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Locally Generated Semigroups.

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LOCALLY GENERATED SEMIGROUPS

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DEDICATION

To my wife Cecilia, and my children Andrea, Veronica, and Eduardo.

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ABSTRACT

For a topological semigroup S, Lawson constructed a semigroup $\Gamma(S)$ with the property that any local homomorphism defined in a neighborhood of the identity of S to a topological semigroup T extends uniquely to a global homomorphism defined on $\Gamma(S)$. In this work we obtain conditions on S to topologize the semigroup $\Gamma(S)$ via an uniformity such that the extended homomorphism is continuous and such that $\Gamma(S)$ is a topological semigroup. We also investigate a different approach of the problem via the relatively free semigroup $RF(U)$ where U is a suitable neighborhood of the identity of S and show that $RF(U)$ is isomorphic to $\Gamma(S)$.

CHAPTER 1

1.1 Introduction.

A very interesting property of the universal covering Lie group $\tilde{G}$ is that any continuous local homomorphism defined in a neighborhood of the identity of the group G into a topological group H extends uniquely to a continuous global homomorphism from $\tilde{G}$ into H. One way of constructing the universal covering group $\tilde{G}$ is by forming the homotopy classes of paths originating from the identity. In [9] this construction is modified for a topological semigroup S. A certain subclass of paths originating from the identity, called causal paths, are considered and the notion of causal homotopy is defined. The semigroup $\Gamma(S)$ of causal homotopic classes of paths is considered and it is showed that this semigroup satisfies a universal property.

In this work we obtain conditions on S to topologize the semigroup $\Gamma(S)$ via a uniformity. With this topology any continuous local homomorphism defined in a neighborhood of the semigroup S into a topological semigroup T extends uniquely to a continuous homomorphism from $\Gamma(S)$ into T. We also establish a functorial property of $\Gamma(S)$ and show that it preserves the semidirect product of semigroups.

We investigate a different approach to the problem of topologizing $\Gamma(S)$ by constructing the relatively free semigroup $RF(U)$, where U is a suitable neighborhood of the identity of S. The relatively free semigroup is obtained from the free semigroup $Fr(U)$ by dividing out a closed congruence relation. We prove that $\Gamma(S)$ is isomorphic to $RF(U)$ and show some techniques to construct $\Gamma(S)$ for some semigroups.

This is an introductory chapter. Here, we present a recapitulation of the fundamental results that are necessary to develop the main part of this work, which is given in the next two chapters. In chapter 2, we deal with the problem of topologizing $\Gamma(S)$, and in chapter 3, we construct the relatively free semigroup $RF(U)$ and show that it is actually isomorphic to $\Gamma(S)$.

1.2 Relations

A *relation* on a set X is a subset of the cartesian product $X \times X$. If U is a relation the inverse relation U^{-1} is the set of all pairs (x, y) such that $(y, x) \in U$. It is easy to see that $(U^{-1})^{-1} = U$. If $U = U^{-1}$ then U is called *symmetric*. If U and V are relations, then the composition $U \circ V$ is the set formed by all the pairs (x, z) such that for some y it is true that $(x, y) \in U$ and $(y, z) \in V$. The composition is an associative operation on the set of all the relations on X, in other words, $U \circ (V \circ W) = (U \circ V) \circ W$, and it is always true that $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$. The set of all pairs (x, x) for $x \in X$ is called the *diagonal*, or the *identity relation* and it is denoted by Δ. If $x \in X$ we define $U[x]$ as the set of all $y \in X$ such that $(x, y) \in U$. The relation U is called *transitive* if $(x, y) \in U$ and $(y, z) \in U$ implies that $(x, z) \in U$, i.e., U is transitive if and only if $U \circ U \subset U$. If $\Delta \subset U$ then U is called *reflexive*. A relation U is called an *equivalence relation* if it is symmetric, reflexive, and transitive. The subset $\bar{x} = \{y \in X : (x, y) \in U\}$ is called the *class* of x *modulo* U. The set X is a disjoint union of all the classes modulo U, and the set of all classes, denoted by X/U, is called the *quotient set of* X *modulo* U.

1.3 Semigroups and congruence relations

1.3.1 Definition. A semigroup is a non-empty set S together with an associative multiplication $(x, y) \to xy$ from $S \times S$ into S. If S has a Hausdorff topology such that multiplication is continuous, with the product topology on $S \times S$, then S is called a *topological semigroup*.

1.3.2 Definition. If S and T are semigroups, a function $h : S \to T$ is called a *homomorphism* if $h(xy) = h(x)h(y)$ for each $x, y \in S$. If h is one-to-one and onto,

h is called an *isomorphism*. If S and T are topological semigroups and h is both an isomorphism and a homeomorphism, then h is called a *topological isomorphism*.

1.3.3 Definition. A relation R on a semigroup S is said to be *left* [*right*] *compatible* if $(a,b) \in R$ and $x \in S$ implies that $(xa, xb) \in R$ [$(ax, bx) \in R$], and *compatible* if it is both left and right compatible. A compatible equivalence relation on a semigroup is called a *congruence relation*.

1.3.4 Proposition. Let S be a semigroup and let R be a congruence on S. Then S/R is a semigroup under multiplication defined by $(\pi(x), \pi(y)) \to \pi(xy)$, and $\pi : S \to S/R$ is onto.

The following result is known as the Lawson–Madison theorem. For a proof see [2].

1.3.5 Theorem. Let S be a locally compact σ-compact topological semigroup and let R be a closed congruence relation on S. Then S/R is a topological semigroup.

If S and T are semigroups and $\phi : S \to T$ is a homomorphism, we denote by $K(\phi)$ the relation $\{(x,y) \in S \times S : \phi(x) = \phi(y)\}$.

The following theorem is known as the first isomorphism theorem. The proof is straightforward and is left to the reader.

1.3.6 Theorem. Let S and T be semigroups and let $\phi : S \to T$ be a surjective homomorphism. Then $K(\phi)$ is a congruence on S and there exist a unique algebraic isomorphism $\psi : S/K(\phi) \to T$ such that $\psi \circ \pi = \phi$ where $\pi : S \to S/K(\phi)$ is the canonical map.

1.4 Uniformities and the uniform topology

In this section we recall standard facts about uniform spaces (see [7]).

1.4.1 Definition. A *uniformity* for a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ such that:

a) each member of $\mathcal{U}$ contains the diagonal,

b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,

c) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some V in $\mathcal{U}$,

d) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$,

e) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a *uniform space*.

1.4.2 Definition. A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

If $\mathcal{B}$ is a base for $\mathcal{U}$, then $\mathcal{B}$ determines $\mathcal{U}$ entirely, for a subset U of $X \times X$ belongs to $\mathcal{U}$ iff U contains a member of $\mathcal{B}$.

1.4.3 Definition. A subfamily $\mathcal{S}$ is a *subbase* for $\mathcal{U}$ if the finite intersections of members of $\mathcal{S}$ form a base for $\mathcal{U}$.

1.4.4 Theorem. A family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if:

a) each member of $\mathcal{B}$ contains the diagonal Δ,

b) if $U \in \mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$,

c) if $U \in \mathcal{B}$, then $V \circ V \subset U$ for some $V \in \mathcal{B}$,

d) the intersection of two members of $\mathcal{B}$ contains a member of $\mathcal{B}$.

Proof. The proof of this theorem is straightforward and is left to the reader. For the if part, consider the set $\mathcal{U}$ formed by all subsets of $X \times X$ that contains a member of $\mathcal{B}$, show now that $\mathcal{U}$ is a uniformity having $\mathcal{B}$ as a basis. ∎

1.4.5 Theorem. A family $\mathcal{S}$ of subsets of $X \times X$ is a subbase for some uniformity for X if:

a) each member of $\mathcal{S}$ contains the diagonal Δ,

b) for each $U \in \mathcal{S}$ the set U^{-1} contains a member of $\mathcal{S}$,

c) for each $U \in \mathcal{S}$ there is $V \in \mathcal{S}$ such that $V \circ V \subset U$.

In particular, the union of any collection of uniformities for X is the subbase for a uniformity for X.

Proof. Let $\mathcal{B}$ be the set of all finite intersections of members of $\mathcal{S}$. Since each member of $\mathcal{S}$ contains the diagonal Δ, the same is true for any finite intersection of members of $\mathcal{S}$.

Take $U = U_1 \cap U_2 \cap \ldots \cap U_n$ with $U_i \in \mathcal{S}$ for all i, then there exists $V_i \in \mathcal{S}$ with $V_i \subset U_i^{-1}$, if $V = \cap V_i$ then $V \subset U^{-1}$.

Take again $U = U_1 \cap U_2 \cap \ldots \cap U_n$ with $U_i \in \mathcal{S}$, there exists $V_i \in \mathcal{S}$ such that $V_i \circ V_i \subset U$. If $V = \cap V_i$ then $V \circ V \subset U$.

Finally the intersection of any two members of $\mathcal{B}$ is again a member of $\mathcal{B}$.

We have proved that all the conditions of last theorem are satisfied. Therefore the family $\mathcal{B}$ is a base for some uniformity for $\mathcal{U}$. ∎

If $(X,\mathcal{U})$ is a uniform space, the topology $\mathcal{T}$ of the uniformity $\mathcal{U}$, or the *uniform topology*, is the family of all subsets T of X such that for each $x \in T$ there is a $U \in \mathcal{U}$ such that $U[x] \subset T$.

1.4.6 Proposition. The family $\mathcal{T}$ defined above is indeed a topology for X.

Proof. Consider the family $\{T_\alpha : \alpha \in \Omega\}$ where $T_\alpha \in \mathcal{T}$ for each $\alpha \in \Omega$, if $x \in \cup_{\alpha\in\Omega} T_\alpha$ then $x \in T_\alpha$ for some $\alpha \in \Omega$, hence there exists $U \in \mathcal{U}$ such that $U[x] \subset T_\alpha$ and therefore $U[x] \subset \cup_{\alpha\in\Omega} T_\alpha$. If T_1 and T_2 are members of $\mathcal{T}$, and if $x \in T_1 \cap T_2$ there exists U and V elements of $\mathcal{U}$ such that $U[x] \in T_1$ and $V[x] \in T_2$. Therefore, $U[x] \cap V[x] = (U \cap V)[x] \subset T_1 \cap T_2$. Which proves that $\mathcal{T}$ is a topology for X. ∎

1.4.7 Theorem. The interior of a subset A of X relative to the uniform topology is the set of all points x such that $U[x] \subset A$ for some $U \in \mathcal{U}$.

Proof. We have to show that the set

$$B = \{x \in A : U[x] \subset A, \text{ for some } U \text{ in } \mathcal{U}\}$$

is open relative to the uniform topology. It is clear that that set B surely contains any open subset of A, if we show that B is open, then it must necessarily be the interior of A. If $x \in B$, then there is a member U of $\mathcal{U}$ such that $U[x] \subset A$ and there is V in $\mathcal{U}$ such that $V \circ V \subset U$. If $y \in V[x]$, then $V[y] \subset (V \circ V)[x] \subset U[x] \subset A$, and hence $y \in B$. Therefore $V[x] \subset B$ and B is open.

