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LOCALLY HOMOGENEOUS AFFINE HYPERSPHERES WITH CONSTANT SECTIONAL CURVATURE

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Abstract

In this paper, we study the n-dimensional locally homogeneous affine hyperspheres with constant sectional curvature, vanishing Pick invariant and the difference tensor K satisfying $K^{n-1} \neq 0$. As main results, we classify such hyperspheres for dimension $n \leq 5$.

1. Introduction

An important problem in unimodular-affine differential geometry is to classify all the affine hyperspheres with affine metric of constant sectional curvature. The following results are well known.

THEOREM 1.1 ([28]). *Let M be a locally strongly convex affine hypersphere in $\mathbf{R}^{n+1}$ with constant sectional curvature. Then M is locally affine equivalent to either a hyperquadric or the hyperbolic affine hypersphere*

$$x_1 x_2 \cdots x_{n+1} = 1,$$

where $(x_1, \ldots, x_{n+1})$ is the standard coordinate of $\mathbf{R}^{n+1}$.

THEOREM 1.2 ([29]). *Let M be an affine hypersphere in $\mathbf{R}^{n+1}$ with constant sectional curvature c and nonzero Pick invariant. Then $c = 0$ and M is locally affine equivalent to*

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2m-1}^2 \pm x_{2m}^2) = 1$$

if $n = 2m - 1$, or

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2m-1}^2 \pm x_{2m}^2) x_{2m+1} = 1$$

if $n = 2m$, where $(x_1, \ldots, x_{n+1})$ is the standard coordinate of $\mathbf{R}^{n+1}$.

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However, the classification problem becomes much more difficult if the affine metric is indefinite and the Pick invariant vanishes [2]. Up to now, the following problem of L. Vrancken remains open.

PROBLEM 1 ([1]). Classify all affine hyperspheres with indefinite affine metric of constant sectional curvature and vanishing Pick invariant.

We remark from [6, 30] that affine hyperspheres with constant sectional curvature and nonzero Pick invariant are homogeneous under unimodular affine transformations. However, affine hyperspheres with constant sectional curvature and vanishing Pick invariant are not necessarily homogeneous [25]. The solution of Problem 1 for dimension 2 and 3 has been obtained in [27] and [2] respectively, but there appear many implicit examples.

On the other hand, different from Euclidean geometry, the class of higher dimensional homogeneous affine hypersurfaces is very large (cf. [6]), and one is far from a complete classification. For homogeneous affine surfaces one obtained the classification [21, 22]. The classification of 3-dimensional case has been completed except for Lorentzian affine hyperspheres (cf. [3, 20, 23, 24, 25]). For general dimension only partial results are known, see [4, 5, 15] for details. In particular for locally strongly convex homogeneous affine hyperspheres, Sasaki [26] reduced the classification to that of homogeneous convex cones. We notice that affine hypersurfaces with parallel cubic form are locally homogeneous affine hyperspheres [8], and recently, such homogeneous affine hyperspheres have been classified for the locally strongly convex case [14], the Lorentzian case [12], 3 and 4-dimensional case [11, 13], and the other subcases [10, 19].

Recently, a whole family of homogeneous affine hypersurfaces constructed by M. Eastwood and V. Ezhov in [9], called the generalized Cayley hypersurfaces, were found in [18] with the following properties: they are improper affine hypersphere with flat indefinite affine metric, zero Pick invariant and the difference tensor K satisfying $\nabla^{(\alpha)}K = 0$ and $K^{n-1} \neq 0$. For each constant $\alpha \in \mathbf{R}$, it is defined by the graph immersion of $x_{n+1} = \Phi(x_1, \dots, x_n; \alpha)$, where

$$(1.1) \qquad \Phi(x_1, \dots, x_n; \alpha) = \sum_{d=2}^{n+1} \frac{(-1)^d}{d!} \prod_{s=0}^{d-3} [(1-\alpha)s + 2] \sum_{j_1 + \cdots + j_d = n+1} x_{j_1} \cdots x_{j_d},$$

and $(x_1, \dots, x_{n+1})$ is the standard coordinate of $\mathbf{R}^{n+1}$. This is the Cayley surface, the Cayley hypersurface (1) of [9] and the hypersurface (6.3) of [7] corresponding to $n = 2$, $\alpha = 0$ and $\alpha = 1$, respectively. Note that the 3-dimensional generalized Cayley hypersurfaces, explicitly given by

$$(1.2) \qquad x_4 = \Phi(x_1, x_2, x_3; \alpha) = x_1 x_3 + \frac{1}{2} x_2^2 - x_1^2 x_2 + \frac{3-\alpha}{12} x_1^4,$$

are characterized by M. Ooguri as follows:

THEOREM 1.3 (cf. Proposition 4.2 of [25]). *Let M^3 be a locally homogeneous affine hypersphere in $\mathbf{R}^4$ with constant sectional curvature κ and vanishing Pick invariant. If $K^2 \neq 0$, then $\kappa = 0$ and M^3 is locally affine equivalent to the 3-dimensional generalized Cayley hypersurfaces.*

Without the condition of homogeneity, both the characterization of the Cayley hypersurface and that of the generalized Cayley hypersurfaces are obtained in [16] and [18], respectively.

Motivated by above facts, it is nature to consider the following problem:

PROBLEM 2. Besides the n-dimensional generalized Cayley hypersurfaces, do there exist other examples, which are locally homogeneous affine hypersphere with constant sectional curvature, vanishing Pick invariant and $K^{n-1} \neq 0$?

Remark 1.1. For dimension $n = 2$, there indeed exists such affine sphere [21], namely $x_3 = x_1 x_2 + \log x_1$. However, for $n = 3$ it follows from Theorem 1.3 that there doesn't exist such affine hypersphere.

In this paper, we give a positive answer to Problem 2 for $n = 4$ and 5, namely

MAIN THEOREM. *Let M^n be a locally homogeneous affine hypersphere in $\mathbf{R}^{n+1}$ with constant sectional curvature and vanishing Pick invariant. If $K^{n-1} \neq 0$, then M^n is an improper affine hypersphere with flat indefinite affine metric for $n \leq 5$. Furthermore, M^4 is locally affine equivalent to the graph immersions of*

$$x_5 = \Phi(x_1, \dots, x_4; \alpha) - \frac{\beta}{4} x_1^4, \tag{1.3}$$

and M^5 is locally affine equivalent to one of the two graph immersions:

$$x_6 = \Phi(x_1, \dots, x_5; \alpha) - \frac{\gamma}{4} x_1^4 - \frac{2\beta}{3} x_1^3 x_2 + \frac{(25 - 11\alpha)\beta}{75} x_1^5, \tag{1.4}$$

$$\begin{aligned} x_6 = \Phi(x_1, \dots, x_5; 0) &- \alpha x_1^3 x_3 + \frac{\alpha}{2} x_1^2 x_2^2 + \frac{\alpha(\alpha + 1)}{2} x_1^4 x_2 + \frac{\alpha(\alpha^2 - 3)}{12} x_1^6 \\ &- \frac{\gamma}{4} x_1^4 - \frac{2\beta}{3} x_1^3 x_2 + \frac{(1 - \alpha)\beta}{3} x_1^5, \end{aligned} \tag{1.5}$$

where α, β, γ are arbitrary constant, and $\Phi(x_1, \dots, x_n; \alpha)$ is given by (1.1).

This paper is organized as follows. In Section 2, we introduce the theory of local affine hypersurfaces. In Section 3, we study the locally homogeneous affine hyperspheres of Main Theorem to obtain a canonical local frame. The proof of Main Theorem is given in Section 4 for dimension $n = 4$ and in Section 5 for dimension $n = 5$, respectively.

2. Preliminaries

We briefly recall the theory of local equiaffine hypersurfaces in [17, 22]. Let $\mathbf{R}^{n+1}$ be the standard $(n+1)$-dimensional real affine space, i.e., $\mathbf{R}^{n+1}$ endowed with the standard flat connection D and its parallel volume form ω, given by the determinant. Let $F : M \hookrightarrow \mathbf{R}^{n+1}$ be a non-degenerate affine hypersurface. It is well known that on such hypersurface there exists a canonical transversal vector field ξ called the *affine normal*. Then we can write

$$D_X F_*(Y) = F_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.1}$$

$$D_X \xi = -F_*(SX). \tag{2.2}$$

This affine normal induces the following invariants on M: the *affine connection* ∇, the *affine metric*, or *Berwald-Blaschke metric* h, the *affine shape operator* S and the *cubic form*, or *Fubini-Pick form* $C := \nabla h$. Moreover, the *affine mean curvature* of M is defined by $H = \frac{1}{n}$ trace S. The hypersurface M is called an *affine hypersphere* if $S = H\, id$, then one easily proves that $H = const$ if $n \geq 2$. M is called a *proper affine hypersphere* if $H \neq 0$ and an *improper affine hypersphere* if $H = 0$. For a proper affine hypersphere the affine normal satisfies $\xi = -H(F - c)$, where c is a fixed point in $\mathbf{R}^{n+1}$, called the *center* of $F(M)$, for simplicity, we choose c as the origin of $\mathbf{R}^{n+1}$. For an improper affine hypersphere the affine normal ξ is constant.

The classical Pick-Berwald theorem states that the affine connection coincides with the Levi-Civita connection $\hat{\nabla}$ of affine metric h if and only if the hypersurface is a hyperquadric. For that reason, the difference tensor $K(X, Y) := \nabla_X Y - \hat{\nabla}_X Y$, related to the cubic form by $C(X, Y, Z) = -2h(K(X, Y), Z)$, plays a fundamental role in affine differential geometry. Denote by $\hat{R}$ the curvature tensor of $\hat{\nabla}$, by difference tensor K we have the Gauss and Codazzi equations:

$$\hat{R}(X, Y)Z = \frac{1}{2}[h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y] - [K_X, K_Y]Z,$$

$$(\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) = \frac{1}{2}[h(Y, Z)SX - h(X, Z)SY - h(SY, Z)X + h(SX, Z)Y],$$

Contracting Gauss equation twice we have

$$\chi = H + J, \tag{2.3}$$

where $J = \frac{1}{n(n-1)} h(K, K)$ is the *Pick invariant* and χ is the *normalized scalar curvature* of h. Moreover, we have the apolarity condition $\operatorname{tr} K_Z = 0$, and the property that $h(K(X, Y), Z)$ is totally symmetric for all X, Y and Z.

For an affine hypersphere with constant affine sectional curvature and $J = 0$, we have $\chi = H$ and the Gauss and Codazzi equations reduce to

$$\hat{R}(X, Y)Z = H[h(Y, Z)X - h(X, Z)Y], \tag{2.4}$$

$$[K_X, K_Y]Z = 0, \tag{2.5}$$

$$(\hat{\nabla}_X K)(Y, Z) = (\hat{\nabla}_Y K)(X, Z). \tag{2.6}$$

We prepare the following definitions and lemmas.

