a : \langle a^\dagger \rangle \to \langle a^* \rangle, \text{ with } xa = a\alpha_a(x),$$

for each $x \in \langle a^\dagger \rangle$. It is known that an abundant semigroup S is IC if and only if $\leq_r = \leq_l$ on S [16, Theorem 2.6].

A semigroup S is said to *satisfy the regularity condition* [16] if for all idempotents e and f of S the element ef is regular. If this is the case, the *sandwich set* $S(e, f) = \{g \in V(ef) \cap E(S) \mid ge = fg = g\}$ of idempotents e and f is non-empty, and takes the form

$$S(e, f) = \{g \in E(S) \mid ge = fg = g, egf = ef\}.$$

A semigroup S is said to be *concordant* if S is IC abundant and satisfies the regularity condition, see [18]. It is known that regular semigroup is concordant, and in this case $\leq$ coincide with the natural partial order defined by Nambooripad [19].

An abundant semigroup with commutative idempotents is called an *adequate semigroup*. If each local submonoid eSe ($e \in E(S)^*$) of a semigroup S is adequate (resp., inverse), then the semigroup S is said to be *locally adequate* (resp., *locally inverse*). We say a semigroup *locally adequate concordant* if it is both concordant and locally adequate.

By [20, Corollary 5.6], an IC abundant semigroup is locally adequate if and only if $\leq$ is compatible with multiplication. It is well known that inverse (resp., locally inverse) semigroups are regular adequate (resp., locally adequate) semigroups and conversely, so that locally adequate concordant semigroups generalize locally inverse semigroups, and hence generalize inverse semigroups.

Refer to [13, Chapter 8] for the definitions of a left (resp., right) S-system and an (S, T)-bisystem for monoids S, T. Let M be a (S, T)-bisystem. Then the mapping $s \otimes m \mapsto sm$ (resp., $m \otimes t \mapsto mt$) is an (S, T)-isomorphism from $S \otimes_S M$ (resp., $M \otimes_T T$) onto M, and we call it a *canonical isomorphism*.

We recall the definition of blocked Rees matrix semigroups [14]. Let J and Λ be non-empty sets and Γ be a non-empty set indexing partitions $P(J) = \{J_\lambda : \lambda \in \Gamma\}$, $P(\Lambda) = \{\Lambda_\lambda : \lambda \in \Gamma\}$ of J and Λ, respectively. We make a convention that i, j, k, l will denote members of J; s, t, m, n will denote members of Λ, and $\lambda, \mu, \nu, \kappa$ will denote members of Γ.

By the (λ, μ)*-block* of a $J \times \Lambda$ matrix we mean those (j, s)-positions with $j \in J_\lambda$ and $s \in \Lambda_\mu$. The (λ, λ)-blocks are called the *diagonal blocks* of the matrix.

For each pair $(\lambda, \mu) \in \Gamma \times \Gamma$, let $M_{\lambda\mu}$ be a set such that for each λ, $M_{\lambda\lambda} = T_\lambda$ is a monoid and for $\lambda \neq \mu$, either $M_{\lambda\mu} = \emptyset$ or $M_{\lambda\mu}$ is a (T_λ, T_μ)-bisystem. Moreover, we impose the following condition on $\{M_{\lambda\mu} : \lambda, \mu \in \Gamma\}$.

(M) For all $\lambda, \mu, \nu \in \Gamma$, if $M_{\lambda\mu}$, $M_{\mu\nu}$ are both non-empty, then $M_{\lambda\nu}$ is non-empty and there is a (T_λ, T_ν)-homomorphism $\varphi_{\lambda\mu\nu} : M_{\lambda\mu} \otimes M_{\mu\nu} \to M_{\lambda\nu}$ such that if $\lambda = \mu$ or $\mu = \nu$, then $\varphi_{\lambda\mu\nu}$ is the canonical isomorphism and such that the square

$$\begin{array}{ccc} M_{\lambda\mu} \otimes M_{\mu\nu} \otimes M_{\nu\kappa} & \xrightarrow{I_{\lambda\mu} \otimes \varphi_{\mu\nu\kappa}} & M_{\lambda\mu} \otimes M_{\mu\kappa} \\ {\scriptstyle \varphi_{\lambda\mu\nu} \otimes I_{\nu\kappa}} \big\downarrow & & \big\downarrow {\scriptstyle \varphi_{\lambda\mu\kappa}} \\ M_{\lambda\nu} \otimes M_{\nu\kappa} & \xrightarrow[\varphi_{\lambda\nu\kappa}]{} & M_{\lambda\kappa} \end{array}$$

is commutative.

Here, for $a \in M_{\lambda\mu}$, $b \in M_{\mu\nu}$, we denote $(a \otimes b)\varphi_{\lambda\mu\nu}$ by ab. On the other hand, let 0 (zero) be a symbol not in any $M_{\lambda\mu}$ with the convention that $0x = x0 = 0$ for every element x of $\{0\} \cup \bigcup\{M_{\lambda\mu} : \lambda, \mu \in \Gamma\}$.

Denote by $(a)_{js}$ the $J \times \Lambda$-matrix with entry a in the (j,s)-position and zeros elsewhere. Let M be the set consisting all $J \times \Lambda$-matrix $(a)_{js}$, where (j,s) is in some (λ,μ)-block and $a \in M_{\lambda\mu}$, and the zero matrix (denoted by θ). Define a $\Lambda \times J$ sandwich matrix $P = (p_{si})$ where a nonzero entry in the (λ,μ)-block of P is a member of $M_{\lambda\mu}$.

Let $A = (a)_{is}$, $B = (b)_{jt} \in M$, by condition (M), the product $A \circ B = APB = (ap_{sj}b)_{it}$ makes M be a semigroup, which we denote by $\mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$ and call a *blocked Rees matrix semigroup*.

In addition, we call M a *PA blocked Rees matrix semigroup* if it satisfies the following conditions (C), (U) and (R):

(C) If $a, a_1, a_2 \in M_{\lambda\mu}$, $b, b_1, b_2 \in M_{\mu\kappa}$, then $ab_1 = ab_2$ implies $b_1 = b_2$; $a_1b = a_2b$ implies $a_1 = a_2$;

(U) For each $\lambda \in \Gamma$ and each $s \in \Lambda_\lambda$ (resp., $j \in J_\lambda$), there is a member j of J_λ (resp., $s \in \Lambda_\lambda$) such that p_{sj} is a unit in $M_{\lambda\lambda}$;

(R) If $M_{\lambda\mu}$, $M_{\mu\lambda}$ are both non-empty where $\lambda \neq \mu$, then $aba \neq a$ for all $a \in M_{\mu\lambda}$. $b \in M_{\lambda\mu}$.

We record some elementary properties of PA blocked Rees matrix semigroups in the following lemma.

Lemma 2.1 ([14, Proposition 2.4]). *Let $M = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$ be a PA blocked Rees matrix semigroup. Then*

(i) a non-zero element $(a)_{is}$ of M is an idempotent if and only if there is an element $\lambda \in \Gamma$ such that $(i,s) \in J_\lambda \times \Lambda_\lambda$ and a is a unit in T_λ with inverse p_{si};

(ii) all nonzero idempotents of M are primitive;

(iii) the non-zero elements $(a)_{is}$ and $(b)_{jt}$ of M are $\mathcal{R}^*$-related if and only if $i = j$;

(iv) the non-zero elements $(a)_{is}$ and $(b)_{jt}$ of M are $\mathcal{L}^*$-related if and only if $s = t$;

(v) M is abundant;

(vi) the non-zero idempotents $e = (a)_{is}$ and $f = (b)_{jt}$ of M with $(i,s) \in J_\lambda \times \Lambda_\lambda$ and $(j,t) \in (i,s) \in J_\mu \times \Lambda_\mu$ are $\mathcal{D}$-related if and only if $\lambda = \mu$;

(vii) the non-zero element $(a)_{is}$ of M is regular if and only if there is an element $\lambda \in \Gamma$ such that $(i,s) \in J_\lambda \times \Lambda_\lambda$ and a is a unit in T_λ.

Let $M = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$ be a PA blocked Rees matrix semigroup. Then we can always assume that there exists $1_\lambda \in J_\lambda \cap \Lambda_\lambda$ such that $H^*_{1_\lambda 1_\lambda} = T_\lambda$ is a cancellative monoid with an identity e_λ $(\lambda \in \Gamma)$.

Recall that a *Munn algebra* is an algebra $\mathscr{M}(A; I, \Lambda; P)$ of matrix type over an algebra A [21] such that each row and each column of the sandwich matrix P contains a unit of A. Let $M = \mathcal{M}^0(G; J, \Lambda; P)$ be a completely 0-simple semigroup. It is known that $R_0[M] \cong \mathscr{M}(R[G]; J, \Lambda; P)$, see [22, Lemma 5.17].

Let $M = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$ be a PA blocked Rees matrix semigroup. Define the *generalized Munn algebra* $\mathscr{M}(R[M_{\lambda\mu}]; J, \Lambda, \Gamma; P)$ *of* M to be the vector space consisting of all the $J \times \Lambda$-matrices (a_{is}) with only finitely many nonzero entries, where $a_{is} \in R[M_{\lambda\mu}]$ if $(i,s) \in J_\lambda \times \Lambda_\mu$, with multiplication defined by the formula $(a_{is}) \circ (b_{jt}) = (a_{is})P(b_{jt})$.

In particular, if $|\Gamma| = 1$, the generalized Munn algebra is a Munn algebra.

The proof of the following result is similar to that of [22, Lemma 5.17].

Lemma 2.2. $R_0[M] \cong \mathscr{M}(R[M_{\lambda\mu}]; J, \Lambda, \Gamma; P)$.

If $(a_{is}) \in \mathscr{M}(R[M_{\lambda\mu}]; J, \Lambda, \Gamma; P)$ has only one nonzero entry a_{jt}, we will write (j,a,t) or $(a)_{jt}$ instead of (a_{is}).

Now we recall the definition of primitive abundant semigroups. Let S be an abundant semigroup. If $e \in E(S)^*$ is minimal under the natural order $\leq$ defined on S, e is said to be *primitive*. It is known that an idempotent $e \in S$ is primitive if and only if e has the property that for each idempotent $f \in E(S)$, $fe = ef = f \neq \theta \Longrightarrow f = e$. The semigroup S is said to be *primitive abundant* if all its nonzero idempotents are primitive.

By Lemma 2.1(ii) and (v), PA blocked Rees matrix semigroups are primitive abundant. Conversely, if S is a primitive abundant, then S is isomorphic to a PA blocked Rees matrix semigroup $\mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$; furthermore, there is variability in the sandwich matrix P on the choice of data in constructing the isomorphism.

We can simply take $r_{1_\lambda} = q_{1_\lambda} = e_\lambda$, and thus for all $\lambda \in \Gamma$, $p_{1_\lambda 1_\lambda} = q_{1_\lambda} r_{1_\lambda} = e_\lambda$, see [14, Theorem 3.8]. The sandwich matrix attaching to a PA blocked Rees matrix will be always assumed to be of such form.

A semigroup S is called *0-$\mathcal{J}^*$-simple* if $S^2 \neq \{\theta\}$ and S, $\{\theta\}$ are the only $\mathcal{J}^*$-classes of S. It is known that a primitive abundant semigroup is a 0-direct union of primitive abundant 0-$\mathcal{J}^*$-simple semigroups. Recall that a semigroup S is said to be *primitive adequate* if S is adequate and all its nonzero idempotents are primitive.

We say that a semigroup S is a *weak Brandt semigroup* if the following conditions are satisfied:

(B1) if a, b, c are elements of S such that $ac = bc \neq 0$ or $ca = cb \neq 0$, then $a = b$;
(B2) if a, b, c are elements of S such that $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$;
(B3) for each element a of S there is an element e of S such that $ea = a$ and an element f of S such that $af = a$;
(B4) if e and f are nonzero idempotents of S, then there are nonzero idempotents $e_1, \ldots, e_n$ of S with $e_1 = e$, $e_n = f$ such that for each $i = 1, \ldots, n-1$, one of $e_i S e_{i+1}$, $e_{i+1} S e_i$ is nonzero.