It follows immediately that $U[x]$ is a neighborhood of x for each U in the uniformity $\mathcal{U}$, and consequently the family of all sets $U[x]$ for $U \in \mathcal{U}$ is a base for the neighborhood system of x. Therefore we have the following result.

1.4.8 Theorem. If $\mathcal{B}$ is a base (or subbase) for a uniformity $\mathcal{U}$, then for each x the family of sets $U[x]$ for $U \in \mathcal{B}$ is a base (subbase respectively) for the neighborhood system of x.

1.4.9 Lemma. If V is symmetric, then $V \circ U \circ V = \cup\{V[x] \times V[y] : (x, y) \in U\}$.

Proof. By definition

$$\begin{aligned} V \circ U \circ V &= \{(u, v) : (u, x) \in V, (x, y) \in U, (y, v) \in V; \quad \text{for some} \quad x, y \in X\} \\ &= \{(u, v) : u \in V[x], v \in V[y], (x, y) \in U\}. \end{aligned}$$

But $u \in V[x]$ and $v \in V[y]$ if and only if $(u, v) \in V[x] \times V[y]$, hence

$$\begin{aligned} V \circ U \circ V &= \{(u, v) : (u, v) \in V[x] \times V[y] \quad \text{for some} \quad (x, y) \in U\} \\ &= \cup\{V[x] \times V[y] : (x, y) \in U\}. \end{aligned}$$

∎

1.4.10 Theorem. If U is a member of the uniformity $\mathcal{U}$, then the interior of U is also a member of $\mathcal{U}$; consequently the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

Proof. If $M \subset X \times X$; then the interior of M is the set of all pairs (x, y) such that $U[x] \times V[y] \subset M$ for some $V, U \in \mathcal{U}$. Since $U \cap V \in \mathcal{U}$ it is easy to see that

$$\text{int}(M) = \{(x, y) : V[x] \times V[y] \subset M \quad \text{for some} \quad V \in \mathcal{U}\}.$$

If $U \in \mathcal{U}$ there exists a symmetric member V of $\mathcal{U}$ such that $V \circ V \circ V \subset U$ and according to the previous lemma $V \circ V \circ V = \cup\{V[x] \times V[x] : (x, y) \in V\}$. Therefore $V \subset \text{int}(U)$. Which implies that $\text{int}(U) \in \mathcal{U}$. ∎

1.4.11 Theorem. The closure relative to the uniform topology of a subset A of X is $\cap\{U[A] : U \in \mathcal{U}\}$. The closure of a subset M of $X \times X$ is $\cap\{U \circ M \circ U : U \in \mathcal{U}\}$.

Proof. $x \in \overline{A}$ if and only if $U[x] \cap A \neq \emptyset \quad \forall U \in \mathcal{U}$. But $U[x] \cap A \neq \emptyset$ iff $x \in U^{-1}[A]$, and since each member of $\mathcal{U}$ contains a symmetric member, $x \in \overline{A}$ iff $x \in U[A]$ for each $U \in \mathcal{U}$. The first statement is then proved. Similarly, if U is a symmetric member of $\mathcal{U}$, then $U[x] \times U[y] \cap M \neq \emptyset$, $M \subset X \times X$, iff $(x, y) \in U[u] \times U[v]$ for some $(u, v) \in M$, that is, iff $(x, y) \in \cup\{U[u] \times U[v] : (u, v) \in M\}$. By the lemma, this last set is equal to $U \circ M \circ U$. So $(x, y) \in \overline{M}$ iff $(x, y) \in U \circ M \circ U$ for each $U \in \mathcal{U}$, i.e.,

$$\overline{M} = \cap\{U \circ M \circ U : u \in \mathcal{U}\}.$$

∎

1.4.12 Theorem. The family of closed symmetric members of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$.

Proof. If $U \in \mathcal{U}$, and V is a member of $\mathcal{U}$ such that $V \circ V \circ V \subset U$, then $\overline{V} \subset V \circ V \circ V$ in view of the preceding theorem; hence U contains a closed member W, take for example $W = \overline{V}$, and $W \cap W^{-1}$ is a closed symmetric member of $\mathcal{U}$ contained in U. ∎

1.4.13 Theorem. A uniform space is a regular topological space.

Proof. By the preceding theorem, the family

$$\{V[x] : V \text{ is a closed symmetric member of } \mathcal{U}\},$$

is a basis for the neighborhood system of x. ∎

It follows immediately for this last theorem that a uniform space is a Hausdorff topological space if and only if each set consisting of a single point is closed.

1.5 Uniform structure on groups

In a topological group G we can define a couple of uniformities, called the left and right uniformities respectively, such that the uniform topologies induced by them are compatible with the original topology of the group. Let's define the right uniformity. The left uniformity can be defined similarly.

Let's denote by $\mathcal{U}$ the neighborhood system of the identity of the group G. For $V \in \mathcal{U}$ we write $V_d = \{(x, y) \in G \times G : xy^{-1} \in V\}$. Consider the set $\mathcal{A} = \{V_d : V \in \mathcal{U}\}$. It is clear that the diagonal Δ is contained in V_d for each $V \in \mathcal{U}$. Since the relations $yx^{-1} \in V$ and $xy^{-1} \in V^{-1}$ are equivalent, we have that $V_d^{-1} = (V^{-1})_d$, and hence $V_d^{-1} \in \mathcal{A}$. It is also clear that $(U \cap V)_d \subset U_d \cap V_d$. Finally, given $U \in \mathcal{U}$, pick $V \in \mathcal{U}$ such that $V^2 \subset U$ then it is easy to show that $V_d \circ V_d \subset U_d$. Therefore the set $\mathcal{A}$ satisfies the conditions of definition 1.3.1, and hence it is an uniformity for the group G. Since $V_d[x] = Vx$ we have that the topology induced by the right uniformity coincides with the original topology of the group G. In general the right and the left uniformities of a group G are different, but they define the same topology on the group G.

The following is a very useful result about topological groups.

1.5.1 Proposition. Let G be a topological group. The uniformity defined by $(\{(x, y) : Vx \cap Vy \neq \emptyset\})$ $\{(x, y) : xV \cap yV \neq \emptyset\}$ is the left uniformity (is the right uniformity) respectively.

Proof. For $V \in \mathcal{U}$, set $V_l = \{(x,y) : y^{-1}x \in V\}$; $V_{l'} = \{(x,y) : xV \cap yV \neq \emptyset\}$; $\mathcal{A} = \{V_l : v \in \mathcal{U}\}$; and $\mathcal{B} = \{V_{l'} : v \in \mathcal{U}\}$. We prove that $\mathcal{A} = \mathcal{B}$ by showing that any element of $\mathcal{A}$ contains an element of $\mathcal{B}$ and that any element of $\mathcal{B}$ contains an element of $\mathcal{A}$. Indeed, pick $V \in \mathcal{U}$, if $(x,y) \in V_l$, then $y^{-1}x \in V$, this implies that $xV \cap yV \neq \emptyset$. Therefore, $V_l \subset V_{l'}$. Hence $\mathcal{B} \subset \mathcal{A}$. On the other hand, if $V \in \mathcal{U}$, pick $W \in \mathcal{U}$ such that $WW^{-1} \subset V$. If $(x,y) \in W_{l'}$ then $xW \cap yW \neq \emptyset$. This implies that $y^{-1}x \in WW^{-1} \subset V$. Therefore $W_{l'} \subset V_l$. Hence $\mathcal{A} \subset \mathcal{B}$. ∎

1.5.2 Proposition. Let G be a topological group and let $S \subset G$ be a subsemigroup such that the identity of G is in $\overline{\mathrm{int}(S)}$. Then the uniformities defined by the following bases are equal:

1) $\{V_a : V \in \mathcal{U}\}$, where $V_a = \{(x,y) \in S \times S : x^{-1}y \in V\}$,
2) $\{V_b : V \in \mathcal{U}\}$, where $V_b = \{(x,y) \in S \times S : xV \cap yV \neq \emptyset\}$,
3) $\{V_c : V \in \mathcal{U}\}$, where $V_c = \{(x,y) \in S \times S : x(V \cap S) \cap y(V \cap S) \neq \emptyset\}$.

Proof. That the uniformities defined by the bases 1) and 2) are equal follows immediately from 1.5.1. Let's prove that that the uniformities defined by the bases 2) and 3) are equal. Indeed, pick $V \in \mathcal{U}$; if $x(V \cap S) \cap y(V \cap S) \neq \emptyset$, then $xV \cap yV \neq \emptyset$. Therefore $V_c \subset V_b$ this implies that V_b is in the uniformity generated by $\{V_c : V \in \mathcal{U}\}$. Hence, the uniformity generated by the base defined by 2) is contained in the uniformity generated by the base 3). To show the other inclusion, pick $V \in \mathcal{U}$. Set $Q = V \cap \mathrm{int}(S) \neq \emptyset$, since $1_G \in \overline{\mathrm{int}(S)}$. Take $s \in Q$ and set $W = Qs^{-1}$, then $W \in \mathcal{U}$. We show that $W_b \subset V_c$. If $(x,y) \in W_b$ then $xW \cap yW \neq \emptyset$, i.e., there exist $q_1, q_2 \in Q$ such that $xq_1s^{-1} = yq_2s^{-1}$. This implies that $xq_1 = yq_2$, hence $xQ \cap yQ \neq \emptyset$. But $Q = V \cap \mathrm{int}(S) \subset V \cap S$, therefore $x(V \cap S) \cap y(V \cap S) \neq \emptyset$, and therefore $W_b \subset V_c$. This proves that $\{V_c : V \in \mathcal{U}\} \subset \{V_b : V \in \mathcal{U}\}$ and therefore the uniformity generated by the base defined by 3) is contained in the uniformity generated by the base defined by 2). ∎

1.6 Wallace's lemma

The following theorem is one of the most useful tools in the area of topological semigroups.

1.6.1 Theorem. Let X, Y and Z be topological spaces, A a compact subset of X, B a compact subset of Y, $f : X \times Y \to Z$ a continuous function, and W an open subset of Z containing $f(A \times B)$. Then there exists an open set U in X and an open set V in Y such that $A \subset U$, $B \subset V$, and $f(U \times V) \subset W$.

