# LOCALLY CONFORMAL SKT STRUCTURES

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ABSTRACT. A Hermitian metric on a complex manifold is called SKT (strong Kähler with torsion) if the Bismut torsion 3-form H is closed. As the conformal generalization of the SKT condition, we introduce a new type of Hermitian structure, called *locally conformal* SKT (or shortly LCSKT). More precisely, a Hermitian structure (J, g) is said to be LCSKT if there exists a closed non-zero 1-form  $\alpha$  such that  $dH = \alpha \wedge H$ . In the paper we consider non-trivial LCSKT structures, i.e. we assume that  $dH \neq 0$  and we study their existence on Lie groups and their compact quotients by lattices.

In particular, we classify 6-dimensional nilpotent Lie algebras admitting a LCSKT structure and we show that, in contrast to the SKT case, there exists a 6-dimensional 3-step nilpotent Lie algebra admitting a non-trivial LCSKT structure. Moreover, we show a characterization of even dimensional almost abelian Lie algebras admitting a non-trivial LCSKT structure, which allows us to construct explicit examples of 6-dimensional unimodular almost abelian Lie algebras admitting a non-trivial LCSKT structure. The compatibility between the LCSKT and the balanced condition is also discussed, showing that a Hermitian structure on a 6-dimensional nilpotent or a 2n-dimensional almost abelian Lie algebra cannot be simultaneously LCSKT and balanced, unless it is Kähler.

### 1. INTRODUCTION

Among Hermitian metrics, an important class is given by the strong Kähler with torsion (SKT) (or pluriclosed) which may be characterized by the condition  $\partial \overline{\partial} \Omega = 0$ , where  $\Omega$  is the fundamental form. An equivalent geometrical meaning is given in terms of the Bismut (or Strominger) connection  $\nabla^B$  [46, 6, 41], which is the unique connection compatible with the Hermitian structure (J,g) (i.e.  $\nabla^B g = 0 = \nabla^B J$ ) such that its torsion H is a totally skew symmetric tensor. The torsion can therefore be identified with a 3-form which is given by  $H = Jd\Omega$  or  $H = d^c\Omega$  with  $d^c = i(\overline{\partial} - \partial)$ . The Hermitian structure is called SKT if H is d-closed i.e. dH = 0.

Initially, SKT metrics appeared in theoretical physics, applied widely in 2-dimensional sigma models [23, 27], having also emerged from superstring theories with torsion [41]. From a mathematical point of view, they are deeply related to generalized Kählerian geometry [23, 26, 16].

Concerning the existence problem for SKT metrics on Hermitian manifolds, there are no general conditions. All the results obtained in the literature indicate that the existence of an SKT metric depends on the type of manifold considered and its dimension.

For instance, in dimension four, any compact complex surface is always an SKT manifold, since each conformal class admits a unique SKT standard metric in the sense of Gauduchon

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[24]. It is proven in [32], that on a 4-dimensional unimodular solvable Lie group a leftinvariant Hermitian structure is SKT and a classification of 4-dimensional SKT Lie algebras is also given.

However, for higher dimensions the existence of a left-invariant SKT structure on unimodular (non-compact) Lie groups is not guaranteed anymore, unless certain selective conditions are required.

For example, by [19] only 4 isomorphism classes of 6-dimensional (non-abelian) nilpotent Lie algebras admit an SKT structure, see also [43] for a classification up to equivalence of complex structures. An interesting property is that the existence of an SKT structure (J, g)on nilpotent Lie algebras of dimension 6 depends only on the complex structure J.

The 8-dimensional SKT nilpotent Lie algebras are also classified in [13]. In a recent article [5], it was shown that every nilpotent Lie algebra admitting a SKT structure is at most 2-step nilpotent.

For the more general class of SKT solvable Lie algebras, the only complete classification is in dimension four [32]. Classifications of SKT solvable Lie algebras of six and higher dimensions are established only under some extra conditions. For example, in [15] the classification of six-dimensional SKT unimodular solvable Lie algebras admitting complex structures with non-zero closed holomorphic (3, 0)-form is given. Moreover, a general characterization of SKT structures on 2n-dimensional almost abelian Lie algebras is established in [4], and a classification for the 6-dimensional case is given in [16]. The case of almost nilpotent solvmanifolds whose nilradical has one-dimensional commutator was considered in [18]. Also, a wide range of two-step solvable Lie groups admitting a left-invariant SKT structure was classified in [21], the same authors have recently obtained the full classification of left-invariant SKT structures on two-step solvable Lie groups in dimension six, [22].

In this article we introduce a new type of Hermitian structure, called *locally conformal* SKT (shortly LCSKT), as a generalization of SKT structures, inspired by ideas of locally conformal Kähler (LCK) geometry [44, 45, 3] and we study the existence of this type of structure on Lie groups and their compact quotients by lattices.

Let H be the Bismut torsion 3-form of a Hermitian manifold (M, J, g). In Section 2, we define an LCSKT Hermitian manifold (M, J, g) as a Hermitian manifold, such that the 3-form H satisfies the condition  $dH = \alpha \wedge H$ , where  $\alpha$  is a non-zero d-closed 1-form. Notice that  $\alpha$  is not necessarily unique and, if  $\alpha = 0$ , then (J, g) is SKT. In the case where the 3-form H is non-degenerate, i.e. ker  $H \neq \{0\}$ , we prove that the definition of an LCSKT structure is equivalent to the d-closeness of each local 3-form  $(e^{-f_i}H)|_{U_i}$ , such that  $\{U_i\}$  is an open cover in M and  $\{f_i\}$  a family of smooth real functions  $f_i: U_i \to \mathbb{R}$  (Propositions 2.1 and 2.2).

In Section 3, we study the existence of LCSKT structures (J, g) on six-dimensional nilpotent Lie algebras. In particular we provide a complete classification of nilpotent Lie algebras admitting an LCSKT structure (Theorem 3.1 and Corollary 3.1), showing that up to isomorphism only one admits a non-trivial LCSKT structure. In contrast to the SKT case this nilpotent Lie algebra is 3-step. Moreover, we prove that a 6-dimensional nilpotent Lie algebra, admitting an LCSKT structure and a balanced metric, has to be abelian.

We devoted Section 4 to the study of the existence of LCSKT structures on almost abelian Lie algebras, i.e. on solvable Lie algebras admitting an abelian ideal of codimension one, determining a characterization of LCSKT almost abelian Lie algebras in arbitrary dimension  $(2n \ge 6)$ . In particular, we construct some explicit examples of unimodular (non-nilpotent) almost abelian Lie algebras admitting LCSKT structures, which allow to give examples of 6-dimensional LCSKT solvmanifolds. Moreover, we prove that on an almost abelian Lie algebra a Hermitian structure cannot be simultaneously LCSKT and balanced, unless it is Kähler.

### 2. Locally conformal SKT structures

Let (M, J, g) be a 2*n*-dimensional Hermitian manifold, such that J is a complex structure on M, which is orthogonal relative to the Riemannian metric g. The 2-fundamental form  $\Omega$ is given by  $\Omega(X, Y) = g(JX, Y)$ , for any vector fields X, Y on M. If  $d\Omega = 0$ , the metric gis said to be Kähler and the Levi-Civita connection  $\nabla^g$  is Hermitian, i.e. J and g are both parallel with respect to  $\nabla^g$ .

In [25], Gauduchon proved that on (M, J, g) there exists an affine line  $\{\nabla^t\}_{t \in \mathbb{R}}$  of canonical Hermitian connections, passing through the Chern connection  $\nabla^C$  and the Bismut connection  $\nabla^B$ . The connections  $\nabla^t$  preserve both g and J and they are completely determined by their torsion. In particular, the Chern connection  $\nabla^C$  is the unique Hermitian connection whose torsion has trivial (1, 1)-component and the Bismut connection  $\nabla^B$  is the unique Hermitian connection with totally skew-symmetric torsion H ([6]).

Since  $\nabla^B$  and  $\nabla^C$  preserve the Hermitian structure (J, g), they induce unitary connections on the anticanonical bundle  $K^{-1}$ , with curvatures respectively  $i\rho^C$  and  $i\rho^B$ , where

$$\rho^{C}(X,Y) := \frac{1}{2} \sum_{i=1}^{2n} g(R^{C}(X,Y)e_{i}, Je_{i})$$

is the Ricci form of  $\nabla^C$  and the Ricci form  $\rho^B$  of  $\nabla^B$  is defined in a similar way. Here  $\{e_i\}_{i=1}^{2n}$  is a local orthonormal frame of TM and for the curvature we use the following convention:

$$R^C(X,Y)Z = [\nabla^C_X, \nabla^C_Y]Z - \nabla^C_{[X,Y]}Z.$$

The two Ricci forms  $\rho^B$  and  $\rho^C$  are related by the relation (see[1, (2.7)])

$$\rho^C = \rho^B + (n-1)dJ\theta,\tag{1}$$

where  $\theta$  is the so-called *Lee form*  $\theta$ . The 1-form  $\theta$  is defined as the trace of the torsion of the Chern connection  $\nabla^C$ 

$$\theta = \frac{1}{(1-n)} J d^{\dagger} \Omega,$$

where  $d^{\dagger}$  is the formal adjoint of d with respect to g, or quivalently, it is the unique 1-form such that

$$d\Omega^{n-1} = (n-1)\theta \wedge \Omega^{n-1}.$$

Concretely, the Bismut torsion 3-form has the following expression

$$H(X, Y, Z) = d^{c}\Omega(X, Y, Z) = -d\Omega(JX, JY, JZ)$$

for every vector fields X, Y, Z on M, where  $d^c = i(\overline{\partial} - \partial)$  is the real Dolbeault operator associated to the complex structure J.

**Remark 2.1.** Note that H is a real 3-form of type (2,1) + (1,2) and we can write it as

$$H = H^{(2,1)} + H^{(1,2)},$$

where  $H^{(2,1)} = -i\partial\Omega$  and  $H^{(1,2)} = \overline{H^{(2,1)}}$ .

If the 3-form H is d-closed i.e.  $dH = dd^c \Omega = 0$ , the Hermitian metric g is called *strong* Kähler with torsion (SKT) (or pluriclosed). Other types of Hermitian metrics can be defined in terms of the Lee form  $\theta$ . A Hermitian manifold (M, J, g) is called *balanced* or *co-symplectic*, respectively *locally conformal balanced* (LCB), if and only if the Lee form is vanishing ( $\theta = 0$ ), respectively closed. Note that for n = 2 a balanced metric is automatically Kähler. Moreover, if  $d\Omega = \theta \wedge \Omega$  with closed Lee form  $\theta$ , the Hermitian manifold is said *locally conformal Kähler* (LCK) [44].

By imposing a similar relation for the Bismut torsion 3-form, we can introduce the following definition.

**Definition 2.1.** A Hermitian structure (J, g) on a 2n-dimensional complex manifold (M, J)with fundamental 2-form  $\Omega$  is called a locally conformal SKT (LCSKT for brevity) structure if and only if there exists a d-closed (non-zero) 1-form  $\alpha$  on M such that

 $dH = \alpha \wedge H,$ 

where  $H = d^{c}\Omega$ . A manifold (M, J, g) is said to be LCSKT, if it admits an LCSKT structure.

