LOCALLY FINITE AND SOLVABLE SUBGROUPS OF SFIELDS

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In this paper we will determine all locally finite subgroups of a sfield and give a criterion for determining when the group algebra of an ascending solvable torsion free group has a quotient sfield.

NOTATION AND DEFINITIONS. A group G has property E if it can be embedded in a sfield D and property EE if every automorphism of the group G can be extended to be an automorphsim of D. For a more complete discussion see [5].

The result on locally finite groups depends mainly on a result of Amitsur (see [1]). A complete discussion of the following notation can be found in his paper.

π will denote the set of all primes and π_1 the set of all odd primes p such that 2 has odd order mod p. Let m and r be relatively prime integers. Put $s=(r-1, m)$, $t=m/s$ and $n=$ minimal integer satisfying $r^n\equiv 1 \bmod m$. Denote by $G_{m,r}$ a group generated by two elements A and B satisfying $A^m=1$, $B^n=A^t$ and $BAB^{-1}=A^r$. Denote by $G^{\infty}_{m,r}$ a group G which has a countable ascending tower of subgroups $\{H_i : 0\leqq i<\infty\}$ such that $G=\bigcup_{i=1}^{\infty} H_i$ and each H_i is isomorphic to G_{m_i,r_i}. T^*, O^*, I^* will denote the binary tetrahedral, octahedral and icosahedral groups.

Let p be a fixed prime dividing m.

$\alpha=\alpha_p$ is the highest power of p dividing m.

η_p is the minimal integer satisfying $r^{\eta_p}\equiv 1 \bmod (mp^{-\alpha})$.

μ_p is the minimal integer satisfying $r^{\mu_p}\equiv p^{\mu'} \bmod (mp^{-\alpha})$ for some integer μ'.

δ_p is the minimal integer such that $p^{\delta_p}\equiv 1 \bmod (mp^{-\alpha})$.

$\delta'_p=\mu_p\delta_p/\eta_p$.

CONDITION C. Integers m and r satisfy Condition C if either

(I) $(n, t)=(s, t)=1$, or

(II) $n=2n'$, $m=2^{\alpha}m'$, $s=2s'$ where $\alpha\geqq 2$, m', s' and n' are odd integers; $(n, t)=(s, t)=2$ and $r\equiv -1 \bmod 2^{\alpha}$. And either

(III) $n=s=2$ and $r\equiv -1 \pmod m$, or

(IV) for every q/n there exists a prime $p\mid m$ such that $q\nmid\eta_p$ and that either

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(1) $p\neq 2$ and $(q, (p^{\delta'_p}-1)/s)=1$, or

(2) $p=q=2$, (II) holds and $m/4\equiv\delta_2'\equiv 1 \pmod 2$.

Amitsur proved the following (see [1]):

THEOREM 1. *A finite group has property* E *if and only if G is isomorphic to one of the following groups*:

(i) *cyclic group*,

(ii) $G_{m,r}$ *where m and r satisfy Condition* C,

(iii) $T^*\times G_{m,r}$ *where* $G_{m,r}$ *is either cyclic of order m or of type* (2), $(G, |G_{m,r}|)=1$ *and* $p|m$ *implies p in* π_1,

(iv) O^* *and* I^*.

We will prove the following generalization of Amitsur's Theorem.

THEOREM 2. *Let G be a locally finite group. The following are equivalent:*

(1) *G has property* E,

(2) *G has property* EE,

(3) *G is isomorphic to*

(i) *a subgroup of* $\prod_{p\in\pi} Z(p^\infty)$,

(ii) $G^\infty_{m,r}$ *where* m_i *and* r_i *satisfy Condition* C,

(iii) $T^*\times H$ *where either* (a) *H is a subgroup of* $\prod_{p\in\pi_1} Z(p^\infty)$, (b) $H=G^\infty_{m,r}$ *where* $(|G_{m_i,r_i}|, 6)=1$, $p|m_i$ *implies p is in* π_1 *and* m_i *and* r_i *satisfy Condition* C,

(iv) O^* *and* I^*.

Theorem 2 for countable locally finite groups was proved in [6]. Thus to prove Theorem 2 it is sufficient to show that any locally finite subgroup of a sfield is countable.

Let G be a locally finite group which can be embedded in a division ring. If S is a subset of G then $\langle S\rangle$ will denote the subgroup of G generated by S.

LEMMA 1. *If G has a subgroup isomorphic to* O^* *or* I^* *then G is isomorphic to* O^* *or* I^*.

PROOF. O^* and I^* are the only groups satisfying Theorem 1 which are not solvable of length 3 or less.

In the remaining lemmas it will be assumed that G has no subgroups isomorphic to O^* or I^*.