Proof. Since f is continuous, $f^{-1}(W)$ is an open set in $X \times Y$ containing $A \times B$. For each $(x, y) \in A \times B$, there exist open sets M and N in X and Y, respectively, such that $x \in M$, $y \in N$, and $M \times N \subset f^{-1}(W)$. Since B is compact, for fixed $x \in A$, there are open sets $M_1, \ldots, M_n$ in X containing x and corresponding open sets $N_1, \ldots, N_n$ in Y such that $B \subset Q = N_1 \cup \ldots \cup N_n$. Let $P = M_1 \cap \ldots \cap M_n$. Then P is open in X, Q is open in Y, $x \in P$, $B \subset Q$, and $P \times Q \subset f^{-1}(W)$. Since A is compact, there exist open sets $P_1, \ldots, P_m$ in X and corresponding $Q_1, \ldots, Q_m$ open in Y such that $B \subset V = Q_1 \cap \ldots \cap Q_m$ and $A \subset U = P_1 \cup \ldots \cup P_m$. It follows that U and V are the required open sets. ∎

CHAPTER 2

2.1 Causal paths

We begin this section with the following definition:

2.1.1 Definition. Let S be a topological semigroup with identity 1_S. Let $\alpha : [0,1] \to S$ be a path on S such that $\alpha(0) = 1_S$. The path α is called a *causal path* if the following property is satisfied: Given U a neighborhood of 1_S, there exists $\epsilon > 0$ such that whenever $s, t \in [0,1]$ with $s < t < s + \epsilon$, then $\alpha(t) \in \alpha(s) \cdot U$, i.e., $\alpha(t) = \alpha(s)u$ for some $u \in U$. Given causal paths $\alpha : [0,1] \to S$ and $\beta : [0,1] \to S$, we define the *concatenation* $\alpha * \beta : [0,1] \to S$ by

$$\alpha * \beta = \begin{cases} \alpha(2t) & \text{for } 0 \le t \le 1/2, \\ \alpha(1)\beta(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

2.1.2 Definition. A subset W of a real topological vector space L is called a *cone* if it satifies the following conditions:

(i) $W + W \subset W$,

(ii) $\mathbf{R}^+ \cdot W \subset W$,

(iii) $\overline{W} = W$, that is, W is closed in L.

2.1.3 Example. In the additive semigroup $C = \{(x,y) \in \mathbf{R}^2 : x \ge 0, y \ge 0\}$ consider the path $\alpha(t) = tX$ where X is a unit vector in C. Given a neighborhood U of the identity of C pick a positive real number ϵ such that the intersection of C and the ball $B(0,\epsilon)$ is contained in U. If $s, t \in [0,1]$ with $s < t < s + \epsilon$, then $\alpha(t) = \alpha(s) + (t-s)X$. Clearly the vector $(t-s)X$ is in U. Observe that a similar argument shows that rays through the origin are causal paths in an arbitrary cone in $\mathbf{R}^n$.

The next example generalizes the preceding one.

2.1.4 Example. Let S be a topological semigroup, and $\alpha : [0,\infty] \to S$ a one parameter subsemigroup. Then $\alpha|_{[0,1]}$ is a causal path of S. Indeed, given a neighborhood U of the identity, pick $\epsilon > 0$ such that if $0 < x < \epsilon$, then $\alpha(x) \in U$. That can be done by the continuity at zero of the path α. If $s, t \in [0,1]$ with $s < t < s + \epsilon$, then $\alpha(t) = \alpha(s)\alpha(t-s)$. Since $t - s < \epsilon$, then $\alpha(t-s) \in U$.

2.1.5 Example. Let $\alpha : I \to G$ be any path in a group G such that $\alpha(0) = 1$, where I is the closed unit interval. Given a neighborhood of the identity U of G we can choose a positive number ϵ such that if $s, t \in I$ with $s < t < s + \epsilon$, then $\alpha(s)^{-1}\alpha(t) \in U$. Therefore in a group any path is a causal path.

2.1.6 Proposition. Let $h : S \to T$ be a continuous homomorphism from the topological semigroup S to the topological semigroup T. If α is a causal path in S, then $h(\alpha)$ is a causal path in T.

Proof. Let U be a neighborhood of 1_T. Then $h^{-1}(U)$ is a neighborhood of 1_S. Let $\epsilon > 0$ be such that if $s, t \in [0,1]$ with $s < t < s + \epsilon$, then $\alpha(t) = \alpha(s)u$ for some $u \in U$. Since h is a homomorphism we have that $(h \circ \alpha)(t) = (h \circ \alpha)(s)h(u)$ with $h(u) \in U$. ∎

2.1.7 Proposition. The concatenation of two causal paths in a topological semigroup is again a causal path.

Proof. Let α and β be causal paths in the topological semigroup S, and let U be a neighborhood of the identity of S. Pick a neighborhood W of the identity of S such that $W^2 \subset U$. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be chosen corresponding to W, in the definition of causal paths for α and β respectively. Take $0 \leq \epsilon < \min(\epsilon_1/2, \epsilon_2/2)$. Suppose that $0 \leq s < t < s + \epsilon$, then we have the following three cases:

1) $0 \leq s < t \leq 1/2$, 2) $1/2 \leq s < t$, and 3) $s < 1/2 < t$.

In case 1) $(\alpha * \beta)(s) = \alpha(2s)$ and $(\alpha * \beta)(t) = \alpha(2t)$. Since $2s < 2t < 2s + \epsilon_1$, we have that $\alpha(2t) = \alpha(2s)w$ for some $w \in W$ and therefore $(\alpha * \beta)(t) \in (\alpha * \beta)(s)U$.

In case 2) $(\alpha*\beta)(t) = \alpha(1)\beta(2t-1)$ and $(\alpha*\beta)(s) = \alpha(1)\beta(2s-1)$, again since $2s < 2t < 2s + \epsilon_2$, subtracting 1 from this inequality we get that $2s - 1 < 2t - 1 < 2s - 1 + \epsilon_2$, hence $\beta(2t-1) = \beta(2s-1)w$ for some $w \in W$. Multiplying this last identity by $\alpha(1)$ we get $(\alpha*\beta)(t) = (\alpha*\beta)(s)w$, i.e., $(\alpha*\beta)(t) \in (\alpha*\beta)(s)U$.

In case 3) we have that $2s < 1 < 2t < 2s+2\epsilon < 1+2\epsilon$, which implies that $2s < 1 < 2s + \epsilon_1$. Hence, $\alpha(1) = \alpha(2s)w_1$ for some $w_1 \in W$. Also, since $1 < 2t < 1 + 2\epsilon$ which is equivalent to $0 < 2t - 1 < \epsilon_2$ we have that $\beta(2t-1) = \beta(0)w_2 = w_2$ for some $w_2 \in W$. Now: $(\alpha*\beta)(t) = \alpha(1)\beta(2t-1) = \alpha(2s)w_1w_2 \in \alpha(2s)W^2 \subset \alpha(2s)U$, which completes the proof of the proposition. ∎

We introduce now the notion of causal homotopy.

2.1.8 Definition. Let $\alpha, \beta : [0,1] \to S$ be causal paths in S with the same end point i.e., $\alpha(1) = \beta(1)$. A *causal homotopy* between α and β is a continuous function $H : [0,1] \times [0,1] \to S$ satisfying:

a) $H(t,0) = \alpha(t)$ for all $t \in [0,1]$,

b) $H(t,1) = \beta(t)$ for all $t \in [0,1]$,

c) $H(0,s) = 1_S$, and $H(1,s) = \alpha(1) = \beta(1)$ for all $s \in [0,1]$,

d) the path $\gamma_s(t) = H(t,s)$ is a causal path for all $s \in [0,1]$.

Two paths are said to be *causally homotopic* if there exists a causal homotopy between them. If H is a causal homotopy between the causal paths α and β, we write $H : \alpha \sim \beta$.

2.1.9 Proposition. The relation of causal homotopy is an equivalence relation on the set of causal paths, and the concatenation operation induces a well defined associative operation on the set $\Gamma(S)$ of causal homotopy classes of causal paths.

Proof. Let S be a semigroup and let α be a causal path in S; then the map $H : [0,1] \times [0,1] \to S$ defined by $H(t,s) = \alpha(t)$ satisfies $H : \alpha \sim \alpha$. In other words, the relation of causal homotopy is reflexive. Suppose now that H is a causal homotopy map between α and β. Then the map $F(t,s) = H(t, 1-s)$ is a causal homotopy map between β and α, which means that the relation of causal homotopy is symmetric. Suppose now that $F : \alpha \sim \beta$ and $G : \beta \sim \gamma$, then the map defined by

$$H(t,s) = \begin{cases} F(t, 2s) & \text{for } 0 \le s \le 1/2, \\ G(t, 2s-1) & \text{for } 1/2 \le s \le 1 \end{cases}$$

is a causal homotopy map between α and γ. So the relation of causal homotopy is transitive. We have proved that the relation of causal homotopy is an equivalence relation. We denote by $[\alpha]$ the causal homotopy class of the causal path α. We define a product in $\Gamma(S)$ by $[\alpha][\beta] = [\alpha * \beta]$. If $F : \alpha \sim \alpha'$ and $G : \beta \sim \beta'$ then the map defined by

$$H(t,s) = \begin{cases} F(2t, s) & \text{for } 0 \le t \le 1/2, \\ \alpha(1)G(2t-1, s) & \text{for } 1/2 \le t \le 1, \end{cases}$$

is a causal homotopy map between $\alpha * \beta$ and $\alpha' * \beta'$, i.e., $[\alpha][\beta] = [\alpha'][\beta']$, and therefore the concatenation induces a well defined product in $\Gamma(S)$.

For the last part of the proposition, consider the causal paths σ, τ, and ω. Define

$$F(s,t) = \begin{cases} \sigma\left(\frac{4s}{t+1}\right) & \text{for } 0 \le s \le 1/4(t+1), \\ \sigma(1)\tau(4s-t-1) & \text{for } 1/4(t+1) \le s \le 1/4(t+2), \\ \sigma(1)\tau(1)\omega\left(\frac{4s-t-2}{2-t}\right) & \text{for } 1/4(t+2) \le s \le 1, \end{cases}$$

to establish $(\sigma * \tau) * \omega \sim \sigma * (\tau * \omega)$ in $\Gamma(S)$.

So $\Gamma(S)$ has the structure of a semigroup with an identity. ∎

2.2 Universal properties of $\Gamma(S)$

Let $\Gamma(S)$ denote the semigroup of causal homotopy classes of causal paths in the semigroup S, with the semigroup operation of concatenation.

2.2.1 Definition. A local homomorphism on S is a function σ from a neighborhood of the identity U of S into a semigroup T endowed with a Hausdorff topology for which left translations are continuous satisfying:

i) If $a, b, ab \in U$ then $\sigma(ab) = \sigma(a)\sigma(b)$,

ii) σ is continuous on U.

The next theorem is a major one. For a proof see [9]

2.2.2 Theorem. Let S be a subsemigroup of a topological group G which contains the identity e in the closure of its interior, and let U be an open set of G containing e. Let $\sigma : S \cap U \to T$ be a local homomorphism. Then there exists a unique homomorphism $\tilde{\sigma} : \Gamma(S) \to T$, such that $\tilde{\sigma}([\alpha]) = \sigma(\alpha(1))$ whenever $\alpha : [0,1] \to G$ is a causal path such that $\alpha([0,1]) \subset U \cap S$.

2.3 The uniform topology of $\Gamma(S)$

Our goal now is to define a suitable topology on $\Gamma(S)$ that makes it a topological semigroup. To do this, we define a uniformity on $\Gamma(S)$ and then we will consider the topology induced by that uniformity which we will call the uniform topology of $\Gamma(S)$.

2.3.1 Definition. A topological semigroup is called *locally causally simply connected* if there exists a neighborhood U of the identity such that any two causal paths with the same end point and completely contained in U are causally homotopic.