Definition 2.1 (cf. Remark 3.3 of [7]). For given positive integer k, a $(1, k+1)$-tensor field K^k is defined by

$$K^k(X_1, \ldots, X_{k+1}) = K_{X_1} \cdots K_{X_k} X_{k+1}$$

for any $X_1, \ldots, X_{k+1}$. If $[K_X, K_Y] = 0$ for all X and Y, the tensor field K^k is totally symmetric. Hence K^k vanishes identically if and only if $(K_v)^k v = 0$ for all vectors v. Denote by m the smallest number such that the symmetric tensor K^m is identically zero at the point p. Then for any tangent vector v at p, we have $(K_v)^m v = 0$ and there exists a tangent vector u at p such that $h((K_u)^{m-1}u, u) \neq 0$.

Lemma 2.1 (cf. Lemma 3.3 of [7]). *If $[K_Y, K_Z] = 0$ for all Y and Z, then K_X is nilpotent for each X. In particular, $K^n = 0$.*

An affine hypersurface M in $\mathbf{R}^{n+1}$ is called locally homogeneous if for all points p and q, there exists a neighborhood U of p and an equiaffine transformation A of $\mathbf{R}^{n+1}$, i.e., $A \in SL(n+1, \mathbf{R}^{n+1}) \ltimes \mathbf{R}^{n+1}$, such that $A(p) = q$ and $A(U) \subset M$. If $U = M$ holds for any $p \in M$, then M is homogeneous. Recall the following

Lemma 2.2 (cf. Lemma 2.1 of [25]). *If M is locally homogeneous affine hypersurface, then for any $p, q \in M$ there exists a neighborhood U of p and $A \in SL(n+1, \mathbf{R}^{n+1}) \ltimes \mathbf{R}^{n+1}$ such that $A(F(p)) = F(q)$, $A(F(U)) \subset F(M)$ and $A_*(\xi(p)) = \xi(q)$. Such transformation A preserves ∇, h, S and K.*

3. A canonical local frame

Let M be a locally homogeneous affine hypersphere of $\mathbf{R}^{n+1}$ with constant sectional curvature and zero Pick invariant, i.e., $J = 0$. Then we have $\chi = H$, (2.4), (2.5) and (2.6). Moreover, if $K^{n-1} \neq 0$ we can choose a canonical local frame as follows.

Lemma 3.1. *If $[K_X, K_Y] = 0$ for each X, Y and $K^{n-1} \neq 0$, then there exists a local frame $\{X_1, \ldots, X_n\}$ such that*

$$K(X_i, X_j) = \begin{cases} X_{i+j}, & i + j \leq n, \\ 0, & \text{otherwise}, \end{cases} \qquad h(X_i, X_j) = \begin{cases} 1, & i + j = n + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.1}$$

The frame is uniquely up to a sign determined.

Proof. The proof of the first part is the same as the proof of Lemma 6.1 in [7], see also Lemma 3.2 in [16], although we obtain the local frame instead of basis. Note that Lemma 3.1 has been proved for $n = 3$ (cf. Lemma 3.4 and Remark 3.4 of [2]). For the uniqueness of frame, we assume that there exist two frames $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_n\}$ satisfying (3.1). Set $Y_1 = a_1 X_1 + \cdots + a_n X_n$. Since $Y_{k+1} = K(Y_1, Y_k)$ for $k = 1, \dots, n-1$, we see from (3.1) that

$$h(Y_1, Y_\ell) = h\left(a_1 X_1 + \cdots + a_n X_n, \sum_{i_1+\cdots+i_\ell \le n} a_{i_1} \cdots a_{i_\ell} X_{i_1+\cdots+i_\ell}\right)$$
$$= a_{n+1-\ell} \sum_{i_1+\cdots+i_\ell = \ell} a_{i_1} \cdots a_{i_\ell} + \cdots + a_1 \sum_{i_1+\cdots+i_\ell = n} a_{i_1} \cdots a_{i_\ell},$$

where $\ell = 2, \dots, n$. Denote by δ the standard Kronecker delta. Solving the equations $h(Y_1, Y_\ell) = \delta_n^\ell$ for $\ell = n, \dots, 1$ respectively, we obtain that

$$a_1^{n+1} = 1, \quad a_2 = a_3 = \cdots = a_n = 0.$$

Hence $Y_1 = \pm X_1$, then by (3.1) the second conclusion is attained. □

Set $\hat\nabla_{X_i} X_j = \sum_{k=1}^n \Gamma_{i,j}^k X_k$ for the frame $\{X_1, \dots, X_n\}$. Then, by local homogeneity of M we see from Lemma 2.2 and 3.1 that $\Gamma_{i,j}^k$ are constant. From now on we follow the convention:

$$i, j, k, \ell \in \{1, \dots, n\}; \quad \Gamma_{p,q}^t = 0 \quad \text{if } \{p, q, t\} \not\subseteq \{1, \dots, n\}.$$

From (3.1) we obtain

$$h(\hat\nabla_{X_i} X_j, X_{n-k+1}) + h(X_j, \hat\nabla_{X_i} X_{n-k+1}) = 0, \tag{3.2}$$

which shows

$$\Gamma_{i,j}^k + \Gamma_{i,n-k+1}^{n-j+1} = 0, \quad \forall i, j, k. \tag{3.3}$$

In particular, $\Gamma_{i,n-k+1}^k = 0$. By (2.6) there holds

$$(\hat\nabla_{X_i} K)(X_j, X_k) = (\hat\nabla_{X_j} K)(X_i, X_k) = (\hat\nabla_{X_k} K)(X_j, X_i).$$

A simple computation shows

$$\sum_{\ell=1}^n \Gamma_{i,j+k}^\ell X_\ell - \sum_{p=1}^{n-k} \Gamma_{i,j}^p X_{p+k} - \sum_{q=1}^{n-j} \Gamma_{i,k}^q X_{q+j}$$
$$= \sum_{\ell=1}^n \Gamma_{j,i+k}^\ell X_\ell - \sum_{p=1}^{n-k} \Gamma_{j,i}^p X_{p+k} - \sum_{q=1}^{n-i} \Gamma_{j,k}^q X_{q+i}$$
$$= \sum_{\ell=1}^n \Gamma_{k,j+i}^\ell X_\ell - \sum_{p=1}^{n-i} \Gamma_{k,j}^p X_{p+i} - \sum_{q=1}^{n-j} \Gamma_{k,i}^q X_{q+j}.$$

Then we can rewrite above formulas as

$$(3.4)\qquad \Gamma^{\ell}_{i,j+k} - \Gamma^{\ell-k}_{i,j} - \Gamma^{\ell-j}_{i,k} = \Gamma^{\ell}_{j,i+k} - \Gamma^{\ell-k}_{j,i} - \Gamma^{\ell-i}_{j,k} = \Gamma^{\ell}_{k,j+i} - \Gamma^{\ell-i}_{k,j} - \Gamma^{\ell-j}_{k,i},$$

which immediately imply that

$$(3.5)\qquad \Gamma^{\ell}_{i,j+k} = \Gamma^{\ell}_{j,i+k} = \Gamma^{\ell}_{k,j+i}, \quad \ell \le \min\{i,j,k\}.$$

Together with (3.3) there hold

$$(3.6)\qquad \Gamma^{1}_{i,n} = \Gamma^{1}_{j,n-j+i} = \Gamma^{j-i+1}_{j,n} = 0, \quad \forall i,j.$$

Taking $i = 1,\ k = n$ in (3.4), we have

$$(3.7)\qquad \Gamma^{\ell-j}_{1,n} = \Gamma^{\ell-1}_{j,n} = \Gamma^{\ell-1}_{n,j} + \Gamma^{\ell-j}_{n,1} - \Gamma^{\ell}_{n,j+1}.$$

For $\ell \le j$ we get $\Gamma^{\ell-1}_{j,n} = \Gamma^{\ell-1}_{n,j} - \Gamma^{\ell}_{n,j+1} = 0$. By induction and (3.3) there hold

$$\Gamma^{\ell}_{j,n} = \Gamma^{\ell}_{n,j} = 0, \quad \ell < j.$$

While for $\ell = j+1 \le n$, by (3.3) we obtain

$$0 = \Gamma^{j}_{j,n} = \Gamma^{j}_{n,j} + \Gamma^{1}_{n,1} - \Gamma^{j+1}_{n,j+1}.$$

Thus $\Gamma^{j+1}_{n,j+1} = (j+1)\Gamma^{1}_{n,1}$ for $j \le n-1$. Together with $\Gamma^{n}_{n,n} = -\Gamma^{1}_{n,1}$ (cf. (3.3)) we see that

$$(3.8)\qquad \Gamma^{j}_{j,n} = \Gamma^{j}_{n,j} = 0, \quad \forall j.$$

For $\ell \ge j+2$ we set $\ell = j+m+1$ for $m = 1,\dots,n-j-1$ in (3.7) to obtain

$$\Gamma^{m+1}_{1,n} = \Gamma^{j+m}_{j,n} = \Gamma^{j+m}_{n,j} + \Gamma^{m+1}_{n,1} - \Gamma^{j+m+1}_{n,j+1},$$

which by induction gives that $\Gamma^{j+m+1}_{n,j+1} = (j+1)\Gamma^{m+1}_{n,1} - j\Gamma^{m+1}_{1,n}$ for $m+j \le n-1$. Together with $\Gamma^{n}_{n,n-m} = -\Gamma^{m+1}_{n,1}$ (cf. (3.3)) we get

$$\Gamma^{m+1}_{1,n} = \frac{n-m+1}{n-m-1}\Gamma^{m+1}_{n,1}, \quad \Gamma^{j+m+1}_{n,j+1} = \left(1 - \frac{2j}{n-m-1}\right)\Gamma^{m+1}_{n,1}.$$

Summing above we have proved the following

LEMMA 3.2.

$$(3.9)\qquad \Gamma^{\ell}_{j,n} = \Gamma^{\ell}_{n,j} = 0, \quad \ell \le j,$$

$$(3.10)\qquad \Gamma^{j+m+1}_{n,j+1} = \left(1 - \frac{2j}{n-m-1}\right)\Gamma^{m+1}_{n,1}, \quad \Gamma^{j+m}_{j,n} = \frac{n-m+1}{n-m-1}\Gamma^{m+1}_{n,1},$$

$$m + j \le n-1.$$

Next, taking $i = 1$, $k = n-1$ in (3.4) we have

$$\begin{aligned}\delta_1^j\Gamma_{1,n}^{\ell} - \delta_n^{\ell}\Gamma_{1,j}^1 - \Gamma_{1,n-1}^{\ell-j} &= \Gamma_{j,n}^{\ell} - \delta_n^{\ell}\Gamma_{j,1}^1 - \Gamma_{j,n-1}^{\ell-1}\\ &= \Gamma_{n-1,j+1}^{\ell} - \Gamma_{n-1,j}^{\ell-1} - \Gamma_{n-1,1}^{\ell-j}.\end{aligned} \tag{3.11}$$