If dH = 0 then (J, g) is an SKT structure and we will call it a *trivial LCSKT structure*.

**Remark 2.2.** Note that a Hermitian structure (J,g) with g conformal to an SKT metric  $\tilde{g}$  is not automatically an LCSKT structure. Indeed, if  $\tilde{\Omega}$  is the fundamental form associated to  $(J,\tilde{g})$  and  $g = e^f \tilde{g}$ , then  $\Omega = e^f \tilde{\Omega}$  and

$$d^c\Omega = e^f(Jdf) \wedge \tilde{\Omega} + e^f d^c \tilde{\Omega}$$

and then

$$dd^{c}\Omega = df \wedge d^{c}\Omega + e^{f}d(Jdf) \wedge \tilde{\Omega} - e^{f}(Jdf) \wedge d\tilde{\Omega}$$

**Remark 2.3.** If (J,g) is a balanced Hermitian structure on a complex manifold (M, J) of complex dimension  $n \ge 2$ , then by [1, (2.13)] we have

$$< dd^c \Omega, \Omega^2 > = -2|d\Omega|^2$$

Therefore, g is SKT if and only if it is Kähler. If we impose the LCSKT condition i.e.  $dd^c\Omega = \alpha \wedge d^c\Omega$ , it is not clear in general if one still has  $d\Omega = 0$ , but we will see that for particular classes of examples the two conditions are complementary.

**Proposition 2.1.** Let (M, J, g) be an LCSKT manifold. Then M has an open cover  $\{U_i\}$  and a family  $\{f_i\}$  of smooth functions  $f_i : U_i \to \mathbb{R}$   $(f_i \in \mathbb{C}^{\infty}(U_i))$  such that each local 3-form  $(e^{-f_i}H)_{|U_i}$  is d-closed.

*Proof.* Since  $\alpha$  is closed then it is locally exact, i.e. for every  $p \in M$  there is a neighborhood  $U_i$  such that  $\alpha = df_i$  for some function  $f_i : U_i \to \mathbb{R}$ . Then, in  $U_i$ , we have

$$d(e^{-f_i}H)_{|U_i} = e^{-f_i}(dH - df_i \wedge H) = e^{-f_i}(dH - \alpha \wedge H) = 0.$$

The converse of the previous proposition is not always verified because of the eventual non-degeneracy of H. For this reason, we introduce the notion of kernel (ker  $\omega$ ) of a differential form  $\omega$  on M,

$$\ker \omega = \{ X \in \Gamma(TM) | \iota_X \omega = 0 \}$$

where  $\iota_X \omega$  is the interior product of the differential form  $\omega$  by the vector field X. If  $\ker \omega = 0$ , then  $\omega$  is said to be non-degenerate, ortherwise  $\omega$  is said to be degenerate.

**Proposition 2.2.** Let (M, J, g) be a Hermitian manifold with fundamental 2-form  $\Omega$  such that  $H = d^c \Omega$  is non-degenerate. Suppose that M has an open cover  $\{U_i\}$  and a family  $\{f_i\}$  of smooth functions  $f_i : U_i \to \mathbb{R}$  such that each local 3-form  $(e^{-f_i}H)_{|U_i|}$  is d-closed, then (M, J, g) is an LCSKT manifold.

Proof. From the condition  $d(e^{-f_i}H)|_{U_i} = 0$ , we have that, in  $U_i$ ,  $dH = df_i \wedge H$ . Then, for every point  $p \in M$  there is a neighborhood  $U_i$  and a closed one-form  $\alpha_i = df_i$  such that  $dH = \alpha_i \wedge H$ . Let  $U_i$  and  $U_j$  be two such neighborhoods. Then on  $U_i \cap U_j$ , we get  $(\alpha_i - \alpha_j) \wedge H = 0$ . Since ker H = 0, it follows that  $\alpha_i = \alpha_j$  and so  $\alpha$  is a globally defined closed one-form such that  $dH = \alpha \wedge H$ . Thus, M is LCSKT.

We have adopted Definition 2.1 as the base definition of LCSKT manifolds rather than the property stated in the Proposition 2.1, in order to cover the general case, including degenerate 3-forms H.

**Remark 2.4.** Note that, given an LCSKT manifold  $(M, J, g, \Omega)$ , the Chern-Ricci flow (see for instance [42])

$$\begin{cases} \partial_t \Omega(t) = -\rho^C(\Omega(t)), \\ \Omega(0) = \Omega, \end{cases}$$

preserves the LCSKT condition. Here  $\rho^C(\Omega(t))$  denotes the Chern Ricci form of  $\Omega(t)$ . Indeed, the solution  $\Omega(t)$  of the previous flow is of the form  $\Omega(t) = \Omega - t\rho^C + i\partial\overline{\partial}\varphi(t)$ , where  $\rho^C$  is the Chern Ricci form of  $\Omega$  and  $\varphi(t)$  solves the system (2.1) in [42], and so  $d\Omega(t) = d\Omega$ .

Since J does not evolve along the flow, we have that  $d^c\Omega(t) = d^c\Omega$ . Thus  $dd^c\Omega(t) = dd^c\Omega$ and the LCSKT condition is preserved.

In the paper we will study the existence of invariant LCSKT structures on compact manifolds given by the quotients  $\Gamma \setminus G$  of simply connected Lie groups G by co-compact discrete subgroups  $\Gamma$ .

By invariant LCSKT structure (J, g) on  $M = \Gamma \backslash G$  we mean a structure induced by a leftinvariant one on G or equivalently by an LCSKT structure on its Lie algebra  $\mathfrak{g}$ . Therefore we can study the existence of invariant LCSKT structures on M working at the level of the Lie algebra  $\mathfrak{g}$  of G. Furthermore, note that the form  $\alpha$  cannot be exact (see the proof of Proposition 4.6 in [36]).

We will now shortly introduce the definition of SKT and LCSKT structures on Lie algebras. Recall that an almost complex structure J on 2n-dimensional real Lie algebra  $\mathfrak{g}$  is defined as an endomorphism of  $\mathfrak{g}$  such that  $J^2 = -\mathrm{Id}_{\mathfrak{g}}$ . If J is integrable, i.e. if

$$N(X,Y) = [JX, JY] - J[JX,Y] - J[X, JY] - [X,Y] = 0,$$

for any  $X, Y \in \mathfrak{g}$ , then J is called a complex structure on  $\mathfrak{g}$  and the *i*-eigenspace  $\mathfrak{g}^{1,0}$  of Jin  $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}$  is a complex subalgebra of  $\mathfrak{g}^{\mathbb{C}}$ . If  $\mathfrak{g}^{1,0}$  is an abelian subalgebra of  $\mathfrak{g}^{\mathbb{C}}$ , or equivalently [JX, JY] = [X, Y], for all  $X, Y \in \mathfrak{g}$ , then the complex structure J is said to be *abelian*. When  $\mathfrak{g}^{1,0}$  is a complex ideal we say that J is bi-invariant, i.e. J[X, Y] = [JX, Y], for every  $X, Y \in \mathfrak{g}$ .

A Hermitian metric g on  $(\mathfrak{g}, J)$  is a (positive definite) inner product which is J-orthogonal, i.e. such that g(JX, JY) = g(X, Y), for every  $X, Y \in \mathfrak{g}$ . Let  $\Omega(\cdot, \cdot) := g(J \cdot, \cdot)$  be the associated fundamental form and  $H = d^c \Omega$  be the torsion Bismut form. Then the Hermitian structure (J, g) is SKT if dH = 0. Moreover, (J, g) is LCSKT if there is a closed 1-form  $\alpha$ on  $\mathfrak{g}$  such that

$$dH = \alpha \wedge H$$

**Remark 2.5.** By using that the compact quotient  $M = \Gamma \backslash G$  admits a bi-invariant volume form and the invariance of J one can show that if (M, J) admits a balanced (resp. SKT) metric, then M admits an invariant balanced (resp. SKT) metric defined by  $\Omega_{inv}$  (see [14, 43]). This follows from the fact that given any covariant k-tensor field T on  $\Gamma \backslash G = M$ one can define a covariant k-tensor  $T_{inv}$  on  $\mathfrak{g}$  as

$$T_{inv}(X_1,\ldots,X_k) = \int_{p \in M} T_p(X_1|_p,\ldots,X_k|_p)\nu, \quad \text{for } X_1,\cdots,X_k \in \mathfrak{g}$$

and where  $X_i|_p$  denotes the value at  $p \in M$  of the projection on M of the left-invariant vector field on G defined by  $X_i$ .

In the case of LCKT structures we can apply symmetrization under the assumption that the closed 1-form  $\alpha$  is invariant. Concretely, suppose that  $M = \Gamma \setminus G$  is equipped with an invariant complex structure J. Let g be a J-Hermitian metric (not necessarily invariant) with fundamental 2-form  $\Omega$ . If (J,g) is an LCSKT structure with invariant closed 1-form  $\alpha$  then  $(J, g_{inv})$  is an invariant LCSKT structure. Indeed, using similar arguments as in [14, 43], we have

$$d(d^{c}(\Omega_{inv})) = dJd(\Omega_{inv}) = (dJd(\Omega))_{inv} = (d(d^{c}(\Omega)))_{inv} = (\alpha \wedge d^{c}(\Omega))_{inv} = \alpha \wedge d^{c}(\Omega)_{inv}$$

and our claim follows.

## 3. Invariant LCSKT structures on 6-dimensional nilmanifolds

In this section we study the existence of invariant LCSKT structures on nilmanifolds, i.e. on compact quotients  $M = \Gamma/G$ , where G is a connected and simply connected nilpotent Lie group and  $\Gamma$  is a lattice in G, i.e. a discrete co-compact subgroup  $\Gamma \subset G$ .

We recall that a Lie algebra  $\mathfrak{g}$  is *nilpotent* if its lower central series  $\{\mathfrak{g}^i\}$  terminates, i.e. if  $\mathfrak{g}^k = \{0\}$ , for some  $k \in \mathbb{N}$ , where

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^j = [\mathfrak{g}^{j-1}, \mathfrak{g}], \ j \ge 1.$$

In his work [39], Salamon proves that the existence of a complex structure J on a 2*n*dimensional nilpotent Lie algebra  $\mathfrak{g}$  is equivalent to the existence of a basis of (1,0)-forms  $\{\omega^j\}_{j=1}^n$  satisfying the following complex structure equations

$$d\omega^1 = 0$$
, and  $d\omega^j \in \mathcal{I}(\omega^1, ..., \omega^{j-1})$  for  $j = 2, ..., n$ ,

where  $\mathcal{I}(\omega^1, ..., \omega^{j-1})$  is the ideal in  $\Lambda^* \mathfrak{g}^*_{\mathbb{C}}$  generated by  $\{\omega^1, ..., \omega^{j-1}\}$ .

A complex structure J on a nilpotent Lie algebra  $\mathfrak{g}$  is called *nilpotent* in the sense of [11], if there exists a basis  $\{\omega^j\}_{j=1}^n$  of (1,0)-forms satisfying

$$d\omega^1 = 0$$
 and  $d\omega^j \in \Lambda^2 \langle \omega^1, ..., \omega^{j-1}, \omega^{\overline{1}}, ..., \omega^{\overline{j-1}} \rangle$  for  $j = 2, ..., n$ .