2.3.2 Example. Let α be an arbitrary causal path in the semigroup C defined in the example 2.1.2. Consider the causal path defined by $\beta(t) = t\alpha(1)$. Then α and β are two causal paths in C with the same end point. We show that α and β are causally homotopic. Indeed, we define the map $H : I \times I \to C$ where I is the closed unit interval by:

$$H(s,t) = \begin{cases} \frac{s}{t}\alpha(t) & \text{for } s < t, \\ \alpha(s) & \text{for } t \leq s. \end{cases}$$

Clearly H is a causal homotopy map between α and β. So the semigroup C is causally simply connected.

2.3.3 Definition. A topological semigroup is called *locally causally path connected* if there exist a basis of neighborhoods $\{U_\alpha : \alpha \in \Omega\}$ of the identity such that any point in U_α can be connected with the identity by a causal path completely contained in U_α for any $\alpha \in \Omega$.

2.3.4 Definition. A topological semigroup is said to be *locally right divisible* if given a neighborhood U of the identity, there exist a neighborhood V of the identity such that $\forall a, b \in V$ there exist $x, y \in U$ such that $ax = by$.

2.3.5 Example. Consider again the semigroup C given in the example 2.1.2. Clearly the family $\{B(0,\epsilon) \cap C : \epsilon > 0\}$ is a basis of neighborhoods of the identity of C that satisfies the condition of the definition 2.3.3, therefore C is a locally causally path connected semigroup. Let U be a neighborhood of the identity of C. Set $V = U$. If $X, Y \in V$ take $A = Y$ and $B = X$, clearly $X + A = Y + B$, and $A, B \in U$. So C is a locally right divisible topological semigroup. The same argument shows that any commutative semigroup is locally right divisible.

2.3.6 Notation. For a neighborhood U of the identity of a topological semigroup S we denote by $\tilde{U}$ and $[\tilde{U}]$ the following sets:

$$\tilde{U} = \{[\alpha] \in \Gamma(S) : \alpha([0,1]) \subset U\},$$

and

$$[\tilde{U}] = \{([\alpha],[\beta]) : [\alpha][\tilde{U}] \cap [\beta][\tilde{U}] \neq \emptyset\}.$$

2.3.7 Theorem. Let S be a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup. Then the family

$$\mathcal{A} = \{[\tilde{U}] : U \subset S \text{ with } U \text{ open and } 1_S \in U\}$$

is a basis for a uniformity of $\Gamma(S)$.

Proof. We will show that the family $\mathcal{A}$ satisfies the conditions of Theorem 1.4.5. For $[\gamma] \in \Gamma(S)$, we have that $[\gamma]\tilde{U} \cap [\gamma]\tilde{U} = [\gamma]\tilde{U} \neq \emptyset$ since $[\gamma] \in [\gamma]\tilde{U}$, hence $([\gamma],[\gamma]) \in [\tilde{U}]$, i.e., the diagonal Δ is contained in $[\tilde{U}]$ for each open subset of S that contains the identity. Clearly $[\tilde{U}]^{-1} = [\tilde{U}]$. If U and V are neighborhoods of the identity of S, then $[U \tilde{\cap} V] \subset [\tilde{U}] \cap [\tilde{V}]$, so the intersection of two members of $\mathcal{A}$ contains a member of $\mathcal{A}$. So conditions a), b), and d) of theorem 1.4.5 are satisfied. Let's prove now that condition c) is also satisfied. Let U be a neighborhood of the identity of S. Since S is locally causally simply connected, we can pick a neighborhood of the identity $W \subset U$ such that any two causal paths in W with the same end point are causally homotopic. Pick a neighborhood V of the identity such that $V^2 \subset W$ and such that any point in V can be joined with the identity by means of a causal path completely contained in V. This is possible since S is locally causally path connected, and by the continuity of multiplication. Finally, since S is locally right divisible, we can pick $V' \subset V$ such that for all $a, b \in V'$ there exist $x, y \in V$ such that $ax = by$. We claim that $[\tilde{V}'] \circ [\tilde{V}'] \subset [\tilde{U}]$. To prove the claim, pick $([\alpha],[\beta]) \in [\tilde{V}'] \circ [\tilde{V}']$. Let $[\gamma] \in \Gamma(S)$ such that $([\alpha],[\gamma]) \in [\tilde{V}']$ and $([\gamma],[\beta]) \in [\tilde{V}']$. Therefore, there exist σ_i, ρ_i with $i = 1, 2$, and $\sigma_i([0,1]) \subset V'$, $\rho_i([0,1]) \subset V'$ such that

$$[\alpha][\rho_1] = [\gamma][\rho_2] \tag{1}$$

and

$$[\gamma][\sigma_1] = [\beta][\sigma_2]. \tag{2}$$

Now take $x, y \in V$ such that $\sigma_1(1)x = \sigma_2(1)y$. By the way V was chosen, there exist τ_1, τ_2 causal paths in V such that $\tau_1(1) = y$ and $\tau_2(1) = x$. Hence, $(\sigma_1 * \tau_2)(1) = (\sigma_2 * \tau_1)(1)$. Therefore, $(\sigma_1 * \tau_2)([0,1]) \subset V^2 \subset W$ and $(\sigma_2 * \tau_1)([0,1]) \subset V^2 \subset W$. Since any two causal paths in W with the same end point are causally homotopic, we conclude that

$$[\sigma_1 * \tau_2] = [\sigma_2 * \tau_1]. \tag{3}$$

Multiplying equations (1) and (2) on the right by $[\tau_1]$ and $[\tau_2]$ respectively, we get

$$[\gamma][\rho_2][\tau_1] = [\alpha][\rho_1][\tau_1] \tag{4}$$

and

$$[\gamma][\sigma_1][\tau_2] = [\beta][\sigma_2][\tau_2]. \tag{5}$$

Now combining equations (4) and (5) we get that $[\beta][\sigma_2 * \tau_2] = [\alpha][\rho_1 * \tau_1]$, but $(\sigma_2 * \tau_2)([0,1]) \subset V^2 \subset W \subset U$ and $(\rho_1 * \tau_1)([0,1]) \subset V^2 \subset W \subset U$, which means that $[\beta]\tilde{U} \cap [\alpha]\tilde{U} \neq \emptyset$, thus $([\alpha],[\beta]) \in [\tilde{U}]$, and therefore $[\tilde{V}'] \circ [\tilde{V}'] \subset [\tilde{U}]$. ∎

2.3.8 Theorem. Let S be a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup. With the uniform topology, multiplication is continuous at the identity of $\Gamma(S)$.

Proof. Let $[\tilde{U}]([e])$ be a neighborhood of $[e]$, the identity of the semigroup $\Gamma(S)$. We may assume that U is a neighborhood of the identity of S with the property that any two causal paths in U with the same end point are causally homotopic. Let V be a neighborhood of the identity of S with $V^2 \subset U$ and such that for any $x \in V$ there exists a causal path $\gamma : [0,1] \to V$ with $\gamma(1) = x$. Let W be a neighborhood of the identity of S such that $W \subset V$ and such that for all $a, b \in W$ there exist $x, y \in V$ with $ax = by$. Finally, pick W' with $W'^2 \subset W$. Consider $[\tilde{W}']([e])$; if $[\sigma],[\tau] \in [\tilde{W}']([e])$ then there exits $\sigma_1, \sigma_2, \tau_1, \tau_2 : [0,1] \to W'$ such that

$$[\sigma * \sigma_1] = [\sigma_2], \tag{1}$$

and

$$[\tau * \tau_1] = [\tau_2]. \tag{2}$$

Now, observe that $(\sigma_1 * \tau * \tau_1)(1) = \sigma_1(1)(\tau * \tau_1)(1) = \sigma_1(1)\tau_2(1)$, but $\sigma_1(1)\tau_2(1) \in W'^2 \subset W$. Also, $(\tau * \tau_1)(1) = \tau_2(1) \in W' \subset W'^2 \subset W$. So there exist $x, y \in V$ such that $(\sigma_1 * \tau * \tau_1)(1)x = (\tau * \tau_1)(1)y$. There exist $\alpha, \beta : [0,1] \to V$ causal paths such that $\alpha(1) = x$ and $\beta(1) = y$. Therefore $(\sigma_1 * \tau * \tau_1 * \alpha)(1) = (\tau * \tau_1 \ * \beta)(1)$, i.e., $(\sigma_1 * \tau_2 * \alpha)(1) = (\tau_2 * \beta)(1)$ and $(\tau_2 * \beta)([0,1]) \subset WV \subset V^2 \subset U$. Also, we have that $(\sigma_1 * \tau_2 * \alpha)([0,1]) \subset W'W'V \subset WV \subset V^2 \subset U$.

Since any two causal paths in U with the same end point are causally homotopic, we have that

$$[\tau_2 * \beta] = [\sigma_1 * \tau_2 * \alpha]. \tag{3}$$

Multiplying equations (1) and (2) we get: $[\sigma * \sigma_1 * \tau * \tau_1] = [\sigma_2 * \tau_2]$; therefore we conclude that

$$[\sigma * \sigma_1 * \tau * \tau_1 \ * \alpha] = [\sigma_2 * \tau_2 * \alpha]. \tag{4}$$

Combining equations (2), (3), and (4) we obtain $[\sigma * \tau * \tau_1 * \beta] = [\sigma_2 * \tau_2 * \alpha]$.

Now, $(\tau_1 * \beta)([0,1]) \subset W'V \subset V^2 \subset U$ and $(\sigma_2 * \tau_2 * \alpha)([0,1]) \subset W'^2V \subset V \subset U$, i.e., $[\sigma * \tau] \in [\tilde{U}]([e])$. This means that multiplication is continuous at the identity of $\Gamma(S)$. ∎

2.3.9 Theorem. Let S a be a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup. Then multiplication is continuous in the second variable.

Proof. Let $[\alpha], [\beta] \in \Gamma(S)$, consider $[\tilde{U}]([\alpha * \beta])$, a neighborhood of $[\alpha * \beta]$ in $\Gamma(S)$. Consider now $[\tilde{U}]([\beta])$, a neighborhood of $[\beta]$ in $\Gamma(S)$. If $[\gamma] \in [\tilde{U}]([\beta])$, there exist $\gamma_1, \gamma_2 : [0,1] \to U$ such that $[\gamma * \gamma_1] = [\beta * \gamma_2]$, therefore $[\alpha * \gamma * \gamma_1] = [\alpha * \beta * \gamma_2]$, i.e., $[\alpha * \gamma] \in [\tilde{U}]([\alpha * \beta])$. ∎

2.4 Uniformity structure on semigroups

We saw in 1.5 that the topology of a topological group can be described in terms of the right and left uniformities. The same technique does not work for topological semigroups due to the absence of inverses for the elements of the semigroup. However, it is possible for some semigroups to define a uniformity closely related to the right and left uniformities of a group. We investigate in this section the kind of semigroups for which this is possible and study the relationship between the original topology of the semigroup and the topology induced by the uniformity.

Let S be a locally right divisible semigroup. For a neighborhood V of the identity of S we define V_d as the set of all pairs $(x,y) \in S \times S$ such that $xV \cap yV \neq \emptyset$. Consider now $\mathcal{A} = \{V_d : V \text{ is a neighborhood of } 1_S\}$.

2.4.1 Theorem. Let S be a locally right divisible topological semigroup. The family $\mathcal{A}$ of subsets of $S \times S$ defined above is a uniformity for S.