Obviously, a Brandt semigroup is a weak Brandt semigroup.

By [14, Corollary 5.6], a weak Brandt semigroup is just a 0-$\mathcal{J}^*$-simple primitive adequate semigroup, or just a 0-$\mathcal{J}^*$-simple PA blocked Rees matrix semigroup $\mathcal{M}^0(M_{\lambda\mu}; J, J, \Gamma; P)$ with the properties that the sandwich matrix P is diagonal and p_{jj} is equal to the identity e_λ of the monoid $M_{\lambda\lambda}$ for each $\lambda \in \Gamma$ and each $j \in J_\lambda$.

Finally we list some basic definitions concerning semigroup algebras and the module theory of algebras which can be found in [21, 23].

Let S be a semigroup and let $R[S]$ denote the *semigroup algebra* of S over R. If T is a subset of the semigroup S, then denote the set of all finite R-linear combinations of elements of T by $R[T]$.

By the *contracted semigroup algebra* of S over R, denoted by $R_0[S]$, we mean the factor algebra $R[S]/R[\theta]$. If $a = \sum r_i s_i \in R_0[S]$, then the set $\operatorname{supp} a = \{s_i \in S \backslash \{\theta\} \mid r_i \neq 0\}$ is called the *support* of a.

Obviously, $S\backslash\{\theta\}$ is a *multiplicative basis* of the contracted semigroup algebra $R_0[S]$, because it is a R-basis of $R_0[S]$ and 0-closed ($S^2 \subseteq S \cup \{0\}$).

Let A be a R-algebra. A right A-module M is said to be *indecomposable* if $M \neq 0$ and M has no direct sum decomposition $M = N \oplus L$, where N and L are nonzero right A-modules.

An idempotent $e \in A$ is called *primitive* if eA is an indecomposable A-module. By [24, Corollary 6.4a], e is primitive in the algebra A if and only if e is primitive in the multiplicative semigroup $\mathrm{Mult}(A)$.

Suppose that A is a R-algebra with an identity. If the right A-module A_A is a direct sum $I_1 \oplus \cdots \oplus I_n$ of indecomposable right A-modules, then we call such a decomposition an *indecomposable decomposition* of A. It is known that this is the case if and only if there exists a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents of A such that $I_i = e_i A$ $(i = 1, \ldots, n)$.

Assume that A is a R-algebra with an identity and $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A. The algebra A is called *basic* if $e_i A \ncong e_j A$, for all $i \neq j$.

The *basic algebra associated to* A is the algebra $A^b = e_A A e_A$, where $e_A = e_{j_1} + \cdots + e_{j_m}$, and $e_{j_1}, \ldots, e_{j_m}$ are chosen such that $e_{j_t} A$, $1 \leq t \leq m$, are all the non-isomorphic projective indecomposable right A-modules.

It is known that A^b is basic and $\mathrm{mod}\, A^b \cong \mathrm{mod}\, A$ as categories (see, for example [23, Corollary 6.10]).

A right artinian algebra A is defined to be *representation-finite* if there are finitely many isomorphism classes of finitely generated, indecomposable right A-modules.

3 Rukolaĭne idempotents

In this section, we first recall the concept of Rukolaĭne idempotents of inverse semigroup algebras which was first introduced by Rukolaĭne [3]. Then we extend the Rukolaĭne idempotents to certain locally adequate concordant semigroup algebras.

Let E be a semilattice and $e, f \in E$. Then f is said to be *maximal under* e [25] or e *covers* f [4] if $e > f$ and there is no $g \in E$ such that $e > g > f$. Denote by $\hat{e}$ the set $\{f \in E : e \text{ covers } f\}$. E is said to be *pseudofinite* if

(i) for $e, f \in E$ with $e > f$, there exists an element g such that e covers g and $e > g \geq f$;
(ii) $\hat{e}$ is a finite set for each $e \in E$.

It is clear that finite semilattices are pseudofinite.

Let S be a finite inverse semigroup and $e \in E(S)^*$. Rukolaĭne [3] defined an element $\sigma(e) \in R_0[S]$ by

$$\sigma(e) = e + \sum_{\{e_{i_1},\ldots,e_{i_j}\}\subseteq \hat{e}} (-1)^j e_{i_1} \cdots e_{i_j}, \tag{1}$$

where $\{e_{i_1}, \ldots, e_{i_j}\}$ takes over all non-empty subset of $\hat{e}$. He proved that the set $\{\sigma(e) \mid e \in E(S)^*\}$ collects a family of orthogonal idempotents of $R_0[S]$.

Let S be an inverse semigroup with $E(S)$ pseudofinite. Then $\hat{e}$ is a finite set whose elements are commutative, and hence $\sigma(e) \in R_0[S]$ is well defined for each $e \in E(S)^*$. In this case, Munn [4] gave an obvious alternative definition of $\sigma(e)$ as

$$\sigma(e) = \prod_{g\in\hat{e}} (e - g).$$

It is shown that $\{\sigma(e) \mid e \in E(S)^*\}$ is a set of orthogonal idempotents of $R_0[S]$, and $\sigma(e)$, $e \in E(S)^*$, are called the *Rukolaĭne idempotents* of $R_0[S]$.

Remark 3.1. *Let S be an inverse semigroup with $E(S)$ pseudofinite. If $e \in S$ is a minimal nonzero idempotent, that is e covers θ, then $\hat{e} = \emptyset$. In this case, we make the convention that $\sigma(e) = e$.*

The idempotents set $E(S)$ of a semigroup S is said to be *locally pseudofinite* (resp., *locally finite*) if $E(eSe)$ is a pseudofinite (resp., a finite) semilattice for each $e \in E(S)$.

Let S be a locally adequate IC abundant semigroup with $E(S)$ locally pseudofinite. Then $E(eSe)$ is a pseudofinite semilattice for each $e \in E(S)^*$ and so $\hat{e}$ is a finite set with elements commutative since $\hat{e} \subseteq E(eSe)$. As in [4], for each $e \in E(S)^*$, let

$$\sigma(e) = \prod_{g\in\hat{e}} (e - g) \in R_0[S].$$

We shall show that $\sigma(e)$ is an idempotent of $R_0[S]$ for each $e \in E(S)^*$. To this aim, we need the following results.

Lemma 3.2. *Let S be a locally adequate IC abundant semigroup with $E(S)$ locally pseudofinite. Then for each $a \in S^*$ we have*
(i) $\alpha_a(a^\dagger) = a^*$, where α_a is the isomorphism from $\omega(a^\dagger)$ to $\omega(a^*)$;
(ii) if we still denote by α_a the extension of α_a to $R_0[\omega(a^\dagger)]$ by R-linearly, then $\alpha_a(\sigma(a^\dagger)) = \sigma(a^*)$.

Proof. (i) By the hypothesis that S is IC abundant, there exists a semigroup isomorphism $\alpha_a : \langle a^\dagger\rangle \to \langle a^*\rangle$. Since S is locally adequate, $\langle a^\dagger\rangle = \omega(a^\dagger)$ is a subsemilattice with identity $a^\dagger$ and $\langle a^*\rangle = \omega(a^*)$ is a subsemilattice with identity a^*. It follows that $\alpha_a(a^\dagger) = a^*$ and (i) holds.

(ii) Note that α_a is a semilattice isomorphism. It follows from the definition of $\widehat{a^\dagger}$ and $\widehat{a^*}$ that $\alpha_a|_{\widehat{a^\dagger}}$ is a bijection from $\widehat{a^\dagger}$ onto $\widehat{a^*}$. Which together with the fact that $\alpha_a(a^\dagger) = a^*$ implies that

$$\alpha_a(\sigma(a^\dagger)) = \prod_{g\in\widehat{a^\dagger}} \Big(\alpha_a(a^\dagger) - \alpha_a(g)\Big) = \prod_{f\in\widehat{a^*}} (a^* - f) = \sigma(a^*). \qquad \square$$

Proposition 3.3. *Let S be a locally adequate IC abundant semigroup with $E(S)$ locally pseudofinite. Then*
(i) for each $e \in E(S)^*$, $\sigma(e)$ is an idempotent and $e\sigma(e) = \sigma(e)e = \sigma(e)$;
(ii) $\sigma(a^\dagger)a = a\sigma(a^*)$ for each $a \in S^*$;
(iii) for $a \in S^*$, $h \in R^*_a \cap E(S)$, $f \in L^*_a \cap E(S)$, $\sigma(h)a = a\sigma(f)$.

Proof. (i) Let $g, h \in \hat{e}$. Note that $\hat{e} \subseteq E(eSe)$. Since S is locally adequate, $E(eSe)$ is a semilattice, hence $(e-g)^2 = e-g$ and $(e-g)$ commutes with $(e-h)$. It follows that

$$\sigma(e)^2 = \prod_{g\in\hat{e}}(e-g)^2 = \prod_{g\in\hat{e}}(e-g) = \sigma(e).$$

This shows that $\sigma(e)$ is an idempotent. The rest is obvious.

(ii) By Lemma 3.2(ii), we have $\alpha_a(\sigma(a^\dagger)) = \sigma(a^*)$. It follows that

$$\begin{aligned}
a\sigma(a^*) &= a\alpha_a(\sigma(a^\dagger)) = a\alpha_a\Big(\prod_{g\in\hat{a^\dagger}}(a^\dagger - g)\Big) = a\prod_{g\in\hat{a^\dagger}}\Big(\alpha_a(a^\dagger)-\alpha_a(g)\Big)\\
&= \Big(a\alpha_a(a^\dagger) - a\alpha_a(t)\Big)\cdot\prod_{g\in\hat{a^\dagger}\setminus\{t\}}\Big(\alpha_a(a^\dagger)-\alpha_a(g)\Big) && \text{(choose } t\in\hat{a^\dagger}\text{)}\\
&= \Big(a^\dagger a - ta\Big)\cdot\prod_{g\in\hat{a^\dagger}\setminus\{t\}}\Big(\alpha_a(a^\dagger)-\alpha_a(g)\Big) && \text{(since } S \text{ is IC)}\\
&= (a^\dagger - t)\cdot a\prod_{g\in\hat{a^\dagger}\setminus\{t\}}\Big(\alpha_a(a^\dagger)-\alpha_a(g)\Big)\\
&= \cdots = \prod_{g\in\hat{a^\dagger}}(a^\dagger - g)\cdot a\\
&= \sigma(a^\dagger)a,
\end{aligned}$$

as required.

(iii) It follows directly from (ii). □

Remark 3.4. *(i) If S is an adequate semigroup, then $\{\sigma(e) \mid e \in E(S)^*\} \subseteq R_0[S]$ is a set of pairwise orthogonal idempotents. Indeed, let $e, f \in E(S)^*$, and there is no loss of generality in assuming $e \not\leq f$. By Proposition 3.3(i), $\sigma(e)\sigma(f) = \sigma(e)ef\sigma(f)$, thus it suffices to show $\sigma(e)ef = 0$. By hypothesis, $e > ef \in E(S)$. If $ef = \theta$ (in S), this is trivial. If $ef \neq \theta$, there exists an idempotent $g \in \hat{e}$ such that $g > ef$. Then*

$$\sigma(e)ef = \Big(\prod_{h\in\hat{e}\setminus\{g\}}(e-h)\Big)(e-g)ef = 0.$$

In either case, $\sigma(e)ef = 0$. Therefore $\sigma(e)\sigma(f) = 0$.

