Locally and globally riddled basins in two coupled piecewise-linear maps

Yu. Maistrenko,[1] T. Kapitaniak[2] and P. Szuminski[2]
[1]*Institute of Mathematics, Academy of Sciences of Ukraine, 3 Tereshchenkivska St., Kiev 252601, Ukraine*
[2]*Division of Dynamics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland*
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The chaos synchronization and riddled basins phenomena are discussed for a family of two-dimensional piecewise linear endomorphisms that consist of two linearly coupled one-dimensional maps. Rigorous conditions for the occurrence of both phenomena are given. Different scenarios for the transition from locally to globally riddled basins and blowout bifurcation have been identified and described. [S1063-651X(97)09511-1]

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I. INTRODUCTION

Two identical chaotic systems $x_{n+1}=f(x_n)$ and $y_{n+1}=f(y_n)$, $x,y\in\mathbb{R}$, evolving on an asymptotically stable chaotic attractor A, when one-to-one coupling

$$x_{n+1}=f(x_n)+d_1(y_n-x_n),$$
$$y_{n+1}=f(y_n)+d_2(x_n-y_n) \qquad (1)$$

is introduced, can be synchronized for some ranges of $d_{1,2}\in\mathbb{R}$, i.e., $|x_n-y_n|\to 0$ as $n\to\infty$ [1–13].

In the synchronized regime the dynamics of the coupled system (1) is restricted to one-dimensional invariant subspace $x_n=y_n$, so the problem of synchronization of chaotic systems can be understood as a problem of stability of the one-dimensional chaotic attractor A in two-dimensional phase space [14,15].

The basin of attraction $\beta(A)$ is the set of points whose ω-limit set is contained in A. In Milnor's definition [16] of an attractor the basin of attraction need not include the whole neighborhood of the attractor, i.e., we say that A is a weak Milnor attractor if $\beta(A)$ has a positive Lebesgue measure. For example, a riddled basin [14,15,17–20], which has recently been found to be typical for a certain class of dynamical systems with one-dimensional invariant subspace [such as $x_n=y_n$ in the example (1)], has positive Lebesgue measure but does not contain any neighborhood of the attractor. In this case the basin of attraction $\beta(A)$ may be a fat fractal, so that any neighborhood of the attractor intersects the basin with positive measure, but may also intersect the basin of another attractor with positive measure.

The dynamics of the system (1) is described by two Lyapunov exponents. One of them describes the evolution on the invariant manifold $x=y$ and is always positive. The second exponent characterizes the evolution transverse to this manifold and is called transversal. If the transversal Lyapunov exponent is negative, the set A is an attractor, at least in the weak Milnor sense.

When the transversal Lyapunov exponent is negative and there exist trajectories in the attractor A, which are transversally repelling, A is a weak Milnor attractor with a locally riddled basin, i.e., there is a neighborhood U of A such that in any neighborhood V of any point in A, there is a set of points in $V\cap U$ of positive measure that leave U in a finite time. The trajectories that leave neighborhood U can either go to the other attractor (attractors) or after a finite number of iterations be diverted back to A. If there is a neighborhood U of A such that in any neighborhood V of any point in U there is a set of points of positive measure that leave U and go to the other attractor (attractors), then the basin of A is globally riddled.

In this paper we identify and describe different ways of transition from locally to globally riddled basins and discuss the conditions for the basin of attraction to be one or the other. We consider the dynamics of a four-parameter family of a two-dimensional piecewise linear noninvertible map

$$F=\begin{cases} f_{l,p}(x_n)+d(y_n-x_n): & x_{n+1}=px_n+\dfrac{l}{2}\left(1-\dfrac{p}{l}\right)\left(\left|x_n+\dfrac{1}{l}\right|-\left|x_n-\dfrac{1}{l}\right|\right)+d_1(y_n-x_n)\\ f_{l,p}(y_n)+d(x_n-y_n): & y_{n+1}=py_n+\dfrac{l}{2}\left(1-\dfrac{p}{l}\right)\left(\left|y_n+\dfrac{1}{l}\right|-\left|y_n-\dfrac{1}{l}\right|\right)+d_2(x_n-y_n),\end{cases} \qquad (2)$$

where $l,p,d_{1,2}\in\mathbb{R}$, which consists of two identical linearly coupled one-dimensional maps being the generalization of the skew tent map. Chaotic attractors of the skew tent map have been considered in [21–27]. In comparison with the maps studied in [14,15,17,18], our map (2) has the advantage that, as we show below, conditions for the occurrence of riddled basins can be given analytically. We use the map (2) as a test model of coupled chaotic systems.

