LOCALLY HOMOGENEOUS COMPLEX MANIFOLDS

BY

PHILLIP GRIFFITHS and WILFRIED SCHMID

Princeton University, Princeton, N.J. and *Columbia University, New York, N.Y.* (1)

In this paper we discuss some geometric and analytic properties of a class of locally homogeneous complex manifolds. Our original motivation came from algebraic geometry where certain non-compact, homogeneous complex manifolds arose naturally from the period matrices of general algebraic varieties in a similar fashion to the appearance of the Siegel upper-half-space from the periods of algebraic curves. However, these manifolds are generally *not* Hermitian symmetric domains and, because of this, several interesting new phenomena turn up.

The following is a description of the manifolds we have in mind. Let $G_{\mathbf{C}}$ be a connected, complex semi-simple Lie group and $B \subset G_{\mathbf{C}}$ a parabolic subgroup. Then, as is well known, the coset space $X = G_{\mathbf{C}}/B$ is a compact, homogeneous algebraic manifold. If $G \subset G_{\mathbf{C}}$ is a connected real form of $G_{\mathbf{C}}$ such that $G \cap B = V$ is compact, then the G-orbit of the origin in X is a connected open domain $D \subset X$, and $D = G/V$ is therefore a *homogeneous complex manifold*. Let $\Gamma \subset G$ be a discrete subgroup such that the normalizer $N(\Gamma)$ intersects V only in the identity. Since Γ acts properly discontinuously without fixed points on D, the quotient space $Y = \Gamma \backslash D$ inherits the structure of a complex manifold. We shall refer to a manifold of this type as a *locally homogeneous complex manifold*.

One case is when $G = M$ is a maximal compact subgroup of $G_{\mathbf{C}}$. Then necessarily $\Gamma = \{e\}$, and $D = X$ is the whole compact algebraic manifold. These varieties have been the subject of considerable study, and their basic properties are well known. The opposite extreme occurs when G has no compact factors. These non-compact homogeneous domains D then include the Hermitian symmetric spaces, about which quite a bit is known, and also include important and interesting non-classical domains which have been discussed relatively little. It is these manifolds which are our main interest; however, since the

(1) During the preparation of this paper, the first named author was partially supported by NSF grant GP-7952X at the Institute for Advanced Study, and the second named author by NSF grant GP–8008 at the University of California, Berkeley.

methods we use apply more or less uniformly, we get out the classical results on X along with new information on the locally homogeneous manifolds $Y = \Gamma\backslash D$.

Here is a more detailed summary of the results in this paper. Recall that an irreducible unitary representation π: $V \to \mathrm{GL}(E)$ gives rise to a *homogeneous vector bundle* $G \times_V E = \mathbf{E}$ lying over $D = G/V$. This bundle has an (essentially unique) G-invariant complex structure and metric, and so induces a Hermitian, holomorphic bundle $\mathbf{F} \to Y$ ($Y = \Gamma\backslash D$, $\mathbf{F} = \Gamma\backslash\mathbf{E}$). Our first job is to compute the curvature Θ of the metric connection in the *locally homogeneous bundle* $\mathbf{F} \to Y$, and this is done in § 4. The main result (Theorems $(4.13)_X$ and $(4.13)_D$) is that the curvature Θ has a canonical expression as a difference of disjoint positive terms; by this we mean that there is a matrix equation $\Theta = A \wedge {}^t\bar{A} - B \wedge {}^t\bar{B}$ where A, B are matrices of $(1, 0)$ forms involving mutually disjoint subspaces of the cotangent space. As an application of this formula, we recall that the curvature form $\Theta(\xi) = i(\xi, \Theta\xi)$ ($\xi \in \mathbf{F}$) is a real (1,1)-form which controls the cohomology $H^*(Y, \mathcal{O}(\mathbf{F}))$ in case Y is compact (cf. (4.14) and (4.16)). We will see that $\Theta(\xi)$ is non-singular if the highest weight λ of π is non-singular, and that the signature of $\Theta(\xi)$ is determined by the Weyl chamber in which λ lies. This leads to a crude vanishing theorem (cf. (4.21)), but one which suggests the following behavior: (a) For all Y, the cohomology $H^k(Y, \mathcal{O}(\mathbf{F})) = 0$ for $k \neq k(\pi)$ where $k(\pi)$ is determined by π; and (b) if D is non-compact and does *not* fibre holomorphically over an Hermitian symmetric space, then $H^0(Y, \mathcal{O}(\mathbf{F})) = 0$ for all representations π. Thus, for some domains arising quite naturally in algebraic geometry, there is no theory of automorphic forms.