For $\ell \le j \le n-1$, by (3.9) we obtain $\Gamma_{j,n-1}^{\ell-1} = \Gamma_{n-1,j}^{\ell-1} - \Gamma_{n-1,j+1}^{\ell} = 0$. By induction and (3.3) there hold

$$\Gamma_{j,n-1}^{\ell} = \Gamma_{n-1,j}^{\ell} = 0, \quad \ell < j \le n-1. \tag{3.12}$$

While for $\ell = j+1 \le n$ in (3.11), by (3.3) and (3.9) we get

$$\begin{aligned}(1 + \delta_1^j + \delta_{n-1}^j)\Gamma_{1,n-1}^1 &= \Gamma_{j,n-1}^j + \delta_{n-1}^j\Gamma_{n-1,1}^1 - \Gamma_{j,n}^{j+1}\\ &= \Gamma_{n-1,j}^j + \Gamma_{n-1,1}^1 - \Gamma_{n-1,j+1}^{j+1}.\end{aligned}$$

Then there hold

$$\begin{aligned}&\Gamma_{n-1,j+1}^{j+1} = \Gamma_{n-1,j}^j + \Gamma_{n-1,1}^1 - (1 + \delta_1^j + \delta_{n-1}^j)\Gamma_{1,n-1}^1, && j \le n-1;\\ &\Gamma_{j,n-1}^j = \Gamma_{j,n}^{j+1} + (1 + \delta_1^j)\Gamma_{1,n-1}^1, && j \le n-2.\end{aligned} \tag{3.13}$$

By induction in the first line equations of (3.13), we obtain

$$\Gamma_{n-1,j+1}^{j+1} = (j+1)(\Gamma_{n-1,1}^1 - \Gamma_{1,n-1}^1) - \delta_{n-1}^j\Gamma_{1,n-1}^1, \quad j \le n-1.$$

In particular, if $j = n-1$, by $\Gamma_{n-1,n}^n = -\Gamma_{n-1,1}^1$ we see that $\Gamma_{n-1,1}^1 = \Gamma_{1,n-1}^1$, thus

$$\Gamma_{n-1,k}^k = 0, \quad 2 \le k \le n-1.$$

Considering the second line equations of (3.13), by (3.3) and (3.10) we have

$$\Gamma_{j,n-1}^j = 0, \quad 2 \le j \le n-2.$$

When $\ell \ge j+2$, set $\ell = j+m+1$ for $m = 1, \ldots, n-j-1$ in (3.11), we obtain

$$\begin{aligned}\delta_1^j\Gamma_{1,n}^{m+2} - \delta_{n-m-1}^j\Gamma_{1,n-m-1}^1 - \Gamma_{1,n-1}^{m+1} &= \Gamma_{j,n}^{j+m+1} - \delta_{n-m-1}^j\Gamma_{n-m-1,1}^1 - \Gamma_{j,n-1}^{j+m}\\ &= \Gamma_{n-1,j+1}^{j+m+1} - \Gamma_{n-1,j}^{j+m} - \Gamma_{n-1,1}^{m+1},\end{aligned}$$

which by (3.3) and (3.10) reduce to

$$\begin{aligned}&\Gamma_{n-m-1,n-1}^{n-1} = \Gamma_{1,n-1}^{m+1} - \Gamma_{1,n}^{m+2} - 2\Gamma_{n-m-1,1}^1, && m \le n-3,\\ &\Gamma_{j,n-1}^{j+m} = \Gamma_{1,n-1}^{m+1} + (1 - \delta_1^j)\Gamma_{1,n}^{m+2}, && j < n-m-1,\\ &\Gamma_{n-1,j+1}^{j+m+1} = \Gamma_{n-1,j}^{j+m} + \Gamma_{n-1,1}^{m+1} - \Gamma_{1,n-1}^{m+1} + (\delta_1^j + \delta_{n-m-1}^j)\Gamma_{1,n}^{m+2}, && j \le n-m-1.\end{aligned}$$

The last equations imply that

$$\Gamma^{j+m+1}_{n-1,j+1} = (j+1)\Gamma^{m+1}_{n-1,1} - j\Gamma^{m+1}_{1,n-1} + (1+\delta^j_{n-m-1})\Gamma^{m+2}_{1,n}, \quad j \le n-m-1.$$

Thus by (3.3) we have $(n-m+1)\Gamma^{m+1}_{n-1,1} + 2\Gamma^{m+2}_{1,n} = (n-m-1)\Gamma^{m+1}_{1,n-1}$ and

$$\Gamma^{j+m+1}_{n-1,j+1} = \left(1 - \frac{2j}{n-m-1}\right)(\Gamma^{m+1}_{n-1,1} + \Gamma^{m+2}_{1,n}), \quad j < n-m-1.$$

Summing above by Lemma 3.2 we proved the following

LEMMA 3.3.

$$\begin{aligned}
&\Gamma^1_{n-1,1} = \Gamma^1_{1,n-1}, \quad \Gamma^k_{n-1,k} = \Gamma^k_{k,n-1} = 0, && 2 \le k \le n-1;\\
&\Gamma^\ell_{j,n-1} = \Gamma^\ell_{n-1,j} = 0, && \ell < j;\\
&\Gamma^{j+m}_{j,n-1} = \Gamma^{m+1}_{1,n-1} + (1-\delta^j_1)\Gamma^{m+2}_{1,n}, && m+j < n-1;\\
&\Gamma^{n-1}_{n-m-1,n-1} = \Gamma^{m+1}_{1,n-1} - \Gamma^{m+2}_{1,n} - 2\Gamma^1_{n-m-1,1}, && m \le n-3;\\
&\Gamma^{j+m+1}_{n-1,j+1} = \left(1 - \frac{2j}{n-m-1}\right)(\Gamma^{m+1}_{n-1,1} + \Gamma^{m+2}_{1,n}), && m+j < n-1;\\
&\Gamma^{m+1}_{1,n-1} = \frac{n-m+1}{n-m-1}\Gamma^{m+1}_{n-1,1} + \frac{2}{n-m-1}\Gamma^{m+2}_{1,n}, && m \le n-2.
\end{aligned}$$

Since M has constant sectional curvature H, from (2.4) and the definition of $\hat{R}$ we have

$$\begin{aligned}
H(\delta^{j+k}_{n+1}X_i - \delta^{i+k}_{n+1}X_j) &= \hat{R}(X_i, X_j)X_k \\
&= \hat{\nabla}_{X_i}\hat{\nabla}_{X_j}X_k - \hat{\nabla}_{X_j}\hat{\nabla}_{X_i}X_k - \hat{\nabla}_{[X_i,X_j]}X_k \\
&= \sum_{p,q=1}^{n}(\Gamma^p_{j,k}\Gamma^q_{i,p} - \Gamma^p_{i,k}\Gamma^q_{j,p} + \Gamma^p_{j,i}\Gamma^q_{p,k} - \Gamma^p_{i,j}\Gamma^q_{p,k})X_q.
\end{aligned} \tag{3.14}$$

First, taking $i < j = k = n$ in (3.14), by Lemma 3.2 we have

$$\delta^i_1 H X_n = \sum_{i<p<q\le n}(\Gamma^p_{i,n}\Gamma^q_{n,p} + \Gamma^p_{i,n}\Gamma^q_{p,n} - \Gamma^p_{n,i}\Gamma^q_{p,n})X_q. \tag{3.15}$$

When $n = 2$, $H = 0$. For $n \ge 3$, looking at the components of X_n we have

$$\begin{aligned}
\delta^i_1 H &= \sum_{i<p<n}(\Gamma^p_{i,n}\Gamma^n_{n,p} + \Gamma^p_{i,n}\Gamma^n_{p,n} - \Gamma^p_{n,i}\Gamma^n_{p,n}) \\
&= \sum_{m=1}^{n-i-1}(\Gamma^{i+m}_{i,n}\Gamma^n_{n,i+m} + \Gamma^{i+m}_{i,n}\Gamma^n_{i+m,n} - \Gamma^{i+m}_{n,i}\Gamma^n_{i+m,n}).
\end{aligned} \tag{3.16}$$

Note from (3.10) that

$$
(3.17)\qquad
\begin{aligned}
\Gamma_{n,i}^{i+m} &= \left(\frac{n-m+1}{n-m-1} - \frac{2i}{n-m-1}\right)\Gamma_{n,1}^{m+1}, && m+i \le n;\\
\Gamma_{i,n}^{i+m} &= \frac{n-m+1}{n-m-1}\Gamma_{n,1}^{m+1}, && m+i \le n-1.
\end{aligned}
$$

Then (3.16) reduce to

$$
(3.18)\qquad \delta_1^i H = \sum_{m=1}^{n-i-1} \frac{1}{n-m-1}\Gamma_{n,1}^{m+1}[(n-m+1)\Gamma_{n,i+m}^{n} + 2i\Gamma_{i+m,n}^{n}].
$$

For $i = n-2$, by (3.3) we obtain $\delta_3^n H = -\frac{1}{n-2}\Gamma_{n,1}^{2}[n\Gamma_{n,1}^{2} + 2(n-2)\Gamma_{n-1,1}^{1}]$. The first equation of Lemma 3.3 and (3.17) give $-\Gamma_{n-1,1}^{1} = \Gamma_{1,n}^{2} = \frac{n}{n-2}\Gamma_{n,1}^{2}$, thus

$$
(3.19)\qquad \delta_3^n H = \frac{n}{n-2}(\Gamma_{n,1}^{2})^2.
$$

Thus $\Gamma_{n,1}^{2} = \Gamma_{n,n-1}^{n} = \Gamma_{n-1,n}^{n} = 0$ for $n \ge 4$, and (3.18) reduce to

$$
(3.20)\qquad \delta_1^i H = \sum_{m=2}^{n-i-2} \frac{1}{n-m-1}\Gamma_{n,1}^{m+1}[(n-m+1)\Gamma_{n,i+m}^{n} + 2i\Gamma_{i+m,n}^{n}].
$$

For $i = n-3$ in (3.20) we obtain $\delta_4^n H = 0$, i.e., $H = 0$ when $n = 4$. For $i = n-4$, by (3.3) we have

$$
(3.21)\qquad \delta_5^n H = -\frac{1}{n-3}\Gamma_{n,1}^{3}[(n-1)\Gamma_{n,1}^{3} + 2(n-4)\Gamma_{n-2,1}^{1}], \quad n \ge 5.
$$

