LOCALLY COMPACT CONVERGENCE SPACES

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INTRODUCTION

Since local compactness plays a central role in topology and analysis, it is natural to investigate this concept in the more general realm of convergence spaces. Recent results (see for example Theorem 3.2 and 3.6 of [5]) indicate that local compactness will be of increasing importance in the study of convergence spaces.

In this paper, we show that the topological spaces known as k-spaces (see [1], [2], and [9]) are precisely the topological modifications of the locally compact convergence spaces. We establish, as a special case of a more general theorem on products of convergence spaces, that the product of a k-space with a locally compact is a k-space. In response to a question by A. Arhangel'skiĭ and S. P. Franklin [1], we construct a locally compact convergence space X_0 whose topological modification is a k-space of arbitrarily large ordinal index. We use the same example to extend the results of [8] by showing, among other things, that a function space $C_c(X)$ can have an arbitrarily long decomposition series when X is a locally convex topological linear space.

1. PRELIMINARIES

The reader is asked to refer to [6] for definitions, notation, and terminology pertaining to convergence spaces. As in [6], the term *space* will always mean "convergence space," and the term "ultrafilter" will be abbreviated to "u.f.". Unlike [6], the present paper will not make the assumption that all spaces are Hausdorff spaces.

Some additional definitions and terminology are needed for our present investigation. Let X be a space. If $A \subseteq X$ and σ is an ordinal number, then we denote by $\mathrm{cl}_X^\sigma A$ the σth iteration of the closure of A; this is defined to be $\mathrm{cl}_X \mathrm{cl}_X^{\sigma-1} A$ if $\sigma - 1$ exists, and $\bigcup \{\mathrm{cl}_X^\rho A: \rho < \sigma\}$ if σ is a limit ordinal. The smallest ordinal α such that $\mathrm{cl}_X^{\alpha+1} A = \mathrm{cl}_X^\alpha A$ for all $A \subseteq X$ is called the *length of the decomposition series of* X, and we denote it by $\ell_D(X)$.

A *pseudo-topological* space is a space with the property that $\mathfrak{F} \to x$ whenever each u.f. finer than $\mathfrak{F}$ converges to x. For any space X, let ρX be the space defined on the same underlying set as follows: $\mathfrak{F} \to x$ in ρX if and only if $\mathfrak{G} \to x$ in X for each u.f. $\mathfrak{G} \geq \mathfrak{F}$. The space ρX is the finest pseudo-topological space coarser than X, and it is called the *pseudo-topological modification* of X. Note that X and ρX have the same u.f. convergence.

A space is said to be *pretopological* if the X-neighborhood filter $\mathcal{V}_X(x)$ at x (obtained by intersecting all filters that converge to x) converges to x for all $x \in X$. Any set in $\mathcal{V}_X(x)$ is called an *X-neighborhood* of x. The *pretopological*

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modification of X, denoted by πX, is characterized by the fact that X and πX have the same neighborhood filters at each point. Note that $\pi X \leq \rho X \leq X$; all three spaces have the same closure operator.

As in [6], we denote by λX the *topological modification* of X; λX is characterized by the fact that X and λX have exactly the same closed sets.

We conclude this section with two lemmas that will be useful later; Lemma 1.2 is established in [8].

LEMMA 1.1. *Let* $\mathfrak{H}$ *be a filter on the set* X, *and let* $\{\mathfrak{F}_\alpha\}$ *denote the set of all* u.f.*'s finer than* $\mathfrak{H}$. *For each* α, *choose* $F_\alpha \in \mathfrak{F}_\alpha$. *Then there is a finite subset* $\{\alpha_1, \cdots, \alpha_n\}$ *such that* $\bigcup\{F_{\alpha_i}: i = 1, \cdots, n\} \in \mathfrak{H}$.

Proof. If the assertion were false, then the collection of all sets of the form $H - \left(\bigcup\{F_{\alpha_i}: i = 1, \cdots, n\}\right)$, for $H \in \mathfrak{H}$, would constitute a filter base $\mathscr{B}$ with the property that no u.f. containing $\mathscr{B}$ could be finer than $\mathfrak{H}$, a contradiction. ∎

LEMMA 1.2. *Let* X *be a pretopological space,* $x \in X$. *Then* $\mathscr{V}_{\lambda X}(x)$ *has a filter base of sets of the form* $\bigcup\{V_n: n \in N\}$, *where* $V_0 \in \mathscr{V}_X(x)$, *and* V_n *is defined recursively as a union of* X*-neighborhoods of points* z *in* V_{n-1}.