The outline of this paper is as follows. In Sec. II we recall the definitions of different types of attractor stability and give rigorous conditions for asymptotic and weak Milnor stability of the synchronized chaotic attractor of map (2). These analytical conditions allow us to obtain a two-dimensional bifurcation diagram of coupled maps (2). Two different types of transitions from locally to globally riddling basins and their connection with global bifurcations of the basins of attraction that we identify are studied in Sec. III. Section IV describes recently identified features of *blowout bifurcation* from locally riddled and globally riddled basins. Finally, we summarize our results in Sec. V.

II. BIFURCATION DIAGRAM FOR STABILITY OF ATTRACTORS OF CHAOTIC SYNCHRONIZATION

Consider the two-dimensional map of the plane (x,y) into itself (2). When

$$(l,p)\in\Pi=\left\{l>1,-\frac{l}{l-1}<p\leqslant-1\right\},$$

the one-dimensional map $f_{l,p}$ has two symmetrical attractors $\Gamma^{(-)}\subset[-1,0]$ and $\Gamma^{(+)}\subset[0,1]$, which are cycles of 2^m chaotic intervals (so-called 2^m-piece chaotic attractors). Depending on parameters l and p, m can be any positive integer. Denote Π_m as a subregion of Π, where $\Gamma_m^{(\pm)}$ is a period-2^m cycle of chaotic intervals. Bifurcation curves for the transition $\Gamma_m^{(\pm)}\to\Gamma_{m+1}^{(\pm)}$ can be found in [28].

For the map $F_{l,p}$, each set

$$A=A_m^{(\pm)}=\{x=y\in\Gamma_m^{(\pm)}\}$$

is a one-dimensional chaotic invariant set that may or may not be an attractor in the plane (x,y). We distinguish three types of attraction. The various notions of attractors involve two properties: (i) that it attracts nearby trajectories and (ii) that it cannot be decomposed into smaller attractors. We shall concentrate on the first property since A has everywhere dense trajectories. Thus the definitions given here should be completed by some minimality condition in order to be generally valid.

Definition 1. The set A is an *asymptotically stable attractor* if for any sufficiently small its neighborhood $U(A)$ there exists its neighborhood $V(A)$ such that if $(x,y)\in V(A)$ then $F^n(x,y)\in U(A)$ for any $n\in\mathbb{Z}^+$ and distance $\rho(F^n(x,y);A)\to 0$, $n\to\infty$.

Definition 2. The set A is a *weak Milnor attractor* if its basin of attraction $\beta(A)$ has a positive Lebeague measure in $\mathbb{R}^2$.

Note that in [28], an asymptotically stable attractor is referred to as an attractor with the property of strong stability and a weak Milnor attractor is referred to as an attractor with the property of weak stability. Among weak Milnor attractors (which are not asymptotically stable) we can distinguish two classes depending on whether or not the basin $\beta(A)$ has a full measure in a neighborhood $U(A)$. In the first case, i.e., when measure $[\beta(A)\cap U]$=measure U, the attractor A can be referred to as a *Milnor attractor*.

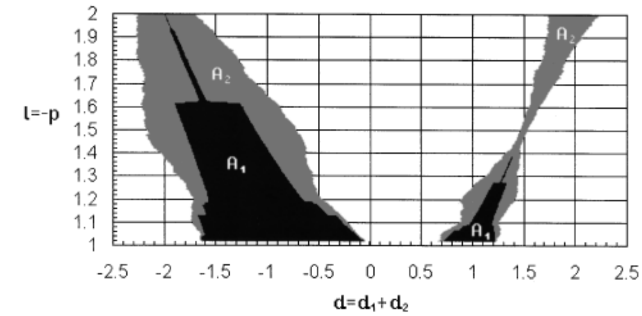

FIG. 1. Regions of asymptotic $\mathcal{A}_1$ and weak Milnor $\mathcal{A}_2$ stability for map (2); $p=-l$. Regions $\mathcal{A}_1$ and $\mathcal{A}_2$ are shown in black and gray, respectively.