Next, taking $i < j = n-1 < k = n$ in (3.14), by Lemma 3.2 and 3.3 we have

$$
(3.22)\qquad
\begin{aligned}
\delta_1^i H X_{n-1} = &\sum_{i<p<q\le n} \Gamma_{i,n}^{p}\Gamma_{n-1,p}^{q} X_q - \Gamma_{n-1,n}^{n}\sum_{i<q\le n}\Gamma_{i,n}^{q}X_q\\
&+ \sum_{i<p<q\le n}(\Gamma_{i,n-1}^{p} - \Gamma_{n-1,i}^{p})\Gamma_{p,n}^{q}X_q.
\end{aligned}
$$

Looking at the components of X_{n-1} we have

$$
\delta_1^i H = \sum_{i<p<n-1}(\Gamma_{i,n}^{p}\Gamma_{n-1,p}^{n-1} + \Gamma_{i,n-1}^{p}\Gamma_{p,n}^{n-1} - \Gamma_{n-1,i}^{p}\Gamma_{p,n}^{n-1}) - \Gamma_{n-1,n}^{n}\Gamma_{i,n}^{n-1}.
$$

For $i = n-2$ we get $\delta_3^n H = 0$. Then (3.19) shows $\Gamma_{n,1}^{2} = 0$ for all $n \ge 3$. Summing above, by previous lemmas we have the following

LEMMA 3.4.

$$\begin{aligned} &\Gamma^{i+1}_{n,i} = \Gamma^{i+1}_{i,n} = 0, && n \geq 3, \\ &\Gamma^{\ell}_{j,n-1} = \Gamma^{\ell}_{n-1,j} = 0, && \ell \leq j, \\ &H = 0, && n \leq 4. \end{aligned}$$

Now, we continue to consider the totally symmetry of $\hat{\nabla} K$. Taking $(i,k) = (1, n-2)$ in (3.4) we have

$$\begin{aligned} \Gamma^{\ell}_{1,n+j-2} - \Gamma^{\ell-n+2}_{1,j} - \Gamma^{\ell-j}_{1,n-2} &= \Gamma^{\ell}_{j,n-1} - \Gamma^{\ell-n+2}_{j,1} - \Gamma^{\ell-1}_{j,n-2} \\ &= \Gamma^{\ell}_{n-2,j+1} - \Gamma^{\ell-1}_{n-2,j} - \Gamma^{\ell-j}_{n-2,1}. \end{aligned} \tag{3.23}$$

For $\ell \leq j \leq n-2$ in (3.23), by Lemma 3.4 we get

$$0 = \Gamma^{\ell-1}_{j,n-2} = \Gamma^{\ell-1}_{n-2,j} - \Gamma^{\ell}_{n-2,j+1},$$

which imply that

$$\Gamma^{\ell}_{j,n-2} = \Gamma^{\ell}_{n-2,j} = 0, \quad \ell < j \leq n-1. \tag{3.24}$$

Similarly, for $\ell = j+1 \leq n-1$ in (3.23) there hold

$$\begin{aligned} (1 + \delta^j_2 + \delta^j_{n-2})\Gamma^1_{1,n-2} &= \Gamma^j_{j,n-2} - \Gamma^{j+1}_{j,n-1} + \delta^j_{n-2}\Gamma^1_{n-2,1} \\ &= \Gamma^j_{n-2,j} + \Gamma^1_{n-2,1} - \Gamma^{j+1}_{n-2,j+1}. \end{aligned} \tag{3.25}$$

On the one hand, there hold $\Gamma^2_{n-2,2} = 2\Gamma^1_{n-2,1} - (1 + \delta^n_3)\Gamma^1_{1,n-2}$ and

$$\Gamma^{j+1}_{n-2,j+1} = (j+1)(\Gamma^1_{n-2,1} - \Gamma^1_{1,n-2}) - \delta^j_{n-2}\Gamma^1_{1,n-2}, \quad 2 \leq j \leq n-2.$$

Combining with $\Gamma^{n-1}_{n-2,n-1} = -\Gamma^2_{n-2,2}$ we see that

$$\begin{aligned} &\Gamma^1_{n-2,1} = \Gamma^1_{1,n-2}, \quad \Gamma^2_{n-2,2} = (1 - \delta^n_3)\Gamma^1_{1,n-2}, \\ &\Gamma^{j+1}_{n-2,j+1} = 0, \quad 2 \leq j < n-2. \end{aligned} \tag{3.26}$$

The first equation of (3.26) and (3.10) give that

$$\Gamma^1_{n-2,1} = \Gamma^1_{1,n-2} = -\Gamma^3_{1,n} = -\frac{n-1}{n-3}\Gamma^3_{n,1},$$

then we obtain from (3.21) that $\delta^n_5 H = \dfrac{(n-1)(n-5)}{(n-3)^2}(\Gamma^3_{n,1})^2$, which show that $H = 0$ when $n = 5$ and $\Gamma^3_{n,1} = 0$ for $n > 5$. On the other hand, for $j \leq n-2$ in (3.25) there hold

$$\Gamma^j_{j,n-2} - \Gamma^{j+1}_{j,n-1} = (1 + \delta^j_2)\Gamma^1_{1,n-2}.$$

By the third line equation of Lemma 3.3 and (3.26) we obtain

$$\Gamma^j_{j,n-2} = \delta^j_2 \Gamma^1_{1,n-2}, \quad 2 \le j \le n-2.$$

As before, for $\ell \ge j+2$, set $\ell = j+m+1$ for $m = 1, \dots, n-j-1$ in (3.23), by (3.3) we obtain

$$\begin{aligned}(\delta^j_1 + \delta^j_{n-m-1})\Gamma^{m+2}_{1,n-1} + (\delta^j_2 + \delta^j_{n-m-2})\Gamma^{m+3}_{1,n} - \Gamma^{m+1}_{1,n-2} \\ = \Gamma^{j+m+1}_{j,n-1} - \Gamma^{m+j}_{j,n-2} - \delta^j_{n-m-2}\Gamma^1_{n-m-2,1} - \delta^j_{n-m-1}\Gamma^2_{n-m-1,1} \\ = \Gamma^{j+m+1}_{n-2,j+1} - \Gamma^{j+m}_{n-2,j} - \Gamma^{m+1}_{n-2,1}, \quad m+j \le n-1.\end{aligned} \tag{3.27}$$

Summing above, by previous lemmas we have proved the following

Lemma 3.5. *There hold* (3.27) *and*

$$\begin{aligned}
&\Gamma^{i+2}_{n,i} = \Gamma^{i+2}_{i,n} = 0, && n > 5,\\
&\Gamma^\ell_{j,n-2} = \Gamma^\ell_{n-2,j} = 0, && \ell < j,\\
&\Gamma^1_{n-2,1} = \Gamma^1_{1,n-2} = -\frac{n-1}{n-3}\Gamma^3_{n,1}, && n \ge 4,\\
&\Gamma^j_{n-2,j} = \Gamma^j_{j,n-2} = \delta^j_2\Gamma^1_{1,n-2}, && 2 \le j \le n-2,\\
&H = 0, && n = 5.
\end{aligned}$$

4. The 4-dimensional case

In this section, for dimension $n = 4$ we completely determine the affine hyperspheres by proving the following

Theorem 4.1. *Let M be a locally homogeneous* 4*-dimensional affine hypersphere of* $\mathbf{R}^5$ *with constant sectional curvature and vanishing Pick invariant. If* $K^3 \ne 0$, *then M is an improper affine hypersphere with flat indefinite affine metric, and M is locally affine equivalent to one of the graph immersions of polynomials*

$$x_5 = \Phi(x_1, \dots, x_4; \alpha) - \frac{\beta}{4}x_1^4, \tag{4.1}$$

where α, β are arbitrary constant.

Proof. By Lemma 3.2–3.5, for $n = 4$ we see that $H = 0$, thus $\hat{R} = 0$. Moreover,

$$\begin{aligned}
&\Gamma^q_{k,p} + \Gamma^{5-p}_{k,5-q} = 0, && \Gamma^i_{j,4} = \Gamma^i_{4,j} = \Gamma^i_{j,3} = \Gamma^i_{3,j} = 0, \quad i \le j \le 4,\\
&\Gamma^{i+1}_{i,4} = \Gamma^{i+1}_{4,i} = 0, \quad i \le 3, && \Gamma^4_{2,4} = \Gamma^3_{1,4} = \Gamma^3_{2,3} = 2\Gamma^4_{3,3} = 3\Gamma^3_{4,1},\\
&\Gamma^3_{1,3} = 3\Gamma^3_{3,1} + 2\Gamma^4_{1,4}, && 2\Gamma^4_{2,3} = 2\Gamma^3_{1,3} + \Gamma^4_{1,4}, \quad 3\Gamma^4_{2,2} = 2\Gamma^4_{1,3}.
\end{aligned}$$

For $(i, j, k) = (1, 3, 1)$ in (3.14), a direct computation gives

$$\begin{aligned} 0 &= \hat{R}(X_1, X_3)X_1 \\ &= 9(\Gamma_{4,1}^3)^2 X_1 + \frac{3}{4}\Gamma_{4,1}^3(16\Gamma_{3,1}^3 + 11\Gamma_{1,4}^4)X_2 + \Gamma_{3,1}^3(\Gamma_{3,1}^3 + \Gamma_{1,4}^4)X_3. \end{aligned}$$

Thus $\Gamma_{4,1}^3 = 0$ and $\Gamma_{3,1}^3(\Gamma_{3,1}^3 + \Gamma_{1,4}^4) = 0$. Now, taking $(i, j, k) = (1, 2, 1)$ in (3.14) we further obtain

$$0 = \hat{R}(X_1, X_2)X_1 = \frac{3}{4}(6\Gamma_{3,1}^3 + 5\Gamma_{1,4}^4)(2\Gamma_{3,1}^3 + \Gamma_{1,4}^4)X_2 - \frac{1}{3}(8\Gamma_{3,1}^3 + 5\Gamma_{1,4}^4)\Gamma_{1,3}^4 X_3.$$