(ii) There exists a locally adequate IC abundant semigroup S with the property that the idempotents $\sigma(e)$ ($e \in E(S)^*$) are not pairwise orthogonal. To see this, let $S = \mathcal{M}^0(G; I, I; P)$ be a completely 0-simple semigroup, where G is a group with identity e, $I = \{1, 2\}$ and P is a $I \times I$-matrix with $p_{21} = 0$ and $p_{ij} = e$ otherwise. Obviously, S is a locally adequate IC abundant semigroup. Since $g = (1, e, 1)$ and $f = (2, e, 1)$ are primitive idempotents of S, we have $\sigma(g) = g$ and $\sigma(f) = f$. Then

$$\sigma(g)\sigma(f) = gf = (1, e, 1)(2, e, 1) = (1, e, 1) \neq 0.$$

Consequently, $\{\sigma(e) \mid e \in E(S)^*\}$ is not a set containing pairwise orthogonal idempotents.

4 Multiplicative basis $\overline{\mathcal{B}}$ and semigroup $\overline{S}$

Let S be a locally adequate concordant semigroup with $E(S)$ locally finite. In this section, first we construct a multiplicative basis $\overline{\mathcal{B}}$ of $R_0[S]$ by means of the Rukolaĭne idempotents defined in Section 3. Then we provide some properties of the semigroup $\overline{\mathcal{B}} \cup \{0\}$.

For each $a \in S^*$, in view of Lemma 3.3 (ii) and (iii),

$$\sigma(a^\dagger)a\sigma(a^*) = \big(a\sigma(a^*)\big)\,\sigma(a^*) = a\sigma(a^*) = \sigma(a^\dagger)a$$

and $\sigma(a^\dagger)a\sigma(a^*)$ does not depend on the choice of the elements a^* and $a^\dagger$. Denote

$$\bar{a} = \sigma(a^\dagger)a\sigma(a^*).$$

Then by (1) we have

$$\bar{a} = a\sigma(a^*) = a + \sum_{\{e_{i_1},\dots,e_{i_j}\}\subseteq \widehat{a^*}} (-1)^j a e_{i_1}\cdots e_{i_j}. \tag{2}$$

Note that $e_{i_1}\cdots e_{i_j} \le e_{i_t} < a^*$ for $t = 1,\dots,j$. Then $ae_{i_1}\cdots e_{i_j} \le aa^* = a$ since $\le$ is compatible with the multiplication of S. Moreover, $ae_{i_1}\cdots e_{i_j} < a$. Otherwise, suppose that $ae_{i_1}\cdots e_{i_j} = a$. Since the elements of $\widehat{a^*}$ commute, $f = e_{i_1}\cdots e_{i_j}$ is an idempotent and $f < a^*$. Now $a\mathcal{L}^*a^*$ implies $a^*f = a^*$, hence $f = a^*f = a^*$. This is a contradiction. Therefore

$$\bar{a} \in a + \sum_{b<a} Rb, \quad a \in S^*. \tag{3}$$

In particular, we have $\bar{a} \neq 0$ for each $a \in S^*$. Now let

$$\overline{\mathcal{B}} = \{\bar{a} \mid a \in S^*\}.$$

We will show that $\overline{\mathcal{B}}$ is a multiplicative basis of $R_0[S]$.

Lemma 4.1. *Let S be a locally adequate concordant semigroup with $E(S)$ locally pseudofinite. Then for $a, b \in S^*$*

$$\bar{a}\bar{b} = \begin{cases} \overline{ab}, & \text{if } E(S)\cap L_a^* \cap R_b^* \neq \emptyset,\\ 0, & \text{otherwise}.\end{cases}$$

In particular, $\overline{\mathcal{B}}$ is 0-closed.

Proof. Suppose that $E(S)\cap L_a^*\cap R_b^* \neq \emptyset$. Let $g \in E(S)\cap L_a^*\cap R_b^*$. Then $a\ \mathcal{L}^*\ g\ \mathcal{R}^*\ b$. Since $\mathcal{L}^*$ (resp., $\mathcal{R}^*$) is a right (resp., left) congruence on S, we have $ab\ \mathcal{L}^*\ gb = b$ and $ab\ \mathcal{R}^*\ ag = a$. Hence $ab \in L_b^* \cap R_a^*$ and so $ab \neq 0$. On the other hand, $\bar{a} = a\sigma(g)$ and $\bar{b} = \sigma(g)b$. It follows from Proposition 3.3 that

$$\begin{aligned}
\bar{a}\bar{b} &= a\sigma(g)\sigma(g)b\\
&= a\sigma(g)b && \text{(since } \sigma(g) \text{ is an idempotent)}\\
&= ab\sigma(b^*) && \text{(by Proposition 3.3 (ii))}\\
&= ab\sigma\left((ab)^*\right) && \text{(since } ab\mathcal{L}^*b)\\
&= \overline{ab}.
\end{aligned}$$

Suppose that $E(S)\cap L_a^*\cap R_b^* = \emptyset$. Take $e \in E(S)\cap L_a^*$ and $f \in E(S)\cap R_b^*$. Then $\bar{a} = a\sigma(e)$ and $\bar{b} = \sigma(f)b$. Note that $\sigma(e)e = \sigma(e)$ and $f\sigma(f) = \sigma(f)$.

If $ef = \theta$, then $ef = 0$ in $R_0[S]$, and hence

$$\bar{a}\bar{b} = (a\sigma(e))(\sigma(f)b) = (a\sigma(e)e)(f\sigma(f)b) = a\sigma(e)(ef)\sigma(f)b = 0.$$

If $ef \neq \theta$, then $\theta \notin S(e,f) = \{g \in E(S) \mid ge = fg = g, egf = ef\}$. Since S satisfies the regularity condition, $S(e,f) \neq \emptyset$. Thus there exists a nonzero idempotent $g \in S(e,f)$ and $eg, gf \in E(S)$. Moreover, $eg \le e$ and $gf \le f$. We claim that either $gf < f$ or $eg < e$. Otherwise, suppose that $gf = f$ and $eg = e$. Then $g\ \mathcal{R}^*\ f$ and $g\ \mathcal{L}^*\ e$. So $g \in L_e^* \cap R_f^* \cap E(S) = \emptyset$, which is a contradiction. Without loss of generality, assume that $eg < e$. Then there exists $h_g \in \hat{e}$ such that $eg \le h_g$ since $E(S)$ is pseudofinite. It follows that

$$\begin{aligned}
\sigma(e)ef &= \sigma(e)egf\\
&= \left(\prod\nolimits_{h\in\hat{e}}(e-h)\right)egf\\
&= \left(\prod\nolimits_{h\in\hat{e}\setminus\{h_g\}}(e-h)\right)\left((e-h_g)eg\right)f
\end{aligned}$$

$$= \left(\prod_{h\in\hat{e}\setminus\{h_g\}}(e-h)\right)\left(eeg - h_g eg\right) f$$
$$= \left(\prod_{h\in\hat{e}\setminus\{h_g\}}(e-h)\right)\left(eg - eg\right) f$$
$$= 0.$$

Therefore

$$\bar{a}\bar{b} = (a\sigma(e)e)\,(f\sigma(f)b) = a\,(\sigma(e)ef)\,\sigma(f)b = 0. \qquad \square$$

Remark 4.2. *For $e, f \in E(S^*)$, either $E(S) \cap L_e^* \cap R_f^* = \emptyset$ or $E(S) \cap L_e^* \cap R_f^* = S(e, f)$.*

In fact, if $E(S) \cap L_e^* \cap R_f^* \neq \emptyset$, then there is a unique idempotent $g \in E(S) \cap L_e^* \cap R_f^*$ such that $ge = g$, $fg = g$ and $egf = ef$. Hence $g \in S(e, f)$ and so $E(S) \cap L_e^* \cap R_f^* \subseteq S(e, f)$.

To prove the reverse inclusion, suppose that $h \in S(e, f)$, we shall show that $h\ \mathcal{H}^*\ g$. Note that $h \in V(ef)$. Then

$$e_1 = efh \in R_{ef}^* \cap L_h^* \cap E(S), \quad e_2 = g_1 ef \in R_{g_1}^* \cap L_{ef}^* \cap E(S).$$

It follows from $he = h$ that $ee_1 = e_1 = e_1 e$. Since $g \in L_e^* \cap R_f^*$, we have $ef \in R_e^* \cap L_f^*$. Thus $e_1\ \mathcal{R}^* ef\ \mathcal{R}^* e$ and $e_1 = e_1 e = e$. Hence $h\ \mathcal{L}^*\ e_1 = e\ \mathcal{L}^*\ g$. Similarly, we may show that $h\ \mathcal{R}^*\ g$. Therefore $h\ \mathcal{H}^*\ g$ and $h = g \in E(S) \cap L_e^* \cap R_f^*$. We have shown that $S(e, f) \subseteq E(S) \cap L_e^* \cap R_f^*$. Consequently, $E(S) \cap L_e^* \cap R_f^* = S(e, f)$.

Lemma 4.3. *Let S be a locally adequate concordant semigroup with $E(S)$ locally pseudofinite. Then $\overline{\mathcal{B}}$ is linearly independent in $R_0[S]$.*

Proof. Suppose to the contrary that $\overline{\mathcal{B}}$ is linearly dependent in $R_0[S]$. Then there exist an nonzero integer n, distinct $\bar{x_1}, \ldots, \bar{x_n} \in \overline{\mathcal{B}}$, and $r_1, \ldots, r_n \in R\setminus\{0\}$ such that

$$r_1\bar{x_1} + \cdots + r_n\bar{x_n} = 0.$$

Let x_l be a maximal element of $\{x_1, x_2, \ldots, x_n\}$ under the natural partial order $\leq$ on S. By (3) suppose that $\bar{x_l} = x_l + \sum_{i=1}^{k_l} r_{il}b_{il}$ with $r_{il} \neq 0$ and $b_{il} < x_l$ for $i = 1, \ldots, k_l$, $l = 1, 2, \ldots, n$. Then

$$r_1(x_1 + \sum_{i=1}^{k_1} r_{i1}b_{i1}) + \cdots + r_n(x_n + \sum_{i=1}^{k_n} r_{in}b_{in}) = 0.$$

Since $S \setminus \{\theta\}$ is a basis of $R_0[S]$ and $r_l \neq 0$, there exists at least an element b_{ij} for some $j \neq l$ and some i such that $b_{ij} = x_l$. Thus $x_l = b_{ij} \leq x_j$, which is a contradiction. Therefore $\overline{\mathcal{B}}$ is linearly independent. $\square$

The next result, which is due to Lawson [16], gives a characterization of the natural partial order on an abundant semigroup.

Lemma 4.4 ([16, Proposition 2.5])**.** *Let S be an abundant semigroup and $a, c \in S^*$. Then $c \leq a$ if and only if there exists an idempotent $f \in \omega(a^*)$ such that $f \in L_c^*$, $af = c$.*

Let S be an abundant semigroup. By Lemma 4.4, if $b \leq g \in E(S)$, then $b \in E(S)$. For each $e \in E(S)^*$, if $\omega(e) = E(eSe)$ is finite, then the element $\sum_{\theta\neq f\leq e} \bar{f} \in R_0[S]$ is well defined. Whenever this can be done without ambiguity we shall use the notation $\sum_{f\leq e} \bar{f}$ instead of $\sum_{\theta\neq f\leq e} \bar{f}$.