2. LOCALLY COMPACT SPACES

A space X is said to be *locally compact* if each convergent filter contains a compact set. In Proposition 2.1 of [6], it is shown that a regular, locally compact Hausdorff space is T-regular; in such a space, each convergent filter has a filter base of closed, compact sets. Some results pertaining to the category of locally compact Hausdorff convergence spaces may be found in Section 3 of [7].

PROPOSITION 2.1. *A space* X *is locally compact if and only if each convergent* u.f. *contains a compact set. Consequently,* X *is locally compact if and only if* ρX *is locally compact.*

Proof. Suppose that each convergent u.f. contains a compact subset, and let $\mathfrak{H}$ be any filter converging to x in X. Let $\{\mathfrak{F}_\alpha\}$ be the set of all u.f.'s finer than $\mathfrak{H}$. From each u.f. $\mathfrak{F}_\alpha$, choose a compact subset F_α. By Lemma 1.1, $\mathfrak{H}$ contains a compact subset. Thus X is locally compact. The second assertion follows immediately from the first, since X and ρX have the same u.f. convergence. ∎

For each space X, define $X^\wedge$ to be the set X with the following convergence structure: $\mathfrak{F} \to x$ in $X^\wedge$ if and only if $\mathfrak{F} \to x$ in X and $\mathfrak{F}$ contains an X-compact subset. The space $X^\wedge$ is called the *locally compact modification of* X; it is the coarsest locally compact space finer than X. The next result is an immediate consequence of the preceding proposition.

COROLLARY 2.2. *If* X *is a pseudo-topological space, then* $X^\wedge$ *is pseudo-topological.*

PROPOSITION 2.3. *If* X *is either a topological Hausdorff space or a regular space, then* $X^\wedge$ *is regular.*

Proof. Let X be a topological Hausdorff space. If $\mathfrak{F} \to x$ in X and $\mathfrak{F}$ contains a compact set A, then $\mathrm{cl}_X \mathfrak{F}$, when restricted to A, converges to x, since A is a compact T_2-topological space. Consequently, $\mathrm{cl}_X \mathfrak{F} \to x$ in X, $A \in \mathrm{cl}_X \mathfrak{F}$, and $\mathrm{cl}_X \mathfrak{F}$ thus $X^\wedge$-converges to x. Since $\mathrm{cl}_{X^\wedge} \mathfrak{F} \geq \mathrm{cl}_X \mathfrak{F}$, it follows that $X^\wedge$ is regular. A similar argument establishes the result when X is a regular space. ∎

We define two new terms that will be useful in our study of k-spaces. A space X will be called a *λ-Hausdorff space* if λX is a Hausdorff space. (This concept is weaker than the notion "ω-Hausdorff space" used in [5].) A locally compact space X is said to be of *Type* T if X is a λ-Hausdorff space and $X = (\lambda X)^\wedge$.

PROPOSITION 2.4. *A locally compact λ-Hausdorff space* X *is of Type* T *if and only if* X *is regular and pseudo-topological and has exactly the same compact subsets as does* λX.

Proof. Let X be a locally compact space of Type T. It follows from Corollary 2.2 and Proposition 2.3 that X is regular and pseudo-topological. Furthermore, it is a simple matter to verify that Y and $Y^\wedge$ always have the same compact subsets.

Conversely, assume the conditions are satisfied. Since X is locally compact, $X \geq (\lambda X)^\wedge$. Let $\mathfrak{F} \to x$ in $(\lambda X)^\wedge$, and let $\mathfrak{G}$ be any u.f. finer than $\mathfrak{F}$. By definition, $\mathfrak{F}$ contains a λX-compact set A, and $\mathfrak{F} \to x$ in λX. Since $A \in \mathfrak{G}$ and X is a λ-Hausdorff space, $\mathfrak{G} \to x$ in X. But X is pseudo-topological, and therefore $\mathfrak{F} \to x$ in X. ∎

The decomposition series of a compact, regular Hausdorff space has length at most 1 (see [4, Corollary 2.4]). On the other hand, Example 2.10 of [4] shows that it is possible to construct a locally compact, regular Hausdorff space with an arbitrarily long decomposition series; the space used in this example is not, however, of Type T (indeed, it is not a λ-Hausdorff space). In Section 5 of this paper, we construct a locally compact space X_0 of Type T (which has some other nice properties to be mentioned later) with an arbitrarily long decomposition series. The existence of such an example has ramifications in the theory of k-spaces and also in the study of the function space $C_c(X)$ (see Section 6).