One can easily check that all abelian complex structures are necessarily nilpotent and that in this case  $d\omega^j$  are of type (1,1).

Nilpotent Lie algebras of dimension 4 and 6 admitting a complex structure have been classified in [39], with detailed list up to isomorphism in [43, Theorem 8] for the 6-dimensional case, see also [8] for a classification up to equivalence of the complex structures. In particular, Ugarte [43, Proposition 2] showed the following result.

**Proposition 3.1** ([43]). Let J be a complex structure on a 6-dimensional nilpotent Lie algebra  $\mathfrak{g}$ .

(a) If J is non-nilpotent, then there is a basis  $\{\omega^j\}_{j=1}^3$  of (1,0)-forms such that

$$\begin{cases} d\omega^{1} = 0, \\ d\omega^{2} = E \,\omega^{13} + \omega^{1\overline{3}}, \\ d\omega^{3} = A \,\omega^{1\overline{1}} + ib \,\omega^{1\overline{2}} - ib\overline{E} \,\omega^{2\overline{1}}, \end{cases}$$
(2)

where  $A, E \in \mathbb{C}$  with |E| = 1, and  $b \in \mathbb{R} - \{0\}$ . (b) If J is nilpotent, then there is a basis  $\{\omega^j\}_{j=1}^3$  of (1,0)-forms satisfying

$$\begin{cases} a\omega^{-} = 0, \\ d\omega^{2} = \epsilon\omega^{1\bar{1}}, \\ d\omega^{3} = \rho\omega^{12} + (1-\epsilon)A\omega^{1\bar{1}} + B\omega^{1\bar{2}} + C\omega^{2\bar{1}} + (1-\epsilon)D\omega^{2\bar{2}}, \end{cases}$$
(3)  
where  $A, B, C, D \in \mathbb{C}$ , and  $\epsilon, \rho \in \{0, 1\}$ .

The abelian complex structures correspond to the case  $\rho = 0$  in the complex structure equations (3). A bi-invariant complex structure is automatically nilpotent and corresponds to  $\epsilon = A = B = C = D = 0$  in (3).

The following Lemma provides a further reduction of the complex structure equations on 2-step nilpotent Lie algebras (for  $\epsilon = 0$ ) (see Proposition 10 and Lemma 11 in [43]).

**Lemma 3.1** ([43]). Let J be a complex structure on a 2-step nilpotent Lie algebra  $\mathfrak{g}$  of dimension 6 with first Betti number  $b_1(\mathfrak{g}) \geq 4$ . If J is not bi-invariant, then there is a basis  $\{\omega^j\}_{j=1}^3$  of (1,0)-forms such that

$$\begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \rho\omega^{12} + \omega^{1\overline{1}} + B\omega^{1\overline{2}} + D\omega^{2\overline{2}}, \end{cases}$$
(4)

where  $B, D \in \mathbb{C}$ , and  $\rho \in \{0, 1\}$ .

**Remark 3.1.** The Lie algebras in Lemma 3.1 are those for which J is nilpotent with  $\epsilon = 0$  and at least one of A, B, C, D not zero in structure equations (3).

Henceforth we will be assuming that J is not a bi-invariant complex structure on  $\mathfrak{g}$ . The bi-invariant case will be dealt with in Remark 3.2.

Using Proposition 3.1 we can suppose, without further restrictions, that a 6-dimensional nilpotent Lie algebra  $\mathfrak{g}$  admitting a complex structure J has complex structure equations (2) if J is non-nilpotent and (3) if J is nilpotent (with reduction (4) if  $\epsilon = 0$ ).

A 1-form  $\alpha$  on  $\mathfrak{g}$  can be written as

$$\alpha = \alpha^{(1,0)} + \alpha^{(0,1)} = \sum_{j=1}^{3} \lambda_j \omega^j + \sum_{j=1}^{3} \overline{\lambda}_j \overline{\omega}^j,$$
(5)

where  $\lambda_j \in \mathbb{C}$  and  $\{\omega^j\}_{j=1}^3$  is the basis of (1.0)-forms satisfying the complex structure equations (2) (resp. (3)) for J non-nilpotent (resp. J nilpotent).

**Lemma 3.2.** Let  $\alpha$  be a closed 1-form on a 6-dimensional nilpotent Lie algebra  $\mathfrak{g}$  with complex structure J. If we write  $\alpha$  with respect to the basis of (1,0)-forms as in (5), then we get the following constraints on the coefficients  $\lambda_j$ : (a) for J non-nilpotent

$$\lambda_2 = 0 \tag{6}$$

$$\lambda_3 - \overline{\lambda}_3 E = 0 \tag{7}$$

$$\lambda_3 A - \overline{\lambda}_3 \overline{A} = 0 \tag{8}$$

(b) for J nilpotent ( $\epsilon = 0$ )

$$\rho\lambda_3 = 0 \tag{9}$$

$$\lambda_3 = \lambda_3 \tag{10}$$

$$\lambda_3 \text{Im} D = 0 \tag{11}$$

$$\lambda_3 B = 0 \tag{12}$$

and for J nilpotent ( $\epsilon = 1$ )

$$\rho \,\lambda_3 = 0 \tag{13}$$

$$\lambda_2 = \overline{\lambda}_2 \tag{14}$$

$$\lambda_3 B - \overline{\lambda}_3 \overline{C} = 0 \tag{15}$$

*Proof.* The constraints imposed by the condition  $d\alpha = 0$  are given by direct computation, using equations (2) in the non-nilpotent case, (3) in the nilpotent case with  $\epsilon = 1$  and (4) in the nilpotent case with  $\epsilon = 0$ .

To investigate the existence of *J*-Hermitian metrics satisfying the LCSKT condition on  $(\mathfrak{g}, J)$  we can use the fact that the generic *J*-Hermitian metric g on  $\mathfrak{g}$  is expressed, in terms of the basis  $\{\omega^j\}$ , as

$$g = r(\omega^{1} \otimes \overline{\omega}^{1} + \overline{\omega}^{1} \otimes \omega^{1}) + s(\omega^{2} \otimes \overline{\omega}^{2} + \overline{\omega}^{2} \otimes \omega^{2}) + t(\omega^{3} \otimes \overline{\omega}^{3} + \overline{\omega}^{3} \otimes \omega^{3}) -iu(\omega^{1} \otimes \overline{\omega}^{2} + \overline{\omega}^{2} \otimes \omega^{1}) + i\overline{u}(\omega^{2} \otimes \overline{\omega}^{1} + \overline{\omega}^{1} \otimes \omega^{2}) -iv(\omega^{2} \otimes \overline{\omega}^{3} + \overline{\omega}^{3} \otimes \omega^{2}) + i\overline{v}(\omega^{3} \otimes \overline{\omega}^{2} + \overline{\omega}^{2} \otimes \omega^{3}) -iz(\omega^{1} \otimes \overline{\omega}^{3} + \overline{\omega}^{3} \otimes \omega^{1}) + i\overline{z}(\omega^{3} \otimes \overline{\omega}^{1} + \overline{\omega}^{1} \otimes \omega^{3})$$

where  $r, s, t \in \mathbb{R}$  and  $u, v, z \in \mathbb{C}$ ;  $r, s, t > 0, rs - |u|^2 > 0, st - |v|^2 > 0, rt - |z|^2 > 0$ ,  $rst + 2Re(i\overline{uv}z) - t|u|^2 - r|v|^2 - s|z|^2 > 0$ . These last conditions guarantee that the metric g is positive definite, i.e.,  $g(Z, \overline{Z}) > 0$  for any nonzero  $Z \in \mathfrak{g}^{\mathbb{C}}$ .

Furthermore, the fundamental form  $\Omega$  of the generic Hermitian structure (J, g) is then given by

$$\Omega = i(r\omega^{1\overline{1}} + s\omega^{2\overline{2}} + t\omega^{3\overline{3}}) + u\omega^{1\overline{2}} - \overline{u}\omega^{2\overline{1}} + v\omega^{2\overline{3}} - \overline{v}\omega^{3\overline{2}} + z\omega^{1\overline{3}} - \overline{z}\omega^{3\overline{1}}$$
(16)

For the 3-form  $H = d^c \Omega$ , we will be using the notation

$$H = H^{(2,1)} + H^{(1,2)} = \sum_{l < m,n=1}^{3} \left( H_{lm\overline{n}} \,\omega^{lm\overline{n}} + H_{n\overline{l}\overline{m}} \,\omega^{n\overline{l}\overline{m}} \right) = (-i\partial\Omega) + (i\overline{\partial}\Omega).$$

Notice also that  $dH = dd^c \Omega = 2i\partial \overline{\partial} \Omega$ .

The following result is proved by a direct calculation, so we omit the proof.

**Lemma 3.3.** Let  $\Omega$  as in (16). The (2,1)-part  $H^{(2,1)}$  of the 3-form  $H = d^c \Omega$  and its derivative dH are given respectively by:

(a) for J non-nilpotent

$$H^{(2,1)} = i(\overline{A}v + ibz)\omega^{12\overline{1}} - bEv\omega^{12\overline{2}} + i(i\overline{A}t - u + E\overline{u})\omega^{13\overline{1}} - i(is + bt)E\omega^{13\overline{2}} -iEv\omega^{13\overline{3}} - i(is - bt)\omega^{23\overline{1}} dH = -4(b^2t\omega^{12\overline{12}} + s\omega^{13\overline{13}})$$
(17)

where  $\{\omega^j\}_{j=1}^3$  is the basis of (1,0)-forms satisfying (2). (b) For J nilpotent ( $\epsilon = 0$ )

$$H^{(2,1)} = i(\rho\overline{z} + v - \overline{B}z)\omega^{12\overline{1}} + i(\rho\overline{v} - \overline{D}z)\omega^{12\overline{2}} + \rho t\omega^{12\overline{3}} - t\omega^{13\overline{1}} - \overline{B}t\omega^{23\overline{1}} - \overline{D}t\omega^{23\overline{2}},$$
  
$$dH = -2t[\rho + |B|^2 - 2Re(D)]\omega^{12\overline{12}}$$
(18)

where  $\{\omega^j\}_{j=1}^3$  is the basis of (1,0)-forms satisfying (4). For J nilpotent ( $\epsilon = 1$ )

$$H^{(2,1)} = i(is + \rho\overline{z} - \overline{B}z)\omega^{12\overline{1}} + i(\rho\overline{v} + \overline{C}v)\omega^{12\overline{2}} + \rho t\omega^{12\overline{3}} - i\overline{v}\omega^{13\overline{1}} - \overline{C}t\omega^{13\overline{2}} - \overline{B}t\omega^{23\overline{1}},$$

$$dH = -2t[\rho + |B|^2 + |C|^2]\omega^{12\overline{12}}$$
(19)

where  $\{\omega^j\}_{j=1}^3$  is the basis of (1,0)-forms satisfying (3).