Lemma 4.5. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite and $e \in E(S)^*$. Then*

$$e = \sum_{f\leq e} \bar{f}.$$

Proof. Since S is a locally adequate semigroup with $E(S)$ locally finite, we have $E(eSe)$ is a finite semilattice. It is clear that an idempotent $f \leq e$ if and only if $f \in E(eSe)$. We prove the lemma by induction. If e is a minimal idempotent of S under the natural partial order, the lemma is obvious by Remark 3.1. Suppose the lemma is true for all idempotent $g < e$. Let $\hat{e} = \{e_1, e_2, \ldots, e_n\}$ with $n \geq 1$. Then $f < e$ if and only if $f \leq e_s$ for some $1 \leq s \leq n$. By (1) we have

$$\sum_{f \leq e} \bar{f} = \bar{e} + \sum_{f<e} \bar{f} = e + \sum_{f<e} \bar{f} + \sum_{\{e_{i_1},\ldots,e_{i_j}\} \subseteq \hat{e}} (-1)^j e_{i_1} \cdots e_{i_j},$$

where $\{e_{i_1}, \ldots, e_{i_j}\}$ takes over all the non-empty subset of $\hat{e}$. It follows from the hypothesis that

$$\sum_{f \leq e} \bar{f} = e + \sum_{f<e} \bar{f} + \sum_{\{e_{i_1},\ldots,e_{i_j}\} \subseteq \hat{e}} (-1)^j \sum_{f \leq e_{i_1} \cdots e_{i_j}} \bar{f}.$$

Fix some $f < e$. Let $e_{t_1} \cdots e_{t_m}$ be a smallest (under the natural partial order) product of $e_1, e_2, \ldots, e_n$ such that $f \leq e_{t_1} \cdots e_{t_m}$. Then $\bar{f}$ appears in the sum

$$\sum_{\{e_{i_1},\ldots,e_{i_j}\} \subseteq \hat{e}} (-1)^j \sum_{f \leq e_{i_1} \cdots e_{i_j}} \bar{f}$$

with coefficient $(-1)^m + C_m^1(-1)^{m-1} + C_m^2(-1)^{m-2} + \cdots + C_m^{m-1}(-1) = -1$. Thus

$$\sum_{\{e_{i_1},\ldots,e_{i_j}\} \subseteq \hat{e}} (-1)^j \sum_{f \leq e_{i_1} \cdots e_{i_j}} \bar{f} = -\sum_{f<e} \bar{f}$$

and $e = \sum_{f \leq e} \bar{f}$. □

Let S be a locally adequate IC abundant semigroup and $a, c \in S^*$ with $c \leq a$. Then by Lemma 4.4 there exists an idempotent $f \in \omega(a^*)$ such that $f \in L_c^*$ and $af = c$. We claim that such an idempotent f is unique. Suppose that g is another such an idempotent. Then $g \mathcal{L}^* f$, and hence $fg = f, gf = g$. Since $f, g \leq a^*$, we have $f, g \in a^*Sa^*$. It follows that $gf = fg$, and so that $g = f$. Denote by e_c such unique idempotent.

Lemma 4.6. *Let S be a locally adequate IC abundant semigroup and $a \in S^*$. Denote $e = a^*$. Then*
(i) the mapping defined by

$$\varphi : \{b \in S^* \mid b \leq a\} \to \{e_{af} \mid \theta \neq f \leq e\}$$
$$b \mapsto e_b$$

is a bijection;
(ii) $\{b \in S^* \mid b \leq a\} = \{ae_{af} \mid \theta \neq f \leq e\}$.

Proof. (i) To show (i) holds, define a mapping by

$$\psi : \{e_{af} \mid \theta \neq f \leq e\} \to \{b \in S^* \mid b \leq a\},$$
$$e_{af} \mapsto ae_{af}.$$

We shall show that φ and ψ are mutually inverse. Let $b \leq a$. Then $b = ae_b$ and so $e_b = e_{ae_b} \in \{e_{af} \mid f \leq e\}$. Thus $\psi\varphi(b) = \psi(e_b) = ae_b = b$. On the other hand, let $f \leq e$. Since $ae_{af} = af$, we have $\varphi\psi(e_{af}) = \varphi(ae_{af}) = e_{ae_{af}} = e_{af}$. Consequently φ is a bijection.

(ii) It is obvious. □

Let S be a locally adequate IC abundant semigroup with $E(S)$ locally finite. Then the set $\{b \in S^* \mid b \leq a\}$ is finite. Hence the element $\sum_{\theta \neq b \leq a} \bar{b} \in R_0[S]$ is well defined. In what follows, we write $\sum_{b \leq a} \bar{b}$ instead of $\sum_{\theta \neq b \leq a} \bar{b}$.

Lemma 4.7. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite and $a \in S^*$. Then*

$$a = \sum_{b \leq a} \bar{b}.$$

Proof. Let $e = a^*$. By Lemma 4.5, we have

$$a = ae = a \sum_{f \leq e} \bar{f} = \sum_{f \leq e, f \in L_{af}^*} a\bar{f} + \sum_{f \leq e, f \notin L_{af}^*} a\bar{f}.$$

Now

$$\sum_{f \leq e, f \notin L^*_{af}} a\bar{f} = \sum_{f \leq e, f \notin L^*_{af}} (af)\sigma(f) = \sum_{f \leq e, f \notin L^*_{af}} ae_{af}\sigma(f).$$

Let $f \in E(S)^*$ with $f \leq e$ and $f \notin L^*_{af}$. Since $e_{af}\ \mathcal{L}^*\ af$ and $af \cdot 1 = af \cdot f$, we have $e_{af} \cdot 1 = e_{af} \cdot f$, that is, $e_{af} = e_{af} f$. Note that $E(eSe)$ is a semilattice. Then $e_{af} f = fe_{af}$ since $e_{af}, f \in E(eSe)$. Thus $e_{af} \leq f$. But $e_{af} \neq f$ because $f \notin L^*_{af}$, hence $e_{af} < f$. By the fact $E(eSe)$ is finite that there exists $h \in \hat{f}$ such that $e_{af} \leq h$. Hence

$$\begin{aligned} e_{af}\sigma(f) &= e_{af}\Big(\prod\nolimits_{t\in\hat{f}}(f-t)\Big) \\ &= \Big(e_{af}(f-h)\Big)\Big(\prod\nolimits_{t\in\hat{f}\setminus\{h\}}(f-t)\Big) \\ &= \Big(e_{af}f - e_{af}h\Big)\Big(\prod\nolimits_{t\in\hat{f}\setminus\{h\}}(f-t)\Big) \\ &= (e_{af}-e_{af})\Big(\prod\nolimits_{t\in\hat{f}\setminus\{h\}}(e-t)\Big) \\ &= 0. \end{aligned}$$

Therefore $\sum_{f \leq e, f \notin L^*_{af}} a\bar{f} = 0$. It follows that

$$\begin{aligned} a &= \sum_{f \leq e, f \in L^*_{af}} a\bar{f} = \sum_{f \leq e, f \in L^*_{af}} af\bar{f} && (\text{since } \bar{f} = f\sigma(f)) \\ &= \sum_{f \leq e, f \in L^*_{af}} af\,\overline{(af)^*} && (\text{since } f\ \mathcal{L}^*\ (af)^*, \text{ by Proposition 3.3 (iii)}) \\ &= \sum_{f \leq e, f \in L^*_{af}} \overline{af} \\ &= \sum_{f \leq e} \overline{ae_{af}} && (\text{since } af = ae_{af}) \\ &= \sum_{b \leq a} \bar{b}. && (\text{by Lemma 4.6 (ii)}) \end{aligned}$$

□

Summing up, we have

Theorem 4.8. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite. Then $\overline{\mathcal{B}}$ is a multiplicative basis of $R_0[S]$ with multiplication given by*

$$\bar{a}\bar{b} = \begin{cases} \overline{ab}, & \text{if } E(S) \cap L^*_a \cap R^*_b \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It follows from Lemmas 4.1, 4.3 and 4.7 directly. □

Let

$$\overline{S} = \overline{\mathcal{B}} \cup \{0\}.$$

Then, by Theorem 4.8, $\overline{S}$ is a subsemigroup of the multiplicative semigroup of $R_0[S]$ such that $R_0[S] = R_0[\overline{S}]$.

In order to study $R_0[S]$ better via $\overline{S}$, we need to give more properties of $\overline{S}$. In the remainder of this section, we always assume that S is a locally adequate concordant semigroup with $E(S)$ locally pseudofinite.

Lemma 4.9. *The map $\phi : S \to \overline{S}$ given by $a \mapsto \bar{a}$ and $\theta \mapsto 0$, where $a \in S^*$, is a bijection.*

Proof. Obviously, ϕ is surjective. It suffices to show that ϕ is injective. Suppose to the contrary that there exist $a, c \in S$ such that $a \neq c$ and $\bar{a} = \bar{c}$. By (3), there exist $a_1, \ldots, a_s, c_1, \ldots, c_t \in S^*$ with $a_1, \ldots, a_s < a$,

$c_1, \ldots, c_t < c$ such that $\bar{a} = a + \sum_{i=1}^{s} r_i a_i$ and $\bar{c} = c + \sum_{i=1}^{t} r_i' c_i$ for some $r_1, \ldots, r_s, r_1', \ldots, r_t' \in R^*$. Thus

$$a + \sum_{i=1}^{s} r_i a_i = c + \sum_{i=1}^{t} r_i' c_i.$$

Because S^* is a basis of $R_0[S]$ and $a \neq c$, a must cancel with some c_i, hence $a = c_i < c$. Similarly, $c = a_j < a$ for some a_j. Now $a < c < a$, a contradiction. Therefore ϕ is injective. □

Lemma 4.10. $E(\overline{S})\backslash\{0\} = \{\bar{e} \mid e \in E(S)^*\}$.

Proof. Let $e \in E(S)^*$. Note $e \in L_e^* \cap R_e^*$. Then by Theorem 4.1, $\bar{e}\bar{e} = \bar{e} \neq 0$ and hence $\{\bar{e} \mid e \in E(S)^*\} \subseteq E(\overline{S})\backslash\{0\}$. To prove the reverse inclusion, assume that $\bar{a} \in E(\overline{S})^*$. Then $\bar{a}\bar{a} = \bar{a} \neq 0$ and so

$$\overline{a^2} = \bar{a}\bar{a} = \bar{a}$$

by Theorem 4.1. By Lemma 4.9 we have $a^2 = a$, that is $a \in E(S)^*$. Hence $E(\overline{S}) \setminus \{0\} \subseteq \{\bar{e} \mid e \in E(S)^*\}$, as required. □

Lemma 4.11. *Let $a \in S^*$. Then $\bar{a^*}\ \mathcal{L}^*(\overline{S})\ \bar{a}$ and $\bar{a^\dagger}\ \mathcal{R}^*(\overline{S})\ \bar{a}$.*

Proof. Note that $L_a^* \cap R_z^* \cap E(S) = L_{a^*}^* \cap R_z^* \cap E(S)$ for any $z \in S$. Then by Theorem 4.1 $\bar{a}\bar{z} = 0$ if and only if $\bar{a^*}\bar{z} = 0$. Suppose that $\bar{a}\bar{x} = \bar{a}\bar{y}$ for some $\bar{x}, \bar{y} \in \overline{S}^1$. If $\bar{a}\bar{x} = \bar{a}\bar{y} = 0$, then $\bar{a^*}\bar{x} = 0 = \bar{a^*}\bar{y}$. On the other hand, if $\bar{a}\bar{x} = \bar{a}\bar{y} \neq 0$, then $\overline{ax} = \bar{a}\bar{x} = \bar{a}\bar{y} = \overline{ay}$. Hence $ax = ay$. Which together with $a^*\mathcal{L}^*a$ implies that $a^*x = a^*y$. Therefore $\bar{a^*}\bar{x} = \overline{a^*x} = \overline{a^*y} = \bar{a^*}\bar{y}$. Dually, if $\bar{a^*}\bar{x} = \bar{a^*}\bar{y}$ for some $\bar{x}, \bar{y} \in \overline{S}^1$, we may show that $\bar{a}\bar{x} = \bar{a}\bar{y}$. Consequently $\bar{a^*}\ \mathcal{L}^*\ \bar{a}$. The case for $\mathcal{R}^*$ is a dual. □

The following result describes the relationship between the Green $*$-relations of S and the Green $*$-relations of $\overline{S}$.