3. k-SPACES

A k-*space* is a topological space X with the property that a subset A is closed whenever $A \cap K$ is closed in the subspace K, for each compact subset K of X.

THEOREM 3.1. *The following statements about a topological space* X *are equivalent.*

(1) X *is a* k-*space.*

(2) $X = \lambda(X^\wedge)$.

(3) $X = \lambda Y$ *for some locally compact space* Y.

Proof. (1) ⇒ (2). It is sufficient to show that each $X^\wedge$-closed subset is X-closed. Let A be an $X^\wedge$-closed set, K a compact subset of X. If $\mathfrak{F}$ is an u.f. that contains K and X-converges to $x \in K$, then $\mathfrak{F}$ also $X^\wedge$-converges to x, and, since A is $X^\wedge$-closed, $x \in K \cap A$. Thus, by hypothesis, A is closed in X.

(3) ⇒ (1). Let A be a subset of X with the property that $A \cap K$ is closed in K for each compact subset K of X. Let $\mathfrak{F}$ be an u.f. containing A and converging to x in Y. Since Y is locally compact, $\mathfrak{F}$ contains a compact subset B of Y. Let $C = B \cup \{x\}$; then C is also λY-compact, and therefore, by assumption, $A \cap C$ is closed in C. Thus $x \in A \cap C$. It follows that A is closed in Y, and consequently in $\lambda Y = X$. ∎

COROLLARY 3.2. *A topological Hausdorff space is a* k*-space if and only if it is the topological modification of a locally compact space of Type* T.

Corresponding to each pair of spaces X and Y, we shall assume that the product space $X \times Y$ has the structure of pointwise convergence; that is, $\mathfrak{F} \to (x, y)$ in $X \times Y$ if and only if $P_1 \mathfrak{F} \to x$ in X and $P_2 \mathfrak{F} \to y$ in Y, where P_1 and P_2 are the respective projection maps. We omit the routine proof of the next proposition.

PROPOSITION 3.3. *For any spaces* X *and* Y, $(X \times Y)^\wedge = X^\wedge \times Y^\wedge$.

If, in Proposition 3.3, one replaces "locally compact modifications" by "topological modifications," then the assertion is no longer true. A pair of spaces X and Y are said to be *topologically coherent* if $\lambda(X \times Y) = \lambda X \times \lambda Y$. This concept was introduced in Section 1 of [4], but it has not been studied extensively.

THEOREM 3.4. *Let* X *and* Y *be* k*-spaces. Then* $X \times Y$ *is a* k*-space if and only if* $X^\wedge$ *and* $Y^\wedge$ *are topologically coherent.*

Proof. If $X \times Y$ is a k-space, then, by Theorem 3.1,

$$\lambda(X \times Y)^\wedge = X \times Y = \lambda X^\wedge \times Y^\wedge .$$

But $\lambda(X \times Y)^\wedge = \lambda(X^\wedge \times Y^\wedge)$, by Proposition 3.3, and therefore $X^\wedge$ and $Y^\wedge$ are topologically coherent. Conversely, if $\lambda(X^\wedge \times Y^\wedge) = \lambda X^\wedge \times \lambda Y^\wedge$, then the same reasoning leads to the conclusion that $\lambda(X \times Y)^\wedge = X \times Y$, so that $X \times Y$ is a k-space, by Theorem 3.1. ∎

Let X be a Hausdorff k-space, and let $\kappa(X)$ denote the length of the decomposition series of $X^\wedge$; we shall refer to $\kappa(X)$ as the *ordinal index of* X. This concept was introduced in [1], and it is an easy matter to verify that the definition of $\kappa(X)$ given in [1] coincides with the one given here. The next theorem answers a question posed in [1]; we postpone the proof to Section 5.

THEOREM 3.5. *For each ordinal number* σ, *there is a Hausdorff* k*-space* X *such that* $\kappa(X) \geq \sigma$.