We will now classify 6-dimensional nilpotent Lie algebras  $(\mathfrak{g}, J)$ , for which there exists a non-vanishing closed 1-form  $(\alpha \neq 0)$  such that  $dH = \alpha \wedge H$ , where  $H = d^c \Omega$  and  $\Omega$  is given by (16). The case  $\alpha = 0$  was already studied in [19].

**Theorem 3.1.** Let  $(\mathfrak{g}, J)$  be a 6-dimensional nilpotent Lie algebra admitting a complex structure J. Then

(a) If J is non-nilpotent, then  $(\mathfrak{g}, J)$  does not admit any LCSKT structure.

(b) If J is nilpotent, then  $(\mathfrak{g}, J)$  admits an LCSKT structure if and only if it has either complex structure equations

$$\begin{cases} d\omega^{j} = 0, \quad j = 1, 2, \\ d\omega^{3} = \omega^{1\overline{1}}, \end{cases}$$

$$(20)$$

or

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{1\overline{1}}, \\ d\omega^3 = \omega^{12} - \omega^{1\overline{2}} \end{cases}$$
(21)

Moreover, if  $(\mathfrak{g}, J)$  has complex structure equations (20) and (21), every Hermitian structure is LCSKT. In the first case every LCSKT structure is trivial since dH = 0 and the 1-form  $\alpha$  is independent of the parameters defining the Hermitian structure in (16)

$$\alpha = 2Re(\lambda_1\omega^1), \quad \lambda_1 \in \mathbb{C} - \{0\}.$$

In the second case every LCSKT structure is non-trivial and the 1-form  $\alpha$  is given by

$$\alpha = \frac{2it\overline{v}}{ts - |v|^2}\omega^1 - \frac{2itv}{ts - |v|^2}\overline{\omega}^1 - \frac{4t^2}{ts - |v|^2}Re(\omega^2)$$

and by [19] the nilpotent Lie algebra with complex structure equations (21) does not admit any SKT structure.

*Proof.* Firstly, we can rewrite the equation  $dH = \alpha \wedge H$  as

$$dH - \alpha \wedge H = 0 \tag{22}$$

In the proof we will study separately the two cases: J non-nilpotent and J nilpotent.

(a) For J non-nilpotent: by imposing the vanishing of the coefficient of  $\omega^{23\overline{13}}$  in (22) and by using the constraints (6) (i.e  $\lambda_2 = 0$ ) and (7), we obtain  $\overline{\lambda}_3(-EH_{2\overline{13}} - H_{23\overline{1}}) = 0$ , i.e.  $\overline{\lambda}_3(is - bt) = 0$ , which implies  $\lambda_3 = 0$  since s, t > 0 and  $b \in \mathbb{R} \setminus \{0\}$ .

Then, by the vanishing of the coefficient of  $\omega^{12\overline{13}}$  in (22), we get  $\lambda_1 H_{2\overline{13}} = 0$ , which implies

 $\lambda_1 = 0$ , since  $H_{2\overline{13}} = i(-is + bt)\overline{E} \neq 0$ .

Therefore, if J is non-nilpotent, then  $\alpha = 0$  and  $(\mathfrak{g}, J)$  cannot admit any non-trivial LCSKT neither any SKT structure, since  $dH \neq 0$  from (17).

(b) For J nilpotent, we consider separately the two cases: (b1)  $\epsilon = 0$  and (b2)  $\epsilon = 1$ .

(b1) From the constraint (10), i.e.  $\lambda_3 \in \mathbb{R}$ , and by the vanishing of the coefficient of  $\omega^{13\overline{13}}$  in (22), we obtain  $2t\lambda_3 = 0$  so  $\lambda_3 = 0$   $(t \neq 0)$ .

By the vanishing of the coefficient of  $\omega^{123\overline{2}}$  and of  $\omega^{123\overline{1}}$  in (22), we get  $\lambda_1\overline{D} = 0$  and  $\lambda_2 = \overline{B}\lambda_1$ . If  $D \neq 0$ , we have  $\lambda_1 = 0$  and  $\lambda_2 = 0$ , so  $\alpha = 0$ . If D = 0, we distinguish the two cases  $\rho = 0$  (i.e. J abelian) and  $\rho = 1$  (i.e. J non-abelian).

If  $\rho = 0$ , by the vanishing of the coefficient of  $\omega^{12\overline{13}}$  in (22), we obtain  $t\lambda_2 = 0$ , so  $\lambda_2 = 0$ . By the vanishing of the coefficient of  $\omega^{12\overline{12}}$  in (22), we get  $-2t|B|^2 = 0$ , so for B = 0 with an arbitrary  $\lambda_1 \neq 0$ , we can deduce that every  $\Omega$  defines a locally conformal SKT structure on the Lie algebra with complex structure equations

$$\begin{cases} d\omega^j = 0, \, j = 1, 2 \\ d\omega^3 = \omega^{1\overline{1}}, \end{cases}$$

with a non-unique  $\alpha = 2Re(\lambda_1\omega_1), \lambda_1 \neq 0$ . Remark that *H* is degenerate with dH = 0.

If  $\rho = 1$ , by the vanishing of the coefficient of  $\omega^{12\overline{13}}$  and of  $\omega^{12\overline{23}}$  in (22), we get  $\overline{\lambda}_1 = -\lambda_2$  and  $\overline{\lambda}_2 = -B\lambda_2$ , which implies that  $\lambda_2(|B|^2 - 1) = 0$ .

If  $|B| \neq 1$ , we have  $\lambda_2 = 0$  and  $\lambda_1 = 0$ , so  $\alpha = 0$ .

If |B| = 1, by the vanishing of the coefficient of  $\omega^{12\overline{12}}$  in (22) we get -4t = 0 which is impossible  $(t \neq 0)$ , so in this case  $(\mathfrak{g}, J)$  cannot admit any locally conformal SKT structure.

(b2) Since  $\epsilon = 1$ , by the constraint (14) we get  $\lambda_2 \in \mathbb{R}$ . We distinguish two cases: i)  $\rho = 0$ , i.e. J abelian and ii)  $\rho = 1$ , i.e J non-abelian.

In the case i), if we have B = C = 0, from (19) we get dH = 0.

By the vanishing of the coefficients of  $\omega^{12\overline{12}}$  and of  $\omega^{12\overline{13}}$  in (22), we get  $\lambda_2 s = 0$ and  $\lambda_3 = i \frac{\overline{v}}{s} \lambda_2$ , so  $\lambda_2 = \lambda_3 = 0$ . As a consequence, for arbitrary  $\lambda_1 \neq 0$  we have a trivial LCSKT structure. Also, notice that and after interchanging  $\omega^2$  with  $\omega^3$  we obtain again the same structure equations (20).

If B = 0 and  $C \neq 0$ , by (15) we get  $\lambda_3 = 0$ . By the vanishing of the coefficients of  $\omega^{12\overline{3}}$  and of  $\omega^{12\overline{13}}$  in (22) we get  $\lambda_2 t \overline{C} = 0$ ,  $Ct\lambda_1 + iv\lambda_2 = 0$ , so  $\lambda_2 = 0$  and  $\lambda_1 = 0$ , i.e.  $\alpha = 0$ .

If C = 0 and  $B \neq 0$ , by (15) we obtain  $\lambda_3 = 0$ . By the vanishing of the coefficients of  $\omega^{12\overline{23}}$  and of  $\omega^{12\overline{31}}$  in (22) we get  $\lambda_2 tB = 0$ ,  $\lambda_1\overline{B} = 0$  so  $\lambda_2 = 0$  and  $\lambda_1 = 0$ , i.e  $\alpha = 0$ .

If  $C \neq 0$  and  $B \neq 0$ , by the vanishing of the coefficients of  $\omega^{123\overline{2}}$ ,  $\omega^{13\overline{13}}$  and of  $\omega^{123\overline{1}}$  in (22), we get the following system

$$\begin{cases} \lambda_2 = -i\frac{v}{t}\lambda_3, \\ \overline{\lambda}_3 \overline{v} = \lambda_3 v, \\ -\overline{B}t\lambda_1 + i\overline{v}\lambda_2 + i(is - \overline{B}z)\lambda_3 = 0 \end{cases}$$

By using the first and second equations, we obtain

$$\overline{\lambda}_2 = -i\frac{\overline{v}}{\overline{t}}\overline{\lambda}_3 = -i(\frac{v}{\overline{t}}\lambda_3) = -\lambda_2$$

since  $\lambda_2 = \overline{\lambda}_2$  (constraint (14)), which implies that  $\lambda_2 = 0$ , so either  $\lambda_3 = 0$  or v = 0. If  $\lambda_3 = 0$ , the third equation implies that  $-\overline{B}t\lambda_1 = 0$  so  $\lambda_1 = 0$ , i.e.  $\alpha = 0$ . If v = 0, by the vanishing of the coefficients of  $\omega^{12\overline{12}}$  in (22), we obtain  $2t[|B|^2 + |C|^2] = 0$  so B = C = 0 and we get a contradiction.

In the case ii), i.e ( $\rho = 1$ ), then from the constraint (13) we have  $\lambda_3 = 0$ . By the vanishing of the coefficients of  $\omega^{123\overline{2}}$ ,  $\omega^{12\overline{23}}$ ,  $\omega^{12\overline{13}}$  and of  $\omega^{123\overline{1}}$  in (22), we obtain the following system

$$\begin{cases} \lambda_2 t\overline{C} = 0, \\ \lambda_2 t(B+1) = 0, \\ -\lambda_1 tC - i\lambda_2 v + \overline{\lambda}_1 t = 0, \\ -\lambda_1 t\overline{B} + i\lambda_2 \overline{v} = 0 \end{cases}$$

In the first case  $C \neq 0$ , by the first equation we get  $\lambda_2 = 0$  and if  $B \neq 0$  by the fourth equation we have  $\lambda_1 = 0$ , so  $\alpha = 0$ . If B = 0 and  $\lambda_1 \neq 0$ , from the third equation we get  $\overline{\lambda}_1 = C\lambda_1$ , which implies that  $|C|^2 = 1$ . Moreover, by the vanishing of the coefficient of  $\omega^{12\overline{12}}$  in (22) we obtain -4t = 0 and we get a contradiction  $(t \neq 0)$ .

In the second case C = 0, for  $B \neq -1$ , by the second equation we get  $\lambda_2 = 0$  and the third equation implies that  $\lambda_1 = 0$ , so  $\alpha = 0$ .

If C = 0 and B = -1, by the third and the fourth equations we obtain

$$\lambda_1 = -i\frac{\overline{v}}{\overline{t}}\lambda_2, \quad \overline{\lambda}_1 = i\frac{v}{\overline{t}}\lambda_2$$

By the vanishing of the coefficient of  $\omega^{12\overline{12}}$  in (22) and by substituting the last expressions of  $\lambda_1$  and  $\overline{\lambda}_1$ , we obtain

$$-4t - 2\left(\frac{ts - |v|^2}{t}\right)\lambda_2 = 0$$

so  $\lambda_2 = \overline{\lambda}_2 = \frac{-2t^2}{ts - |v|^2} \neq 0$  and  $\lambda_1 = \frac{2it\overline{v}}{ts - |v|^2}$ , since  $ts - |v|^2 \neq 0$ .