Proof. It is straightforward to see that $V_d^{-1} = V_d$ and that $\Delta \subset V_d$ for any neighborhood of the identity of the semigroup S. It is also clear that $(U \cap V)_d \subset U_d \cap V_d$ for U and V neighborhoods of the identity of S. Let U be an arbitrary neighborhood of the identity of the semigroup S. By the continuity of multiplication, we can choose a neighborhood W of the identity of S such that $W^2 \subset U$. Now, since S is a locally right divisible semigroup, we can pick a neighborhood $V \subset W$ of the identity such that $\forall a, b \in V$ there exist $x, y \in W$ such that $ax = by$. Finally pick a neighborhood V' of the identity of S such that $V'^2 \subset V$. Take $(x,y) \in V'_d \circ V'_d$, there exists $z \in S$ such that (x,z) and (z,y) are elements of V'_d. Therefore, there exist x', z', z'', y' elements of V' such that $xx' = zz'$ and $zz'' = yy'$. By the way V' and V were chosen, there exists $x_1, y_1 \in W$ such that $z'x_1 = z''y_1$. Combining this relation with the previous two relations, we get that $xx'x_1 = zz'x_1 = zz''y_1 = yy'y_1$. Now, $x'x_1 \in V'W \subset W^2 \subset U$ and $y'y_1 \in V'W \subset W^2 \subset U$ and, therefore $(x,y) \in U_d$.

We have proved that $V' \circ V' \subset U$. We have shown that the family $\mathcal{A}$ satisfies the conditions of the definition 1.4.1, and therefore it is a uniformity for S. ∎

The uniform topology of the semigroup S could be different from the original topology of S. In the rest of this chapter we are assuming that they are equal.

2.4.2 Definition. A topological semigroup S is called *nice* if its topology is compatible with the uniform topology and it satisfies definitions 2.3.1, 2.3.3, and 2.3.4.

2.4.3 Example. For a subsemigroup S of a group G such that $1_G \in \overline{\text{int}(S)}$ we saw in the proposition 1.5.2 that the uniform topology of S is compatible with the relative topology of S. Therefore the class of these semigroups that satisfies the conditions given in the definitions 2.3.1, 2.3.3, and 2.3.4 are examples of nice semigroups. Particularly, cones in $\mathbf{R}^n$ are nice semigroups.

2.4.4 Proposition. If S is a nice topological semigroup, the map $[\alpha] \mapsto \alpha(1) : \Gamma(S) \to S$ is continuous.

Proof. Let V be a neighborhood of $\alpha(1)$. Pick a neighborhood U of the identity such that $U_d(\alpha(1)) \subset V$. Consider $[\tilde{U}][\alpha]$, which is a neighborhood of α in $\Gamma(S)$. If $[\beta] \in [\tilde{U}]([\alpha])$, then there exist $\sigma_1, \sigma_2 : [0,1] \to U$ such that $[\beta * \sigma_1] = [\alpha * \sigma_2]$. Therefore, $\beta(1)\sigma_1(1) = \alpha(1)\sigma_2(1)$. Since both $\sigma_1(1)$ and $\sigma_2(1)$ are elements of U, we have that $\beta(1) \in U_d(\alpha(1))$. ∎

2.5 Functorial properties of $\Gamma(S)$

2.5.1 Theorem. Let S and T be nice topological semigroups and let $h : S \to T$ be a continuous homomorphism. Then the map $\overline{h} : \Gamma(S) \to \Gamma(T)$ defined by $\overline{h}([\alpha]) = [h(\alpha)]$ is a continuous homomorphism.

Proof. We have that $\overline{h}([\alpha][\beta]) = \overline{h}([\alpha * \beta]) = [h(\alpha * \beta)]$, but

$$\alpha * \beta = \begin{cases} \alpha(2t) & \text{for } 0 \le t \le 1/2, \\ \alpha(1)\beta(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Then

$$h((\alpha * \beta(t)) = \begin{cases} h(\alpha(2t)) & \text{for } 0 \le t \le 1, \\ h(\alpha(1))h(\beta(2t-1)) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Therefore, $\overline{h}([\alpha][\beta]) = [(h \circ \alpha) * (h \circ \beta)] = [h \circ \alpha][h \circ \beta] = \overline{h}([\alpha])\overline{h}([\beta])$. This proves that $\overline{h}$ is a homomorphism.

Let's see now that $\overline{h}$ is continuous. Take $[\alpha] \in \Gamma(S)$, and consider $[\tilde{U}]([h \circ \alpha])$ which is a neighborhood of $\overline{h}([\alpha])$ in $\Gamma(T)$, where U is a neighborhood of the identity of T. By continuity of h, there exists a neighborhood V of the identity of S such that $h(V) \subset U$. Now, if $[\beta] \in [\tilde{V}]([\alpha])$, then there exist $\sigma_1, \sigma_2 : [0,1] \to V$ such that $[\beta][\sigma_1] = [\alpha][\sigma_2]$. Therefore, $\overline{h}([\beta])\overline{h}([\sigma_1]) = \overline{h}([\alpha])\overline{h}([\sigma_2])$, but $\overline{h}([\sigma_1])$ and $\overline{h}([\sigma_2])$ maps the interval $[0,1]$ into U. In other words, $\overline{h}([\beta]) \in [\tilde{U}]([\overline{h}([\alpha])$).

Clearly if $i : S \to S$ is the identity homomorphism, then $\overline{i} : \Gamma(S) \to \Gamma(S)$ is the identity homomorphism of the semigroup $\Gamma(S)$.

Also, if $h : S \to T$ and $g : T \to U$ are continuous homomorphisms, then $\overline{(h \circ g)}([\alpha]) = [(h \circ g)(\alpha)] = [h(g(\alpha))] = \overline{h}([g(\alpha)]) = (\overline{h} \circ \overline{g})([\alpha])$, therefore $\overline{(h \circ g)} = \overline{h} \circ \overline{g}$.

If $h : S \to S$ is an invertible homomorphism, then by the preceding results, we have that $\overline{i} = \overline{(h \circ h^{-1})} = \overline{h} \circ \overline{(h^{-1})}$. In other words, $(\overline{h})^{-1} = \overline{(h^{-1})}$. ∎

2.6 Direct and semidirect products

Let S_1 and S_2 be topological semigroups. Suppose we have a homomorphism h from S_1 to the semigroup $\mathrm{Aut}(S_2)$ of automorphisms of S_2 such that the maps $(s_1, s_2) \mapsto h(s_1)(s_2) : S_1 \times S_2 \to S_2$ and $(s_1, s_2) \mapsto (h(s_1))^{-1}(s_2) : S_1 \times S_2 \to S_2$ are continuous. Then in the cartesian product $S_1 \times S_2$ we define a new operation given by $(x_1, y_1)(x_2, y_2) = (x_1 h(y_1)(x_2), y_1 y_2)$. The set $S_1 \times S_2$ with this new product is called the semidirect product between S_1 and S_2 and it is denoted by $S_1 \ltimes S_2$. Then we can define a structure of a semidirect product between the semigroups $\Gamma(S_1)$

and $\Gamma(S_2)$ via the composition $\Gamma(S_2)\xrightarrow{\pi}S_2\xrightarrow{h}\mathrm{Aut}(S_1)\xrightarrow{\phi}\mathrm{Aut}(\Gamma(S_1))$, where π is the endpoint homomorphism and ϕ is defined by $(\phi(f))([\gamma]) = [f \circ \gamma]$. If we set $\psi = \phi \circ h \circ \pi$ then it is clear that $\Gamma(S_2)\xrightarrow{\psi}\mathrm{Aut}(\Gamma(S_1))$ is a homomorphism. This homomorphism is given by the formula $(\psi([\beta]))([\gamma]) = [h(\beta(1)) \circ \gamma]$. So therefore, we can define $\Gamma(S_1) \ltimes \Gamma(S_2)$ as it was done above.

In what follows we prove that the semigroups $\Gamma(S_1 \ltimes S_2)$ and $\Gamma(S_1) \ltimes \Gamma(S_2)$ are actually isomorphic. But first, let's prove the following:

2.6.1 Theorem Let S_1 and S_2 be topological semigroups. Suppose we have defined a semidirect product $S_1 \ltimes S_2$ via the homomorphism $S_2\xrightarrow{h}\mathrm{Aut}(S_1)$. Then $\alpha \times \beta$ is a causal path in $S_1 \times S_2$ if and only if α is a causal path in S_1 and β is a causal path in S_2.

Proof. Suppose that $\alpha \times \beta$ is a causal path in $S_1 \ltimes S_2$. Since the projection map on the second coordinate $\pi_2 : S_1 \times S_2 \to S_2$ is a continuous homomorphism, by proposition 2.1.5 we have that β is a causal path in S_2. Let's see that α is also a causal path in S_1. Let U be a neighborhood of e_1, the identity of S_1. The map $F : [0,1] \times \{e_1\} \to S_1$ defined by $F(s, e_1) = h(\beta(s))(e_1) = e_1 \in U$ is continuous. Therefore, by Wallace's lemma, there exists V open in S_1 with $e_1 \in V$ such that $F([0,1] \times V) \subset U$. In other words, $h(\beta(s))(v) \in U$ for all $v \in V$. Consider now the set $V \times S_2$, which is a neighborhood of (e_1, e_2). Then there exist $\epsilon > 0$ such that if $s < t < s + \epsilon$ then $(\alpha(t), \beta(t)) = (\alpha(s), \beta(s))(v_1, v_2)$ with $(v_1, v_2) \in V \times S_2$. Therefore, $\alpha(t) = \alpha(s)h(\beta(s))(v_1) \in \alpha(s)U$.

Suppose now that α and β are causal paths in S_1 and S_2 respectively. Let's prove that $\alpha \times \beta$ is a causal path in $S_1 \ltimes S_2$. Let $U_1 \times U_2$ be a neighborhood of (e_1, e_2) the identity element of $S_1 \ltimes S_2$. Consider the map $G : [0,1] \times \{e_1\} \to S_1$ defined by $G(s, e_1) = h(\beta(s))^{-1}(e_1) = e_1 \in U_1$. Since G is continuous, by Wallace's lemma, there exists a neighborhood V of e_1 in S_1 such that $G([0,1] \times V) \subset U_1$. In other words, $h(\beta(s))^{-1}(v) \in U_1$ for all $s \in [0,1]$ and for all $v \in V$. Now, pick $\epsilon > 0$ such that if $s < t < s+\epsilon$ then $\alpha(t) = \alpha(s)v$ and $\beta(t) = \beta(s)u$ for some $v \in V$ and some $u \in U$. Then $(\alpha(s), \beta(s)(h(\beta(s))^{-1}(v), u) = (\alpha(s)v, \beta(s)u) = (\alpha(t), \beta(t))$ and $(h(\beta(s))^{-1}(v), u) \in U_1 \times U_2$. ∎

2.6.2 Theorem. Let $S_2 \xrightarrow{h} \mathrm{Aut}(S_1)$ be a homomorphism, where S_1 and S_2 are topological semigroups; and suppose that the maps $(s_1, s_2) \to h(s_1)(s_2) : S_1 \times S_2 \to S_2$ and $(s_1, s_2) \to (h(s_1))^{-1}(s_2) : S_1 \times S_2 \to S_2$ are continuous. Then the map $\Psi : \Gamma(S_1 \ltimes S_2) \to \Gamma(S_1) \ltimes \Gamma(S_2)$ defined by $\Psi([\alpha, \beta]) = ([\alpha], [\beta])$ is a semigroup isomorphism.