These equations show that $\Gamma_{3,1}^3 = \Gamma_{1,4}^4 = 0$. Set $\Gamma_{1,3}^4 = \frac{\alpha}{2}$ and $\Gamma_{1,2}^4 = \beta$, there hold

$$\begin{aligned} &\hat{\nabla}_{X_4} X_i = \hat{\nabla}_{X_i} X_4 = \hat{\nabla}_{X_3} X_i = \hat{\nabla}_{X_2} X_3 = 0, \quad \forall i, \\ &\hat{\nabla}_{X_1} X_3 = \frac{\alpha}{2} X_4, \quad \hat{\nabla}_{X_1} X_2 = \beta X_4, \quad \hat{\nabla}_{X_2} X_1 = -\frac{\alpha}{3} X_3, \\ &\hat{\nabla}_{X_1} X_1 = -\frac{\alpha}{2} X_2 - \beta X_3, \quad \hat{\nabla}_{X_2} X_2 = \frac{\alpha}{3} X_4. \end{aligned} \tag{4.2}$$

The only nonzero components of $\hat{\nabla}K$ are

$$\begin{aligned} \hat{\nabla}K(X_2, X_1, X_1) &= \alpha X_4, \\ \hat{\nabla}K(X_1, X_1, X_1) &= \alpha X_3 + 3\beta X_4, \end{aligned}$$

and the only nonzero Lie brackets are

$$[X_1, X_2] = \frac{\alpha}{3} X_3 + \beta X_4, \quad [X_1, X_3] = \frac{\alpha}{2} X_4.$$

Now, we look at the following system of differential equations of (ρ_1, ρ_2).

$$\begin{cases} X_1(\rho_1) = 0, \quad X_2(\rho_1) = \frac{\alpha}{3}, \quad X_3(\rho_1) = X_4(\rho_1) = 0, \\ X_1(\rho_2) = -\frac{\alpha}{2}\rho_1, \quad X_2(\rho_2) = \beta, \quad X_3(\rho_2) = \frac{\alpha}{2}, \quad X_4(\rho_2) = 0. \end{cases} \tag{4.3}$$

A direct computation shows that for $k = 1, 2$

$$(X_i X_j - X_j X_i - [X_i, X_j])\rho_k = 0, \quad \forall i, j.$$

Hence, for instance by introducing coordinates, it is clear that the system of differential equations (4.3) has a unique solution (ρ_1, ρ_2) with initial conditions $\rho_1(0) = \rho_2(0) = 0$. Then, by straightforward computation, using (4.3) we verify the following lemma.

Lemma 4.1. *The linear independent vector fields*

$$Y_1 = X_1 + \rho_1 X_3 + \rho_2 X_4, \quad Y_2 = X_2, \quad Y_3 = X_3, \quad Y_4 = X_4$$

satisfy $[Y_i, Y_j] = 0$ *for all* i, j. *Hence, there exist local coordinates* $\{u_1, u_2, u_3, u_4\}$ *on* M *such that* $\frac{\partial}{\partial u_i} = Y_i$ *for* $i = 1, 2, 3, 4$ *and* $\rho_1 = \frac{\alpha}{3}u_2$, $\rho_2 = \beta u_2 + \frac{\alpha}{2}u_3$.

Expressing the Levi-Civita connection in terms of the frame Y_i we see that

$$\begin{aligned} &\hat{\nabla}_{Y_3} Y_3 = \hat{\nabla}_{Y_2} Y_3 = \hat{\nabla}_{Y_3} Y_2 = 0, \quad \hat{\nabla}_{Y_4} Y_i = \hat{\nabla}_{Y_i} Y_4 = 0, \quad \forall i, \\ &\hat{\nabla}_{Y_1} Y_1 = -\frac{\alpha}{2} Y_2 - \beta Y_3 + \frac{1}{6}\alpha^2 u_2 Y_4, \quad \hat{\nabla}_{Y_2} Y_2 = \frac{\alpha}{3} Y_4, \\ &\hat{\nabla}_{Y_1} Y_2 = \hat{\nabla}_{Y_2} Y_1 = \beta Y_4, \quad \hat{\nabla}_{Y_1} Y_3 = \hat{\nabla}_{Y_3} Y_1 = \frac{\alpha}{2} Y_4. \end{aligned} \tag{4.4}$$

Also the only nonzero components of the affine metric are

$$h(Y_1, Y_1) = 2\beta u_2 + \alpha u_3, \quad h(Y_1, Y_2) = \frac{\alpha}{3}u_2, \quad h(Y_1, Y_4) = h(Y_2, Y_3) = 1,$$

and the only nonzero components of the difference tensor are

$$K_{Y_1} Y_1 = Y_2 + \frac{2}{3}\alpha u_2 Y_4, \quad K_{Y_1} Y_2 = Y_3, \quad K_{Y_1} Y_3 = K_{Y_2} Y_2 = Y_4.$$

Note that the affine normal field ξ of improper affine hypersphere M is constant. From $D_X Y = K_X Y + \hat{\nabla}_X Y + h(X, Y)\xi$, it follows that the immersion F is determined by the following system of differential equations:

$$\begin{cases} F_{u_1u_1} = \frac{2-\alpha}{2} F_{u_2} - \beta F_{u_3} + \frac{\alpha(4+\alpha)}{6} u_2 F_{u_4} + (2\beta u_2 + \alpha u_3)\xi, \\ F_{u_1u_2} = F_{u_3} + \beta F_{u_4} + \frac{\alpha}{3} u_2 \xi, \quad F_{u_1u_3} = \frac{2+\alpha}{2} F_{u_4}, \\ F_{u_2u_2} = \frac{3+\alpha}{3} F_{u_4}, \quad F_{u_1u_4} = F_{u_2u_3} = \xi, \\ F_{u_3u_3} = F_{u_2u_4} = F_{u_3u_4} = F_{u_4u_4} = 0, \end{cases}$$

where $F_{u_i} := F_* \frac{\partial}{\partial u_i}$. Solving above system of differential equations, up to an affine transformation, we obtain

$$\begin{aligned} F = {} & A + u_1 A_1 + \left(u_2 + \frac{2-\alpha}{4} u_1^2\right) A_2 + \left(u_3 + u_1 u_2 + \frac{2-\alpha}{12} u_1^3 - \frac{\beta}{2} u_1^2\right) A_3 \\ & + \left[u_4 + \frac{2+\alpha}{2} u_1 u_3 + \frac{3+\alpha}{6} u_2^2 + \frac{2+\alpha}{4} u_1^2 u_2 + \frac{4-\alpha^2}{4! \times 4} u_1^4 + \beta u_1 u_2 - \frac{1}{6}\alpha\beta u_1^3\right] A_4 \\ & + \left[u_1 u_4 + u_2 u_3 + \frac{3+\alpha}{6} u_1 u_2^2 + \frac{2+\alpha}{4} u_1^2 u_3 + \frac{2+\alpha}{12} u_1^3 u_2 \right. \\ & \left. + \frac{4-\alpha^2}{5! \times 4} u_1^5 + \frac{\beta}{2} u_1^2 u_2 - \frac{\alpha\beta}{4!} u_1^4\right] \xi, \end{aligned}$$

where $A_i = F_{u_i}(0)$, $A = F(0)$ are constant vectors of $\mathbf{R}^5$. Because of non-degenerate, M lies linearly full in $\mathbf{R}^5$. Hence $\xi, A_1, \dots, A_4$ are linearly independent vectors. By an equiaffine transformation we can write

$$F = \Big(u_1, u_2 + \frac{2-\alpha}{4}u_1^2, u_3 + u_1u_2 + \frac{2-\alpha}{12}u_1^3 - \frac{\beta}{2}u_1^2, u_4 + \frac{2+\alpha}{2}u_1u_3 + \frac{3+\alpha}{6}u_2^2 + \frac{2+\alpha}{4}u_1^2u_2 + \frac{4-\alpha^2}{4!\times 4}u_1^4 + \beta u_1u_2 - \frac{1}{6}\alpha\beta u_1^3, u_1u_4 + u_2u_3 + \frac{3+\alpha}{6}u_1u_2^2 + \frac{2+\alpha}{4}u_1^2u_3 + \frac{2+\alpha}{12}u_1^3u_2 + \frac{4-\alpha^2}{5!\times 4}u_1^5 + \frac{\beta}{2}u_1^2u_2 - \frac{\alpha\beta}{4!}u_1^4\Big) \hookrightarrow \mathbf{R}^5.$$

Set $x_1 = u_1$, $x_2 = u_2 + \frac{2-\alpha}{4}u_1^2$, $x_3 = u_3 + u_1u_2 + \frac{2-\alpha}{12}u_1^3 - \frac{\beta}{2}u_1^2$ and

$$x_4 = u_4 + \frac{2+\alpha}{2}u_1u_3 + \frac{3+\alpha}{6}u_2^2 + \frac{2+\alpha}{4}u_1^2u_2 + \frac{4-\alpha^2}{4!\times 4}u_1^4 + \beta u_1u_2 - \frac{1}{6}\alpha\beta u_1^3,$$

$$x_5 = u_1u_4 + u_2u_3 + \frac{3+\alpha}{6}u_1u_2^2 + \frac{2+\alpha}{4}u_1^2u_3 + \frac{2+\alpha}{12}u_1^3u_2 + \frac{4-\alpha^2}{5!\times 4}u_1^5 + \frac{\beta}{2}u_1^2u_2 - \frac{\alpha\beta}{4!}u_1^4,$$

we see that M lies on the graph immersion of polynomial

$$x_5 = x_1x_4 + x_2x_3 - x_1^2x_3 - x_1x_2^2 + \frac{3-\alpha}{3}x_1^3x_2 - \frac{(2-\alpha)(3-\alpha)}{30}x_1^5 - \frac{\beta}{4}x_1^4.$$

This is exactly the hypersurface (4.1). Obviously, if $\beta = 0$ these are exactly the 4-dimensional generalized Cayley hypersurfaces. □

5. The 5-dimensional case

In this section, for dimension $n = 5$ we completely determine the affine hyperspheres by proving the following

THEOREM 5.1. *Let M be a locally homogeneous 5-dimensional affine hypersphere of $\mathbf{R}^6$ with constant sectional curvature and vanishing Pick invariant. If $K^4 \neq 0$, then M is an improper affine hypersphere with flat indefinite affine metric, and M is locally affine equivalent to one of the two graph immersions*:

$$x_6 = \Phi(x_1, \dots, x_5; \alpha) - \frac{\gamma}{4}x_1^4 - \frac{2\beta}{3}x_1^3x_2 + \frac{(25-11\alpha)\beta}{75}x_1^5, \tag{5.1}$$

$$x_6 = \Phi(x_1, \dots, x_5; 0) - \alpha x_1^3x_3 + \frac{\alpha}{2}x_1^2x_2^2 + \frac{\alpha(\alpha+1)}{2}x_1^4x_2 + \frac{\alpha(\alpha^2-3)}{12}x_1^6 - \frac{\gamma}{4}x_1^4 - \frac{2\beta}{3}x_1^3x_2 + \frac{(1-\alpha)\beta}{3}x_1^5, \tag{5.2}$$

where α, β, γ are arbitrary constant.