Since the curvature gives only crude vanishing theorems, in § 5 we compute the Laplace-Beltrami operator $\square$ acting on the space $C^k(Y, \mathbf{F})$ of C^∞, $\mathbf{F}$-valued $(0, k)$-forms on Y. This calculation is somewhat involved, but does yield fairly precise vanishing theorems together with some information on the non-zero group $H^{k(\pi)}(Y, \mathcal{O}(\mathbf{F}))$. For example, in § 6 we use the calculation of $\square$ to give a proof of Bott's result that $H^k(X, \mathcal{O}(\mathbf{E})) = 0$ for $k \neq k(\pi)$, and that $H^{k(\pi)}(X, \mathcal{O}(\mathbf{E}))$ is an irreducible $G_{\mathbf{C}}$-module W_π whose highest weight has a simple determination. In particular we obtain the usual Borel-Weil theorem.

In case $Y = \Gamma\backslash D$ where D is non-compact, we consider the *square-integrable cohomology* $\mathcal{H}^k_\Gamma(D, \mathbf{E})$. By definition, $\mathcal{H}^k_\Gamma(D, \mathbf{E})$ is the space of Γ-invariant, harmonic forms φ in $C^k(D, \mathbf{E})$ such that $\int_{\mathcal{F}} \|\varphi\|^2 < \infty$, where $\mathcal{F} \subset D$ is a fundamental domain for Γ. In case $\Gamma = \{e\}$, $\mathcal{H}^k_\Gamma(D, \mathbf{E}) = \mathcal{H}^k(D, \mathbf{E})$ is a unitary G-module (which may be zero). In the opposite extreme when $Y = \Gamma\backslash D$ is compact, $\mathcal{H}^k_\Gamma(D, \mathbf{E}) \cong H^k(Y, \mathcal{O}(\mathbf{F}))$. In § 7 we apply the computation of $\square$ to prove that $\mathcal{H}^k_\Gamma(D, \mathbf{E}) = 0$ for $k \neq k(\pi)$, provided that the highest weight λ of π is at least a fixed distance from the walls of the Weyl chamber in which λ lies. This essentially gives the non-existence part of a conjecture of Langlands [24], as well as the non-existence of automorphic forms of any type unless D fibres holomorphically over a Hermitian symmetric space.

The existence problem for $\mathcal{H}_\Gamma^{k(\pi)}(D, \mathbf{E})$ is extremely interesting. In case $Y = \Gamma\backslash D$ is compact, we use the Atiyah-Singer theorem to write: $\dim \mathcal{H}_\Gamma^{k(\pi)}(D, \mathbf{E}) = \dim H^{k(\pi)}(Y, \mathcal{O}(\mathbf{F})) = (-1)^{k(\pi)} \sum_{k=0}^{m} (-1)^k \dim H^k(Y, \mathcal{O}(\mathbf{F})) = (-1)^{k(\pi)} T(Y, \mathbf{F})$ where $T(Y, \mathbf{F})$, the Todd genus of $\mathbf{F} \to Y$, is a topological expression involving the Chern classes of $\mathbf{F} \to Y$ and the tangent bundle $\mathbf{T}(Y) \to Y$. These Chern classes are given by differential forms involving the curvature, which has been computed in § 4, so that we finally get (cf. Theorem 7.2): $\dim \mathcal{H}_\Gamma^{k(\pi)}(D, \mathbf{E}) = c \cdot \dim W_\pi \cdot \mu(\mathcal{F})$; here $c > 0$ is a constant independent of π and Γ, W_π is the irreducible $G_{\mathbf{C}}$-module appearing in Bott's theorem above, and $\mu(\mathcal{F})$ is the volume of a fundamental domain $\mathcal{F}$ of Γ. In the opposite extreme $\Gamma = \{e\}$, according to Langlands' conjecture, the unitary G-module $\mathcal{H}^{k(\pi)}(D, \mathbf{E})$ should be irreducible and should occur discretely in $L^2(G)$; Langlands has also predicted the character. We give a precise formulation of the conjecture in § 7.