Proof. Take $([\alpha], [\beta]) \in \Gamma(S_1) \ltimes \Gamma(S_2)$. Then by the previous theorem, $[\alpha \times \beta]$ is an element of $\Gamma(S_1 \ltimes S_2)$. Therefore Ψ is clearly onto. Let's see now that Ψ is one–to–one. Suppose that $([\alpha], [\beta]) = ([\sigma], [\rho])$. Then there exists a continuous function $F : [0,1] \times [0,1] \to S_1$ such that $F(t,0) = \alpha(t)$ for all $t \in [0,1]$, $F(t,1) = \sigma(t)$ for all $t \in [0,1]$, $F(0,s) = e_1$ for all $s \in [0,1]$, $F(1,s) = \alpha(1) = \sigma(1)$ for all $s \in [0,1]$ and the path $F_s(t) = F(t,s)$ is a causal path for all fixed $s \in [0,1]$.

Also, there exists a continuous function $G : [0,1] \times [0,1] \to S_2$ such that $G(t,0) = \beta(t)$ for all $t \in [0,1]$, $G(t,1) = \rho(t)$ for all $t \in [0,1]$, $G(0,s) = e_2$ for all $s \in [0,1]$, $G(1,s) = \beta(1) = \rho(1)$ for all $s \in [0,1]$, and the path $G_s(t) = G(t,s)$ is a causal path for all s fixed in $[0,1]$.

Define now the map $F \times G : [0,1] \times [0,1] \to S_1 \times S_2$ by $F \times G(t,s) = (F(t,s), G(t,s))$ then clearly,

$$F \times G(t,0) = (F(t,0), G(t,0)) = (\alpha(t), \beta(t)),$$

$$F \times G(t,1) = (F(t,1), G(t,1)) = (\sigma(t), \rho(t)),$$

$$F \times G(0,s) = (F(0,s), G(0,s) = (e_1, e_2),$$

$$F \times G(1,s) = (F(1,s), G(1,s)) = (\alpha(1), \beta(1)) = (\sigma(1), \rho(1)).$$

Also, according with the above theorem, $(F \times G)_s(t) = (F_s(t), G_s(t))$ is a causal path in $S_1 \ltimes S_2$. So we have proved that (α, β) is causally homotopic to (σ, ρ). In other words $[(\alpha, \beta)] = [(\sigma, \rho)]$, which proves that Ψ is one to one.

Let's see now that Ψ is a semigroup homomorphism. By the definition of the product of causal paths we have that $[(\alpha, \beta)][(\sigma, \rho)] = [(\alpha, \beta) * (\sigma, \rho)]$, but

$$\begin{aligned}
(\alpha, \beta) * (\sigma, \rho) &= \begin{cases} (\alpha, \beta)(2t) & 0 \leq t \leq 1/2, \\ (\alpha, \beta)(1)(\sigma, \rho)(2t-1) & 1/2 \leq t \leq 1/2. \end{cases} \\
&= \begin{cases} (\alpha, \beta)(2t) & 0 \leq t \leq 1/2. \\ (\alpha(1)h(\beta(1)) \circ \sigma, \beta(1)\rho)(2t-1) & 1/2 \leq t \leq 1. \end{cases} \\
&= (\alpha, \beta) * (\sigma, \rho)(t) = (\alpha * h(\beta(1) \circ \sigma, \beta * \rho)(t),
\end{aligned}$$

so therefore,

$$\begin{aligned}
\Psi([(\alpha, \beta)][(\sigma, \rho)]) &= ([(\alpha * h(\beta(1)) \circ \sigma)], [(\beta * \rho)]) \\
&= ([\alpha][h(\beta(1)) \circ \sigma)], [\beta][\rho]) = ([\alpha], [\beta])([\sigma], [\rho]) \\
&= \Psi([(\alpha, \beta)])\Psi([(\sigma, \rho)]).
\end{aligned}$$

This proves that Ψ is a homomorphism. ∎

CHAPTER 3

3.1 The free and relatively free semigroups

3.1.1 Definition. Let X be a Hausdorff topological space. The free topological semigroup $Fr(X)$ on X is defined as the set

$$Fr(X) = X \cup X^2 \cup X^3 \cup \ldots,$$

where the union is a disjoint union, the topology on X^n is the product topology, and each X^n is open in $Fr(X)$. So a set $A \subset Fr(X)$ is open iff $A \cap X^n$ is open for all n. Members of X^n are viewed as words of length n, and the semigroup operation on $Fr(X)$, denoted by $\circ$, is the juxtaposition of words,

$$(a_1, \ldots, a_m) \circ (b_1, \ldots, b_n) = (a_1, \ldots, a_m, b_1, \ldots, b_n).$$

3.1.2 Proposition. The set $Fr(X)$ with the topology and multiplication of words defined above is indeed a topological semigroup.

Proof. We need to prove that multiplication is continuous. Let $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ be elements of $Fr(X)$, and let U be open in $Fr(X)$ such that $ab = (a_1, \ldots, a_m, b_1, \ldots, b_n) \in U$. By the definition of the topology of $Fr(X)$ the set $U \cap X^{n+m}$ is an open subset of X^{n+m}. Therefore, there exists $U_1, U_2, \ldots, U_{m+n}$ open subsets of X such that $a \in W = U_1 \times U_2 \times \ldots \times U_m$, $b \in V = U_{m+1} \times U_{m+2} \times \ldots \times U_{m+n}$ and $WV = U \cap X^{m+n} \subset U$. ∎

3.1.3 Proposition. If X is locally compact, σ-compact, and Hausdorff, so is the space $Fr(X)$.

Proof. If X is locally compact then by the Tychonoff Theorem we have that X^n is locally compact for all n. Since the disjoint union of a countable family of locally compact spaces is locally compact, we have that $Fr(X)$ is locally compact. Similarly, if X is σ-compact so is X^n for all n. Since the disjoint union of a countable family of σ–compact spaces is again σ–compact, we have that $Fr(X)$ is σ–compact.

Let $a, b \in Fr(X)$ with $a \neq b$. Then $a \in X^n$ and $b \in X^m$ for some integers n, m. If $n \neq m$ then X^n and X^m are disjoint neighborhoods of a and b respectively. If $n = m$, then since X^n is Hausdorff there exists U and V disjoint open subsets of X^n with $a \in U$ and $b \in V$. By the definition of the topology of $Fr(X)$ it is clear that U and V are also open in $Fr(X)$. ∎

3.1.4 Proposition. The topological semigroup $Fr(X)$ is the free topological semigroup on X in the sense that any function into a topological semigroup S extends uniquely to a homomorphism from $Fr(X)$ into S. The extension is continuous if and only if the original function on X is continuous.

Proof. Given a function $f : X \to S$, then the function $\hat{f} : Fr(X) \to S$ define by $\hat{f}(x_1, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)$ is a homomorphism that extends f. Clearly $\hat{f}$ is the unique extension of f to a homomorphism define on $Fr(X)$. Suppose that f is continuous. Let U be an open subset of S containing the product $f(x_1)f(x_2)\ldots f(x_n)$. Since multiplication is continuous, there exists open subsets $U_1, U_2, \ldots, U_n$ in S such that $f(x_i) \in U_i$ for each i, and $U_1 \times U_2 \times \ldots \times U_n \subset U$. Since f is continuous, there exists for each i a set V_i open in X with $x_i \in V_i$ and such that $f(V_i) \subset U_i$ for each i. Now $(x_1, \ldots, x_n) \in V_1 \times V_2 \times \ldots \times V_n$ and $\hat{f}(V_1 \times V_2 \times \ldots \times V_n) \subset U$. Thus $\hat{f}$ is continuous.

Since f is the restriction of $\hat{f}$ to the open set X, it turns out that f is continuous provided that $\hat{f}$ is continuous. ∎

3.1.5 Definition. Let S be a semigroup, U a subset of S containing e, the identity of S, and A a subset of S such that $U \cap A \neq \emptyset$ and $(U \cap A)^2 \subset A$. We define the relatively free (A, U) semigroup, denoted by $RF(A, U)$, to be the one obtained by forming the free semigroup $Fr(A \cap U)$ and then dividing out the smallest congruence relation σ on $Fr(A \cap U)$ which identifies words (a_1, a_2) of length two with the corresponding word $a_1 a_2$ of length one in the case $a_1, a_2, a_1 a_2 \in A \cap U$.

In this section we are mainly interested in the case when $A = S$ and U is an open neighborhood of the identity of S. In this case we denote $RF(A \cap U)$ by $RF(U)$. Since σ is a congruence relation we have that $RF(U)$ is indeed a semigroup. If U is a locally compact and σ-compact neighborhood of the identity of S and σ is a closed congruence, then by Proposition 3.1.4 and Theorem 1.3.5 we have that $R(U)$ is actually a topological semigroup. Therefore we want to find conditions that make σ a closed congruence.

3.1.6 Definition. Let U be a neighborhood of the identity of the topological semigroup S. A word $w \in Fr(U)$ is *derivable by a contraction* from the word $w' = a_1 * a_2 * \ldots * a_n$ if $w = a_1 * a_2 * \ldots a_{i-2} * (a_{i-1} a_i) * a_{i+1} * \ldots * a_n$ for some $1 < i \leq n$, where $a_{i-1} a_i \in U$. We say that the word w is *derivable by an expansion* from the word $w' = a_1 * a_2 * \ldots * a_n$ if $w = a_1 * a_2 * \ldots * a_{i-1} * a * a' * a_{i+1} * \ldots * a_n$, where $a_i = aa'$ with $a, a' \in U$. We say that the word w is *directly derivable* from the word w' if w is derivable from w' by a contraction or an expansion.

Note that if w is derivable by a contraction from the word w' then $\text{length}(w) = \text{length}(w') - 1$. If w is derivable from the word w' by an expansion then $\text{length}(w) = \text{length}(w') + 1$.

3.1.7 Definition. The word w is *derivable* from the word w' if there exists a finite sequence $w_0, w_1, w_2, \ldots, w_n$ with $w = w_0$, $w' = w_n$ and w_i is directly derivable from w_{i-1}.

We define now the following relation ρ on $Fr(U)$ where U is a neighborhood of the identity of the topological semigroup S. The words w and w' are ρ related (we write $w \rho w'$) if w is derivable from w'. It is easy to see that ρ is an equivalence relation.

3.1.8 Theorem. Let U be an arbitrary neighborhood of the identity of a topological semigroup S. The relation ρ defined above is a congruence relation.