Proof. By Lemma 3.2–3.5, $H = 0$ (thus $\hat{R} = 0$) and (3.14) we see that

$$0 = h(\hat{R}(X_3, X_4)X_3, X_2) = (\Gamma^4_{4,3})^2,$$

thus $\Gamma^4_{4,3} = 0$. Moreover,

$$(5.3)\quad \begin{aligned} &\Gamma^q_{k,p} + \Gamma^{6-p}_{k,6-q} = 0, \quad \Gamma^i_{j,5} = \Gamma^i_{5,j} = \Gamma^i_{j,4} = \Gamma^i_{4,j} = 0, \quad i \le j, \\ &\Gamma^{i+1}_{i,5} = \Gamma^{i+1}_{5,i} = 0, \quad i \le 4, && \Gamma^k_{j,3} = \Gamma^k_{3,j} = 0, \quad k < j, \\ &\Gamma^{i+1}_{i,4} = \Gamma^{i+1}_{4,i} = 0, \quad i \le 4, && \Gamma^{j+2}_{j,5} = \Gamma^{j+2}_{5,j} = 0, \quad j \le 3, \\ &3\Gamma^4_{5,1} = \Gamma^4_{1,5} = \Gamma^3_{1,4} + 2\Gamma^5_{4,3}, && 2\Gamma^5_{2,5} = \Gamma^4_{2,4} + 2\Gamma^5_{4,3}, \\ &\Gamma^4_{2,4} + \Gamma^5_{2,5} = 2\Gamma^4_{1,5} + \Gamma^3_{1,4} = 2\Gamma^4_{3,3} + \Gamma^5_{3,4}, && 2\Gamma^5_{3,4} = 2\Gamma^3_{1,4} + \Gamma^4_{3,3}, \\ &\Gamma^4_{1,4} + \Gamma^5_{1,5} = 2\Gamma^5_{2,4} - \Gamma^4_{2,3} = 2\Gamma^5_{3,3}, && \Gamma^4_{1,4} = 3\Gamma^4_{4,1} + 2\Gamma^5_{1,5}, \\ &2\Gamma^5_{1,4} = \Gamma^4_{1,3} + 3\Gamma^5_{3,2}. \end{aligned}$$

For $(i, j, k) = (2, 5, 2)$ and $(3, 4, 2)$ in (3.14), the direct computations show that

$$(5.4)\quad \begin{aligned} 0 &= \hat{R}(X_2, X_5)X_2 = \Gamma^5_{5,2}(\Gamma^5_{5,2} + \Gamma^4_{2,4})X_5, \\ 0 &= \hat{R}(X_3, X_4)X_2 = (\Gamma^4_{3,3}\Gamma^5_{4,3} + \Gamma^5_{5,2}\Gamma^5_{4,3} - \Gamma^5_{5,2}\Gamma^5_{3,4})X_5. \end{aligned}$$

Claim: $\Gamma^4_{5,1} = 0$. *Otherwise, assume that* $\Gamma^4_{5,1} = a \neq 0$, *by* (3.3) *we see from the first equation of* (5.4) *and* (5.3) *that*

$$\Gamma^4_{2,4} = a, \quad \Gamma^5_{3,4} = -\frac{4}{5}a, \quad \Gamma^4_{3,3} = \frac{12}{5}a, \quad \Gamma^3_{1,4} = -2a,$$
$$\Gamma^5_{2,5} = \Gamma^4_{1,5} = 3a, \quad \Gamma^5_{4,3} = \frac{5}{2}a.$$

Then the second equation of (5.4) implies that $a = 0$, a contradiction. Claim has been proved. Then it follows from (5.3) and (5.4) that

$$\Gamma^4_{5,1} = \Gamma^4_{2,4} = \Gamma^5_{3,4} = \Gamma^4_{3,3} = \Gamma^3_{1,4} = \Gamma^5_{2,5} = \Gamma^4_{1,5} = \Gamma^5_{4,3} = 0.$$

Similarly, by (3.3) and the sixth line equations of (5.3) there hold

$$(5.5)\quad \begin{aligned} 0 &= h(\hat{R}(X_1, X_2)X_1, X_4) = \Gamma^5_{2,4}(\Gamma^5_{2,4} - \Gamma^5_{1,5}), \\ 0 &= \hat{R}(X_1, X_4)X_1 = \Gamma^4_{4,1}(\Gamma^4_{4,1} + \Gamma^5_{1,5})X_4, \\ 0 &= \hat{R}(X_1, X_3)X_1 = \Gamma^5_{3,3}(\Gamma^5_{3,3} - \Gamma^5_{1,5})X_3 - (\Gamma^5_{3,3} + \Gamma^4_{4,1})(\Gamma^5_{3,2} + \Gamma^4_{1,3})X_4. \end{aligned}$$

Claim: $\Gamma^5_{3,3} = 0$. *Otherwise, assume that* $\Gamma^5_{3,3} = b \neq 0$, *by* (5.5) *we see from the sixth line equations of* (5.3) *that* $\Gamma^5_{1,5} = \Gamma^4_{1,4} = -3\Gamma^4_{4,1} = b$. *Then the second* *equation of* (5.5) *implies that* $b=0$, *a contradiction. Claim has been proved. Moreover, we see from*

$$(\hat{\nabla}_{X_1}K)(X_2,X_2)=(\hat{\nabla}_{X_2}K)(X_1,X_2),\quad (\hat{\nabla}_{X_2}K)(X_1,X_1)=(\hat{\nabla}_{X_1}K)(X_2,X_1)$$

that

$$3\Gamma_{1,4}^4=\Gamma_{2,4}^5+2\Gamma_{2,3}^4,\quad 2\Gamma_{2,3}^5=\Gamma_{1,4}^5+2\Gamma_{1,3}^4,\quad 3\Gamma_{2,2}^5=2\Gamma_{1,3}^5. \tag{5.6}$$

It follows from this and the sixth line equations of (5.3) that

$$\Gamma_{2,3}^4=2\Gamma_{2,4}^5:=2\lambda,\quad \Gamma_{1,4}^4=\Gamma_{4,1}^4=-\Gamma_{1,5}^5=\frac{5}{3}\lambda.$$

Taking this into the first equation of (5.5) we obtain $\lambda=0$, thus

$$\Gamma_{2,4}^5=\Gamma_{2,3}^4=\Gamma_{1,4}^4=\Gamma_{4,1}^4=\Gamma_{1,5}^5=\Gamma_{3,3}^5=0.$$

Set $\Gamma_{1,3}^5=\beta$, $\Gamma_{1,2}^5=\gamma$. Summing above, by the last equation of (5.3) and the last two equations of (5.6) we can express the Levi-Civita connection as follows:

$$\begin{aligned}
&\hat{\nabla}_{X_5}X_i=\hat{\nabla}_{X_i}X_5=\hat{\nabla}_{X_4}X_i=\hat{\nabla}_{X_{i+1}}X_4=\hat{\nabla}_{X_3}X_3=0,\quad \forall i,\\
&\hat{\nabla}_{X_1}X_4=\Gamma_{1,4}^5X_5,\quad \hat{\nabla}_{X_2}X_2=\frac{2}{3}\beta X_5,\\
&\hat{\nabla}_{X_1}X_3=\Gamma_{1,3}^4X_4+\beta X_5,\quad \hat{\nabla}_{X_3}X_1=\left(\frac{1}{3}\Gamma_{1,3}^4-\frac{2}{3}\Gamma_{1,4}^5\right)X_4,\\
&\hat{\nabla}_{X_2}X_3=\left(\Gamma_{1,3}^4+\frac{1}{2}\Gamma_{1,4}^5\right)X_5,\quad \hat{\nabla}_{X_3}X_2=\left(\frac{2}{3}\Gamma_{1,4}^5-\frac{1}{3}\Gamma_{1,3}^4\right)X_5,\\
&\hat{\nabla}_{X_1}X_2=-\Gamma_{1,3}^4X_3+\gamma X_5,\quad \hat{\nabla}_{X_2}X_1=-\left(\Gamma_{1,3}^4+\frac{1}{2}\Gamma_{1,4}^5\right)X_3-\frac{2}{3}\beta X_4,\\
&\hat{\nabla}_{X_1}X_1=-\Gamma_{1,4}^5X_2-\beta X_3-\gamma X_4.
\end{aligned} \tag{5.7}$$

Combining with the Gauss equation $\hat{R}(X_1,X_2)X_1=0$ we obtain that

$$(\Gamma_{1,3}^4+\Gamma_{1,4}^5)\left(\Gamma_{1,3}^4-\frac{1}{3}\Gamma_{1,4}^5\right)=0. \tag{5.8}$$

Hence we can divide our discussion into two cases:

$$\text{Case I: } \Gamma_{1,4}^5=3\Gamma_{1,3}^4,\quad \text{Case II: } \Gamma_{1,4}^5=-\Gamma_{1,3}^4.$$

For Case I, we set $\Gamma_{1,4}^5=3\Gamma_{1,3}^4=\frac{3\alpha}{5}$. We note from (5.7) that the only nonzero components of $\hat{\nabla}K$ are

$$\begin{aligned}
&\hat{\nabla}K(X_1,X_1,X_1)=\alpha X_3+2\beta X_4+3\gamma X_5,\quad \hat{\nabla}K(X_3,X_1,X_1)=\alpha X_5,\\
&\hat{\nabla}K(X_2,X_1,X_1)=\alpha X_4+2\beta X_5,\qquad\quad \hat{\nabla}K(X_1,X_2,X_2)=\alpha X_5.
\end{aligned}$$

The only nonzero Lie brackets are

$$[X_1, X_4] = \frac{3\alpha}{5} X_5, \quad [X_1, X_3] = \frac{8\alpha}{15} X_4 + \beta X_5,$$
$$[X_2, X_3] = \frac{\alpha}{6} X_5, \quad [X_1, X_2] = \frac{3\alpha}{10} X_3 + \frac{2}{3}\beta X_4 + \gamma X_5.$$

Now, we look at the following system of differential equations of $(\rho_1, \dots, \rho_4)$.

$$\begin{cases} X_1(\rho_1) = 0, \quad X_2(\rho_1) = \frac{3\alpha}{10}, \quad X_3(\rho_1) = X_4(\rho_1) = X_5(\rho_i) = 0, \quad \forall i, \\ X_1(\rho_2) = -\frac{\alpha}{6}\rho_1, \quad X_2(\rho_2) = 0, \quad X_3(\rho_2) = \frac{\alpha}{6}, \quad X_4(\rho_2) = 0, \\ X_1(\rho_3) = -\frac{8\alpha}{15}\rho_1, \quad X_2(\rho_3) = \frac{2\beta}{3}, \quad X_3(\rho_3) = \frac{8\alpha}{15}, \quad X_4(\rho_3) = 0, \\ X_1(\rho_4) = -\left(\beta\rho_1 + \frac{3\alpha}{5}\rho_3\right), \quad X_2(\rho_4) = \gamma - \frac{\alpha}{6}\rho_1, \\ X_3(\rho_4) = \beta, \quad X_4(\rho_4) = \frac{3\alpha}{5}. \end{cases} \tag{5.9}$$

Direct computations show that for $k = 1, 2, 3, 4$

$$(X_i X_j - X_j X_i - [X_i, X_j])\rho_k = 0, \quad \forall i, j.$$

Hence, for instance by introducing coordinates, it is clear that the system of differential equations (5.9) has a unique solution $(\rho_1, \dots, \rho_4)$ with initial conditions $\rho_1(0) = \cdots = \rho_4(0) = 0$. Then, by straightforward computation, using (5.7) and (5.9) we verify the following lemma.