Hence, every 2-form  $\Omega$ , given by (16), defines a non-trivial LCSKT structure on the nilpotent Lie algebra  $(\mathfrak{g}, J)$  with the complex structure equations

$$\left\{ \begin{array}{l} d\omega^1=0,\\ d\omega^2=\omega^{1\overline{1}},\\ d\omega^3=\omega^{12}-\omega^{1\overline{2}}, \end{array} \right.$$

with a unique (non-zero) closed 1-form

$$\alpha = \frac{2it\overline{v}}{ts - |v|^2}\omega^1 - \frac{2itv}{ts - |v|^2}\overline{\omega}^1 - \frac{4t^2}{ts - |v|^2}Re(\omega^2).$$

**Remark 3.2.** If  $\mathfrak{g}$  is not abelian and J is bi-invariant, then (g, J) has structure equations

$$\begin{cases} d\omega^1 = d\omega^2 = 0\\ d\omega^3 = \omega^{12}. \end{cases}$$

Analogous computations to those of Theorem 3.1 show that for a fundamental form as in eq. (16), we get  $dH = 2t\omega^{12\overline{12}}$ . If  $\alpha$  as in eq. (5) is a closed 1-form then  $\lambda_3 = 0$ . Imposing the condition  $dH - \alpha \wedge H = 0$  implies, by the vanishing of the coefficients  $\omega_{13\overline{12}}$  and  $\omega_{23\overline{12}}$ , that  $\lambda_1 = \lambda_2 = 0$ . Thus  $\alpha = 0$ , which is a contradiction. Then, if J is bi-invariant, there does not exist any J-Hermitian metric q such that (J, q) is LCSKT.

In order to determine, up to isomorphism, the underlying 6-dimensional LCSKT nilpotent Lie algebras, we will adopt for the list of Lie algebras the notation  $\mathfrak{h}_k$  used in [10, Theorem 8]. Moreover, for instance by (0, 0, 0, 0, 0, 12) we will denote the nilpotent Lie algebra with real structure equations

$$\begin{cases} de^k = 0, \ k = 1, ..., 5, \\ de^6 = e^1 \wedge e^2, \end{cases}$$

where  $\{e^j\}$  is a basis of real 1-forms of the Lie algebra and d denotes the Chevalley-Eilenberg differential.

**Corollary 3.1.** A 6-dimensional nilpotent Lie algebra  $\mathfrak{g}$  admits an LCSKT structure (J,g) if and only if  $\mathfrak{g}$  is isomorphic either to  $\mathfrak{h}_8 = (0,0,0,0,0,12)$  or  $\mathfrak{h}_{16} = (0,0,0,12,14,24)$ . Moreover, if  $\mathfrak{g} \cong \mathfrak{h}_8$ , the underlying complex structure J must be abelian and every LCSKT structure is trivial. If  $\mathfrak{g} \cong \mathfrak{h}_{16}$ , J has to be (non-abelian) nilpotent and every LCSKT structure is non-trivial.

*Proof.* Let  $\{f^j\}_{j=1}^6$  be the basis of real 1-forms on  $\mathfrak{g}$  such that

$$\omega^{1} = f^{1} + if^{2}, \ \omega^{2} = f^{3} + if^{4}, \ \omega^{3} = f^{5} + if^{6}.$$
(23)

To obtain the real structure equations of  $\mathfrak{g}$  we use (23) and we impose (20) and (21), respectively. For case (20), if we consider the change of basis

$$e^1 = -\sqrt{2}f^2, \ e^2 = \sqrt{2}f^1, \ e^k = f^k, \ k = 3, \dots, 6,$$

we obtain the structure equations of  $\mathfrak{h}_8 = (0, 0, 0, 0, 0, 12)$ . For case (21), considering the change of basis

$$e^{1} = \sqrt{2}f^{2}, e^{2} = \sqrt{2}f^{1}, e^{3} = f^{3}, e^{4} = f^{4}e^{5} = -\frac{1}{\sqrt{2}}f^{5}, e^{6} = \frac{1}{\sqrt{2}}f^{6},$$

we obtain the structure equations of the 3-step nilpotent Lie algebra  $\mathfrak{h}_{16} = (0, 0, 0, 12, 14, 24)$ . On  $\mathfrak{h}_8$ , for every LCSKT structure we have dH = 0 and the LCSKT structure is trivial. On the other hand, on  $\mathfrak{h}_{16}$  we have always  $dH \neq 0$  and a non-trivial LCSKT structure.

**Remark 3.3.** Note that  $\mathfrak{h}_{16}$  cannot admit any SKT structure since it is 3-step nilpotent, [19, 5]. Moreover, by [43, Prop. 25 and Th. 31]  $\mathfrak{h}_8$  and  $\mathfrak{h}_{16}$  do not admit any balanced (or LCK) Hermitian structure. The simply connected Lie groups with Lie algebras  $\mathfrak{h}_8$ ,  $\mathfrak{h}_{16}$ both admit lattices, since every simply connected nilpotent Lie group whose Lie algebra has rational structure constants has a lattice (see [38, Theorem 2.1] and [33]). **Remark 3.4.** Nilmanifolds of dimension 8 admitting balanced metrics have recently been constructed in [28]. It would be a natural question to investigate if such manifolds can simultaneously satisfy the LCSKT condition.

As a consequence of Theorem 3.1, Corollary 3.1 and Remark 3.3 we have the following

**Theorem 3.2.** If  $M^6 = \Gamma \setminus G$  is a 6-dimensional nilmanifold, then  $M^6$  admits an invariant LCSKT structure (J,g) if and only if the Lie algebra of  $\mathfrak{g}$  of G is isomorphic either to  $\mathfrak{h}_8$  or  $\mathfrak{h}_{16}$ . The LCSKT structure is non-trivial (resp. trivial) if and only if  $\mathfrak{g}$  is isomorphic to  $\mathfrak{h}_{16}$  (resp.  $\mathfrak{h}_8$ ). Moreover,  $M^6$  does not admit any invariant balanced or invariant LCK structure. If  $\mathfrak{g} \cong \mathfrak{h}_{16}$ ,  $M^6$  cannot have any SKT structure.

**Remark 3.5.** Note that according to [19, Proposition 6.1], the holonomy group of the Bismut connection  $\nabla^B$  of a 6-dimensional nilmanifold  $M^6 = \Gamma/G$  with an invariant LCSKT structure (J,g) is not reduced to SU(3), since the LCSKT structure (J,g) is not balanced.

Using the complex structure equations (20) and (21) associated to  $(M^6, J, g)$  (see Theorem 3.1), one can check that the Hermitian metric with fundamental form

$$\Omega = \frac{i}{2}(\omega^{1\overline{1}} + \omega^{2\overline{2}} + \omega^{3\overline{3}})$$

is LCB with  $\rho^B \neq 0$ . Indeed, if we choose the basis  $\{f^j\}$  of 1-forms such that

$$\omega^1 = f^1 + if^2, \, \omega^2 = f^3 + if^4, \, \omega^3 = f^5 + if^6,$$

we have  $d\Omega = 2f^{125}$  and  $\theta = f^5$  for  $\mathfrak{g} \cong \mathfrak{h}_8$  and for  $\mathfrak{g} \cong \mathfrak{h}_{16}$ , we obtain  $d\Omega = f^{123} - f^{145} - f^{246}$ and  $\theta = f^3$ . As a consequence in both cases  $d\theta = 0$ . To compute the Bismut Ricci form we can use that  $\rho^C$  vanishes by [29]. Therefore, as a consequence of (1)

$$\rho^B = -2dJ\theta$$

and we get for both cases  $\rho^B = 4f^{12}$ .

### 4. INVARIANT LCSKT STRUCTURES ON ALMOST ABELIAN SOLVMANIFOLDS

In this section we study the existence of invariant LCSKT structures on almost abelian solvmanifolds, i.e. on compact quotients  $M = \Gamma/G$ , where G is a connected and simply connected almost abelian Lie group and  $\Gamma$  is a lattice in G. Therefore, we will determine the necessary and sufficient conditions for the existence of an LCSKT structure on an almost abelian Lie algebra  $\mathfrak{g}$ . We recall that a Lie algebra  $\mathfrak{g}$  is called *almost abelian* if it contains a codimension-one abelian ideal  $\mathfrak{n}$ , so it is the next simplest type of Lie algebra after abelian Lie algebras. Let J be a complex structure on  $\mathfrak{g}$ , then  $\mathfrak{n}^1 := \mathfrak{n} \cap J\mathfrak{n}$  is the maximal Jinvariant subspace of  $\mathfrak{n}$ . By [4, 29], if (J, g) is a Hermitian structure on  $\mathfrak{g}$ , then there exists a unitary basis  $\{e_1, \ldots, e_{2n}\}$  of  $(\mathfrak{g}, J, g)$  such that

$$\mathfrak{n} = span \langle e_1, \dots, e_{2n-1} \rangle, \mathfrak{n}_1 = span \langle e_2, \dots, e_{2n-1} \rangle, Je_i = e_{2n+1-i}, i = 1, \dots, n$$

So in particular

 $Je_1 = e_{2n}, \quad J(\mathfrak{n}_1) \subset \mathfrak{n}_1.$ 

We set  $J_1 := J|_{\mathfrak{n}_1}$  and the basis  $\{e_j\}_{j=1}^{2n}$  will be called *adapted* to the orthogonal decomposition (splitting)

$$\mathfrak{g} = \langle e_1 \rangle \oplus \mathfrak{n}_1 \oplus \langle e_{2n} \rangle, \, \mathfrak{n} = \langle e_1 \rangle \oplus \mathfrak{n}_1$$

The matrix B associated to  $(ad_{e_{2n}})|_{\mathfrak{n}}$  determines the whole information about the Lie algebra structure and  $\mathfrak{g}$  is therefore isomorphic to the semidirect product  $\mathbb{R} \ltimes ad_{e_{2n}}|_{\mathfrak{n}} \mathbb{R}^{2n-1}$ .

As proved in [29], the matrix B associated with  $ad_{e_{2n}}|_{\mathfrak{n}}$  in the adapted basis  $\{e_j\}_{j=1}^{2n}$  is of the form

$$B := (ad_{e_{2n}})|_{\mathfrak{n}} = \begin{pmatrix} a & 0 \\ v & A \end{pmatrix}, \quad a \in \mathbb{R}, \, v \in \mathfrak{n}_1, \, A \in \mathfrak{gl}(\mathfrak{n}_1), \, [A, J_1] = 0.$$

Here, we are identifying  $v \in \mathfrak{n}_1$  with its coordinates in the basis  $\{e_2, \dots, e_{2n-1}\}$  freely. In terms of commutation relations, this implies

$$[e_{2n}, e_1] = ae_1 + v, \quad [e_{2n}, X] = AX, \quad X \in \mathfrak{n}_1$$
 (24)

In this way, every Hermitian almost abelian Lie algebra  $(\mathfrak{g}, J, g)$  is characterized by the triplet (a, v, A) or, equivalently, by the real  $(2n-1) \times (2n-1)$  matrix corresponding to B. We will denote such an algebra by  $\mathfrak{g}(a, v, A)$ . Let us recall the conditions imposed on the algebraic data (a, v, A), for which an almost abelian Lie algebra admits a Kähler or an SKT structure.