Lemma 4.12. *Let $a, b \in S$. Then*

(i) $a\ \mathcal{L}^*(S)\ b \iff \bar{a}\ \mathcal{L}^*(\overline{S})\ \bar{b}$;

(ii) $a\ \mathcal{R}^*(S)\ b \iff \bar{a}\ \mathcal{R}^*(\overline{S})\ \bar{b}$;

(iii) $a\ \mathcal{D}^*(S)\ b \iff \bar{a}\ \mathcal{D}^*(\overline{S})\ \bar{b}$.

Proof. Note that $\mathcal{R}^*$ is the dual of $\mathcal{L}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$. It suffices to prove (i) is true.

Assume that $a\ \mathcal{L}^*(S)\ b$. Then $a^*\ \mathcal{L}^*(S)\ b^*$ and so $a^*b^* = a^*$, $b^*a^* = b^*$. Clearly, $a^* \in L_{b^*}^* \cap R_{a^*}^*$ and $b^* \in L_{a^*}^* \cap R_{b^*}^*$. It follows from Theorem 4.1 that $\bar{a^*}\bar{b^*} = \overline{a^*b^*} = \bar{a^*}$ and $\bar{b^*}\bar{a^*} = \overline{b^*a^*} = \bar{b^*}$, that is, $\bar{a^*}\ \mathcal{L}^*(\overline{S})\ \bar{b^*}$. It follows from Lemma 4.11 that $\bar{a}\ \mathcal{L}^*(\overline{S})\ \bar{a^*}\ \mathcal{L}^*(\overline{S})\ \bar{b^*}\ \mathcal{L}^*(\overline{S})\ \bar{b}$. Conversely, suppose that $\bar{a}\ \mathcal{L}^*(\overline{S})\ \bar{b}$. Then by Lemma 4.11 $\bar{a^*}\ \mathcal{L}^*(\overline{S})\ \bar{b^*}$, that is, $\bar{a^*}\bar{b^*} = \bar{a^*}$ and $\bar{b^*}\bar{a^*} = \bar{b^*}$. Hence $\overline{a^*b^*} = \bar{a^*}$ and $\overline{b^*a^*} = \bar{b^*}$. By Lemma 4.9 we have $a^*b^* = a^*$ and $b^*a^* = b^*$. Thus $a^*\ \mathcal{L}^*(S)\ b^*$, hence $a\ \mathcal{L}^*(S)\ b$. □

Let $\mathbb{K} = \{\mathcal{L}^*, \mathcal{R}^*, \mathcal{D}^*\}$. From Lemmas 4.9 and 4.12, for each $\mathcal{K} \in \mathbb{K}$, the mapping

$$\begin{aligned} \phi_{\mathcal{K}}:\ (S/\mathcal{K})^* &\to (\overline{S}/\mathcal{K})^* \\ K_a^* &\mapsto K_{\bar{a}}^*, \end{aligned}$$

where $a \in S^*$, is a bijection. Throughout this paper, we identify the set $(S/\mathcal{D}^*)^*$ (resp., $(S/\mathcal{R}^*)^*$, $(S/\mathcal{L}^*)^*$) with the set $(\overline{S}/\mathcal{D}^*)^*$ (resp., $(\overline{S}/\mathcal{R}^*)^*$, $(\overline{S}/\mathcal{L}^*)^*$), and denote it by Y (resp., I, L). For each $\alpha \in (S/\mathcal{K})^*$, if $K_\alpha^* = K_a^*$ for some $a \in S^*$, then we denote by $\overline{K_\alpha^*}$ or $K_{\bar{a}}^*$ the nonzero $\mathcal{K}$-class $\phi_{\mathcal{K}}(K_\alpha^*)$ of $\overline{S}$, and let $\overline{K_\alpha^*}^0 = \overline{K_\alpha^*} \cup \{0\}$ and $K_{\bar{a}}^{*0} = K_{\bar{a}}^* \cup \{0\}$.

As a direct consequence of Lemma 4.12, we have

Corollary 4.13. *For each $a \in S$, we have*

(i) $R_{\bar{a}}^{*0}$ (resp., $L_{\bar{a}}^{*0}$) is a right (resp., left) ideal of $\overline{S}$;

(ii) $D_{\bar{a}}^{*0}$ is an ideal of $\overline{S}$.

Proof. It follows from Lemma 4.12 and the proof of Lemma 4.1. □

Lemma 4.14. *Let $e \in E(S)^*$. Then $H_e^*(S) \cong H_{\bar{e}}^*(\overline{S})$.*

Proof. By Lemma 4.12, it is easy to see that the map ϕ defined in Lemma 4.9 sends H_e^* onto $H_{\bar{e}}^*$. Let $\phi_e : H_e^* \to H_{\bar{e}}^*$ be the restriction of ϕ to H_e^*. Clearly ϕ_e is a bijection. Let $a, b \in H_e^*$. Then $e \in H_e^* = L_a^* \cap R_b^*$. It follows from Theorem 4.1 that

$$\phi_e(ab) = \overline{ab} = \bar{a}\bar{b} = \phi_e(a)\phi_e(b).$$

Hence ϕ_e is an isomorphism, as required. □

Theorem 4.15. *$\overline{S}$ is primitive abundant.*

Proof. That $\overline{S}$ is abundant follows directly from Lemmas 4.10 and 4.11. To show $\overline{S}$ is primitive, let $\bar{e}, \bar{f} \in E(\overline{S})\backslash\{0\}$ with $\bar{e} \leq \bar{f}$. Thus $\bar{e}\bar{f} = \bar{f}\bar{e} = \bar{e} \neq 0$. By Theorem 4.1 there exists $\bar{g} \in L_{\bar{e}}^* \cap R_{\bar{f}}^* \cap E(\overline{S})$. Hence $\bar{e} = \bar{e}\bar{f} \in L_{\bar{f}}^* \cap R_{\bar{e}}^*$. Similarly, we have $\bar{e} = \bar{f}\bar{e} \in R_{\bar{f}}^* \cap L_{\bar{e}}^*$. Therefore $\bar{e}\ \mathcal{H}^*(\overline{S})\ \bar{f}$ and $\bar{e} = \bar{f}$. Consequently $\overline{S}$ is primitive. □

Recall a semigroup T with zero θ is called a 0-*direct union of semigroups* T_α $(\alpha \in X)$ if $T = \bigcup_{\alpha \in X} T_\alpha$ and $T_\alpha T_\beta = T_\alpha \cap T_\beta = \{\theta\}$ for all $\alpha \neq \beta$.

Theorem 4.16. *(i) For each $\alpha \in Y$, $\overline{D_\alpha^*}^0$ is 0-$\mathcal{J}^*$-simple primitive abundant;*
(ii) $\overline{S}$ is a 0-direct union of $\overline{D_\alpha^*}^0$ $(\alpha \in Y)$.
(iii) On $\overline{S}$, $\mathcal{D}^* = \mathcal{J}^*$.

Proof. (i) Let $\bar{a}, \bar{b} \in \overline{D_\alpha^*}$. If $E(S) \cap L_a^* \cap R_b^* = \emptyset$, then $\bar{a}\bar{b} = 0 \in \overline{D_\alpha^*}^0$. If $E(S) \cap L_a^* \cap R_b^* \neq \emptyset$, then by the proof of Lemma 4.1 we have

$$\bar{a}\bar{b} = \overline{ab} \in R_{\bar{a}}^* \cap L_{\bar{b}}^* \subseteq \overline{D_\alpha^*}.$$

It follows that $\overline{D_\alpha^*}^0$ is a subsemigroup of $\overline{S}$. It follows from the fact $\overline{S}$ is primitive abundant that $\overline{D_\alpha^*}^0$ is primitive abundant. In particular, $\left(\overline{D_\alpha^*}^0\right)^2 \neq 0$. That $\overline{D_\alpha^*}^0$ is 0-$\mathcal{J}^*$-simple is obvious.

(ii) Note that $\{\overline{D_\alpha^*} \mid \alpha \in Y\}$ collects all the nonzero $\mathcal{D}^*$-classes of $\overline{S}$. Thus $\overline{S}$ is a 0-disjoint union of the subsemigroups $\overline{D_\alpha^*}^0 (\alpha \in Y)$. That $\overline{D_\alpha^*}^0 \overline{D_\beta^*}^0 = \{0\}$ whenever $\alpha \neq \beta$ follows from Theorem 4.1. Therefore $\overline{S}$ is a 0-direct union of $\overline{D_\alpha^*}^0 (\alpha \in Y)$.

(iii) Let $\alpha \in S/\mathcal{D}^*$ and $\bar{a} \in \overline{D_\alpha^*}$. Notice that $\overline{D_\alpha^*}$ is a $\mathcal{D}^*$-class of $\overline{S}$. Since $J^*(\bar{a})$ is an ideal of $\overline{S}$ containing $\bar{a}$ which is saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$, we have $\overline{D_\alpha^*}^0 \subseteq J^*(\bar{a})$. On the other hand, by (ii), $\overline{D_\alpha^*}^0$ is an ideal of $\overline{S}$, and hence it is an ideal saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$. Then the fact $J^*(\bar{a})$ is the smallest ideal containing $\bar{a}$ which is saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$ implies that $\overline{D_\alpha^*}^0 = J^*(\bar{a})$. Note that for all $\bar{b}, \bar{c} \in \overline{S}$, $\bar{b}\mathcal{J}^*\bar{c}$ if and only if $J^*(\bar{b}) = J^*(\bar{c})$. It follows that $\bar{b}\mathcal{J}^*\bar{c}$ if and only if $\bar{b}, \bar{c} \in \overline{D_\beta^*}$ for some $\beta \in S/\mathcal{D}^*$. This shows that for each $\alpha \in S/\mathcal{D}^*$, $\overline{D_\alpha^*}$ is a $\mathcal{J}^*$-class of $\overline{S}$. (iii) follows. □

Let T be an abundant semigroup. In [18], S. Armstrong defined the $*$-trace of T to be the partial groupoid $tr^*(T) = (T, \cdot)$ with partial binary operation "$\cdot$" defined by

$$a \cdot b = \begin{cases} ab, & \text{if } E(S) \cap L_a^* \cap R_b^* \neq \emptyset, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

It is clear that $tr^*(T)$ is a disjoint union of $\mathcal{D}^*$-classes of T, which is closed under $\cdot$. The multiplication "$\cdot$" on $tr^*(T)$ can be extended to $tr^*(T)^0 = tr^*(T) \cup \{0\}$ by setting undefined products equal to 0, where 0 is a symbol not in T and acts as zero element. Then $tr^*(T)^0$ is a semigroup under this multiplication. Armstrong [18] studied and characterized the $*$-trace of a concordant semigroup, in particular, he proved that $tr^*(T)^0$ is a primitive abundant semigroup.

Remark 4.17. *Let S be a locally adequate concordant semigroup with $E(S)$ locally pseudofinite. Then $\overline{S}$ is a multiplicative subsemigroup of $R_0[S]$ and a good homomorphism image of $tr^*(S)^0$. Indeed, from Lemmas 4.9, 4.10 and 4.12, one can deduce that $\overline{S}$ is isomorphic to the semigroup obtained from $tr^*(S)^0$ by equating θ (the zero element of S) with 0. And Lemma 4.14, Theorems 4.15 and 4.16 can also be obtained from the results of [18].*

5 Direct product decomposition

Let S be a locally adequate concordant semigroup with $E(S)$ locally finite. We have constructed a new basis for $R_0[S]$ in last section. As an application, we provide a direct product decomposition for $R_0[S]$ in this section.

Theorem 5.1. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite and $\overline{S} = \{a\sigma(a^*) \mid a \in S^*\} \cup \{0\}$. Then*

$$R_0[S] \cong \prod_{\alpha \in Y} R[\overline{D^*_\alpha}]$$

is a direct product decomposition of $R_0[S]$, where $Y = (S/\mathcal{D}^*)^*$ and $\overline{D^*_\alpha}$, $\alpha \in Y$, are all non-zero $\mathcal{D}^*$-classes of $\overline{S}$.