4. A PRODUCT THEOREM

It is not a simple matter to find conditions under which two spaces X and Y are topologically coherent; a weak sufficient condition is given in Theorem 1.8 of [4]. A more useful sufficient condition for topological coherence is obtained in this section.

Recall that, for each space X, we denote by πX, λX, and $\mathscr{V}(x)$ the pretopological modification, the topological modification, and the X-neighborhood filter at x.

Throughout this section, we assume that X and Y are spaces and that $Z = X \times Y$ is the product space.

PROPOSITION 4.1. *If* Y *is a locally compact, regular pretopological Hausdorff space, then* $\lambda Z \leq \lambda X \times Y$.

Proof. Let $(x, y) \in Z$, and let V be a λZ-open neighborhood of (x, y). Choose a compact set $D \in \mathscr{V}_Y(y)$ such that $\{x\} \times D \subseteq V$. Let $\mathfrak{G} \to x$ in X, and let $\{\mathfrak{H}_\alpha\}$ be the collection of all filters on Y that converge to a point in D. If $\mathfrak{H}_\beta \to z$ in D, then $(x, z) \in V$, and, since V is open, there are sets $G_\beta \in \mathfrak{G}$ and $H_\beta \in \mathfrak{H}_\beta$ such that $G_\beta \times H_\beta \subseteq V$. The collection $\mathfrak{K} = \{H_\alpha\}$ of members of $\{\mathfrak{H}_\alpha\}$ obtained in this way is a *covering system for* D (that is, each filter converging to a point in D contains a member of $\mathfrak{K}$). Since D is compact, it follows from Proposition 1.2 in [4], that a finite subcollection $\{H_{\alpha_1}, \cdots, H_{\alpha_n}\}$ of $\mathfrak{K}$ covers D. Let $\{G_{\alpha_1}, \cdots, G_{\alpha_n}\}$ be the collection of the corresponding sets G_α, and let $G_0 = G_{\alpha_1} \cap \cdots \cap G_{\alpha_n}$. Then $G_0 \times D \subseteq V$, where $G_0 \in \mathfrak{G}$.

Now let $\{\mathfrak{G}_\sigma \mid \sigma \in I\}$ be the collection of all filters on X such that $\mathfrak{G}_\sigma \to x$ in X. By the argument in the preceding paragraph, there is $G_\sigma \in \mathfrak{G}_\sigma$ such that $G_\sigma \times D \subseteq V$. Let $V_0 = \bigcup\{G_\sigma \mid \sigma \in I\}$; then $V_0 \in \mathscr{V}_X(x)$ and $V_0 \times D \subseteq V$. Continuing this process, we find that by Lemma 1.2, $\bigcup\{V_n \mid n > 0\} \times D \subseteq V$, where $\bigcup\{V_n \mid n \geq 0\} \in \mathscr{V}_{\lambda X}(x)$. Hence $V \in \mathscr{V}_{\lambda X}(x) \times \mathscr{V}_Y(y)$, and the proof is complete. ■

THEOREM 4.2. *If* X *is a space and* Y *is a locally compact topological Hausdorff space, then* X *and* Y *are topologically coherent.*

Proof. Since $\lambda Z \geq \lambda X \times \lambda Y$ is true in general, and since $Y = \lambda Y$ because Y is topological, the proof follows from Proposition 4.1. ■

COROLLARY 4.3 (D. E. Cohen [2]). *The product of a* k-*space with a locally compact topological Hausdorff space is a* k-*space.*

Proof. Apply Theorems 3.4 and 4.2. ■

5. AN EXAMPLE

Given an ordinal number σ, we shall construct a locally compact space X_0 of Type T such that $\ell_D(X) \geq \sigma$; λX_0 is a completely regular k-space of ordinal index at least σ. The existence of such a space provides a proof of Theorem 3.5.

For simplicity, assume that σ is an infinite limit ordinal. Let

$$A = \{\alpha: \alpha = 0 \text{ or } \alpha \text{ is a limit ordinal}, \ \alpha < \sigma\}.$$

If $0 \leq \rho < \sigma$, define $\gamma_\rho = \sup\{\alpha \in A: \alpha \leq \rho\}$, and define $A_\rho = \{\alpha \in A: \gamma_\rho \leq \alpha < \sigma\}$. Let N be the set of all finite ordinals. For each $\rho < \sigma$, let Q_ρ be the set of all functions from A_ρ into N. The base set for the space X_0 consists of distinct elements of the form $x_k^\rho(s)$, where $0 \leq \rho < \sigma$, $k \in N$, and $s \in Q_\rho$.