**Theorem 4.1** ([30]). A Hermitian almost abelian Lie algebra  $(\mathfrak{g}(a, v, A), J, g)$  of dimension 2n is Kähler if and only if

$$A \in \mathfrak{so}(\mathfrak{n}_1)$$
 and  $v = 0$ 

**Remark 4.1.** Notice that, by Theorem 2.2 in [37], a Hermitian almost abelian Lie algebra  $(\mathfrak{g}(a, v, A), J, g)$  is LCB if and only if  $A^t v = 0$ .

**Theorem 4.2** ([4]). A 2n-dimensional Hermitian almost abelian Lie algebra  $(\mathfrak{g}(a, v, A), J, g)$ is SKT if and only if  $(aA + A^2 + A^tA) \in \mathfrak{so}(\mathfrak{n}_1)$  or, equivalently, if A is normal (i.e  $[A, A^t] = [A, J_1] = 0$ ) and each eigenvalue of A has real part equal to 0 or  $-\frac{a}{2}$ .

Now, according to formula (3.2) in [13] (see also [12]) for every Lie algebra  $\mathfrak{g}$  endowed with a Hermitian structure (J, g) the Bismut torsion 3-form H is given by

$$H(X, Y, Z) = -g([JX, JY], Z) - g([JY, JZ], X) - g([JZ, JX], Y).$$
(25)

and the exterior derivative dH can be computed by the usual formula

$$dH(W, X, Y, Z) = -H([W, X], Y, Z) + H([W, Y], X, Z) - H([W, Z], X, Y) -H([X, Y], W, Z) + H([X, Z], W, Y) - H([Y, Z], W, X),$$
(26)

for any vector  $X, Y, Z, W \in \mathfrak{g}$ .

**Proposition 4.1.** An almost abelian Lie algebra  $\mathfrak{g} := \mathfrak{g}(a, v, A)$  with Hermitian structure (J,g) is LCSKT if and only if there exists a non-zero closed-1-form  $\alpha \in \mathfrak{g}^*$  such that the following conditions are satisfied

$$a\alpha(e_1) + \alpha(v) = 0, \quad A^t \alpha|_{\mathfrak{n}_1} = 0 \tag{27}$$

$$\alpha(X)g(S(A)J_1Y,Z) - \alpha(Y)g(S(A)J_1X,Z) + \alpha(Z)g(S(A)J_1Y,X) = 0$$
(28)

$$g(S((a + \alpha(e_{2n}))A + A^2 + A^t A)J_1Y, Z) = \frac{1}{2}(g(v, Y)\alpha(Z) - g(v, Z)\alpha(Y)),$$
(29)

for every  $X, Y, Z \in \mathfrak{n}_1$ , and where  $S(A) = \frac{1}{2}(A + A^t)$  is the symmetric part of A.

*Proof.* For every 1-form  $\alpha \in \mathfrak{g}^*$ , we have

$$d\alpha(Y, e_{2n}) = -\alpha([Y, e_{2n}]) = \alpha(ad_{e_{2n}}(Y)), \quad \forall Y \in \mathfrak{g}$$

Using the commutation relations (24) we get

$$d\alpha(e_1, e_{2n}) = \alpha(ad_{e_{2n}}(e_1)) = a\alpha(e_1) + \alpha(v)$$

and

$$d\alpha(X, e_{2n}) = \alpha(ad_{e_{2n}}(X)) = \alpha(AX), \quad \forall X \in \mathfrak{n}_1.$$

By imposing  $d\alpha = 0$ , it follows that  $a\alpha(e_1) + \alpha(v) = 0$  and  $\alpha(AX) = 0$ , for every  $X \in \mathfrak{n}_1$ , i.e.  $A^t \alpha|_{\mathfrak{n}_1} = 0$ . Therefore, we obtain the condition (27).

Taking into account that  $\mathfrak{n}_1$  is an abelian ideal preserved by J and by using (25) we clearly get H(X, Y, Z) = 0, whenever  $X, Y, Z \in \mathfrak{n}_1$ . Then, by (26), one quickly checks that dH(W, X, Y, Z) = 0, if three of the entries lie in  $\mathfrak{n}_1$ .

Therefore, by imposing  $dH(e_1, X, Y, Z) = (\alpha \wedge H)(e_1, X, Y, Z)$  we get

$$-\alpha(X)H(e_1, Y, Z) + \alpha(Y)H(e_1, X, Z) - \alpha(Z)H(e_1, X, Y) = 0$$

Now, if we compute the expression of  $H(e_1, Y, Z)$ , for any  $Y, Z \in \mathfrak{n}_1$ , we get

$$H(e_1, Y, Z) = -g([Je_1, JY], Z) - g([JZ, Je_1], Y)$$
  
=  $-g([e_{2n}, J_1Y], Z) + g([e_{2n}, J_1Z], Y)$   
=  $-g(AJY, Z) + g(AJZ, Y)$   
=  $-g(AJ_1Y, Z) - g(J_1A^tY, Z) = -2g(S(A)J_1Y, Z)$  (30)

In the last step of the computation above, we used that  $J_1$  commutes with  $A^t$  since  $J_1$  commutes with A and  $J_1^t = -J_1$ . Applying (30) in all summands of (24), we obtain the condition (28). Moreover, if we use the formula (25) and the commutation relations (24) again, we get

$$H(e_{2n}, Y, Z) = 0, \quad H(e_{2n}, e_1, Z) = -g(v, Z), \quad \forall Y, Z \in \mathfrak{n}_1.$$

Similar computations show that

$$dH(e_{2n}, e_1, Y, Z) = 2g(S(aA + A^2 + A^tA)JY, Z)$$

and

$$\begin{aligned} (\alpha \wedge H)(e_{2n}, e_1, Y, Z) &= \alpha(e_{2n})H(e_1, Y, Z) - \alpha(e_1)H(e_{2n}, Y, Z) \\ &+ \alpha(Y)H(e_{2n}, e_1, Z) - \alpha(Z)H(e_{2n}, e_1, Y) \\ &= -2\alpha(e_{2n})g(S(A)J_1Y, Z) - \alpha(Y)g(v, Z) + \alpha(Z)g(v, Y). \end{aligned}$$

By imposing

$$dH(e_{2n}, e_1, Y, Z) = (\alpha \wedge H)(e_{2n}, e_1, Y, Z)$$

we get the condition (29)

$$g(S((a + \alpha(e_{2n}))A + A^2 + A^t A)J_1Y, Z) = \frac{1}{2}(g(v, Y)\alpha(Z) - g(v, Z)\alpha(Y)).$$

With respect the dual basis  $\{e^i\}_{i=1}^{2n}$  of the adapted basis  $\{e_i\}_{i=1}^{2n}$  of  $\mathfrak{g}$  we can write the 1-form  $\alpha \in \mathfrak{g}^*$  as

$$\alpha = \sum_{i=1}^{2n} \lambda_i e^i = \lambda_1 e^1 + \sum_{k=2}^{2n-1} \lambda_k e^k + \lambda_{2n} e^{2n},$$

where  $\lambda_i = \alpha(e_i)$ .

**Remark 4.2.** Note that the expressions of the Chern- and Bismut-Ricci forms were obtained respectively in [29, Lemma 6.1] and [4, Prop. 4.8] (see also [16] for a correction in the expression of  $\rho^B$ ) and are given by

$$\rho^{C} = -(a^{2} + \frac{1}{2}a\,TrA)e^{1} \wedge e^{2n},$$
  
$$\rho^{B} = -(a^{2} - \frac{1}{2}a\,TrA + \|v\|^{2})e^{1} \wedge e^{2n} - (A^{t}v)^{\flat} \wedge e^{2n},$$

where  $X^{\flat}(.) := g(X,.)$ , for  $X \in \mathfrak{g}$ . Therefore, by [17] it follows that  $\rho^C = \rho^B$  if and only if  $(a \operatorname{Tr} A - |v|^2)e^1 \wedge e^{2n} - (A^t v)^{\flat} \wedge e^{2n} = 0.$ 

$$(a TrA - |v|^2)e^2 \wedge e^{2\pi} - (A^{\circ}v)^{\circ} \wedge e^{2\pi}$$

We can prove the following result.

**Proposition 4.2.** Let (J,g) be a Hermitian structure on a 2n-dimensional almost abelian Lie algebra  $\mathfrak{g}$ . If (J,g) is balanced and LCSKT, then it is necessarily Kähler.

*Proof.* The Lee form  $\theta$  for 2*n*-dimensional almost abelian Lie algebra  $\mathfrak{g}$ , is given by

$$\theta = -(trA)e^{2n} + (Jv)^{\flat} \tag{31}$$

(see [17, Lemma 2.1]). By [17, Th. 3.1] we get that (J,g) is balanced  $(\theta = 0)$  if and only if v = 0 and trA = 0. If (J,g) is also LCSKT, the condition (29) reduces to

$$S((a + \lambda_{2n}))A + A^2 + A^t A)) = 0.$$

Taking traces in this last equation, yields to  $(a + \lambda_{2n})trA + 2trS(A)^2 = 0$  so  $trS(A)^2 = 0$ , which implies that S(A) = 0. In this case v = 0, S(A) = 0 the Hermitian structure (J, g)has to be Kähler ([30]).

We will now study the equations in Proposition 4.1 by imposing the additional assumption that A is non-degenerate, i.e.  $detA \neq 0$ . We will see that we obtain similar results to those in [4] for SKT structures.

**Lemma 4.1.** If  $\mathfrak{g}(\mu(a, v, A), J, g)$  is a 2n-dimensional almost abelian Hermitian Lie algebra with det  $A \neq 0$ , then the Hermitian structure (J, g) is LCSKT with closed 1-form  $\alpha$  if and only if

$$\alpha|_{\mathfrak{n}_1} = 0, \quad a\alpha(e_1) = 0 \tag{32}$$

and

$$((a + \alpha(e_{2n}))A + A^2 + A^t A) \in \mathfrak{so}(\mathfrak{n}_1).$$

$$(33)$$

Proof. If  $detA \neq 0$  by the second condition of (27) we obtain  $\alpha|_{\mathfrak{n}_1} = 0$ , and thus the first condition becomes  $a\alpha(e_1) = 0$ . Consequently, (28) is trivially satisfied and (29) is reduced to  $g(S((a + \alpha(e_{2n}))A + A^2 + A^tA))J_1Y, Z) = 0$  for any  $Y, Z \in \mathfrak{n}_1$  which in turn implies that  $S((a + \alpha(e_{2n}))A + A^2 + A^tA)) = 0$  and so  $((a + \alpha(e_{2n}))A + A^2 + A^tA) \in \mathfrak{so}(\mathfrak{n}_1)$ .  $\Box$ 

**Lemma 4.2.** Let  $\mathfrak{g}(\mu(a, v, A), J, g)$  be an almost abelian Hermitian Lie algebra such that  $\det A \neq 0$ . If (J,g) is LCSKT with closed 1-form  $\alpha$ , then the real part of the eigenvalues of A are either 0 or  $-\frac{1}{2}(a + \lambda_{2n})$ , where  $\lambda_{2n} = \alpha(e_{2n})$ .