A. Asymptotic stability of the attractor

In our previous study [28] of map (2) some preliminary analytical conditions for the $d\overset{\text{def}}{=}d_1+d_2$ parameter regions in which $A_m=A_m^{(\pm)}$, $m\in\mathbb{Z}^+$, is (i) an asymptotically stable attractor and (ii) a weak Milnor attractor were obtained. In the following we generalize these conditions, being both necessary and sufficient.

Theorem. The attractor A_m is asymptotically stable if and only if $(l,p)\in\Pi_m^{(k)}\subset\Pi$ $(m=0,1,\dots;\ k=2,3,\dots)$ and the conditions

$$|l-p|^{\alpha_m}|p-d|^{\alpha_{m+1}}<1,$$

$$|l-d|^{k\alpha_m+(-1)^m(k-1)}|p-d|^{k\alpha_{m+1}+(-1)^{m+1}(k-1)}<0 \quad (3)$$

are fulfilled, where a_m is a sequence of integer numbers defined as $\alpha_0=0$, $\alpha_1=1$, and

$$\alpha_m=\alpha_{m-1}+2\alpha_{m-2}\quad(m=2,3,\dots)$$

and subregions $\Pi_m^{(k)}$ are given as

$$\Pi_m^{(2)}=\left\{(l,p)\in\Pi:\ -\frac{l_m+1}{l_m}<p_m<-\frac{1+\sqrt{1+4l_m^2}}{2l_m}\right\},$$

$$\Pi_m^{(k)}=\left\{(l,p)\in\Pi:\ -\frac{l_m^k-1}{(l_m-1)l_m^{k-1}}<p_m<-\frac{l_m^{k-1}-1}{(l_m-1)l_m^{k-2}}\right\},$$

$$k=3,4,\dots,\quad(4)$$

where

$$l_m=l^{\alpha_m+(-1)^m}p^{\alpha_{m+1}+(-1)^{m+1}},$$

$$p_m=l^{\alpha_m}p^{\alpha_{m+1}}.\quad(5)$$

In Fig. 1 regions $\mathcal{A}_1$ of asymptotic stability of attractor A are indicated in black; the boundaries of $\mathcal{A}_1$ are obtained from relations (3), in which inequality signs were replaced by equality. Numerical calculations have been done for $p=-l$, i.e., for the tent map $f_{l,-l}$, when formulas (4) and (5) are transformed to

$$\Pi_m^{(2)}=\left\{1<l=-p\leq 2:\ \sqrt{2}<l_m<\frac{l_m+1}{l_m}\right\},$$

$$\Pi_m^{(k)}=\left\{1<l=-p\leq 2:\ \frac{l_m^{k-1}-1}{(l_m-1)l_m^{k-2}}<l_m<\frac{l_m^{k}-1}{(l_m-1)l_m^{k-1}}\right\}$$

$$(k=3,4,\ldots),\quad (4')$$

where

$$l_m=-p_m=l^{2^m}. \qquad (5')$$

B. Weak Milnor stability of the attractor

Conditions for the chaotic invariant set $A_m=A_m^{(\pm)}$ to be a weak Milnor attractor were obtained using an invariant measure $\mu=\mu_{l,p}$ of the map $f_{l,p}$ and can be given in the form

$$\lambda_\perp=[\alpha_m+(-1)^m\mu]\ln|l-d| + [\alpha_{m+1}+(-1)^{m+1}\mu)]\ln|p-d|<0, \qquad (6)$$

where

$$\mu=\mu_{l_m,p_m}\left\{|x|<\frac{1}{l}\right\}.$$

Regions $\mathcal{A}_2$ of the weak Milnor stability of the attractor $A=A_m^{(\pm)}$ are shown in Fig. 1 in gray; boundaries of $\mathcal{A}_2$ are obtained from condition (6) by replacing the inequality by an equality.

Unfortunately, generally we do not have analytical expression for the density ρ of the measure μ as it can be explicitly found only in exceptional cases. For example, in the moment of the first homoclinic bifurcation of the fixed point of $f_{l,p}$ [i.e., when $l=p/(1-p^2)$] the density function $\rho=\rho(x)$ is piecewise constant with two break points $x=\pm 1/l$ such that $\mu\{|x|<1/l\}=1/p^2$.