LEMMA 2. *If G has* T^* *as a subgroup then there is a subgroup H of G such that* $G=T^*\times H$ *and every finite subgroup of H satisfies* (I) *of Condition* C.

PROOF. Let $R=C_G(T^*)$. Let $H=\{x\in C_G(T^*)|\ |x| \text{ is odd}\}$. If

x, $y \in H$ and $g \in G$, then $\langle x, y, g, T^* \rangle = T^* \times S$ where S is a subgroup of odd order satisfying (I) of Condition C by Theorem 1. $|Z(T^*)| = 2$, thus x, $y \in S$. Therefore H is a subgroup. Also $g = t \cdot s$ with $t \in T^*$, $s \in S \subseteq H$. Hence $G = T^* \times H$.

A *factor* of a group G is a quotient group of a subgroup of G. The *rank* of an abelian group G will be k if it has an elementary abelian factor of order p^k for some prime p but no elementary abelian factor of order p^{k+1} for some prime p. The *derived factors* of a group G are the quotient groups G_i/G_{i+1} where G_i is the ith derived group.

LEMMA 4. *If G is a locally finite group embeddable in a sfield then $G \cong K \times H$ where K is a finite group and H is solvable of length $\leqq 2$ and each derived factor of K is of rank $\leqq 2$.*

PROOF. By Lemmas 1, 2 and Theorem 1, $G = K \times H$ with $K \cong O^*$, I^* or T^* and every finite subgroup of H is metacyclic. Since a metacyclic group is solvable of length $\leqq 2$ and each derived factor is of rank $\leqq 2$, the same is true of H.

LEMMA 5. *Let G be a solvable group of derived length n. If for some k each of the derived factors has rank at most k, then G is countable.*

PROOF. If A is an uncountable abelian group then it does not have finite rank. Thus G_i/G_{i+1} is countable for each i and hence G is countable.

Theorem 2 is a consequence of Lemmas 4 and 5.

It follows that in a sfield there is a maximal locally finite subgroup and it is countable.

If G is a finite subgroup of a sfield D and generates D then the automorphism group of D is determined by the automorphism group of G modulo the inner automorphism group of D (see [**7**]). It would be interesting to know if this is also true for a locally finite group.

Before proving the next theorem we will give some more definitions and notation.

A group G is *ascending solvable* if G has an ascending normal series such that each factor is abelian [**9**, p. 163]. If K is a sfield and G a group, then $K[G]$ will denote the *group algebra* of G over K. If θ is an automorphism of K and x an indeterminate over K, $K[x, \theta]$ will denote the *Ore polynomial ring* in x over K determined by θ (see [**11**]). A ring R is *regular* if it has no divisors of zero and if for elements a and b in R there are nonzero elements a_1, b_1, a_2 and b_2 in R such that $aa_1 = bb_1$ and $a_2a = b_2b$. F is a *quotient sfield* of R if R is a subring of the sfield F and for any f in F, $f = r_1r_2^{-1} = r_3^{-1}r_4$ for r_i in R $(1 \leqq i \leqq 4)$.

The above are connected by the following proposition proved by Asano [2].

PROPOSITION 1. *A ring R has a quotient sfield if and only if R is a regular ring. The quotient sfield is unique up to isomorphism. Any automorphism θ of R can be extended uniquely to F.*

The following result was proved by Ore [11]:

PROPOSITION 2. *$K[x;\theta]$ is a regular ring.*

The quotient sfield of $K[x;\theta]$ will be denoted by $K(x;\theta)$. If the group algebra $K[G]$ is a regular ring, its quotient sfield will be denoted by $K(G)$.

The following is a special case of a theorem of Bovdi [3]:

PROPOSITION 3. *If G is an ascending solvable group and K a sfield, then the group algebra $K[G]$ has a quotient sfield if and only if $K[G]$ has no divisors of zero.*

We will give a criterion to determine when the group algebra has no divisors of zero. The following definition and proposition is due to Ore [10].

Let R be a regular ring and $A=(a_{ij})$ be an $n\times n$ matrix with coefficients in R. We now give a determinant function $|\ \|$ for A.