For each $\alpha \in A$, the *αth layer* B^α of X_0 is defined to be the set $\{x_k^\rho(s): \gamma_\rho = \alpha, \ k \in N, \ s \in Q_\rho\}$. The αth layer is partitioned into *boxes* of the form $B^\alpha(s) = \{x_k^\rho(s): \gamma_\rho = \alpha, \ k \in N\}$, where $s \in Q_\alpha$. Each box $B^\alpha(s)$ can be visualized as an infinite matrix whose mth row $(m \in N)$ is

$$S^{\alpha+m}(s) = \{x_k^{\alpha+m}(s): k \in N\}$$

and whose kth column $(k \in N)$ is given by $C_k^\alpha(s) = \{x_k^{\alpha+m}(s): m \in N\}$. Figure 1 is a diagram of X_0 for the case where $\sigma = 2\omega$.

If $\rho \in A$ and $\rho \neq 0$, we define

$$L_k^\rho(s) = \{x_k^\rho(s)\} \cup \{x_k^\mu(s'): \gamma_\mu < \rho, \ s'(\alpha) = k \text{ for all } \alpha \in A_\mu - A_\rho, \ s(\alpha) = s'(\alpha) \text{ for all } \alpha \in A_\rho\}.$$

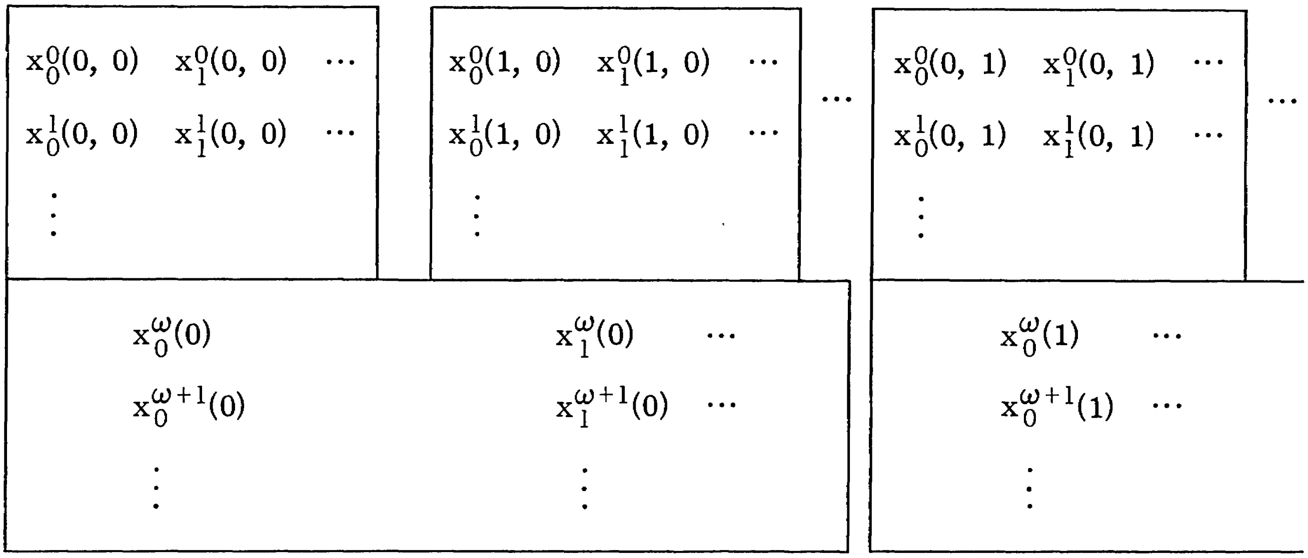

Figure 1

The set $L_k^\rho(s)$, called the *ordinal line of* $x_k^\rho(s)$, is obtained by choosing exactly one column from exactly one box in each layer B^α such that $\alpha < \rho$ and $\alpha \in A$. The construction is accomplished in such a way that if ρ and μ are limit ordinals with $\rho \leq \mu$, and if k and m are in N, and $s \in Q_\rho$ and $s' \in Q_\mu$, then either $L_k^\rho(s) \subseteq L_m^\mu(s')$ or $L_k^\rho(s) \cap L_m^\mu(s') = \emptyset$. We can well-order the elements of each ordinal line $L_k^\rho(s)$, using the natural order on the ordinal superscripts; note that $x_k^\rho(s)$ is the greatest element of $L_k^\rho(s)$ relative to this ordering, and that $L_k^\rho(s)$ is order isomorphic to the set of all ordinals not greater than ρ.