Proof. Since $\overline{S} \setminus \{0\}$ is a basis of $R_0[S]$, we have $R_0[S] = R_0[\overline{S}]$. It follows from Theorem 4.16 (ii) that $R_0[\overline{S}] = \prod_{\alpha \in Y} R_0[\overline{D^*_\alpha}^0]$. Note that $R[\overline{D^*_\alpha}] = R_0[\overline{D^*_\alpha}^0]$. Then we have $R_0[S] \cong \prod_{\alpha \in Y} R[\overline{D^*_\alpha}]$, as required. □

Next we consider the case $R_0[S]$ containning an identity. The following result is essential for us.

Lemma 5.2 ([26, Theorem 1.4]). *Let S be a semigroup. If the semigroup ring $R_0[S]$ contains an identity, then there exists a finite subset U of $E(S)$ and for all $s \in S$, there exist $e, f \in U$ such that $s = es = sf$.*

Lemma 5.3. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite. If $R_0[S]$ contains an identity, then S as well as $\overline{S}$ has finitely many $\mathcal{R}^*$-classes (resp., $\mathcal{L}^*$-classes, $\mathcal{D}^*$-classes). In particular, S as well as $\overline{S}$ has finitely many idempotents.*

Proof. By Lemma 4.12, we only need to consider the case of $\overline{S}$. Suppose that $R_0[S]$ contains an identity. Then by Lemma 5.2 there exists a finite subset U of $E(\overline{S})$ such that for all $\bar{s} \in \overline{S}$, $\bar{s} = \bar{e}\bar{s} = \bar{s}\bar{f}$ for some $\bar{e}, \bar{f} \in U$. Thus, in order to show that $\overline{S}/\mathcal{R}^*$ is finite, it suffices to show that $U \cap R^*_{\bar{s}} \neq \emptyset$ for each $\bar{s} \in \overline{S}$. Suppose to the contrary that $U \cap R^*_{\bar{a}} = \emptyset$ for some $\bar{a} \in \overline{S}$. Then there exists an idempotent $\bar{e} \in U$ such that $\bar{e}\bar{a} = \bar{a}$, but $\bar{e} \notin R^*_{\bar{a}}$. Since $\overline{S}$ is abundant, there exists $\bar{f} \in E(\overline{S}) \cap R^*_{\bar{a}}$. Thus $\bar{e}\bar{f} = \bar{f}$, and hence

$$(\bar{f}\bar{e})(\bar{f}\bar{e}) = \bar{f}(\bar{e}\bar{f})\bar{e} = \bar{f}\bar{e}.$$

This shows that $\bar{f}\bar{e} \in E(\overline{S})$. We claim that $\bar{f}\bar{e} \neq 0$. Otherwise, suppose $\bar{f}\bar{e} = 0$. Then

$$\bar{a} = \bar{f}\bar{a} = \bar{f}(\bar{e}\bar{a}) = (\bar{f}\bar{e})\bar{a} = 0,$$

which is a contradiction. It follows from Lemma 3.3 [14] that $\bar{f}\bar{e} \in R^*_{\bar{f}} \cap L^*_{\bar{e}}$. Thus $\bar{f}\bar{e} = \bar{e}\bar{f}\bar{e} = \bar{e}$, which together with $\bar{e}\bar{f} = \bar{f}$ implies that $\bar{e}\ \mathcal{R}^*\ \bar{f}$. Hence $\bar{e}\ \mathcal{R}^*\ \bar{f}\ \mathcal{R}^*\ \bar{a}$, a contradiction. Therefore $\overline{S}$ has finite many $\mathcal{R}^*$-classes. Dually, $\overline{S}$ has finite many $\mathcal{L}^*$-classes and so finite many $\mathcal{D}^*$-classes.

Since $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$, $\overline{S}$ has finite many $\mathcal{H}^*$-classes. Hence $\overline{S}$ has finite many idempotents since each $\mathcal{H}^*$-class contains at most one idempotent. □

Corollary 5.4. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite. If $R_0[S]$ contains an identity, then*

$$R_0[S] = \oplus_{\alpha \in Y} R[\overline{D^*_\alpha}],$$

where $Y = \overline{S}/\mathcal{D}^*$ and $\overline{D^*_\alpha}$, $\alpha \in Y$, are all non-zero $\mathcal{D}^*$-classes of $\overline{S}$.

Proof. If $R_0[S]$ contains an identity, then S as well as $\overline{S}$ has finitely many $\mathcal{D}^*$-classes. Now it follows from the proof of Theorem 5.1. □

To end this section, we consider two special cases: adequate and regular. As applications of Theorem 5.1, we give a direct product decomposition of IC adequate semigroup algebras and locally inverse semigroup algebras.

Recall that an *IC adequate semigroup* (sometimes called *ample semigroup*) is an adequate semigroup which is IC. Note that the set of idempotents of an adequate semigroup is a semilattice and adequate semigroups are locally adequate. Hence a locally adequate concordant semigroup is adequate if and only if it is IC adequate.

Corollary 5.5. *Let S be an IC adequate semigroup with $E(S)$ locally finite. Then $R_0[S]$ is a direct product of contracted weak Brandt semigroup algebras. Moreover, $R_0[S]$ contains an identity if and only if $S/\mathcal{R}^*$ and $S/\mathcal{L}^*$ are finite.*

Proof. Let S be an IC adequate semigroup with $E(S)$ locally finite and let $\overline{S} = \{\bar{a} \mid a \in S\} \cup \{0\}$. Then

$$R_0[S] \cong \prod_{\alpha \in Y} R[\overline{D^*_\alpha}],$$

where $Y = (\overline{S}/\mathcal{D}^*)^*$ and $\overline{D^*_\alpha}$, $\alpha \in Y$, are all non-zero $\mathcal{D}^*$-classes of $\overline{S}$. Since S is adequate, it follows from Lemma 4.12 that $\overline{S}$ is also adequate. Then by Theorem 4.16, for each $\alpha \in Y = \overline{S}/\mathcal{D}^*$, $\overline{D^*_\alpha}^0$ is a 0-$\mathcal{J}^*$-simple primitive adequate semigroup. So it is a weak Brandt semigroup. Note that $R[\overline{D^*_\alpha}] = R_0[\overline{D^*_\alpha}^0]$. Therefore $R_0[S]$ is a direct product of contracted weak Brandt semigroup algebras.

Suppose that $R_0[S]$ contains an identity. Then by Lemma 5.3 $\overline{S}/\mathcal{R}^*$ and $\overline{S}/\mathcal{L}^*$ are finite. Conversely, Suppose that $\overline{S}/\mathcal{R}^*$ and $\overline{S}/\mathcal{L}^*$ are finite. Then $Y = \overline{S}/\mathcal{D}^*$ is finite. It follows from the proof of Theorem 5.1 that

$$R_0[S] = \oplus_{\alpha \in Y} R[\overline{D^*_\alpha}]$$

where $Y = \overline{S}/\mathcal{D}^*$ and $\overline{D^*_\alpha}$, $\alpha \in Y$, are all non-zero $\mathcal{D}^*$-classes of $\overline{S}$. Similar argument as above, $\overline{D^*_\alpha}^0$ is a weak Brandt semigroup for each $\alpha \in Y$. Let

$$\overline{D^*_\alpha}^0 = \mathcal{M}^0(M^\alpha_{\lambda\mu}; I^\alpha, I^\alpha, \Gamma^\alpha; P^\alpha)$$

for each $\alpha \in Y$, where P^α is a diagonal matrix with $p^\alpha_{ii} = e^\alpha_\lambda$ for each $i \in I^\alpha_\lambda$, and where e^α_λ is the identity of the monoid $M^\alpha_{\lambda\lambda}$ for each $\lambda \in \Gamma^\alpha$. Then the element

$$e = \sum_{\alpha \in Y} \sum_{\lambda \in \Gamma^\alpha, i \in I^\alpha_\lambda} (e^\alpha_\lambda)_{ii} \in R_0[S]$$

is well defined, where $(e^\alpha_\lambda)_{ii}$ is the $|I^\alpha| \times |I^\alpha|$ matrix with entry e^α_λ in the (i, i) position and zeros elsewhere. Clearly, e is the identity of $R_0[S]$. □

It is clear that a locally adequate concordant semigroup is regular if and only if it is locally inverse.

Corollary 5.6. *Let S be a locally inverse semigroup with $E(S)$ locally finite. Then*

$$R_0[S] \cong \prod_{\alpha \in (S/\mathcal{D})^*} \mathscr{M}(R[G_\alpha]; I_\alpha, \Lambda_\alpha; P_\alpha), \tag{4}$$

where G_α is the maximal subgroup in D_α, I_α (resp., Λ_α) is the set of the $\mathcal{R}$-classes (resp., $\mathcal{L}$-classes) contained in D_α, and P is a regular $\Lambda_\alpha \times I_\alpha$-matrix with entries in G^0_α for each $\alpha \in S/\mathcal{D}$.

Proof. It is clear that a regular 0-$\mathcal{J}^*$-simple primitive abundant is completely 0-simple. Note that Green's $*$-relations coincide with Green's relations in regular semigroups. Then $\overline{D^*_\alpha}^0$ is a completely 0-simple semigroup, say, $\overline{D^*_\alpha}^0 = \mathcal{M}^0(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$, for each $\alpha \in (\overline{S}/\mathcal{D}^*)^* = (S/\mathcal{D})^*$. Thus $R_0[\overline{D^*_\alpha}^0] = \mathscr{M}(R[G_\alpha]; I_\alpha, \Lambda_\alpha; P_\alpha)$. It follows from Theorem 5.1 that $R_0[S] \cong \prod_{\alpha \in (S/\mathcal{D})^*} \mathscr{M}(R[G_\alpha]; I_\alpha, \Lambda_\alpha; P_\alpha)$. □

Corollaries 5.5 and 5.6 generalize the results on finite ample semigroups [9] and on finite locally inverse semigroups [8].

Corollary 5.7 ([4, Theorem 6.5]). *Let S be an inverse semigroup with $E(S)$ locally finite. Then*

$$R_0[S] \cong \prod_{\alpha \in (S/\mathcal{D})^*} M_{|I_\alpha|}(R[G_\alpha]),$$

where G_α is the maximal subgroup in D_α and $|I_\alpha|$ denotes the number of the $\mathcal{R}$-classes of D_α for each $\alpha \in S/\mathcal{D}$.

Proof. By hypothesis, Lemmas 4.10 and 4.12, we deduce that $\overline{S}$ is an inverse semigroup. Then the fact $\overline{S}$ is a 0-direct union of $\overline{D_\alpha}^0 (\alpha \in S/\mathcal{D})$ implies that each $\overline{D_\alpha}^0$ is a Brandt semigroup. Say, $\overline{D_\alpha}^0 = \mathcal{M}^0(G_\alpha; I_\alpha, I_\alpha; P_\alpha)$, where G_α is a maximal subgroup of $\overline{S}$ which is contained in $\overline{D_\alpha}$, I_α is the set of $\mathcal{R}$-classes of $\overline{D_\alpha}$, P_α is a diagonal $I_\alpha \times I_\alpha$-matrix with $(p_\alpha)_{ii}$ is equal to the identity e_α of G_α for each $i \in I_\alpha$. Furhtermore, by Lemma 4.14, G_α is isomorphic to any maximal subgroup of S contained in D_α; by Lemma 4.12, I_α is the set of $\mathcal{R}$-classes of D_α. Now it is easily verified that $R[\overline{D_\alpha}] \cong M_{|I_\alpha|}(R[G_\alpha])$. Consequently, by Theorem 5.1, we obatin the desired direct product decomposition. □

6 Projective indecomposable modules

Throughout this section, let S denote a locally adequate concordant semigroup with $E(S)$ locally finite. Since projective indecomposable modules are discussed on algebras with identities, we always assume that the contracted semigroup algebra $R_0[S]$ contains an identity.