Conditions (6) shows the negativeness of the transversal Lyapunov exponent $\lambda_\perp$ of the typical trajectory in the attractor A. In numerical calculations shown in Fig. 1, we used the expression for $\lambda_\perp$ from Birkhoff's ergodic theorem

$$\lambda_\perp=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{\infty}\ln|f'(x_n)-d|,$$

where $\{x_n=f^n(x_0)\}$ is a typical trajectory of the map $f_{l,p}$ in the attractor A.

C. Locally and globally riddled basins

Suppose that the transversal Lyapunov exponent $\lambda_\perp$ of a typical trajectory on the attractor $A=A_m^{(\pm)}$ is negative, but still there exist trajectories on A, which are transversally repelling, i.e., we are in the gray region $\mathcal{A}_2$ in Fig. 1. Then one can simply check that the attractor A is not stable in a Lyapunov sense as there exists a neighborhood U such that any neighborhood V contains a positive Lebesgue measure set of points that leave U under the iterations of $F_{l,p}$. In this case the basin of attraction of A is locally riddled [2].

Definition 3. A set A is an attractor with a locally riddled basin of attraction if there is a neighborhood U of A such that in any neighborhood V of any points in A there is a set of points in $V\cap U$ of positive Lebesgue measure that leaves U in a finite time.

This riddling property has apparently a local character. It takes place in a sufficiently small neighborhood $U=U(A)$ and gives no information about the further behavior of the trajectories after their leave from U. In the model under consideration, different situations related to this global dynamics property take place. Two of the most spread of them are the following (note that the map $F_{l,p}$ is noninvertible).

(i) Locally riddled basin. After leaving neighborhood $U(A)$, almost all (in a measure sense) points return to U. Then some of them, after a finite number of iterations, leave U again and so on. The dynamics of such trajectories presents nonregular temporal ''bursting'': The trajectory spends some time (usually long enough) near attractor A until it goes away; then, after some other time (usually short enough) it returns to the neighborhood. This behavior describes a sort of *on-off intermitency* wide spread in practice. If $\lambda_\perp$ is negative but not too close to zero, the bursts will usually stop: Finally, the trajectory will be attracted by the attractor A. If $\lambda_\perp$ is negative and sufficiently close to zero the bursts can never stop.

(ii) Globally riddled basin. After leaving $U(A)$, a positive measure set of points goes to another attractor(s) or to infinity. This another attractor may be, for instance, an attracting point cycle or attracting cycle of chaotic two-dimensional (2D) sets.

Only in the globally riddled case can the basin of attraction $\beta(A)$ have a riddled structure of a fat fractal as a subset of $\mathbb{R}^2$, which means the following: The neighborhood of any point $(\bar{x},\bar{y})\in\beta(A)$ is filled by a positive Lebesgue measure set of points (x,y) that are attracted by another attractor (or other attractors).

In Figs. 2(a) and 2(b) we show the examples of locally riddled basins for $l=1.3$, $p=-2$, and $d_1=d_2=0.6$ [Fig. 2(a)] and globally riddled basins for $l=1.3$, $p=-2$, and $d_1=d_2=0.725$ [Fig. 2(b)]. The attractor $A^{(+)}$ of Fig. 2(a) attracts almost all points from its neighborhood, but it is not Lyapunov stable. In Fig. 2(b) the attractor $A^{(+)}$ is not asymptotically stable as in any its neighborhood there exists a positive measure set of points that goes to another attractor $A^{(-)}$ or to infinity. These properties are clearly visible at the enlargements shown in Figs. 2(c)–2(e). In Fig. 2(c) we notice that all points from the neighborhood $(0.5,1)\times(0.5,1)$ go to the attractor A^+. However, some of these points temporally leave this neighborhood. In Fig. 2(d) we presented the points in the neighborhood of $A^{(+)}$ that under iterations of map (2) leave the neighborhood $U=(0.4,1.1)\times(0.4,1.1)$ (white area) and points (gray area) that do not leave this neighborhood. Finally, almost all points from both areas converge to $A^{(+)}$, but those from the white region have to follow a longer path. Note that the locally riddled basin contains also a zero-measure set of points (including unstable periodic ones) that are not attracted by $A^{(+)}$ (see [29–31]). But when a computer simulation is processed, points not attracted by $A^{(+)}$ cannot usually be seen on the screen and one can arrive at the wrong conclusion that a locally (but not globally) riddled