Given a box $B^\alpha(s) \subseteq X_0$ and an ordinal $m \in N$, let the mth row $S^{\alpha+m}(s)$ be partitioned into denumerably many denumerable sets. That is, let

$$S^{\alpha+m}(s) = \bigcup \{S_j^{\alpha+m}(s): j \in N\},$$

where each $S_j^{\alpha+m}(s)$ is denumerable and the sets $S_j^{\alpha+m}(s)$ are pairwise disjoint.

We now define a pretopological structure on X_0 by designating the neighborhood filter $\mathscr{V}_{X_0}(x_k^\rho(s))$ at each point $x_k^\rho(s)$. A point $x_k^\rho(s)$ is called a *point of Type* 1 if ρ is not a limit ordinal. If $x_k^\rho(s)$ is a point of Type 1 and $\rho = 0$, define $\mathscr{V}_{X_0}(x_k^0(s))$ to be the fixed u.f. $(x_k^0(s))^\cdot$. If $\rho > 0$ and $\rho - 1$ exists, then $\mathscr{V}_{X_0}(x_k^\rho(s))$ is defined to be $(x_k^\rho(s))^\cdot \cap \mathfrak{S}_j^{\rho-1}(s)$, where $\mathfrak{S}_j^{\rho-1}(s)$ is the filter generated by

$$\{S_k^{\rho-1}(s) - F: F \text{ is a finite subset of } X_0\}.$$

If ρ is a limit ordinal, then $x_k^\rho(s)$ is said to be a *point of Type* 2. If $x_k^\rho(s)$ is a point of Type 2, then $\mathscr{V}_{X_0}(x_k^\rho(s))$ is defined to be the filter generated by sets of the form $\{K_a: a \in L_k^\rho(s),\ a < x_k^\rho(s)\}$, where $K_a = \{y \in L_k^\rho(s): a \leq y \leq x_k^\rho(s)\}$.

If $x_k^\rho(s)$ is a point of Type 2, then a set of the form K_a will be called a *basic* X_0*-neighborhood of* $x_k^\rho(s)$. If $x_k^\rho(s)$ is a point of Type 1, then for $\rho = 0$, $\{x_k^0(s)\}$ is the only basic X_0-neighborhood of $x_k^0(s)$; for $\rho \neq 0$, sets of the form $S_k^{\rho-1}(s) - F$ (F finite) are designated as basic X_0-neighborhoods of $x_k^\rho(s)$. It is easy to see that all of the basic X_0-neighborhoods are compact; in the case of the points of Type 2, we can regard the sets K_a as closed intervals on an ordinal line with the order topology. It is also clear from our construction that X_0 is a Hausdorff space (note that sets of the form $S_k^\rho(s)$ and $L_\ell^\mu(s')$ can intersect in at most one point). Consequently, X_0 is a regular, locally compact pretopological Hausdorff space.

For each ordinal $\rho < \sigma$, let $S^\rho = \bigcup \{S^\rho(s) \colon s \in Q_\rho\}$. From the construction of X_0, we see that

$$\mathrm{cl}_{X_0}^{\alpha}\, S^0 = \begin{cases} \bigcup \{S^\rho \colon 0 \le \rho \le \alpha\} & \text{if } \alpha \text{ is finite,} \\ \bigcup \{S^\rho \colon 0 \le \rho < \alpha\} & \text{if } \alpha \text{ is infinite.} \end{cases}$$

In particular, $\mathrm{cl}_{X_0}^{\sigma}\, S^0 = X_0$. Thus $\ell_D(X_0) \ge \sigma$.