Proof. Suppose that $w\rho w'$ and z is an arbitrary word of $Fr(U)$. There exist a finite sequence $w_0, w_1, \ldots, w_n$ with $w = w_0$, $w' = w_n$ and w_i is directly derivable from w_{i-1}. Consider the sequence $z * w_0, z * w_1, \ldots, z * w_n$. Then $z * w = z * w_0$, $z * w' = z * w_n$ and $z * w_i$ is directly derivable from $z * w_{i-1}$. So $z * w\rho z * w'$. Similarly we can prove that $w * z\rho w' * z$. ∎

Note that if $a_1, a_2, a_1a_2 \in U$ then $a_1 * a_2 \rho a_1 a_2$.

3.1.9 Theorem. The relation ρ defined above is the smallest congruence relation that identifies $a_1 * a_2$ with the product a_1a_2 on the condition that $a_1, a_2, a_1a_2 \in U$. In other words $\rho = \sigma$.

Proof. Let α be a congruence relation on $Fr(U)$ such that $a_1 * a_2 \alpha a_1 a_2$ on the condition that $a_1, a_2, a_1a_2 \in U$. We have to show that if $w\rho w'$ then $w\alpha w'$. Indeed, if $w\rho w'$ then there exist a sequence $w_0, w_1, \ldots, w_n$ such that $w = w_0$, $w' = w_n$, and w_i is directly derivable from w_{i-1} for all i. So by the transitivity of α it is enough to show that $w_i \alpha w_{i-1}$. Suppose that w_i is derivable from w_{i-1} by a contraction, i.e., suppose that $w_{i-1} = a_1 * a_2 * \ldots * a_{k-1} * a_k * \ldots * a_p$ and $w_i = a_1 * a_2 * \ldots * a_{k-2} * (a_{k-1}a_k) * \ldots * a_p$. Since $a_{k-1} * a_k \alpha a_{k-1} a_k$ and α is a congruence relation, we have that $a_1 * a_2 * \ldots * a_{k-2} * a_{k-1} * a_k \alpha a_1 a_2 * \ldots * a_{k-2} * (a_{k-1}a_k)$. Multiplying this relation in the right by $a_{k+1} * \ldots * a_p$ we get that $w_{i-1} \alpha w_i$. ∎

Now we want to find conditions for ρ to be a closed congruence.

3.1.10 Theorem. Let U be a compact neighborhood of the identity of a topological semigroup S. Let σ be the smallest congruence relation that identifies words of length two with their product in $Fr(U)$. Suppose that for every pair of words (w, w') such that $w\sigma w'$ there exists a sequence $w_0, w_1, \ldots, w_n$, with $w = w_0$, $w' = w_n$ such that w_i is directly derivable from w_{i-1} for $1 \le i \le n$ and $n \le M$, where M is an integer that depends only on the length of the words w and w'. Then σ is a closed congruence relation and $RF(U)$ is a topological semigroup.

Proof. Let (w_α, w'_α) be a net in σ that converges to (w, w'). We have to show that $(w, w') \in \sigma$. Since $w_\alpha \sigma w'_\alpha$, there exists a sequence $w_0^\alpha, w_1^\alpha, \dots, w_{n(\alpha)}^\alpha$ such that w_i^α is directly derivable from w_{i-1}^α, $w_\alpha = w_0^\alpha$ and $w'_\alpha = w_{n(\alpha)}^\alpha$.

Since $w \in U^{\text{length}(w)}$, which is an open subset of $Fr(U)$, and w_α converges to w, we have that w_α is eventually in $U^{\text{length}(w)}$. Therefore, there exists β such that $w_\alpha \in U^{\text{length}(w)}\ \forall \alpha \geq \beta$.

Similarly, $w'_\alpha \in U^{\text{length}(w')} \quad \forall \alpha \geq \beta'$. Taking $\beta'' \geq \max(\beta, \beta')$, we have that $w_\alpha \in U^{\text{length}(w)}$ and $w'_\alpha \in U^{\text{length}(w')} \quad \forall \alpha \geq \beta''$. Since $n(\alpha) \leq M \quad \forall \alpha$, the net $n(\alpha)$ converges to some integer $n \leq M$, passing to a subnet if necessary, and since the set of positive integers is discrete we have that $n(\alpha) = n$ eventually. Therefore we can assume that $n(\alpha) = n \quad \forall \alpha \geq \beta''$. Since w_1^α is directly derivable from w_α, we have that $\forall \alpha \geq \beta'' \quad w_1^\alpha \in U^{\text{length}(w)-1} \cup U^{\text{length}(w)+1}$. Since the set $U^{\text{length}(w)-1} \cup U^{\text{length}(w')+1}$ is compact, we can assume that w_1^α converges to a point $w_1 \in U^{\text{length}(w)-1} \cup U^{\text{length}(w)+1}$, passing to a subnet if necessary. If $w_1 \in U^{\text{length}(w)-1}$ then by continuity of multiplication we have that w_1 is directed derivable from w by a contraction, since $w_1^\alpha \in U^{\text{length}(w)-1}$ eventually. Similarly, if $w_1 \in U^{\text{length}(w)+1}$ then w_1 is directly derivable from w by an expansion. A similar argument shows that w_i^α converges to w_i for $i \leq n$ and that w_i is directly derivable from w_{i-1}. Since $w_0 = w$ and $w_n = w'$ we have that $(w, w') \in \sigma$. Since U is compact by proposition 3.1.3, $Fr(U)$ is locally compact and σ-compact, topological semigroup. By the Lawson–Madison theorem, theorem 1.3.5, $RF(U)$ is a topological semigroup. ∎

3.1.11 Definition. Let S be a topological semigroup, T an algebraic semigrop, and $U \subset S$ a neighborhood of the identity of S. A function $f : U \to T$ is called a local homomorphism if whenever $a_1, a_2, a_1a_2 \in U$ we have that $f(a_1a_2) = f(a_1)f(a_2)$.

3.1.12 Proposition. Let S be a topological semigroup, T an algebraic semigroup, and $f : U \to T$ a local homomorphism, where U is a neighborhood of the identity of S. Then f extends uniquely to a homomorphism on $RF(U)$.

Proof. If $\hat{f} : Fr(U) \to T$ is the unique extension of f to a homomorphism defined in $Fr(U)$, define $\tilde{f} : RF(U) \to T$ by $\tilde{f}(\pi(w)) = \hat{f}(w)$ where $\pi : Fr(U) \to RF(U)$ is the canonical map. Suppose that $\pi(w) = \pi(w')$, then there exists a sequence $w_0, w_1, \ldots, w_n$ of elements of $Fr(U)$ such that $w = w_0$, $w' = w_n$, and w_i is directly derivable from w_{i-1} for all $1 \leq i \leq n$. To show that $\tilde{f}$ is well defined is enough to prove that $\hat{f}(w_i) = \hat{f}(w_{i-1})$ for all $1 \leq i \leq n$. Indeed, if w_i is derivable from w_{i-1} from an elementary contraction, then there exist $a_1, a_2, \ldots, a_p$ such that $w_{i-1} = a_1 * a_2 * \ldots * a_{k-1} * a_k * \ldots * a_p$ and $w_i = a_1 * a_2 * \ldots * a_{k-2} * (a_{k-1}a_k) * a_{k+1} * \ldots * a_p$. Now, $\hat{f}(w_i) = f(a_1)f(a_2)\ldots f(a_{k-2})f(a_{k-1}a_k)f(a_{k+1})\ldots f(a_p)$, and since f is a local homomorphism, we have that $f(a_{k-1}a_k) = f(a_{k-1})f(a_k)$, hence $f(w_i) = f(w_{i-1})$. Clearly $\tilde{f}$ is the unique homomorphism that extends f. ∎

It is easy to see that the composition map $U \to Fr(U) \to RF(U)$ is a local homomorphism.

3.1.13 Theorem. Let G be a topological group, $S \subset G$ a locally causal simply connected, locally causal path connected, and locally right divisible topological subsemigroup, and $U \subset S$ a compact neighborhood of the identity of S such that any two causal paths on U with the same end point are causally homotopic and such that any point in U can be joined with the identity with a causal path totally contained in U. If $RF(U)$ is obtained from $Fr(U)$ by dividing out a closed congruence relation, then $RF(U)$ is isomorphic to $\Gamma(S)$.

Proof. The map $i : U \to \Gamma(S)$ defined by $i(x) = [\alpha]$, where $\alpha : [0,1] \to U$ is a causal path such that $\alpha(1) = x$, is a local homomorphism. By proposition 3.1.12, there exist a unique homomorphism $h : RF(U) \to \Gamma(S)$ such that $h \circ j = i$, where $j : U \to RF(U)$ is the canonical map. By Lawson's theorem $RF(U)$ is a topological semigroup. Since the canonical map $j : U \to RF(U)$ is a local homomorphism, by theorem [2.2.2] there exist a unique homomorphism $h' : \Gamma(S) \to RF(U)$ such that $h' \circ i = j$. Again by the universal property of $\Gamma(S)$ the map $h \circ h' =$ identity of $\Gamma(S)$ and by the universal property of $RF(U)$ the map $h' \circ h =$ the identity of $RF(U)$. Therefore $\Gamma(S)$ and $RF(U)$ are isomorphic semigroups. ∎

3.2 Semigroups with compatible homotopy structure.

3.2.1 Definition. Let G be a topological group and let S be a subsemigroup. We say that S has *a compatible homotopy structure* if

(i) S is pathwise connected, pathwise locally connected, and semilocally simply connected;

(ii) the identity e is in S and in the closure of the interior of S;

(iii) two causal paths are causally homotopic if and only if they are homotopic.

It is shown in [9] that for semigroups with compatible homotopic structure, we can identify $\Gamma(S)$ with a certain subsemigroup of the simply connected covering semigroup. It is also shown in [9] that the Ol'shankii semigroups have a compatible homotopic structure. Here we present another example.

3.2.2 Definition. A Lie algebra L is called *almost abelian* if there is a hyperplane ideal N such that:

(i) $[N, N] = \{0\}$,

(ii) there is a functional $\omega \in \hat{L}$ such that $[x, n] = \omega(x)n$ for all $x \in L$ and $n \in N$.

The almost abelian Lie algebras are characterized by the following theorem. For a proof see [5].

3.2.3 Theorem. For a finite dimensional Lie algebra L the following statements are equivalent:

(1) L is almost abelian.

(2) Every hyperplane in L is a subalgebra.

(3) Every vector space in L is a subalgebra.

(4) Every cone W in L is a Lie semialgebra.

(5) Every half space is a Lie semialgebra.

For an almost abelian Lie algebra L, pick a Lie group G such that the exponential map $\exp: L \to G$ is onto. It is a well known fact that the exponential map is a diffeomorphism from a neighborhood of zero in L into a neighborhood of the identity of the group G. The Campbell–Hausdorff multiplication formula extends to L and $(L, *)$, where the symbol * represents C–H multiplication, is a group. Take a cone C in L, then $(C, *)$ is a subsemigroup of $(L, *)$. Furthermore, each ray in C is a one parameter semigroup with respect to the Campbell–Hausdorff multiplication. therefore, arguments similar to those given in examples 2.1.3 and 2.3.2 show that that the semigroup $(C, *)$ satisfies the conditions of the definition 3.2.1 and therefore it has the compatible homotopic structure. Therefore according with the above remark we can identify $\Gamma(C)$ with a subsemigroup of the simply connected covering semigroup of C.