Lemma 5.1. *The linear independent vector fields*

$$Y_1 = X_1 + \rho_1 X_3 + \rho_3 X_4 + \rho_4 X_5, \quad Y_2 = X_2 + \rho_2 X_5,$$
$$Y_3 = X_3, \quad Y_4 = X_4, \quad Y_5 = X_5$$

satisfy $[Y_i, Y_j] = 0$ *for all* i, j. *Hence, there exist local coordinates* $\{u_1, \dots, u_5\}$ *on* M *such that* $\frac{\partial}{\partial u_i} = Y_i$ *and*

$$\rho_1 = \frac{3\alpha}{10} u_2, \quad \rho_2 = \frac{\alpha}{6} u_3, \quad \rho_3 = \frac{2\beta}{3} u_2 + \frac{8\alpha}{15} u_3, \quad \rho_4 = -\frac{\alpha^2}{40} u_2^2 + \gamma u_2 + \beta u_3 + \frac{3\alpha}{5} u_4.$$

Expressing the Levi-Civita connection in terms of the frame Y_i we see that

$$\hat{\nabla}_{Y_3} Y_3 = \hat{\nabla}_{Y_{i+1}} Y_4 = \hat{\nabla}_{Y_4} Y_{i+1} = 0, \quad \hat{\nabla}_{Y_5} Y_j = \hat{\nabla}_{Y_j} Y_5 = 0, \quad \forall i, j,$$
$$\hat{\nabla}_{Y_1} Y_1 = -\frac{3\alpha}{5} Y_2 - \beta Y_3 - \left(\gamma + \frac{\alpha^2}{25} u_2\right) Y_4 + \left(\frac{7\alpha\beta}{10} u_2 + \frac{21\alpha^2}{50} u_3\right) Y_5,$$

$$\hat{\nabla}_{Y_2} Y_1 = \hat{\nabla}_{Y_1} Y_2 = -\frac{\alpha}{5} Y_3 + \left(\gamma + \frac{\alpha^2}{10} u_2\right) Y_5, \quad \hat{\nabla}_{Y_2} Y_3 = \hat{\nabla}_{Y_3} Y_2 = \frac{\alpha}{2} Y_5,$$

$$\hat{\nabla}_{Y_1} Y_3 = \hat{\nabla}_{Y_3} Y_1 = \frac{\alpha}{5} Y_4 + \beta Y_5, \quad \hat{\nabla}_{Y_1} Y_4 = \hat{\nabla}_{Y_4} Y_1 = \frac{3\alpha}{5} Y_5, \quad \hat{\nabla}_{Y_2} Y_2 = \frac{2\beta}{3} Y_5.$$

The only nonzero components of the affine metric are

$$h(Y_1, Y_1) = \frac{\alpha^2}{25} u_2^2 + 2\gamma u_2 + 2\beta u_3 + \frac{6\alpha}{5} u_4, \quad h(Y_1, Y_2) = \frac{2\beta}{3} u_2 + \frac{7\alpha}{10} u_3,$$
$$h(Y_1, Y_3) = \frac{3\alpha}{10} u_2, \quad h(Y_1, Y_5) = h(Y_2, Y_4) = h(Y_3, Y_3) = 1,$$

and the only nonzero components of the difference tensor are

$$K_{Y_1} Y_1 = Y_2 + \frac{3\alpha}{5} u_2 Y_4 + \left(\frac{4\beta}{3} u_2 + \frac{9\alpha}{10} u_3\right) Y_5, \quad K_{Y_1} Y_2 = Y_3 + \frac{3\alpha}{10} u_2 Y_5,$$
$$K_{Y_1} Y_3 = K_{Y_2} Y_2 = Y_4, \quad K_{Y_1} Y_4 = K_{Y_2} Y_3 = Y_5.$$

As before, the immersion F is determined by the following system of differential equations:

$$\begin{cases} F_{u_1u_1} = \dfrac{5-3\alpha}{5} F_{u_2} - \beta F_{u_3} + \left[\dfrac{\alpha(15-\alpha)}{25} u_2 - \gamma\right] F_{u_4} \\ \qquad + \left[\dfrac{(40+21\alpha)\beta}{30} u_2 + \dfrac{(45+21\alpha)\alpha}{50} u_3\right] F_{u_5} \\ \qquad + \left(\dfrac{\alpha^2}{25} u_2^2 + 2\gamma u_2 + 2\beta u_3 + \dfrac{6\alpha}{5} u_4\right)\xi, \\ F_{u_1u_2} = \dfrac{5-\alpha}{5} F_{u_3} + \left[\dfrac{\alpha(\alpha+3)}{10} u_2 + \gamma\right] F_{u_5} + \left(\dfrac{2\beta}{3} u_2 + \dfrac{7\alpha}{10} u_3\right)\xi, \\ F_{u_1u_3} = \dfrac{5+\alpha}{5} F_{u_4} + \beta F_{u_5} + \dfrac{3\alpha}{10} u_2 \xi, \quad F_{u_2u_2} = F_{u_4} + \dfrac{2\beta}{3} F_{u_5}, \\ F_{u_1u_4} = \dfrac{5+3\alpha}{5} F_{u_5}, \quad F_{u_2u_3} = \dfrac{2+\alpha}{2} F_{u_5}, \quad F_{u_1u_5} = F_{u_2u_4} = F_{u_3u_3} = \xi, \\ F_{u_3u_4} = F_{u_4u_4} = F_{u_{i+1}u_5} = 0, \quad \forall i, \end{cases}$$

where $F_{u_i} := F_* \dfrac{\partial}{\partial u_i}$. Solving above system of differential equations, up to an affine transformation, we obtain

$$F = A + u_1 A_1 + \left(u_2 + \frac{5-3\alpha}{10} u_1^2\right) A_2$$
$$+ \left(u_3 + \frac{5-\alpha}{5} u_1 u_2 + \frac{(\alpha-5)(3\alpha-5)}{3! \times 25} u_1^3 - \frac{\beta}{2} u_1^2\right) A_3$$

$$
\begin{aligned}
&+\left[u_4+\frac{5+\alpha}{5}u_1u_3+\frac{1}{2}u_2^2+\frac{25-\alpha^2}{50}u_1^2u_2\right.\\
&\quad\left.+\frac{(\alpha^2-25)(3\alpha-5)}{5!\times 25}u_1^4-\frac{\gamma}{2}u_1^2-\frac{(5+\alpha)\beta}{30}u_1^3\right]A_4\\
&+\left[u_5+\frac{5+3\alpha}{5}u_1u_4+\frac{2+\alpha}{2}u_2u_3+\frac{(3\alpha+5)(\alpha+5)}{50}u_1^2u_3+\frac{5+3\alpha}{10}u_1u_2^2\right.\\
&\quad-\frac{(\alpha^2-25)(3\alpha+5)}{3!\times 125}u_1^3u_2+\frac{(\alpha^2-25)(9\alpha^2-25)}{5!\times 25^2}u_1^5+\beta u_1u_3+\gamma u_1u_2\\
&\quad\left.+\frac{\beta}{3}u_2^2+\frac{(5-\alpha)\beta}{10}u_1^2u_2-\left(\frac{\alpha\gamma}{5}+\frac{\beta^2}{3!}\right)u_1^3-\frac{\alpha\beta}{15}u_1^4\right]A_5\\
&+\left[u_1u_5+u_2u_4+\frac{1}{2}u_3^2+\frac{1}{3!}u_2^3+\frac{2+\alpha}{2}u_1u_2u_3+\frac{5+3\alpha}{10}u_1^2u_4+\frac{3\alpha+5}{20}u_1^2u_2^2\right.\\
&\quad+\frac{(3\alpha+5)(\alpha+5)}{3!\times 25}u_1^3u_3-\frac{(\alpha^2-25)(3\alpha+5)}{5!\times 25}u_1^4u_2+\frac{(\alpha^2-25)(9\alpha^2-25)}{6!\times 25^2}u_1^6\\
&\quad\left.+\frac{\beta}{2}u_1^2u_3+\frac{\beta}{3}u_1u_2^2+\frac{\gamma}{2}u_1^2u_2+\frac{(5-\alpha)\beta}{30}u_1^3u_2-\left(\frac{\alpha\gamma}{20}+\frac{\beta^2}{4!}\right)u_1^4-\frac{\alpha\beta}{75}u_1^5\right]\xi,
\end{aligned}
$$

where $A_i=F_{u_i}(0)$, $A=F(0)$ are constant vectors of $\mathbf{R}^6$. Similar to the proof of Theorem 4.1, by an equiaffine transformation we see that M lies on the graph immersion of polynomial

$$
\begin{aligned}
x_6&=x_1x_5+x_2x_4+\frac{1}{2}x_3^2-x_1^2x_4-\frac{1}{3}x_2^3-2x_1x_2x_3+\frac{3-\alpha}{3}x_1^3x_3+\frac{3-\alpha}{2}x_1^2x_2^2\\
&\quad-\frac{(3-\alpha)(2-\alpha)}{6}x_1^4x_2+\frac{4(3-\alpha)(2-\alpha)(5-3\alpha)}{6!}x_1^6\\
&\quad-\frac{\gamma}{4}x_1^4-\frac{2\beta}{3}x_1^3x_2+\frac{(25-11\alpha)\beta}{75}x_1^5.
\end{aligned}
$$

This is exactly the hypersurface (5.1). Obviously, if $\beta=\gamma=0$ these are exactly the 5-dimensional generalized Cayley hypersurfaces.