If $n=1$, define $|A\|=a_{11}$. If $n=2$, define $|A\|=a_{11}A_{22}-a_{21}A_{12}$ where A_{22} and A_{12} have the property that $a_{12}A_{22}=a_{22}A_{12}$. For n define $|A\|=a_{11}A_1^{(1)}+a_{21}A_1^{(2)}+\cdots+a_{n1}A_1^{(n)}$ where the $A_1^{(j)}$ are a set of solutions to the homogeneous equations

$$\begin{gathered} a_{12}A_1^{(1)}+a_{22}A_1^{(2)}+\cdots+a_{n2}A_1^{(n)}=0 \\ \vdots \\ a_{1n}A_1^{(1)}+a_{2n}A_1^{(2)}+\cdots+a_{nn}A_1^{n}=0. \end{gathered}$$

The function $|\ \|$ has the following important property:

PROPOSITION 4. *The linear system*

$$\sum_{i=1}^{n} x_i a_{ij}=0 \qquad (1\leqq j\leqq n) \quad a_{ij}\in R$$

has a nontrivial solution if and only if $|a_{ij}\|=0$.

THEOREM 3. *Let G be an ascending solvable group with an ascending normal series $\{H_i\}_{i=0}^{\alpha}$ with $H_{i+1}/H_i=\langle x_{i+1}\cdot H_i\rangle$. Let n_{i+1} be the order of x_{i+1}* mod *H_i and let θ_{i+1} be the automorphism of $K[H_i]$ induced by x_{i+1}. $K[G]$ has a quotient sfield if and only if for each i that n_{i+1} is* *finite and for all* n_{i+1} *tuples* $(d_0, \cdots, d_{n_{i+1}})$ *of elements of* $K[H_i]$, $|a_{kj}| \neq 0$ *where*

$$a_{kj} = \theta_{i+1}^{j}(d_{k-j}) \quad \text{if } k \geqq j,$$
$$= \theta_{i+1}^{j}(d_{n_{i+1}+k-j})x_{i+1}^{n_{i+1}} \quad \text{if } k < j.$$

Proof. By Proposition 3 it is sufficient to show that $K[G]$ has no divisors of zero. Assume that $K[H_i]$ has no divisors of zero for all $i<j\leqq\alpha$. If j is a limit ordinal then $K[H_j]$ has no divisors of zero. If j is not a limit ordinal then there are two possibilities, either x_j has infinite order mod H_{j-1} or finite order mod H_{j-1}. In the first case it is obvious that $K[H_j]$ has no divisors of zero.

Therefore we are left with considering the case of G and H groups with H normal in G, $G/H=\langle x\cdot H\rangle$ where x has order n mod H and $K[H]$ has no divisors of zero. Let θ be the automorphism of $K[H]$ induced by x. Every element of $K[G]$ can be written uniquely in the form $\sum_{i=0}^{n-1} d_i x^i$ for d_i in $K[H]$. Let $w=\sum_{i=0}^{n-1} d_i x^i$ be a fixed element of $K[G]$. If w is a divisor of zero, then there is a $y=\sum_{i=0}^{n-1} y_j x^j$ with y_j in $K[H]$ such that $yw=0$. Thus

$$\sum_{j=0}^{n-1} y_j(x^j w) = \sum_{j=0}^{n-1} y_j \left(\sum_{i=0}^{n} \theta^j(d_i)x^{i+j}\right) = 0.$$

This breaks down into the following system of n-equations in the unknowns y_i.

$$\begin{aligned} y_0d_0 + y_1\theta(d_{n-1})x^n + \cdots + y_{n-1}\theta^{n-1}(d_1)x^n &= 0\\ y_0d_1 + y_1\theta(d_0) + \qquad\qquad + y_{n-1}\theta^{n-1}(d_2)x^n &= 0\\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad & \quad \vdots\\ v_0d_{n-1} + y_1\theta(d_{n-2}) + \cdots + y_{n-1}\theta^{n-1}(d_0) &= 0. \end{aligned}$$

Application of Proposition 4 completes the proof.

The determinant given in Theorem 3 is very complicated. For $n=2$ or 3 we will give a simpler form of the same expression. This will be done in the next two lemmas.

Lemma 6. *Let K be a sfield $0\neq b\in K$ and θ an automorphism of K such that $\theta(b)=b$ and θ^n is the same automorphism of K as the automorphism induced by b in K. Consider the ring $K[x;\theta]$. The following are equivalent.*

(1) *x^n-b is irreducible.*

(2) *(x^n-b) is a prime ideal.*

(3) $K[x;\theta]/(x^n-b)$ *is a sfield.*
Conditions (1), (2) *and* (3) *imply that* (x^n-b) *is maximal but the converse is not true.*

PROOF. $K[x;\theta]/(x^n-b)$ is a sfield if and only if it has no divisors of zero [**8**, p. 158]. x^n-b reducible implies that (x^n-b) is not prime so (2) implies (1). If $K[x;\theta]/(x^n-b)$ has no divisors of zero then x^n-b is prime so (3) implies (2). If x^n-b is irreducible and $f(x)$ is not in (x^n-b), then there are elements $f_1(x)$ and $g_1(x)$ in $K[x;\theta]$ such that $f(x)f_1(x)+(x^n-b)g_1(x)=1$ (see [**11**]). Thus $f(x)$ has a right inverse (mod (x^n-b)) so (1) implies (3).