3.3 Examples.

Let G be a topological group, $S \subset G$ a subsemigroup, and U a neighborhood of the identity of G such that $U \cap S$ generates S. Then the injection map $i : U \cap S \to S$ is a local homomorphism and the extension map $\phi : Fr(U \cap S) \to S$ is a surjective homomorphism. Consider the canonical map $\pi : Fr(U \cap S) \to RF(U \cap S)$ and the map $\tilde{\phi} : RF(U \cap S) \to S$ defined by $\tilde{\phi}(\pi(w)) = \phi(w)$. Clearly, the map $\tilde{\phi}$ is also onto, and it is always true that the kernel relation of the map π, $K(\pi)$, is contained in the kernel relation of the map ϕ, $K(\phi)$. It is easy to see that if $K(\pi) = K(\phi)$ then $\tilde{\phi}$ is an isomorphism. In other words, to show that $\tilde{\phi}$ is an isomorphism, we need to show that if $x_1, x_2, \ldots, x_n, y_1 y_2, \ldots, y_m$ are elements of S with $x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_m$ then $\pi(x_1 * x_2 * \ldots * x_n)$=$\pi(y_1 * y_2 * \ldots * y_m)$. To illustrate the above technique, consider the following examples:

3.3.1 Example. Let $G = \mathbb{R}^n$, $S = C =$ a cone in $\mathbb{R}^n$, and $U = B(0, \epsilon)$ the unit ball centered at the origin of $\mathbb{R}^n$. If $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in U \cap C$ and $\sum_{i=1}^n x_i = \sum_{i=1}^m y_i$, then we show $x_1 * x_2 * \ldots * x_n = y_1 * y_2 * \ldots * y_m$ in $RF(U \cap C)$. In this case we can suppose that $n = m$ filling up with zeros if necessary. Observe that for each i, $\pi(x_i) = \pi\left((\frac{1}{n}x_i) * (\frac{1}{n}x_i) * \overset{\text{n times}}{\cdots} * (\frac{1}{n}x_i)\right)$. It is easy to see that the number of operations needed to pass from the word x_i to the word $(\frac{1}{n}x_i) * (\frac{1}{n}x_i) * \overset{\text{n times}}{\cdots} * (\frac{1}{n}x_i)$ is bounded by the integer n. In this particular case we have that for all i, j, $(\frac{1}{n}x_i) * (\frac{1}{n}x_j) = (\frac{1}{n}x_j) * (\frac{1}{n}x_i)$. Observe that to pass from the word $(\frac{1}{n}x_i) * (\frac{1}{n}x_j)$ to the word $(\frac{1}{n}x_j) * (\frac{1}{n}x_i)$ two operations are necessary , first an elementary contraction and then an elementary expansion. Observe also that

$$\pi(x_1 * x_2 * \ldots * x_n) = \pi\Big((\frac{1}{n}x_1) * (\frac{1}{n}x_2) * \ldots * (\frac{1}{n}x_n) * \overset{\text{n times}}{\cdots}$$
$$\ldots * (\frac{1}{n}x_1) * (\frac{1}{n}x_2) * \ldots * (\frac{1}{n}x_n)\Big),$$

and that is possible to pass from one word to the other by a number of elementary operations bounded by $3n^2$.

Observe now that,

$$\begin{aligned}\pi(x_1 * x_2 * \ldots * x_n) &= \pi\left(\left(\frac{1}{n}\sum_{i=1}^n x_i\right) * \left(\frac{1}{n}\sum_{i=1}^n x_i\right) * \overset{\text{n times}}{\cdots} * \left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right)\\ &= \pi\left(\left(\frac{1}{n}\sum_{i=1}^n y_i\right) * \left(\frac{1}{n}\sum_{i=1}^n y_i\right) * \overset{\text{n times}}{\cdots} * \left(\frac{1}{n}\sum_{i=1}^n y_i\right)\right)\\ &= \pi\left(y_1 * y_2 * \ldots * y_n\right).\end{aligned}$$

A simple computation shows that the number of elementary operations needed to pass from the word $x_1 * x_2 * \ldots * x_n$ to the word $y_1 * y_2 * \ldots * y_n$ is bounded by the integer $8n^2$.

In this example we have shown that $K(\pi) = K(\phi)$ and therefore C is isomorphic to $RF(U \cap C)$. We can also choose U to be a compact neighborhood of zero, for example take U to be the closure of $B(0,1)$. We also showed in this example that the conditions of theorem 3.1.13 are satisfied and therefore $RF(U)$ is isomorphic to $\Gamma(C)$.

3.3.2 Example. For another example, consider $S = [1, +\infty)$ with the ordinary multiplication, as a subsemigroup of the positive real numbers, $U = [1, 1+\epsilon]$ where ϵ is a positive real number. If $t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_m \in [1, 1+\epsilon]$ and $t_1 t_2 \ldots t_n = s_1 s_2 \ldots s_m$ then in $RF([1, 1+\epsilon])$ we have that $t_1 * t_2 * \ldots * t_n = s_1 * s_2 * \ldots * s_m$.

Indeed, let's prove the affirmation for the cases $m = 1$ and $m = 2$ and for an arbitrary n. If $t_1 t_2 \ldots t_n = s_1$ then

$$t_1 * t_2 * \ldots * t_n = (t_1 t_2) * t_3 * \ldots * t_n = ((t_1 t_2) t_3) * \ldots * t_n = \ldots = s_1$$

so the case $m = 1$ has been proved. Let's consider now the case $m = 2$.

Suppose that $t_1 t_2 \ldots t_n = s_1 s_2$. Let k be such that $t_1 t_2 \ldots t_{k-1} \leq s_1 < t_1 t_2 \ldots t_k$, let β be such that $t_1 t_2 \ldots t_{k-1} \beta = s_1$. It is clear that $1 \leq \beta \leq 1+\epsilon$. Pick $\gamma \in [1, 1+\epsilon]$ such that $s_1 \gamma = t_1 t_2 \ldots t_k$. Therefore we have that $s_1 \gamma t_{k+1} \ldots t_n = s_1 s_2$ and dividing both sides of this equation by s_1 we get that $\gamma t_{k+1} \ldots t_n = s_2$ which implies by the previous case that $\gamma * t_{k+1} * \ldots * t_n = s_2$ in $RF([1, 1+\epsilon]$. Multiplying both sides of this relation in the left by s_1 we get that

$$s_1 * \gamma * t_{k+1} * \ldots * t_n = s_1 * s_2, \tag{1}$$

but by the previous case, we have that

$$t_1 * t_2 * \ldots * t_{k-1} * \beta = s_1. \tag{2}$$

Combining equations (1) and (2) we get

$$t_1 * t_2 * \ldots * t_{k-1} * \beta * \gamma * t_{k+1} * \ldots * t_n = s_1 * s_2. \tag{3}$$

But since $t_k = \beta * \gamma$ we finally have that

$$t_1 * t_2 * \ldots * t_n = s_1 * s_2.$$

From equation (3) it is clear that the number of steps necessary to pass from $t_1 * t_2 * \ldots * t_n$ to $s_1 * s_2$ by a finite number of elementary contractions and expansions is equal to n.

For the general case, suppose by induction that the claim is true for all integers k such that $k < n$ and $k < m$. Suppose that $t_1t_2 \ldots t_n = s_1s_2 \ldots s_m$. Let k be an integer such that $t_1t_2 \ldots t_{k-1} \leq s_1 < t_1t_2 \ldots t_k$ Pick β and γ such that $t_1t_2 \ldots t_{k-1}\beta = s_1$ and

$$s_1\gamma = t_1t_2 \ldots t_k. \tag{4}$$

Hence $\beta\gamma = t_k$, and therefore $t_1t_2 \ldots t_{k-1}\beta\gamma t_{k+1} \ldots t_n = s_1s_2 \ldots s_m$ or equivalently, $s_1\gamma t_{k+1} \ldots t_n = s_1s_2 \ldots s_m$. Dividing this last relation by s_1 we get that $\gamma t_{k+1} \ldots t_n = s_2 \ldots s_m$ and therefore by inductive hypotheses we have that $\gamma t_{k+1} * \ldots * t_n = s_2 * \ldots * s_m$. Multiplying this last relation by s_1 on the left of both sides of the equation we get

$$s_1 * \gamma * t_{k+1} * \ldots * t_n = s_1 * s_2 * \ldots * s_m. \tag{5}$$

Combining equations (4) and (5) we finally have that

$$t_1 * t_2 * \ldots * t_k * t_{k+1} * \ldots * t_n = s_1 * s_2 * \ldots * s_m.$$

It is easy to see that the number of steps required to pass from the word $t_1 * t_2 * \ldots * t_n$ to the word $s_1 * s_2 * \ldots * s_m$ by a finite number of elementary contractions and expansions is equal to $nm - n - k(m-1) < nm$. Hence the hypotheses of theorem 3.1.13 are satisfied and therefore $RF([0,1])$ is isomorphic to both $[1,\infty)$ and $\Gamma([1,\infty))$.

3.3.3 Example. Let's see now an example in a non–commutative semigroup. Consider the group

$$G = \{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0,\ y \in \mathbf{R}\} \subset GL(2,\mathbf{R}),$$

and the semigroup

$$S = \{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : 1 \le x,\ 0 \le y \le x-1\}.$$

We identify the matrix $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with the pair (x,y). Let $S_1 = \{(x,0) : 1 \le x\}$ and $S_2 = \{(x, x-1) : 1 \le x\}$, then an easy computation shows that $S = S_1S_2 = S_2S_1$. For notational convenience, we write $(x,0) \in S_1$, as $[x]$ and $(y, y-1) \in S_2$ as (y), then we have that $(y)[x] = [x_1](y_1)$ where $x_1 = xy - y + 1$ and $y_1 = \frac{xy}{xy-y+1}$.

Consider now $S_\epsilon = \{[x](y) : xy \le 1+\epsilon\}$, then in $RF(S_\epsilon)$, the element $[x_1](y_1)[x_2](y_2)\ldots[x_n](y_n)$ reduces to $[\tilde{x}_1]\ldots[\tilde{x}_m](\tilde{y}_1)\ldots(\tilde{y}_k)$ and one always obtains the same numbers for $\tilde{x}_1, \ldots, \tilde{x}_m$ and $\tilde{y}_1, \ldots, \tilde{y}_k$.

3.4 Conclusions

For a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup S, we define a uniformity for $\Gamma(S)$ and hence a topology. With the uniform topology, multiplication is continuous at the identity of $\Gamma(S)$ and in the second variable. Furthermore, if S is a nice semigroup then the map $[\alpha] \mapsto \alpha(1){:}\,\Gamma(S) \to S$ is continuous. Therefore $\Gamma(S)$ satisfies a universal property. If U is a compact neighborhood of the identity of the topological semigroup S, then under suitable hypothesis the relatively free semigroup $RF(U)$ is a topological semigroup which is algebraically isomorphic to $\Gamma(S)$. We conjecture that if S is a nice semigroup then that isomorphism is also a homeomorphism, i.e., is a topological isomorphism, and $\Gamma(S)$ is actually a topological semigroup. We also conjecture that if C is a cone in an almost abelian Lie algebra then $\Gamma(C)$ is isomorphic to C.

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