Since the writing of this manuscript, Okamoto and M. S. Narasimhan have verified Langlands' conjecture for Hermitian symmetric domains and vector bundles indexed by "sufficiently nonsingular" highest weights (cf. (7.1) below). Subsequently, the second named author of this paper found a proof of the conjecture in general, though again only for "sufficiently nonsingular" weights. A similar proof gives a related conjecture of Langlands, which asserts that for compact $Y = \Gamma\backslash D$, the dimension of $H^{k(\pi)}(Y, \mathcal{O}(\mathbf{F}))$ equals the multiplicity in $L^2(\Gamma\backslash G)$ of the G-module $\mathcal{H}^{k(\pi)}(D, \mathbf{E})$. Both arguments depend on the vanishing theorems in § 7.

The possible connections between the "automorphic cohomology groups" $\mathcal{H}_\Gamma^{k}(D, \mathbf{E})$ and the problem of periods of algebraic manifolds are also taken up in § 7.

Sections 8 and 9 are devoted to some geometric properties of the noncompact domains $D \subset X$. First, we generalize the well known holomorphic convexity of the bounded, symmetric domains by proving that D has the maximum degree of pseudoconvexity which is allowed by the presence of certain compact analytic subvarieties in D. This result has been used by one of us to show that, with the proper choice of complex structure, the cohomology group $H^{k(\pi)}(D, \mathcal{O}(\mathbf{E}))$ has naturally the structure of a Frechét space on which G acts continuously, and which contains a G-submodule infinitesimally equivalent to $\mathcal{H}^{k(\pi)}(D, \mathbf{E})$. These facts are related to a conjecture of Blattner about Harish-Chandra's discrete series representations and will be pursued in a future paper (cf. also [28]).

In § 9 we prove a generalization of the hyperbolic character of bounded domains by showing that the homogeneous manifolds D are *negatively curved* with respect to the family of holomorphic mappings arising in algebraic geometry. This result has recently been quite useful and leads to interesting generalizations of the Picard theorem.

To conclude the introduction we want to give a few references to background and related material. The compact homogeneous manifolds $X = G_{\mathbf{C}}/B$ were discussed by H. C. Wang [31] and by Borel [4]; a rather complete discussion of homogeneous complex structures is given by Borel and Hirzebruch [6]. The invariant differential forms giving the Chern classes of homogeneous line bundles were given by Borel [4] and later by Bott [7] and Borel–Hirzebruch [6]. In Borel's paper [4] there are the first indications of the curvature properties which the non-classical domains turn out to have.

The expression for the Chern classes of homogeneous line bundles suggested the phenomenon that $H^k(X, \mathcal{O}(\mathbf{E})) \neq 0$ for at most one integer $k = k(\pi)$. For $k(\pi) = 0$ this vanishing theorem was deduced from the Kodaira vanishing theorem by Borel [4], Borel–Weil [5], and Borel–Hirzebruch [6]. The general vanishing theorem was proved by Bott [7], who made only partial use of curvature arguments.

The existence of $H^{k(\pi)}(X, \mathcal{O}(\mathbf{E}))$ was proved for line bundles when $k(\pi) = 0$ by Borel and Weil [5], who used their results to give equivariant projective embeddings of X. By combining the vanishing theorem for the $H^k(X, \mathcal{O}(\mathbf{E}))$ and the Hirzebruch–Riemann–Roch theorem [16], Borel and Hirzebruch were led in [6] to conjecture the main theorem, proved by Bott [7], giving in general the $G_{\mathbf{C}}$-module structure of $H^{k(\pi)}(X, \mathcal{O}(\mathbf{E}))$. In [21] Kostant gave a uniform treatment of the subject using Lie algebra cohomology.