*Proof.* We will proceed in a similar way as in the proof of [4, Lemma 4.8].

Consider the complexification  $\mathfrak{n}_1^{\mathbb{C}}$  of  $\mathfrak{n}_1$  and extend A linearly to  $\mathfrak{n}_1^{\mathbb{C}}$ . Let  $k \in \mathbb{C}$  be an eigenvalue of A corresponding to the eigenvector  $z = x + \sqrt{-1}y = x + iy \in \mathfrak{n}_1^{\mathbb{C}}, x, y \in \mathfrak{n}_1$ . Since  $((a + \lambda_{2n})A + A^2 + A^tA) \in \mathfrak{so}(\mathfrak{n}_1)$ , we have

$$\langle \langle ((a+\lambda_{2n})A+A^2+A^tA)z,\overline{z}\rangle \rangle = 0 = ((a+\lambda_{2n})k+2k^2) \langle \langle z,\overline{z}\rangle \rangle = k(a+\lambda_{2n}+2k) \langle \langle z,\overline{z}\rangle \rangle,$$

where  $\langle \langle ., . \rangle \rangle$  is the Hermitian inner product on  $\mathfrak{n}_1^{\mathbb{C}}$  induced by a fixed inner product  $g = \langle ., . \rangle$ on  $\mathfrak{n}_1$ . We have two cases to consider. Either  $k = 0, -\frac{1}{2}(a + \lambda_{2n})$  (and the lemma follows) or  $k \neq 0, -\frac{1}{2}(a + \lambda_{2n})$ , and in this case  $\langle \langle z, \overline{z} \rangle \rangle = 0$  which implies that ||x|| = ||y|| and  $\langle x, y \rangle = 0$ .

Writing  $k = m + \sqrt{-1}n = m + in$ , with  $m, n \in \mathbb{R}$ , and using that Ax = mx - ny and Ay = nx + my, we get

$$0 = \langle ((a + \lambda_{2n})A + A^2 + A^t A)u, u \rangle$$
  
=  $[(a + \lambda_{2n})m + m^2 - n^2 + m^2 + n^2)] ||x||^2$   
=  $m(a + \lambda_{2n} + 2m) ||x||^2.$ 

Then we can conclude that m = 0 or  $m = -\frac{1}{2}(a + \lambda_{2n})$  from which the lemma follows.

In order to establish our main result in this section, we need to recall the following linear algebra estimate, [4, Corollary A2]. Consider the norm in  $gl_m(\mathbb{R})$ , defined by the equality  $||B||^2 = tr(BB^t)$ .

**Lemma 4.3.** For any  $E \in gl_m(\mathbb{R})$  with eigenvalues  $k_1, ..., k_m \in \mathbb{C}$ , we have that

$$||S(E)||^2 \ge \sum_{i=1}^m Re(k_i)^2$$

with equality if and only if E is normal.

**Theorem 4.3.** Let  $\mathfrak{g} := \mathfrak{g}(\mu(a, v, A), J, g)$  be a Hermitian almost abelian Lie algebra with  $detA \neq 0$ . The Hermitian structure (J,g) is LCSKT with closed 1-form  $\alpha$  if and only if  $[A, A^t] = [A, J_1] = 0$  and each eigenvalue of A has a real part equal to 0 or  $-\frac{1}{2}(a + \lambda_{2n})$ , where  $\lambda_{2n} = \alpha(e_{2n})$ .

*Proof.* The condition  $[A, J_1] = 0$  is automatically satisfied since J is integrable on  $\mathfrak{g}$ .

Suppose that (J, g) is LCSKT with A non-degenerate. From Lemma 4.2, the eigenvalues of  $A, k_1, ..., k_{2n-2}$ , come in pairs and can be rearranged such that

$$Re(k_1) = \dots = Re(k_{2l}) = -\frac{1}{2}(a + \lambda_{2n}), \quad Re(k_{2l+1}) = \dots = Re(k_{2n-2}) = 0.$$

Then  $trA = -l(a + \lambda_{2n})$ . Now, taking traces in (33) one obtains that  $tr(A^2 + AA^t) = -(a + \lambda_{2n})trA$ . Thus  $||S(A)||^2 = \frac{1}{2}l(a + \lambda_{2n})^2$ . This yields equality in Lemma4.3, thus A is a normal endomorphism i.e  $[A, A^t] = 0$ .

For the proof of the converse assertion we consider the matrix  $B = (a + \lambda_{2n})A + A^2 + A^tA$ in the left-hand side of (33). By hypothesis, A is normal and a very simple computation shows that B is also normal. Recall that a normal matrix is skew-symmetric if and only if its eigenvalues are zero or purely imaginary. We make use of the spectral theorem for normal matrices to show that this is the case. The matrix A is unitary conjugate to

diag 
$$\left(-\frac{1}{2}(a+\lambda_{2n})+i\mathrm{Im}(k_1),-\frac{1}{2}(a+\lambda_{2n})+i\mathrm{Im}(k_2),\cdots,i\mathrm{Im}(k_{2n-3}),i\mathrm{Im}(k_{2n-2})\right)$$
.

Another computation will show that B is unitary conjugate to

diag
$$(0, 0, \cdots, i(a + \lambda_{2n}) \operatorname{Im}(k_{2n-3}), i(a + \lambda_{2n}) \operatorname{Im}(k_{2n-2}))$$

and the result follows.

In this situation where  $detA \neq 0$ , the non-vanishing 1-form  $\alpha \in \mathfrak{g}^*$  is given by

$$\alpha = \lambda_1 e^1 + \lambda_{2n} e^{2n} \neq 0,$$

with  $a\lambda_1 = 0$  and  $\lambda_{2n} = -(2Re(k) + a)$ , where k is an eigenvalue of A.

We can see that even if  $\alpha \neq 0$ , we get the trivial case (dH = 0) by setting  $\lambda_{2n} = 0$ , a = 0and  $\lambda_1 \neq 0$  (arbitrary), so

$$(A^2 + A^t A) \in \mathfrak{so}(\mathfrak{n}_1).$$

Thus  $\alpha = \lambda_1 e^1 \neq 0$ , in this case the 3-form H is degenerate.

A classification, up to isomorphism, of 6-dimensional almost abelian Lie algebras was given in [35], corrected and completed by [40] (see also [7, 16]). By Corollary 3.1, the only 6-dimensional nilpotent almost abelian Lie algebra admitting a (trivial) LCSKT structure is  $\mathfrak{h}_8 = (0, 0, 0, 0, 0, 0, f^{12})$ .

The six-dimensional non-nilpotent almost abelian Lie algebras admitting a complex structure are classified, up to isomorphism, in [16, Theorem 3.2] and they are denoted by  $l_i, i = 1, ..., 26$ .

Using the previous results, we will now construct some explicit examples of 6-dimensional (non-nilpotent) almost abelian Lie algebras admitting an LCSKT structure. Recall that a Lie algebra  $\mathfrak{g}$  is unimodular if  $\operatorname{tr}(ad_X) = 0$ , for every  $X \in \mathfrak{g}$ , and the unimodularity is a necessary condition for a Lie group to admit a lattice [34].

**Example 4.1.** Consider the indecomposable six-dimensional almost abelian Lie algebra

$$l_8^{p,q,s} = \mathfrak{g}_{6.8}^{p,q,q,s} = (pf^{16}, qf^{26}, qf^{36}, sf^{46} + f^{56}, sf^{56} - f^{46}, 0)$$

with  $p \neq 0$  and  $0 < |q| \le |p|$ . The Lie algebra is unimodular if and only if  $s = -\frac{1}{2}(p+2q)$ . Let (J,g) be the almost Hermitian structure given by

$$J(f_1) = f_6, J(f_2) = f_3, J(f_4) = f_5, \quad g = \sum_{k=1}^6 f^k \otimes f^k$$

Therefore

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & J_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the basis  $(f_1, \ldots, f_6)$  is adapted to the splitting

$$\mathfrak{l}_8^{p,q,s} = \mathbb{R}_{f_1} \oplus \mathfrak{n}_1 \oplus \mathbb{R}_{f_6}.$$

We have

$$B = (ad_{f_6})|_{\mathfrak{n}} = \left(\begin{array}{cc} p & 0\\ 0 & A \end{array}\right),$$

with

$$A = \left( \begin{array}{cccc} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & s & 1 \\ 0 & 0 & -1 & s \end{array} \right).$$

Therefore, J is integrable, since  $[A, J_1] = 0$  and  $J(\mathfrak{n}_1) \subset \mathfrak{n}_1$ , and the matrix A is nondegenerate, since  $q \neq 0$ . Moreover, we have also  $[A, A^t] = 0$ , i.e. A is normal. From Theorem 4.3, the Lie algebra  $\mathfrak{l}_8^{p,q,s}$  admits an LCSKT structure if either s = q or s = 0. The non-vanishing 1-closed form  $\alpha$  is given by  $\alpha = -(2q+p)f^6$  with  $0 < |q| \le |p|$  and  $q \ne -\frac{p}{2}$ .

We remark that, for generic parameters p, q, s with  $p \neq 0$  and  $0 < |q| \leq |p|$ , the 3-form H is given by  $H = -2(qf^{123} + sf^{145})$  and thus  $dH = -2q(2q + p)f^{1236} - 2s(s + p)f^{1456}$ . In particular, for the unimodular case  $s = q = -\frac{p}{4}$ , we have a non-trivial LCSKT structure with  $dH = \frac{p^2}{4}(f^{1236} + f^{1456}) \neq 0$ . Note that (J, g) is LCB, since v = 0 (see Remark 4.1).

We now construct an explicit example of almost abelian symplectic solvmanifold  $(M = \Gamma/G, J, g)$  endowed with an invariant LCSKT structure (J, g).

In general for a solvable Lie group it is not easy to find a lattice and to establish a general existence criterium, for more details see the references [7, 9, 2]. For almost abelian Lie groups there is the following sufficient criterion ([7, Proposition 2.1]).

**Proposition 4.3** ([7]). Let  $G = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^{m-1}$  be a m-dimensional almost abelian Lie group. Then G admits a lattice if and only if there exists a real number  $t_0 \neq 0$  for which  $\varphi(t_0) = exp(t_0ad_{e_{2n}})$  can be conjugated to an integer matrix.

In this case where there is a matrix P such that  $P\varphi(t_0)P^{-1}$  is an integer matrix, a lattice is given by [2]

$$\Gamma_{t_0} = t_0 \mathbb{Z} \ltimes P^{-1} \mathbb{Z}^{2n-1}.$$

Recall that a 6-dimensional Lie algebra  $\mathfrak{g}$  is called symplectic, if there is a closed 2-form  $\omega \in \Lambda^2 \mathfrak{g}^*$  such that  $\omega^3 \neq 0$ . The symplectic structure on  $\mathfrak{g}$  induces an invariant symplectic structure on the quotient  $\Gamma/G$ .