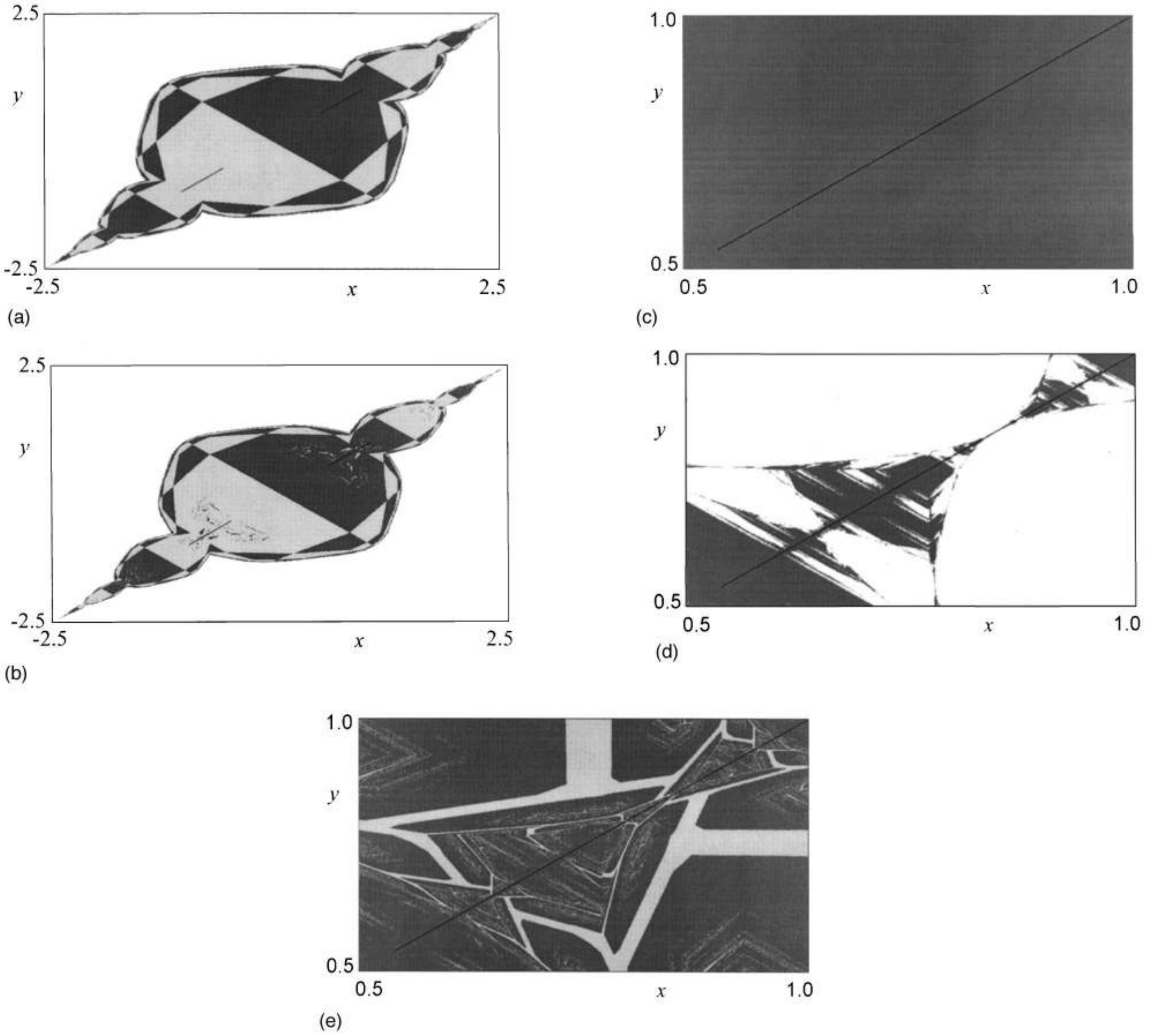

FIG. 2. Attractors A^+ and A^- of map (2); $l=1.3$ and $p=-2$. (a) $d_1=d_2=0.6$, (b) $d_1=d_2=0.725$, and (c) and (d) enlargements of (a) and (b), respectively. Points that under iterations of map (2), temporally leave the neighborhood $(0.4,1.1)\times(0.4,1.1)$ are indicated in white in (c). Basins of attraction of attractors A^+ and A^- are shown, respectively, in darker and lighter gray and the basin of attraction at infinity is indicated in white.

basin can represent a set open in $\mathbb{R}^2$ that includes a neighborhood of the attractor $A^{(+)}$. Finally, in Fig. 2(e) we can observe a typical case of global riddling. The basin of attractor $A^{(+)}$ (darker gray) is riddled by the basin of attractor A^- (lighter gray).