Next we show that λX_0 is a zero-dimensional topological space. It is well known that λX is a T_1-space whenever X is a T_1-space. Since X_0 is a Hausdorff space, it will follow that λX_0 is a Hausdorff space and is completely regular, and, consequently, that X_0 is a λ-Hausdorff space. Let $x_k^\rho(s) \in X_0$; by a *basic* λX_0-*neighborhood* of $x_k^\rho(s)$ we mean one constructed in accordance with Lemma 1.2, where basic X_0-neighborhoods are used at each stage of the iterative construction process. Let W be such a basic λX_0-neighborhood of $x_k^\rho(s)$; then W is open, by Lemma 1.2. We shall show that W is X_0-closed (hence, also λX_0-closed), and this will establish that λX_0 is zero-dimensional. Let $\mathfrak{F}$ be a free u.f. on X_0 that converges to $x_\ell^\mu(s')$ in X_0 and contains W. If $x_\ell^\mu(s')$ is a point of Type 1, then $\mathfrak{F}$ must contain $S_\ell^{\mu-1}(s')$. Therefore, $S_\ell^{\mu-1}(s') - F$ (F finite) must be a subset of W, and this can occur only if $x_\ell^\mu(s')$ is itself in W. If $x_\ell^\mu(s')$ is of Type 2, then a closed interval $K_a \subseteq L_\ell^\mu(s')$ must be a subset W, and, again, this can occur only if $x_\ell^\mu(s') \in W$. Thus W is both open and closed in λX_0.

We have so far established that λX_0 is a completely regular Hausdorff k-space. To show that $\kappa(\lambda X_0) \ge \sigma$, it is sufficient, by Proposition 2.4, to show that X_0 and λX_0 have the same compact sets. Let C be an infinite λX_0-compact set. We assert that C is a subset of a finite union of sets of the form $L_k^\alpha(s)$ and $S_k^\rho(s')$; since such a finite union is X_0-compact, it will follow that C is also X_0-compact. If C were not a subset of such a finite union, then there would exist an infinite subset C_1 of C with the property that no two members of C_1 are in the same ordinal line $L_k^\alpha(s)$, and no two members of C_1 are in the same set of the form $S_k^\beta(s')$. Since we can obtain the λX_0-neighborhoods of each point by taking unions of sets of the form $L_k^\alpha(s) - F_1$ and $S_k^\beta(s') - F_2$, where F_1 and F_2 are finite sets, it would be possible to construct a λX_0-neighborhood of any point in X_0 disjoint from C_1. This implies that no u.f. containing C_1 can λX_0-converge, contrary to the assumption that C is λX_0-compact. We have thus established all of the original assertions about X_0 and λX_0.

6. THE FUNCTION SPACE $C_c(X)$

Let X be a space, and let $C_c(X)$ be the set of all real-valued continuous functions on X with the continuous-convergence structure ($\Phi \to f$ in $C_c(X)$ if and only if $\Phi(\mathfrak{F}) \to f(x)$ in R whenever $\mathfrak{F} \to x$ in X). If X is a space with the property that the natural map from X into $C_c(C_c(X))$ is an embedding, then X is said to be *c-embedded*. A characterization of c-embedded spaces is obtained in [5].

It is shown in [8] that even if X is a topological space with certain nice properties, $C_c(X)$ can have an infinitely long decomposition series; the reverse situation is also shown to be possible. The proof of these and related results is based on an example given in Section 3 of [8], which turns out to be the special case of our example X_0 of Section 5 obtained by taking $\sigma = \omega$. The properties of the space X_0 of Section 5 are precisely those needed to extend the results of [8] pertaining to "infinitely long decomposition series" to statements about "arbitrarily long decomposition series." We summarize these observations in the following theorem.

THEOREM 6.1. *Let σ be an ordinal number.* (1) *There is a locally compact, c-embedded space* X *with* $\ell_D(X) \geq \sigma$ *such that* $C_c(X)$ *is a topological space.* (2) *There is a locally convex topological linear space* X *for which* $\ell_D(C_c(X)) \geq \sigma$. (3) *There is a locally convex linear (convergence) space* X *such that* $\ell_D(X) \geq \sigma$.

Acknowledgment. We recently learned that V. Kannan [3] obtained an earlier solution to the problem posed by Archangel'skiĭ and Franklin. Indeed, Kannan shows that for each ordinal number α, there is a k-space whose ordinal index is exactly α. We thank the referee for calling this result to our attention.

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