Since $K[x;\theta]$ is a principal ideal ring (see [**11**]), certainly (x^n-b) prime implies that (x^n-b) is maximal.

Let ϵ be a primitive 4th root of unity and x a transcendental over Q, the rational numbers. Let $F=Q(\epsilon, x)$, the field obtained by adjoining ϵ and x, and let θ be the automorphism of F determined by $\epsilon\to\epsilon$ and $x\to\epsilon x$. Consider the ring $F[y;\theta]$ and the ideal (y^4-x^4). Direct calculation verifies that (y^4-x^4) is maximal but not prime.

LEMMA 7. *With the same hypothesis as in Lemma* 6, $b\neq\theta^{n-1}(a)\theta^{n-2}(a)\cdots\theta(a)a$ *for all a in K is a necessary condition for* x^n-b *to be irreducible polynomial in* $K[x;\theta]$. *For* $n=2$ *or* 3 *it is also sufficient.*

PROOF. Let $N(a)=\theta^{n-1}(a)\cdots\theta(a)\cdot a$. If $b=N(a)$, then

$$x^n - b = (x^{n-1} + \theta^{n-1}(a)x^{n-2} + \cdots + N(a)\theta(a)^{-1}a^{-1}x + N(a)a^{-1})\cdot(x - a).$$

Thus the condition is necessary. If $n=2$ and $x^2-b=(x-c)(x-a)$ for a and c in K, then $ca=-b$ and $-c-\theta(a)=0$. Thus $b=\theta(a)a$. If $n=3$ and $x^3-b=(x^2+dx+c)(x-a)$ for a, b and c in K, then just in the case $n=2$ one can verify that $b=\theta^2(a)\theta(a)a$.

With these lemmas it is easily seen that Theorem 3 can be modified to give

THEOREM 4. *Let G be an ascending solvable group with an ascending normal series* $\{H_i\}_{i=0}^{\alpha}$ *with* $H_{i+1}/H_i=\langle x_{i+1}\cdot H_i\rangle$. *Let* n_{i+1} *be the order of* x_{i+1} mod H_i *and let* θ_i *be the automorphism of* $K[H_i]$ *induced by* x_{i+1}. *If* $n_{i+1}=2, 3$ *or* ∞ *for each i, then* $K[G]$ *has a quotient sfield if and only if for each i that* n_{i+1} *is finite,* $x_{i+1}^{n_{i+1}}\neq\theta_{i+1}^{n_{i+1}-1}(d)\cdots\theta(d)\cdot d$ *for all d in* $K(H_i)$.

The theory developed in Theorem 4 would be included in the results of [**5**] if the group algebra of an ascending solvable group G having no divisors of zero implies that G has a normal series such that every factor is infinite cyclic. This is not true. Consider the group $G=\langle b_1,\ b_2,\ c \mid c^{-1}b_ic=b_i^{-1}\ (1\leqq i\leqq 2),\ b_1^{-1}b_2^{-1}b_1b_2=c^4\rangle$. G does not have a normal series such that each factor is infinite cyclic (see [4]) but direct calculation verifies that it satisfies Theorem 4.

References

1. S. A. Amitsur, *Finite subgroups of division rings*, Trans. Amer. Math. Soc. **80** (1955), 361–386.

2. K. Asano, *Uber die Quotientenbildung von Schiefringen*, J. Math. Soc. Japan **1** (1949), 73–78.

3. A. A. Bovdi, *Imbedding of crossed products in fields*, Soviet Math. Dokl. **2** (1963), 1157–1159.

4. J. F. Bowers, *On composition series of polycyclic groups*, J. London Math. Soc. **35** (1960), 433–444.

5. R. J. Faudree, *Subgroups of the multiplicative group of a division ring*, Trans. Amer. Math. Soc. **124** (1966), 41–48.

6. ———, *Embedding theorems for ascending nilpotent groups*, Proc. Amer. Math. Soc. **18** (1967), 148–154.

7. ———, *Automorphism groups of finite subgroups of division rings*, Pacific J. Math. **26** (1968), 59–65.

8. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

9. A. G. Kurosh, *Theory of groups*, Vol. II, Chelsea, New York, 1960.

10. O. Ore, *Linear equations in non-commutative fields*, Ann. of Math. **32** (1931), 463–477.

11. ———, *Theory of non-commutative polynomials*, Ann. of Math. **34** (1933), 480–508.

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