III. TRANSITION FROM LOCALLY TO GLOBALLY RIDDLED BASINS

For map (2) we observed two types of bifurcation leading to the transition from locally to globally riddled basins (l-g bifurcation). The examples of these bifurcations are shown in Figs. 3(a)–3(f). The first type of l-g bifurcation is shown in Figs. 3(a)–3(c) for $l=1.95$ and $p=-1.95$. Before the bifurcation for $d_1=d_2=-1$ [Fig. 3(a)] the basins of both attractors $A^{(+)}$ and $A^{(-)}$ contain a neighborhood of attractors except of a zero measure set. After the bifurcation for $d_1=d_2=-0.9$ basins of attractors $A^{(+)}$ and $A^{(-)}$ are riddled by the basin of the attractor at infinity as shown in Figs. 3(b) and 3(c) [particularly in the closeup in Fig. 3(c)]. The similar type of l-g bifurcation occurs in the case shown in Figs. 2(a) and 2(b), where basins of both attractors $A^{(+)}$ and $A^{(-)}$ are riddled by each other.

A different type of l-g bifurcation is shown in Figs. 3(d)–3(f) for $l=-p=\sqrt{2}$. As in the first type before bifurcation [Fig. 3(d), $d_1=d_2=-0.94$], the basins of both attractors $A^{(+)}$ and $A^{(-)}$ are locally but not globally riddled (only attractor $A^{(+)}$ is shown). After the bifurcation, new attractors A_1 and A_2 form [Fig. 3(e) and the closeup in Fig. 3(f), $d_1=d_2=0.935$] in the neighborhood of $A^{(+)}$ and the basin of $A^{(+)}$ becomes globally riddled by the basins of these new attractors.