In the non-compact case, most of the attention seems to have been devoted to the groups $\mathcal{H}^0_\Gamma(D, \mathbf{E})$ where D is a Hermitian symmetric space. In case $Y = \Gamma \backslash D$ is compact, $H^0(Y, \mathcal{O}(\mathbf{F}))$ is a vector space of automorphic forms, and dim $H^0(Y, \mathcal{O}(\mathbf{F}))$ was given, for suitable bundles $\mathbf{F} \to Y$, by Hirzebruch [17], Ise [19], and Langlands [23] (who did not assume that Γ had no fixed points). On the other hand, a rather striking vanishing theorem was given by Calabi–Vesentini [8], and their work gave rise to a series of papers on the groups $H^k(Y, \mathcal{O}(\mathbf{F}))$ when D is a Cartan domain; cf. [25] and [26].

The possibility of realizing Harish-Chandra's discrete series representations on the L^2-cohomology groups $\mathcal{H}^i(D, \mathbf{E})$ was conjectured by Langlands [24]. In the case of those groups which act on Hermitian symmetric spaces, Okamoto and Ozeki [27] have reduced the conjecture to a conjecture of Blattner about the structure of the discrete series representations, which is known to be correct in a few cases. For the groups $G = \mathrm{SO}(2h, 1)$, one of us [28] has proven the Langlands conjecture by a direct construction. The most recent progress on the conjecture has already been mentioned above.

Finally, we remark that many of the results of this paper have been previously announced in [10] and [29].

Table of Contents

1. Kähler C-spaces

We begin our discussion by recalling some facts concerning compact, simply connected, homogeneous complex manifolds. H. C. Wang, who has named these manifolds C-spaces, has classified them in [31]. In this paper, for simplicity, we shall consider only C-spaces which admit a Kähler metric.

Let $G_{\mathbf{C}}$ be a connected complex semisimple Lie group, B a parabolic subgroup. The complex analytic quotient space $X = G_{\mathbf{C}}/B$ is then a Kähler C-space, and every Kähler C-space arises in this fashion. The Lie algebras of $G_{\mathbf{C}}$ and B will be referred to as $\mathfrak{g}$ and $\mathfrak{b}$; both are complex Lie algebras. We choose a maximal compact subgroup M of $G_{\mathbf{C}}$. Its Lie algebra, $\mathfrak{m}_0$, is a real form of $\mathfrak{g}$, and we denote complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{m}_0$ by τ. The algebra $\mathfrak{b}$ has a unique maximal nilpotent ideal $\mathfrak{n}_-$. Since the subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{n}_-$ can be realized as a group of upper triangular matrices, with ones along the diagonal, it has no nontrivial compact subgroups. Thus $\mathfrak{m}_0 \cap \mathfrak{n}_-$, and hence also $\mathfrak{n}_- \cap \tau(\mathfrak{n}_-)$, must be zero. By appealing to Bruhat's lemma, for example, we can conclude that the parabolic subalgebras $\mathfrak{b}$ and $\tau(\mathfrak{b})$ are opposite to each other, i.e. $\mathfrak{g}$ is spanned by $\mathfrak{b}$ and $\tau(\mathfrak{n}_-)$. Moreover, $\mathfrak{v} = \mathfrak{b} \cap \tau(\mathfrak{b})$ is a reductive subalgebra such that

$$\mathfrak{b} = \mathfrak{v} \oplus \mathfrak{n}_- \quad \text{(semidirect product)} \tag{1.1}$$

and

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{n}_- \oplus \tau(\mathfrak{n}_-). \tag{1.2}$$

Since the real span of $\mathfrak{m}_0$ and $\mathfrak{b}$ is all of $\mathfrak{g}$, the M-orbit of $eB \in X = G_{\mathbf{C}}/B$ must be open. On the other hand, this orbit is closed because M is compact. Hence M acts transitively on X, with isotropy group $V = M \cap B$, and we can identify the quotient space M/V with X. The Lie algebra of V is $\mathfrak{v}_0 = \mathfrak{m}_0 \cap \mathfrak{b} = \mathfrak{m}_0 \cap \mathfrak{b} \cap \tau(\mathfrak{b}) = \mathfrak{m}_0 \cap \mathfrak{v}$; V is connected because X is simply connected. It should be remarked that $\mathfrak{v}$, as the intersection of the two parabolic subalgebras $\mathfrak{b}$ and $\tau(\mathfrak{b})$, must have the same rank as $\mathfrak{g}$; this follows again from Bruhat's lemma. Moreover, $\mathfrak{m}_0$ and $\mathfrak{v}_0$ are also of equal rank, since they are real forms of $\mathfrak{g}$ and $\mathfrak{v}$, respectively.