By Corollary 4.13, for $i \in I = \overline{S}/\mathcal{R}^*$, $\overline{R_i^*}^0$ is a right ideal of $\overline{S}$. Note that $R_0[S] = R_0[\overline{S}]$. Then $R[\overline{R_i^*}]$ is a right ideal of $R_0[S]$ and can be considered as a right $R_0[S]$-module for each $i \in I$.

We first give out a direct sum decomposition of $R_0[S]$.

Theorem 6.1. *If $R_0[S]$ has an identity, then*

$$R_0[S]_{R_0[S]} = \bigoplus_{i \in I} R[\overline{R_i^*}] \tag{5}$$

is a finite direct sum decomposition of $R_0[S]$.

Proof. If $R_0[S]$ contains an identity, then S as well as $\overline{S}$ has finitely many $\mathcal{R}^*$-classes and so I is finite. Since $\overline{S}^*$ is a disjoint union of $\overline{R_i^*}$ $(i \in I)$, the right $R_0[S]$-module $R_0[S]_{R_0[S]}$ is a direct sum of $R[\overline{R_i^*}]$ $(i \in I)$. Therefore (5) gives a finite direct sum decomposition of $R_0[S]$. □

By Lemma 4.16, $\overline{D_\alpha^*}^0$ $(\alpha \in Y)$ is a 0-$\mathcal{J}^*$-simple PA blocked Rees matrix semigroup, say, $\overline{D_\alpha^*}^0 = \mathcal{M}^0(M_{\lambda\mu}^\alpha; J^\alpha, \Lambda^\alpha, \Gamma^\alpha; P^\alpha)$.

Next we investigate conditions under which the projective $R_0[S]$-modues $R[\overline{R_i^*}]$ are isomorphic.

Lemma 6.2. *Let $\alpha, \beta \in Y$, $i \in J^\alpha$, $j \in J^\beta$. If $R[\overline{R_i^*}] \cong R[\overline{R_j^*}]$, then $\alpha = \beta$.*

Proof. Let $\psi : R[\overline{R_i^*}] \to R[\overline{R_j^*}]$ be a right $R_0[S]$-module isomorphism. Suppose to the contrary that $\alpha \neq \beta$. Let $\bar{x} \in \overline{R_i^*}$. Since $\bar{S}$ is abundant, there exists an idempotent $\bar{e} \in L_{\bar{x}}^* \subseteq \overline{D_\alpha^*}$. Then $\bar{x}\bar{e} = \bar{x}$ and $\psi(\bar{x}\bar{e}) = \psi(\bar{x}) \neq 0$. On the other hand, $(\psi(\bar{x}), \bar{e}) \notin \mathcal{D}^*$, thus $\psi(\bar{x})\bar{e} = 0$ by Lemma 4.16 (ii). Whence $\psi(\bar{x}\bar{e}) \neq \psi(\bar{x})\bar{e}$, a contradiction. Therefore $\alpha = \beta$, as required. □

Let $\beta \in Y$ and $R[\overline{R_i^*}] \subseteq R[\overline{D_\beta^*}]$. Then $R[\overline{R_i^*}]$ is a right $R_0[S]$-module. By Theorem 5.1, $R_0[S] = \prod_{\alpha \in Y} R[\overline{D_\alpha^*}]$. Thus we only need to consider $R[\overline{R_i^*}]$ as a right $R[\overline{D_\beta^*}]$-module; $M \subseteq R[\overline{R_i^*}]$ is an indecomposable right $R_0[S]$-module if and only if M is an indecomposable right $R[\overline{D_\beta^*}]$-module. Therefore, it suffices to find all the non-isomorphic projective indecomposable right $R[\overline{D_\alpha^*}]$-modules $(\alpha \in Y)$.

Let $M = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$ be a PA blocked Rees matrix semigroup and $\lambda \in \Gamma$, $i, j \in J_\lambda$. For each $\mu \in \Gamma$, define

$$\overline{R_{i\mu}^*} = \bigcup\nolimits_{s \in \Lambda_\mu} \overline{H_{is}^*}, \quad n_{i\mu} = |M_{\lambda\mu}|.$$

Here $n_{i\mu} = n_{j\mu}$. Since $|\overline{H^*_{ks}}| = |M_{\lambda\mu}| = |\overline{H^*_{lt}}|$ for all $(k,s),(l,t)$ in the (λ,μ)-block, we have $|\overline{R^*_{i\mu}}| = n_{i\mu}|\Lambda_\mu|$. We say the semigroup M *satisfies the row-block condition* if for all $\lambda \neq \nu \in \Gamma$, $i \in J_\lambda$ and $j \in J_\nu$, there exists $\mu \in \Gamma$ such that $n_{i\mu} \neq n_{j\mu}$.

Lemma 6.3. *Let $\overline{D^*}$ be a $\mathcal{D}^*$-class of $\overline{S}$ and $\overline{D^*}^0 = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$, $i, j \in J$.*

(i) If $i, j \in J_\lambda$ for some $\lambda \in \Gamma$, then $R[\overline{R^*_i}] \cong R[\overline{R^*_j}]$;

(ii) If $R[\overline{R^*_i}] \cong R[\overline{R^*_j}]$, then $n_{i\mu} = n_{j\mu}$ for each $\mu \in \Gamma$;

(iii) If $\overline{D^*}^0$ satisfies the row-block condition, then $\{R[\overline{R^*_{1_\lambda}}] \mid \lambda \in \Gamma\}$ collects pairwise non-isomorphic projective right $R[\overline{D^*}]$-modules.

Proof. (i) Let $i, j \in J_\lambda$ for some $\lambda \in \Gamma$. Then for any $\mu \in \Gamma$, $n_{i\mu} = n_{j\mu}$, and hence we can define a map $\psi : R[\overline{R^*_i}] \to R[\overline{R^*_j}]$ by $(i, \bar{a}, s) \mapsto (j, \bar{a}, s)$, where $s \in \Lambda_\mu$ and $\bar{a} \in M_{\lambda\mu}$, and extend R-linearly. By definition, ψ restricts to a bijection $\overline{R^*_i} \to \overline{R^*_j}$. Hence ψ is a R-module isomorphism from $R[\overline{R^*_i}]$ to $R[\overline{R^*_j}]$. We claim that ψ is a right $R[\overline{D^*}]$-module homomorphism. For this, let $\bar{x} = (i, \bar{a}, s) \in \overline{R^*_i}$ and $\bar{y} = (k, \bar{b}, t) \in \overline{D^*}^0$, then

$$\psi(\bar{x})\bar{y} = (j, \bar{a}p_{sk}\bar{b}, t) = \psi(i, \bar{a}p_{sk}\bar{b}, t) = \psi(\bar{x}\bar{y}).$$

Therefore ψ is a right $R_0[S]$-module isomorphism, and (i) is proved.

(ii) Without loss of generality, suppose that $\psi : R[\overline{R^*_i}] \to R[\overline{R^*_j}]$ is a $R[\overline{D^*}]$-module isomorphism with $\psi(\overline{R^*_i}) = \overline{R^*_j}$. Let $\mu \in \Gamma$ and $\bar{x} \in \overline{R^*_{i\mu}}$. Since $\overline{D^*}^0$ is abundant and all its idempotents are in the diagonal blocks, there exist an element $l \in J_\mu$ and an idempotent $\bar{e} \in \overline{D^*}$ such that $\bar{e} \in L^*_{\bar{x}} \cap \overline{R^*_l}$. But by the fact $\psi(\bar{x}\bar{e}) = \psi(\bar{x})\bar{e}$ and (4.8), we deduce that

$$E(\overline{S}) \cap L^*_{\bar{x}} \cap \overline{R^*_l} \neq \emptyset \Leftrightarrow E(\overline{S}) \cap L^*_{\psi(\bar{x})} \cap \overline{R^*_l} \neq \emptyset.$$

It follows from the fact $0 \neq \psi(\bar{x}) = \psi(\bar{x}\bar{e})$ that $L^*_{\psi(\bar{x})} \cap \overline{R^*_l}$ contains an idempotent, hence $\psi(\bar{x}) \in \overline{R^*_{j\mu}}$. Therefore $\psi(\overline{R^*_{i\mu}}) \subseteq \overline{R^*_{j\mu}}$. Notice that $\overline{R^*_k} = \cup_{\nu\in\Gamma}\overline{R^*_{k\nu}}$ for each $k \in J$. Because ψ is a bijection from $\overline{R^*_i}$ to $\overline{R^*_j}$, we have $n_{i\mu}|\Lambda_\mu| = |\overline{R^*_{i\mu}}| = |\overline{R^*_{j\mu}}| = n_{j\mu}|\Lambda_\mu|$. This implies that $n_{i\mu} = n_{j\mu}$.

(iii) This follows from (i) and (ii). □

Let $\overline{D^*}$ be a $\mathcal{D}^*$-class of $\overline{S}$ and let $\overline{D^*}^0 = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$ satisfy the row-block condition. By (5), $R[\overline{D^*}]$ is a direct sum of $R[\overline{R^*_j}] (j \in J)$. For each pair $i, j \in J$, according to Lemma 6.3, $R[\overline{R^*_i}] \cong R[\overline{R^*_j}]$ if and only if there exists $\lambda \in \Gamma$ such that $i, j \in J_\lambda$. Thus it suffices to find the non-isomorphic indecomposable direct summands of $R[\overline{R^*_{1_\lambda}}]$ for each $\lambda \in \Gamma$.

Let $\lambda \in \Gamma$ and let $f_{\lambda,1}, \cdots, f_{\lambda,n_\lambda}, \cdots, f_{\lambda,n_\lambda+m_\lambda}$ be a complete set of primitive orthogonal idempotents of $R[T_\lambda]$ such that $f_{\lambda,1}R[T_\lambda], \cdots, f_{\lambda,n_\lambda}R[T_\lambda]$ are all the non-isomorphic projective indecomposable right $R[T_\lambda]$-modules. Notice that

$$R[\overline{R^*_{1_\lambda}}] = (1_\lambda, e_\lambda, 1_\lambda)R[\overline{D^*}] = \bigoplus_{1\leq p\leq n_\lambda+m_\lambda} (1_\lambda, f_{\lambda,p}, 1_\lambda)R[\overline{D^*}].$$

Lemma 6.4. *Let $\overline{D^*}$ be a $\mathcal{D}^*$-class of $\overline{S}$ and $\overline{D^*}^0 = \mathcal{M}^0(M_{\lambda\mu}; J, \Lambda, \Gamma; P)$, $\lambda \in \Gamma$.*

(i) For each pair $u, v \in R[T_\lambda]$, the right $R[T_\lambda]$-modules $uR[T_\lambda] \cong vR[T_\lambda]$ if and only if the right $R[\overline{D^*}]$-modules $(1_\lambda, u, 1_\lambda)R[\overline{D^*}] \cong (1_\lambda, v, 1_\lambda)R[\overline{D^*}]$;

(ii) Let $f \in R[T_\lambda]$ be an idempotent. Then $fR[T_\lambda]$ is an indecomposable right $R[T_\lambda]$-module if and only if $(1_\lambda, f, 1_\lambda)R[\overline{D^*}]$ is an indecomposable right $R[\overline{D^*}]$-module.