These types of l-g bifurcations were observed to be typi-

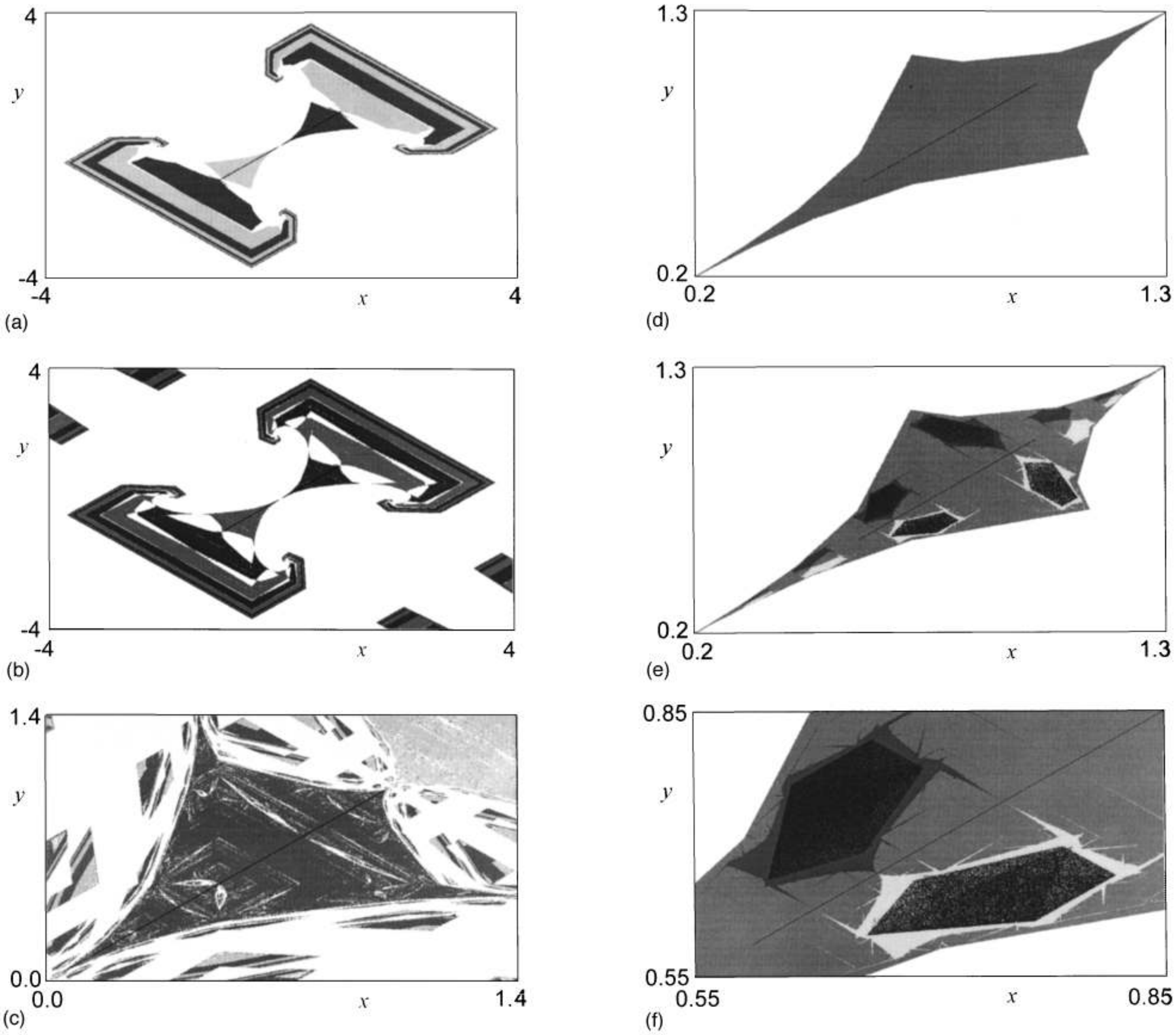

FIG. 3. l-g bifurcations of map (2): (a)–(c) bifurcations of the first type and (d)–(f) bifurcations of the second type. (a)–(c) $l=1.95$ and $p=-1.95$: (a) $d_1=d_2=-1$, (b) $d_1=d_2=-0.9$, and (c) enlargement of (b). (d)–(f) $l=\sqrt{2}$ and $p=-\sqrt{2}$: (d) $d_1=d_2=-0.94$, (e) $d_1=d_2=-0.935$, and (f) enlargement of (e). Basins of attraction of attractors A^+ and A^- are shown, respectively, in darker and lighter gray and the basin of attraction at infinity is indicated in white.

cal for map (2). We can summarize their properties in the following definitions.

Definition 4. The l-g bifurcation is of the first (or inner) type if in the local and global riddling the basins of the same attractors are involved.

Definition 5. The l-g bifurcation is of the second (or outer) type if in the local and global riddling the basins of different attractors are involved.

IV. BLOWOUT BIFURCATION

The bifurcation of losing the weak Milnor stability has been called a blowout bifurcation [15,17]. Recently, Ashwin, Buescu, and Stewart [32] suggested to use the notion of criticality for the classification of blowout bifurcations, analogous to that for the bifurcation of fixed points in invariant subspaces. In [32] the blowout bifurcation at $\nu=\nu_0$ was called subcritical if there exists an unstable invariant set B_ν, namely, the boundary dividing the basins of attractor A and the attractor at infinity for $\nu<\nu_0$, which is destroyed on passing through $\nu=\nu_0$. If for $\nu>\nu_0$ there exists a family of attractors A_ν that correspond to the on-off intermittent attractors, then the blowout bifurcation is called supercritical; see [32] for examples and further discussion.

From our model, roughly speaking, we can conclude that the blowout bifurcation is subcritical if in the bifurcation moment the basin of attractor A is globally riddled by the basin of the attractor at infinity. The blowout bifurcation is supercritical if in the bifurcation moment the basin of A is locally riddled or it is globally riddled by the basins of some attractors among which there are no attractors at infinity.

Note that in the subcritical case the blowout bifurcation is really like an explosion giving rise to the immediate disappearance of attractor A: It cannot be seen in computer simulations, so its basin becomes a zero-measure set. Cardinally different is the scenario of the bifurcation in the supercritical