It will be useful to have a description of the complex structure of $M/V = X$ without reference to $G_{\mathbf{C}}$. Let f be a holomorphic function on some open set $U \subset X$, and $\tilde{f}$ the pullback of f to $G_{\mathbf{C}}$. Since $\tilde{f}$ is B-invariant on the right, every $x \in \mathfrak{b}$, considered as a left-invariant *real* tangent vector field on $G_{\mathbf{C}}$, annihilates $\tilde{f}$. Because $\tilde{f}$ satisfies the Cauchy-Riemann equations on $G_{\mathbf{C}}$, the restriction of $\tilde{f}$ to M, or, equivalently, the pullback of f to M, will be annihilated by every $x \in \mathfrak{b}$ when x is regarded as a left-invariant *complex* tangent vector field on M. Let $p: M \to X = M/V$ be the quotient map. According to what has just been said, for every $m \in M$ the induced mapping p_* from the complexified tangent space of M at m, identified with $\mathfrak{g}$ via left translation, carries $\mathfrak{b}$ into the space of antiholomorphic tangent vectors at $p(m)$. Since the kernel of p_* is precisely $\mathfrak{v}$, a count of dimensions shows $p_*(\mathfrak{n}_-) = p_*(\mathfrak{b})$ to be the full antiholomorphic tangent space. Suppose now that h is a C^∞ function on $p^{-1}(U) \subset M$, with the property that $xh = 0$, for every $x \in \mathfrak{b}$, extended to a left-invariant complex vector field. In particular, such a function h must be constant on each V-coset, and hence drops to a C^∞ function f on U; from our characterization of the space of $(0, 1)$-tangent vectors, we deduce that f satisfies the Cauchy–Riemann equations. Thus we have shown that

$$p^* O(U) = \{h \in C^\infty(p^{-1}(U)) \mid xh = 0 \text{ for all } x \in \mathfrak{b}\}. \tag{1.3}$$

Here $O(U)$ is the ring of holomorphic functions on $U \subset X$, and the elements of $\mathfrak{b}$ act as left-invariant complex tangent vector fields.

We turn our attention to homogeneous holomorphic vector bundles over X, i.e. holomorphic vector bundles to which the action of $G_{\mathbf{C}}$ on X lifts. Let $\mathbf{E} \to X$ be such a vector bundle. The action of the isotropy group B on the fibre of $\mathbf{E}$ over eB, to be denoted by E, determines a holomorphic representation π: $B \to \mathrm{GL}(E)$. This representation associates $\mathbf{E}$ to the holomorphic principal bundle $B \to G_{\mathbf{C}} \to X$. As an example, we mention the holomorphic tangent bundle $\mathbf{T}(X)$, which is clearly a homogeneous vector bundle. Its fibre over the "origin" eB is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$, and under this isomorphism the action of the isotropy group corresponds to the adjoint representation of B on $\mathfrak{g}/\mathfrak{b}$. Conversely, every vector bundle $\mathbf{E}$ associated to the principal bundle $B \to G_{\mathbf{C}} \to X$ by a holomorphic representation π of B on a vector space E is a homogeneous holomorphic vector bundle. As a C^∞ vector bundle, $\mathbf{E}$ is then associated to the principal bundle $V \to M \xrightarrow{p} X$ via the restriction of π to V; the C^∞ sections of $\mathbf{E}$ over an open set $U \subset X$ can be identified with the C^∞ functions F: $p^{-1}(U) \to E$ such that $F(mv) = \pi(v^{-1}) F(m)$ for all $m \in p^{-1}(U)$, $v \in V$; and the space of holomorphic sections of $\mathbf{E}$ over U is isomorphic to the space of E-valued C^∞ functions F on $p^{-1}(U)$ which satisfy

$$xF = -\pi(x)F \quad \text{for every } x \in \mathfrak{b}. \tag{1.4}$$

This description is analogous to (1.3) and can be proven by a similar argument.