**Example 4.2.** Consider the decomposable six-dimensional unimodular almost abelian Lie algebra

$$\mathfrak{l}_{23}^0 = \mathfrak{g}_{5.14}^0 \oplus \mathbb{R} = (f^{26}, -f^{16}, f^{46}, 0, 0, 0)$$

which corresponds to the Lie group  $(G_{5.14}^0 \times \mathbb{R})$  ([16], table 3, The Appendix). The Lie algebra  $\mathfrak{l}_{23}^0$  admits the (integrable) complex structure

$$J(f_1) = f_2, J(f_3) = f_5, J(f_4) = f_6$$

Applying the change of basis  $e^1 = -f^4$ ,  $e^4 = f^1$ ,  $e^k = f^k$  for k = 2, 3, 5, 6, we obtain the complex structure given by

$$J(e_1) = e_6, J(e_3) = e_5, J(e_2) = e_4.$$

Therefore we have the usual orthonormal decomposition  $\mathfrak{l}_{23}^0 = \mathbb{R}e_1 \oplus \mathfrak{n}_1 \oplus \mathbb{R}e_6$ . If we consider the J-Hermitian metric  $g = \sum_{i=1}^6 e^i \otimes e^i$ , the associated fundamental 2-form  $\Omega$  is given by  $\Omega = f^{16} + f^{24} + f^{35}$ . Moreover, by a direct computation, we have  $H = e^{136}$  and thus dH = 0.

By imposing the closure of the 1-form  $\alpha = \sum_{i=1}^{6} \lambda_i e^i$  we get the following constraints

$$\lambda_2 = \lambda_4 = \lambda_3 = 0, \lambda_1 \in \mathbb{R}.$$

By imposing the conditions (28) and (29) we get  $\lambda_5 = 0$ . Consequently, the almost abelian Lie algebra  $l_{23}^0$  admits a trivial LCSKT structure with  $\alpha = \lambda_1 e^1$ , for arbitrary  $\lambda_1 \in \mathbb{R} - \{0\}$ .

Moreover, by [9, Theorem 1.1, Table 1],  $exp(tad_{e_6})$  is conjugated to an integer matrix, for  $t = 2\pi$  and  $t \in \{\pi, \frac{\pi}{2}, \frac{\pi}{3}\}$ , therefore  $(G_{5.14}^0 \times \mathbb{R})$  admits lattices  $\Gamma_t$ . As a consequence the solvmanifolds  $M_t = (\Gamma_t/(G_{5.14}^0 \times \mathbb{R}), J, g)$  are LCSKT and have first Betti number  $b_1 = 5$  for  $t = 2\pi$  and  $b_1 = 3$  for  $t \in \{\pi, \frac{\pi}{2}, \frac{\pi}{3}\}$ . Moreover, by [31, Theorem 1, Appendix B]  $\Gamma_t/(G_{5,14}^0 \times \mathbb{R})$  has the invariant symplectic form

$$\omega = \omega_{1,3}e^{13} + \omega_{1,5}e^{15} + \omega_{1,6}e^{16} + \omega_{2,4}e^{24} + \omega_{2,6}e^{26} + \omega_{3,6}e^{36} + \omega_{4,6}e^{46} + \omega_{5,6}e^{56}$$

where  $\omega_{2,4}(\omega_{1,3}\omega_{5,6} - \omega_{3,6}\omega_{1,5}) \neq 0$ . Note that the 6-dimensional LCSKT almost abelian symplectic solvmanifolds  $M_t = (\Gamma_t/(G_{5,14}^0 \times \mathbb{R}), J, g)$ , are not balanced but they are LCB. Indeed, by using (31), the Lee form  $\theta$  is given by  $\theta = -e^5$ , which is closed. Moreover, the Chern-Ricci form vanishes, but the Bismut Ricci form  $\rho^B = -e^{16}$  is not zero.

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#### References

- B. Alexandrov, S. Ivanov, Vanishing theorems on Hermitian manifolds, Differential Geom. Appl. 14 (2001), no. 3, 251–265.
- [2] A. Andrada, M. Origlia, Lattices in almost abelian Lie groups with locally conformal Kähler or symplectic structures, *Manuscripta Math.* 155 (2018), 389–417.
- [3] D. Angella, L. Ugarte, Locally conformal Hermitian metrics on complex non-Kähler manifolds, *Mediterr. J. Math.* 13 (2016), no. 4, 2105–2145.
- [4] R.M. Arroyo, R.A. Lafuente, The long-time behavior of the homogeneous pluriclosed flow, Proc. Lond. Math. Soc. (3) 119 (2019), no. 1, 266–289.
- [5] R.M. Arroyo, M. Nicolini, SKT structures on nilmanifolds, Math. Z. 302 (2022), no. 2, 1307–1320.
- [6] J.M. Bismut, A local index theorem for non Kähler manifolds, Math. Ann. 284 (1989), 1057–1068.
- [7] C. Bock, On low dimensional solvmanifolds, Asian J. Math. 20 (2016), 199-262.
- [8] M. Ceballos, A. Otal, L. Ugarte, R. Villacampa, Invariant Complex Structures on 6-Nilmanifolds: Classification, Frölicher Spectral Sequence and Special Hermitian Metrics, J. Geom. Anal. 26 (2016), 252–286.
- [9] S. Console, M. Macrí, Lattices, cohomology and models of 6-dimensional almost abelian solvmanifolds, *Rend. Semin. Mat. Univ. Politec. Torino* 74 (2016), 95–119.
- [10] L.A. Cordero, M. Fernandez, A. Gray, L. Ugarte, Nilpotent complex structures on compact nilmanifolds, *Rend. Circ. Mat. Palermo* 49 Suppl. (1997), 83–100.
- [11] L.A. Cordero, M. Fernandez, A. Gray, L. Ugarte, Compact nilmanifolds with nilpotent complex structure: Dolbeault cohomology, *Trans. Amer. Math. Soc.* 352 (2000), 5405–5433.
- [12] I. Dotti, A. Fino, Hyperkähler torsion structures invariant by nilpotent Lie groups, Classical Quantum Gravity 19 (3) (2002), 551–562.
- [13] N. Enrietti, A. Fino, L. Vezzoni, Tamed symplectic forms and strong Kähler with torsion metrics, J. Symplectic Geom. 10 (2012), 203–223.
- [14] A. Fino, G. Grantcharov, Properties of manifolds with skew-symmetric torsion and special holonomy, Adv. Math. 189 (2004), no. 2, 439–450.
- [15] A. Fino, A. Otal, L. Ugarte, Six-dimensional solvmanifolds with holomorphically trivial canonical bundle, Int. Math. Res. Not. 24 (2015), 13757–13799.
- [16] A. Fino, F. Paradiso, Generalized Kähler Almost Abelian Lie Groups, Ann. Mat. Pura Appl. (4) 200 (2021), 1781–1812.

- [17] A. Fino, F. Paradiso, Balanced Hermitian structures on almost abelian Lie algebras, J. Pure Appl. Algebra 227 (2023), no. 2, Paper No. 107186.
- [18] A. Fino, F. Paradiso, Hermitian structures on a class of almost nilpotent solvmanifolds, J. Algebra 609 (2022), 861–925.
- [19] A. Fino, M. Parton, S. Salamon, Families of strong KT structures in six dimensions, Comm. Math. Helv. 79 (2004) no. 2, 317–340.
- [20] A. Fino, L. Vezzoni, A correction to "Tamed symplectic forms and strong Kähler with torsion metrics", J. Symplectic Geom. 17 (2019), 1079–1081.
- [21] M. Freibert, A. Swann, Two-step solvable SKT shears, Math. Z. 299 (2021), no. 3-4, 1703–1739.
- [22] M. Freibert, A. Swann, Compatibility of balanced and SKT metrics on two-step solvable Lie groups, arXiv:2203.16638v1[math.DG].
- [23] S.J. Gates Jr., C.M. Hull, M.Roček, Twisted multiplets and new supersymmetric nonlinear σ-models, Nuclear Phys. B 248 (1984), 157–186.
- [24] P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte, Math. Ann. 267 (1984), 495–518.
- [25] P. Gauduchon, Hermitian connections and Dirac operators, Boll. Un. Mat. Ital. 11 B (1997), no. 2 suppl., 257–288.
- [26] M. Gualtieri, Generalized Kähler geometry, Commun. Math. Phys. 331 (2014), 297–331.
- [27] P.S. Howe, G. Papadopoulos, Further remarks on the geometry of two-dimensional nonlinear  $\sigma$ -models, Classical Quantum Gravity **12** (1988), 1647–1661.
- [28] A. Latorre, L. Ugarte, R. Villacampa, Frölicher spectral sequence of compact complex manifolds with special Hermitian metrics, arXiv:2207.14669v1[math.DG].
- [29] J. Lauret, E.A. Rodriguez-Valencia, On the Chern-Ricci flow and its solitons for Lie groups, Math. Nachr. 288 (2015), no. 13, 1512–1526.
- [30] J. Lauret, C. Will, On the symplectic curvature flow for locally homogeneous manifolds, J. Symplectic Geom. 15 (2017), no.1, 1–49.
- [31] M. Macrì, Cohomological properties of unimodular six dimensional solvable Lie algebras, *Differential Geom. Appl.* 31 (2013), 112–129.
- [32] T.B. Madsen, A. Swann, Invariant strong KT geometry on four-dimensional solvable Lie groups, J. Lie Theory 21 (2011), 55–70.
- [33] A. Malcev, On solvable Lie algebras, Izv. Akad. Nauk SSSR Ser. Mat. 9 (1945), 329–356.
- [34] J. Milnor, Curvature of left invariant metrics on Lie groups, Adv. Math. 21 (1976), 293–329.
- [35] G.M. Mubarakzyanov, Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 4 (1963), 104–116.
- [36] L. Ornea, A. Otiman, M. Stanciu, Compatibility between non-Kähler structures on complex (nil)manifolds, arXiv:2003.10708v2 [math.DG], to appear in *Transform. Groups.*
- [37] F. Paradiso, Locally conformally balanced metrics on almost abelian Lie algebras, Complex Manifolds 8 (2021), no. 1, 196–207.
- [38] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer (1972).
- [39] S. Salamon, Complex structures on nilpotent Lie algebras, J. Pure Appl. Algebra 157 (2001), 311–333.
- [40] A. Shabanskaya, Classification of six dimensional solvable indecomposable Lie algebras with a codimension one nilradical over ℝ, Shabanskaya, Thesis (Ph.D.)–The University of Toledo. 2011. 210 pp. ISBN: 978-1124-69251-7
- [41] A. Strominger, Superstrings with torsion, Nucl. Phys. B 274 (1986), 253–284.
- [42] V. Tosatti, B. Weinkove, The Chern-Ricci flow, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 33 (2022), no.1, 73–107.
- [43] L.Ugarte, Hermitian structures on six-dimensional nilmanifolds, Transform. Groups 12 (2007), 175–202.
- [44] I. Vaisman, On locally conformal almost Kähler manifolds, Israel J. Math. 24 (1976), 338–351.
- [45] I. Vaisman, On locally and globally conformal Kähler manifolds, Trans. Amer. Math. Soc. 262 (1980), 533–542.
- [46] K. Yano, Differential geometry on complex and almost complex spaces, *Pergamon Press* (1965).

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