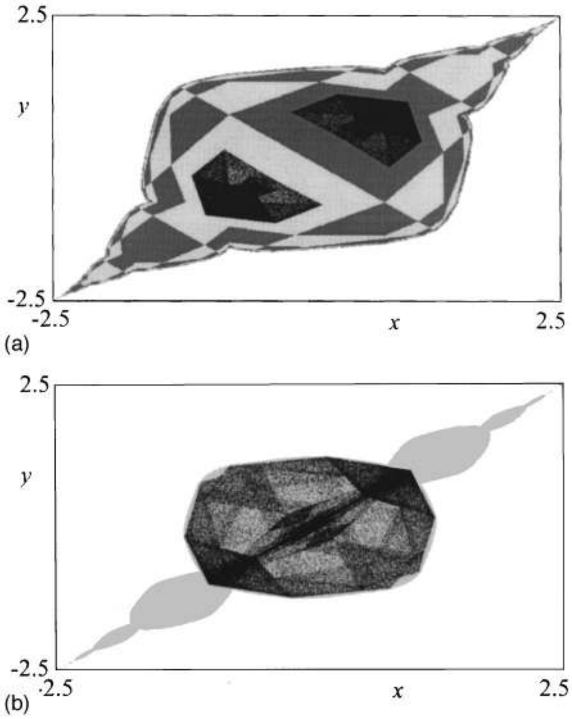

FIG. 4. Attractors after blowout bifurcations: $l=1.3$ and $p=-2$. (a) $d_1=d_2=0.5$, before bifurcation attractors A^+ and A^- are locally riddled as shown in Fig. 2(a), and (b) $d_1=d_2=0.8$, before bifurcation attractors A^+ and A^- are globally riddled as shown in Fig. 2(b).

case. In this case the bifurcation consists in the transition from the 1D to 2D attractor(s) and it does not resemble an explosion, as the density of the new 2D attractor(s) changes ''slowly'' when the parameter is in a neighborhood of a transition point.

The examples of blowout bifurcations for map (2) at $l=1.3$ and $p=-2$ are shown in Figs. 4(a) and 4(b). Our numerical calculations allow identification of the two types of blowout bifurcation. Figure 4(a) shows the blowout bifurcation in the case where before the bifurcation attractors $A^{(+)}$ and $A^{(-)}$ were locally riddled [$d_1=d_2=0.6$, Fig. 2(a)]. After the blowout bifurcation the synchronized state $x=y$ is no longer stable and we observe two hyperchaotic two-dimensional attractors $A^{(+)\prime}$ and $A^{(-)\prime}$ [$d_1=d_2=0.5$, Fig. 4(a)]. An unstable invariant set, the boundary between basins of attraction of the attractors $A^{(+)}$ and $A^{(-)}$ is not destroyed. Shortly after the bifurcation, we observed the intermittence between the unstable synchronized attractors $A^{(+)}$ and $A^{(-)}$ and hyperchaotic attractors $A^{(+)\prime}$ and $A^{(-)\prime}$, respectively, in the form of a ''burst.''

In Fig. 4(b) we observe the blowout bifurcation of attractors $A^{(+)}$ and $A^{(-)}$, the basins of which are mutually globally riddled by each other (i.e., they are intermingled according to the definition in [18]) [$d_1=d_2=0.725$, Fig. 2(b)]. After the bifurcation [$d_1=d_2=0.8$, Fig. 4(b)] we observed a unique two-dimensional hyperchaotic attractor A. In this case the boundary between basins of attraction of the attractors $A^{(+)}$ and $A^{(-)}$ has been destroyed before bifurcation (transition from the locally to the globally riddled basin), but the boundary between the sum of basins $\beta(A^{(+)})\cap\beta(A^{(-)})$ and the basin of the attractor at infinity is not destroyed yet.

V. CONCLUSION

Depending on the coupling parameters, the synchronized state of map (2) is characterized by different types of stability. In the case of weak synchronization, the attractor of the synchronized state is not asymptotically stable and two different states of riddling are possible. Conditions for weak Milnor and asymptotic stability of the synchronized chaotic attractor of map (2) are given analytically in a rigorous form.

We showed that the l-g bifurcation, i.e., the transition from locally to globally riddled basins, can occur in two recently identified different way. In the first (inner) type of bifurcation in local and global riddling the same attractors are involved. When at the transition of the bifurcation point a new attractor(s) is (are) formed and the basins of this (these) new attractor(s) riddle the basins of the initial attractor(s) we have the second type of l-g bifurcation.

The blowout bifurcation, i.e., the transition from weak stability to weak instability, can be sub- or supercritical. If before bifurcation attractors are globally riddled by the attractor at infinity the bifurcation is subcritical; otherwise it is supercritical. The observed properties of l-g and blowout bifurcations seem to be typical for a class of system with a lower-dimensional invariant manifold and important for studies of chaos synchronization problems.

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