If π is an irreducible representation of V on a complex vector space E, the induced representation of the Lie algebra $\mathfrak{v}_0$ determines a unique complex representation of $\mathfrak{v}$. We extend it to all of $\mathfrak{b}$ by letting $\mathfrak{n}_-$ act trivially. This infinitesimal representation can be lifted to B because the fundamental group of B is equal to that of V. The resulting holomorphic extension of π to B is the only possible one: since $\mathfrak{b}$ is the semidirect product of $\mathfrak{v}$ with the nilpotent Lie algebra $\mathfrak{n}_-$, $\mathfrak{n}_-$ must act trivially on any irreducible $\mathfrak{b}$-module. We deduce that every irreducible representation of V leads to one unique homogeneous holomorphic vector bundle. One particular class of examples is furnished by the homogeneous holomorphic line bundles, which arise from one-dimensional representations of V.

In order to study differential forms on X, which will be useful as a computational tool, we choose a basis $e_1, \ldots, e_n$ of $\tau(\mathfrak{n}_-)$, and we set $\bar{e}_i = \tau(e_i)$. According to (1.2), $e_1, \ldots, e_n$, $\bar{e}_1, \ldots, \bar{e}_n$ are linearly independent, and the equations

$$\omega^i(e_j) = \delta^i_j, \quad \omega^i(\bar{e}_j) = 0, \quad \omega^i(\mathfrak{v}) = 0$$

$$\bar{\omega}^i(e_j) = 0, \quad \bar{\omega}^i(\bar{e}_j) = \delta^i_j, \quad \bar{\omega}^i(\mathfrak{v}) = 0$$

define elements of the dual space of $\mathfrak{g}$. We shall regard these as left-invariant complex one-forms on M. A given differential form φ on X pulls back to

$$p^*\varphi = \sum f_{i_1 \ldots i_r j_1 \ldots j_s} \omega^{i_1} \wedge \ldots \wedge \omega^{i_r} \wedge \bar{\omega}^{j_1} \wedge \ldots \wedge \bar{\omega}^{j_s} \tag{1.5}$$

on M, with coeffficients $f_{i_1 \ldots i_r j_1 \ldots j_s} \in C^\infty(M)$. Conversely, if we let V act on $C^\infty(M)$ by right translation and on $\tau(\mathfrak{n}_-)^*$ and $\mathfrak{n}_-^*$ by the dual of the adjoint representation, every V-invariant element of $C^\infty(M) \otimes \Lambda\tau(\mathfrak{n}_-)^* \otimes \Lambda\mathfrak{n}_-^*$ is the pullback to M of a differential form on X. Since p_* maps $\mathfrak{n}_-$ onto the antiholomorphic tangent space, a form φ on X is of type (k, l) precisely when every summand on the right side of (1.5) involves k unbarred and l barred terms.

The exterior differentiation operator, d, is the sum of two operators ∂ and $\bar{\partial}$ of degree (1, 0) and (0, 1), respectively. We shall derive a formula for $\bar{\partial}\varphi$ when φ is a form of type $(0, k)$. For this purpose, we consider the structure constants of the algebra $\mathfrak{n}_-$:

$$[\bar{e}_i, \bar{e}_j] = \sum_l c^l_{ij} \bar{e}_l, \quad c^l_{ij} = -c^l_{ji}.$$

According to the Maurer–Cartan equation on M,

$$d\bar{\omega}^l = -\tfrac{1}{2}\sum_{i,j} c_{ij}^l \bar{\omega}^i \wedge \bar{\omega}^j + \text{ terms annihilating every pair } (\bar{e}_r, \bar{e}_s),\ 1 \leqslant r, s \leqslant n.$$

The adjoint representation of $\mathfrak{n}_-$ on itself by duality determines an action on $\mathfrak{n}_-^*$, which we write as juxtaposition; explicitly, $\bar{e}_i(\bar{\omega}^l) = -\sum_j c_{ij}^l \bar{\omega}^j$. In terms of this notation,

$$d\bar{\omega}^l = \tfrac{1}{2}\sum_i \bar{\omega}^i \wedge \bar{e}_i(\bar{\omega}^l) + \dots$$