Proof. (i) Suppose that $\varphi : uR[T_\lambda] \to vR[T_\lambda]$ is a right $R[T_\lambda]$-module isomorphism. Let $w \in R[T_\lambda]$ and $(i, y, s) \in \overline{D^*}$. Then $(1_\lambda, w, 1_\lambda)(i, y, s) = (1_\lambda, w(p_{1_\lambda,i}y), s)$. If $i = 1_\lambda$, then $p_{1_\lambda,i}y = e_\lambda y = y$ by our assumption on P. Therefore

$$(1_\lambda, w, 1_\lambda)R[\overline{D^*}] = \sum_{\mu\in\Gamma, s\in\Lambda_\mu, x\in M_{\lambda\mu}} R(1_\lambda, wx, s).$$

By condition (C) in the definition of PA blocked Rees matrix semigroups, for all $\mu \in \Gamma$ and $x, y \in M_{\lambda\mu}$, if $wx = wy$, then $x = y$ in $M_{\lambda\mu}$. Thus the R-linear map

$$\widetilde{\varphi} : (1_\lambda, u, 1_\lambda)R[\overline{D^*}] \to (1_\lambda, v, 1_\lambda)R[\overline{D^*}]$$
$$(1_\lambda, ux, s) \mapsto (1_\lambda, \varphi(u)x, s)$$

is well defined and is injective. We claim that $\widetilde{\varphi}$ is a right $R[\overline{D^*}]$-module isomorphism. Indeed, since φ is surjective, $\widetilde{\varphi}$ is also surjective, hence $\widetilde{\varphi}$ is a bijection. Let $(l, y, s) \in \overline{D^*}$. Then $\widetilde{\varphi}((1_\lambda, ux, s)(l, y, t)) = \widetilde{\varphi}(1_\lambda, uxp_{sl}y, t) = \widetilde{\varphi}((1_\lambda, ux, s))(l, y, t)$ and consequently, $\widetilde{\varphi}$ is a $R[\overline{D^*}]$-homomorphism.

Conversely, suppose that $\widetilde{\varphi} : (1_\lambda, u, 1_\lambda)R[\overline{D^*}] \to (1_\lambda, v, 1_\lambda)R[\overline{D^*}]$ is a right $R[\overline{D^*}]$-module isomorphism. For each $w \in uR[T_\lambda]$,

$$\widetilde{\varphi}(1_\lambda, w, 1_\lambda) = \widetilde{\varphi}((1_\lambda, w, 1_\lambda))(1_\lambda, e_\lambda, 1_\lambda) \in (1_\lambda, vR[T_\lambda], 1_\lambda).$$

Thus we can define a map $\varphi : uR[T_\lambda] \to vR[T_\lambda]$ by $(1_\lambda, \varphi(w), 1_\lambda) = \widetilde{\varphi}(1_\lambda, w, 1_\lambda)$. Obviously, φ is a bijection. It thus suffices to show $\varphi(wx) = \varphi(w)x$ for all $x \in R[T_\lambda]$. Indeed,

$$(1_\lambda, \varphi(wx), 1_\lambda) = \widetilde{\varphi}((1_\lambda, wx, 1_\lambda)) = \widetilde{\varphi}((1_\lambda, w, 1_\lambda))(1_\lambda, x, 1_\lambda) = (1_\lambda, \varphi(w)x, 1_\lambda),$$

which implies $\varphi(wx) = \varphi(w)x$, and (i) follows.

(ii) Clearly, $f' = (1_\lambda, f, 1_\lambda)$ is an idempotent of $R_0[M]$. We only need to show that $f' \in \text{Mult}\, R[\overline{D^*}]$ is primitive if and only if $f \in \text{Mult}\, R[T_\lambda]$ is primitive. Indeed, let $e' \in \text{Mult}\, R[\overline{D^*}]$ be an idempotent. Then $e' < f'$ if and only if there exists an idempotent $e \in R[T_\lambda]$ such that $e' = (1_\lambda, e, 1_\lambda)$ and $e < f$, and hence (ii) follows. □

Notice that the results of Lemma 6.4 can be applied to general PA blocked Rees matrix semigroups.

Theorem 6.5. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite and $Y = S/\mathcal{D}^*$. If*
(i) for each $\alpha \in Y$, $\overline{D^*_\alpha}^0 = \mathcal{M}^0(M^\alpha_{\lambda\mu}; J^\alpha, \Lambda^\alpha, \Gamma^\alpha; P^\alpha)$ satisfies the row-block condition,
(ii) for each $\lambda \in \Gamma^\alpha$, $f^\alpha_{\lambda,1}, \cdots, f^\alpha_{\lambda,n^\alpha_\lambda}, \cdots, f^\alpha_{\lambda,n^\alpha_\lambda+m^\alpha_\lambda}$ is a complete set of primitive orthogonal idempotents of $R[T^\alpha_\lambda]$ such that $f^\alpha_{\lambda,1}R[T^\alpha_\lambda], \cdots, f^\alpha_{\lambda,n^\alpha_\lambda} R[T^\alpha_\lambda]$ are all the non-isomorphic projective indecomposable right $R[T^\alpha_\lambda]$-modules,
then the set $\bigcup_{\alpha\in Y,\lambda\in\Gamma^\alpha}\{(1^\alpha_\lambda, f^\alpha_{\lambda,q}, 1^\alpha_\lambda)R_0[S] \mid 1 \le q \le n^\alpha_\lambda\}$ collects all the non-isomorphic projective indecomposable right $R_0[S]$-modules.

Proof. Let $\alpha \in Y$ and $\lambda \in \Gamma^\alpha$. By Lemma 6.4 and the hypotheses, the right $R[\overline{D^*_\alpha}]$-modules $(1^\alpha_\lambda, f^\alpha_{\lambda,q}, 1^\alpha_\lambda)R[\overline{D^*_\alpha}]$ are indecomposable; furthermore, $(1^\alpha_\lambda, f^\alpha_{\lambda,q}, 1^\alpha_\lambda)R[\overline{D^*_\alpha}] \cong (1^\alpha_\lambda, f^\alpha_{\lambda,p}, 1^\alpha_\lambda)R[\overline{D^*_\alpha}]$ if and only if $f^\alpha_{\lambda,q}R[T^\alpha_\lambda] \cong f^\alpha_{\lambda,p}R[T^\alpha_\lambda]$ as right $R[T^\alpha_\lambda]$-modules , where $1 \le q, p \le n^\alpha_\lambda + m^\alpha_\lambda$. Therefore, $(1^\alpha_\lambda, f^\alpha_{\lambda,q}, 1^\alpha_\lambda)R[\overline{D^*_\alpha}]$ $(1 \le q \le n^\alpha_\lambda)$ are all the non-isomorphic projective indecomposable right $R[\overline{D^*_\alpha}]$-modules.

As mentioned before, M is an indecomposable right $R[\overline{D^*_\alpha}]$-module if and only if M is an indecomposable right $R_0[S]$-module. Consequently, $\bigcup(1^\alpha_\lambda, f^\alpha_{\lambda,q}, 1^\alpha_\lambda)R_0[S]$ are all the non-isomorphic projective indecomposable right $R_0[S]$-modules, where the union takes over all $\alpha \in Y$, $\lambda \in \Gamma^\alpha$ and $1 \le q \le n^\alpha_\lambda$, □

For each $\overline{D^*_\alpha}^0 = \mathcal{M}^0(M^\alpha_{\lambda\mu}; J^\alpha, \Lambda^\alpha, \Gamma^\alpha; P^\alpha)$, if $|\Gamma^\alpha| = 1$, then the semigroup $\overline{D^*_\alpha}^0$ is isomorphic to a Rees matrix semigroup [27], say, $\overline{D^*_\alpha}^0 = \mathcal{M}^0(T_\alpha; J_\alpha, \Lambda_\alpha; P_\alpha)$. In the following proposition we specialize to this case.

Proposition 6.6. *Let S be a locally adequate concordant semigroup with $E(S)$ locally finite and for each $\alpha \in Y = S/\mathcal{D}^*$, $\overline{D^*_\alpha}^0 = \mathcal{M}^0(T_\alpha; J_\alpha, \Lambda_\alpha; P_\alpha)$ be a Rees matrix semigroup over a cancellative monoid T_α.*
(i) $R_0[S]^b \cong \prod_{\alpha\in Y} R[T_\alpha]^b$;
(ii) $R_0[S]$ is representation-finite if and only if for each $\alpha \in Y$, $R[T_\alpha]$ is representation-finite.

Proof. (i) It is clear that $R_0[S]$ satisfies the row-block condition. Let $\alpha \in Y = S/\mathcal{D}^*$. Suppose that $f^\alpha_1, \cdots, f^\alpha_{n_\alpha}, \cdots, f^\alpha_{n_\alpha+m_\alpha}$ is a complete set of primitive orthogonal idempotents of $R[T_\alpha]$ such that $f^\alpha_1 R[T_\alpha], \cdots, f^\alpha_{n_\alpha} R[T_\alpha]$ are all the non-isomorphic projective indecomposable right $R[T_\alpha]$-modules. Then $e_{R[T_\alpha]} = f^\alpha_1 + \cdots + f^\alpha_{n_\alpha}$, and thus

$R[T_\alpha]^b = e_{R[T_\alpha]} R[T_\alpha] e_{R[T_\alpha]}$. By Theorem 6.5, we have $e_{R_0[S]} = \sum_{\alpha \in Y} (1_\alpha, e_{R[T_\alpha]}, 1_\alpha)$, where $1_\alpha \in J_\alpha$ denote the element 1^α_λ for each $\alpha \in Y$. This, together with the fact $R_0[S] = \prod_{\alpha \in Y} R[\overline{D^*_\alpha}]$, implies that

$$\begin{aligned} R_0[S]^b &= e_{R_0[S]} R_0[S] e_{R_0[S]} \\ &= \bigoplus_{\alpha \in Y} (1_\alpha, e_{R[T_\alpha]}, 1_\alpha) R_0[\overline{S}] (1_\alpha, e_{R[T_\alpha]}, 1_\alpha) \\ &= \bigoplus_{\alpha \in Y} (1_\alpha, e_{R[T_\alpha]}, 1_\alpha)(1_\alpha, R[T_\alpha], 1_\alpha)(1_\alpha, e_{R[T_\alpha]}, 1_\alpha) \\ &= \bigoplus_{\alpha \in Y} (1_\alpha, e_{R[T_\alpha]} R[T_\alpha] e_{R[T_\alpha]}, 1_\alpha) \\ &\cong \prod_{\alpha \in Y} R[T_\alpha]^b. \end{aligned}$$

(ii) This follows from (i) immediately. □

To end our paper, for regular case, we have the following results.

Corollary 6.7. *Let S be a locally inverse semigroup with idempotents set $E(S)$ locally finite. Suppose that $R_0[S]$ contains an identity. Then $R_0[S]$ is representation-finite if and only if $R[G_\alpha]$ is representation-finite for each $\alpha \in S/\mathcal{D}$.*

Proof. let $\alpha \in Y$. Then

$$\overline{D^*_\alpha}^0 = \overline{D_\alpha}^0 = \mathcal{M}^0(G_\alpha; J_\alpha, \Lambda_\alpha; P_\alpha)$$

is a completely 0-simple semigroup. The result follows from Proposition 6.6 immediately. □

Let G be a finite group and K be a field with characteristic p. If $p \nmid |G|$, then $K[G]$ is semisimple and conversely (Maschke's Theorem). If this is the case, $K[G]$ is representation-finite since semisimple algebra is representation-finite. If $p \big| |G|$, $K[G]$ is representation-finite if and only if the Sylow p-subgroups G_p of G are cyclic (Higman's Theorem [28]). Therefore, $K[G]$ is representation-finite if and only if either $p \nmid |G|$, or all the Sylow p-subgroups G_p of G are cyclic.

Now the next result follows from Corollary 6.7 directly.

Corollary 6.8. *Let S be a locally inverse semigroup with $E(S)$ locally finite and all its maximal subgroups of finite order. Suppose that $K_0[S]$ contains an identity. Then $K_0[S]$ is representation-finite if and only if for each $\alpha \in S/\mathcal{D}$ with $p \big| |G_\alpha|$, the Sylow p-subgroups $(G_\alpha)_p$ of G_α are all cyclic.*

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