For Case II, we set $\Gamma_{1,4}^5=-\Gamma_{1,3}^4=\alpha$. We note from (5.7) that the only nonzero components of $\hat{\nabla}K$ are

$$
\begin{aligned}
&\hat{\nabla}K(X_1,X_1,X_1)=3\alpha X_3+2\beta X_4+3\gamma X_5, && \hat{\nabla}K(X_3,X_1,X_1)=3\alpha X_5,\\
&\hat{\nabla}K(X_2,X_1,X_1)=-\alpha X_4+2\beta X_5, && \hat{\nabla}K(X_1,X_2,X_2)=-\alpha X_5.
\end{aligned}
$$

The only nonzero Lie brackets are

$$
\begin{aligned}
&[X_1,X_4]=\alpha X_5, && [X_1,X_3]=\beta X_5,\\
&[X_2,X_3]=-\frac{3}{2}\alpha X_5, && [X_1,X_2]=\frac{\alpha}{2}X_3+\frac{2}{3}\beta X_4+\gamma X_5.
\end{aligned}
$$

Now, we look at the following system of differential equations of $(\rho_1, \dots, \rho_4)$.

$$
(5.10)\quad \begin{cases} X_1(\rho_1) = 0, \quad X_2(\rho_1) = \dfrac{\alpha}{2}, \quad X_3(\rho_1) = X_4(\rho_1) = X_5(\rho_i) = 0, \quad \forall i, \\ X_1(\rho_2) = \dfrac{3\alpha}{2}\rho_1, \quad X_2(\rho_2) = 0, \quad X_3(\rho_2) = -\dfrac{3\alpha}{2}, \quad X_4(\rho_2) = 0, \\ X_1(\rho_3) = 0, \quad X_2(\rho_3) = \dfrac{2\beta}{3}, \quad X_3(\rho_3) = X_4(\rho_3) = 0, \\ X_1(\rho_4) = -(\beta\rho_1 + \alpha\rho_3), \quad X_2(\rho_4) = \gamma + \dfrac{3\alpha}{2}\rho_1, \\ X_3(\rho_4) = \beta, \quad X_4(\rho_4) = \alpha. \end{cases}
$$

Direct computations show that for $k = 1, 2, 3, 4$

$$
(X_i X_j - X_j X_i - [X_i, X_j])\rho_k = 0, \quad \forall i, j.
$$

As before, by straightforward computation, using (5.7) and (5.10) we verify the following lemma.

LEMMA 5.2. *The linear independent vector fields*

$$
Y_1 = X_1 + \rho_1 X_3 + \rho_3 X_4 + \rho_4 X_5, \quad Y_2 = X_2 + \rho_2 X_5,
$$
$$
Y_3 = X_3, \quad Y_4 = X_4, \quad Y_5 = X_5
$$

satisfy $[Y_i, Y_j] = 0$ *for all* i, j. *Hence, there exist local coordinates* $\{u_1, \dots, u_5\}$ *on* M *such that* $\dfrac{\partial}{\partial u_i} = Y_i$ *and*

$$
\rho_1 = \frac{\alpha}{2}u_2, \quad \rho_2 = -\frac{3\alpha}{2}u_3, \quad \rho_3 = \frac{2\beta}{3}u_2, \quad \rho_4 = \frac{3\alpha^2}{8}u_2^2 + \gamma u_2 + \beta u_3 + \alpha u_4.
$$

Expressing the Levi-Civita connection in terms of Y_i we see that

$$
\hat{\nabla}_{Y_3} Y_3 = \hat{\nabla}_{Y_{i+1}} Y_4 = \hat{\nabla}_{Y_4} Y_{i+1} = 0, \quad \hat{\nabla}_{Y_5} Y_j = \hat{\nabla}_{Y_j} Y_5 = 0, \quad \forall i, j,
$$
$$
\hat{\nabla}_{Y_1} Y_1 = -\alpha Y_2 - \beta Y_3 - (\gamma + \alpha^2 u_2) Y_4 + \left(\frac{7\alpha\beta}{6}u_2 - \frac{3\alpha^2}{2}u_3\right) Y_5,
$$
$$
\hat{\nabla}_{Y_2} Y_1 = \hat{\nabla}_{Y_1} Y_2 = \alpha Y_3 + \left(\gamma + \frac{\alpha^2}{2}u_2\right) Y_5, \quad \hat{\nabla}_{Y_2} Y_3 = \hat{\nabla}_{Y_3} Y_2 = -\frac{\alpha}{2} Y_5,
$$
$$
\hat{\nabla}_{Y_1} Y_3 = \hat{\nabla}_{Y_3} Y_1 = -\alpha Y_4 + \beta Y_5, \quad \hat{\nabla}_{Y_1} Y_4 = \hat{\nabla}_{Y_4} Y_1 = \alpha Y_5, \quad \hat{\nabla}_{Y_2} Y_2 = \frac{2\beta}{3} Y_5.
$$

The only nonzero components of the affine metric are

$$h(Y_1, Y_1) = \alpha^2 u_2^2 + 2\gamma u_2 + 2\beta u_3 + 2\alpha u_4, \quad h(Y_1, Y_2) = \frac{2\beta}{3} u_2 - \frac{3\alpha}{2} u_3,$$
$$h(Y_1, Y_3) = \frac{\alpha}{2} u_2, \quad h(Y_1, Y_5) = h(Y_2, Y_4) = h(Y_3, Y_3) = 1,$$

and the only nonzero components of the difference tensor are

$$K_{Y_1} Y_1 = Y_2 + \alpha u_2 Y_4 + \left(\frac{4\beta}{3} u_2 + \frac{3\alpha}{2} u_3\right) Y_5, \quad K_{Y_1} Y_2 = Y_3 + \frac{\alpha}{2} u_2 Y_5,$$
$$K_{Y_1} Y_3 = K_{Y_2} Y_2 = Y_4, \quad K_{Y_1} Y_4 = K_{Y_2} Y_3 = Y_5.$$

Then, the immersion F is determined by the system of differential equations:

$$\begin{cases} F_{u_1 u_1} = (1-\alpha) F_{u_2} - \beta F_{u_3} + (\alpha u_2 - \alpha^2 u_2 - \gamma) F_{u_4} \\ \qquad + \left[\frac{(7\alpha+8)\beta}{6} u_2 + \frac{3\alpha(1-\alpha)}{2} u_3\right] F_{u_5} + (\alpha^2 u_2^2 + 2\gamma u_2 + 2\beta u_3 + 2\alpha u_4)\xi, \\ F_{u_1 u_2} = (1+\alpha) F_{u_3} + \left[\frac{\alpha(\alpha+1)}{2} u_2 + \gamma\right] F_{u_5} + \left(\frac{2\beta}{3} u_2 - \frac{3\alpha}{2} u_3\right)\xi, \\ F_{u_1 u_3} = (1-\alpha) F_{u_4} + \beta F_{u_5} + \frac{\alpha}{2} u_2 \xi, \quad F_{u_2 u_2} = F_{u_4} + \frac{2\beta}{3} F_{u_5}, \\ F_{u_1 u_4} = (1+\alpha) F_{u_5}, \quad F_{u_2 u_3} = \frac{2-\alpha}{2} F_{u_5}, \quad F_{u_1 u_5} = F_{u_2 u_4} = F_{u_3 u_3} = \xi, \\ F_{u_3 u_4} = F_{u_4 u_4} = F_{u_{i+1} u_5} = 0, \quad \forall i, \end{cases}$$

where $F_{u_i} := F_* \dfrac{\partial}{\partial u_i}$. Solving above system of differential equations, up to an affine transformation, we obtain

$$\begin{aligned} F = {} & A + u_1 A_1 + \left(u_2 + \frac{1-\alpha}{2} u_1^2\right) A_2 + \left[u_3 + (1+\alpha) u_1 u_2 + \frac{1-\alpha^2}{3!} u_1^3 - \frac{\beta}{2} u_1^2\right] A_3 \\ & + \Bigg[u_4 + (1-\alpha) u_1 u_3 + \frac{1}{2} u_2^2 + \frac{1-\alpha^2}{2} u_1^2 u_2 \\ & \quad + \frac{(1-\alpha^2)(1-\alpha)}{4!} u_1^4 - \frac{\gamma}{2} u_1^2 + \frac{(\alpha-1)\beta}{3!} u_1^3\Bigg] A_4 \\ & + \Bigg[u_5 + (1+\alpha) u_1 u_4 + \frac{2-\alpha}{2} u_2 u_3 + \frac{1-\alpha^2}{2} u_1^2 u_3 + \frac{1+\alpha}{2} u_1 u_2^2 \\ & \quad + \frac{(1+\alpha)(1-\alpha^2)}{3!} u_1^3 u_2 + \frac{(1-\alpha^2)^2}{5!} u_1^5 + \beta u_1 u_3 + \gamma u_1 u_2 + \frac{\beta}{3} u_2^2 \\ & \quad + \frac{(1+\alpha)\beta}{2} u_1^2 u_2 - \frac{2\alpha\gamma + \beta^2}{3!} u_1^3\Bigg] A_5 \end{aligned}$$

$$+\left[u_1u_5+u_2u_4+\frac{1}{2}u_3^2+\frac{1}{3!}u_2^3+\frac{2-\alpha}{2}u_1u_2u_3+\frac{1+\alpha}{2}u_1^2u_4+\frac{1+\alpha}{4}u_1^2u_2^2\right.$$
$$+\frac{1-\alpha^2}{3!}u_1^3u_3+\frac{(1-\alpha^2)(1+\alpha)}{4!}u_1^4u_2+\frac{(1-\alpha^2)^2}{6!}u_1^6+\frac{\beta}{2}u_1^2u_3+\frac{\beta}{3}u_1u_2^2$$
$$\left.+\frac{\gamma}{2}u_1^2u_2+\frac{(1+\alpha)\beta}{3!}u_1^3u_2-\frac{2\alpha\gamma+\beta^2}{4!}u_1^4\right]\xi,$$

where $A_i = F_{u_i}(0)$, $A = F(0)$ are constant vectors of $\mathbf{R}^6$. As before, by an equi-affine transformation we see that M lies on the graph immersion of polynomial

$$x_6 = x_1x_5+x_2x_4+\frac{1}{2}x_3^2-x_1^2x_4-\frac{1}{3}x_2^3-2x_1x_2x_3+(1-\alpha)x_1^3x_3+\frac{3+\alpha}{2}x_1^2x_2^2$$
$$-\frac{(1-\alpha)(2+\alpha)}{2}x_1^4x_2+\frac{(1-\alpha)^2(2+\alpha)}{12}x_1^6-\frac{\gamma}{4}x_1^4-\frac{2\beta}{3}x_1^3x_2+\frac{(1-\alpha)\beta}{3}x_1^5.$$

This is exactly the hypersurface (5.2) of Main Theorem. When $\alpha = 0$, we note that above graph immersion coincides with that of (5.1). □

The proof of Main Theorem follows immediately from that $H = 0$ for $n \le 5$ in Section 3, Theorem 4.1 and 5.1.

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