Hence, if φ is a form of type $(0, k)$ on X, such that

$$p^*\varphi = \sum f_{i_1 \dots i_k} \bar{\omega}^{i_1} \wedge \dots \wedge \bar{\omega}^{i_k}, \tag{1.6a}$$

with coefficients $f_{i_1 \dots i_k} \in C^\infty(M)$, then

$$p^*\bar{\partial}\varphi = \sum \bar{e}_j f_{i_1 \dots i_k} \bar{\omega}^j \wedge \bar{\omega}^{i_1} \wedge \dots \wedge \bar{\omega}^{i_k} + \tfrac{1}{2}\sum f_{i_1 \dots i_k} \bar{\omega}^j \wedge \bar{e}_j(\bar{\omega}^{i_1} \wedge \dots \wedge \bar{\omega}^{i_k}). \tag{1.6b}$$

Let us suppose now that $\mathbf{E}_\pi \to X$ is the homogeneous holomorphic vector bundle corresponding to an irreducible representation π of V on a complex vector space E. By combining the description of the holomorphic sections of $\mathbf{E}_\pi$ with that of the forms of type $(0, k)$, one obtains an identification of the space of $\mathbf{E}_\pi$-valued $(0, k)$-forms on X, $A^k(\mathbf{E}_\pi)$, with the subspace of V-invariant elements of $C^\infty(M) \otimes E \otimes \Lambda^k \mathfrak{n}_-^*$; here V must be made to act on $C^\infty(M)$ by right translation, on E by π, and on $\Lambda^k\mathfrak{n}_-^*$ by the dual of the adjoint representation. Every $\varphi \in A^k(\mathbf{E}_\pi)$ has an expression of the form (1.6a), with coefficients $f_{i_1 \dots i_k} \in C^\infty(M) \otimes E$. Since $\mathbf{E}_\pi$ is a holomorphic vector bundle, the operator $\bar{\partial}$: $A^k(\mathbf{E}_\pi) \to A^{k+1}(\mathbf{E}_\pi)$ can be defined; equation (1.6b) remains valid in this context.

2. Dual manifolds of Kähler C-spaces

Let $X = G_{\mathbf{C}}/B$ be a Kähler C-space as in section one, and G a noncompact real form of $G_{\mathbf{C}}$. We make the special assumption, once and for all, that $G \cap B$ be compact. In this case we can choose a maximal compact subgroup K of G which contains $G \cap B$, and a maximal compact subgroup M of $G_{\mathbf{C}}$ containing K. As before, we set $V = M \cap B$. Since $G \cap B$ is the isotropy group of G acting on X at eB, $\dim G \cap B \geqslant \dim G - \dim_{\mathbf{R}} X = \dim M - (\dim M - \dim V) = \dim V$; on the other hand, V is connected and $G \cap B \subset K \cap B \subset M \cap B = V$. Thus $G \cap B = V$, and the quotient space $D = G/V$ can be identified with the G-orbit of $eB \in X$, which is open. In this manner, D becomes a homogeneous complex manifold; we say that D is dual to the Kähler C-space X. The adjective "dual" should not suggest the kind of one-to-one correspondence which exists in the special case of the Hermitian symmetric spaces.

The complex structure of $D = G/V$ again has an intrinsic characterization. Since $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$, we may regard $\mathfrak{g}$ as the algebra of left-invariant complex tangent vector fields on G. If the letter p is now used to designate the projection $G \to D$, (1.3) remains correct; the proof carries over immediately. The restriction to D of a homogeneous holomorphic vector bundle $\mathbf{E}_\pi \to X$ determined by a holomorphic representation $\pi: B \to \mathrm{GL}(E)$ is a G-homogeneous holomorphic vector bundle; its holomorphic sections over an open set $U \subset D$ can be described by (1.4). Finally, the discussion of differential forms on X in § 1 applies to D as well, if the roles of M and τ are assumed by G and complex conjugation of $\mathfrak{g}$ with respect of $\mathfrak{g}_0$.

We shall denote the Lie algebra of K by $\mathfrak{k}_0$, and its complexification by $\mathfrak{k}$. Then $\mathfrak{k}$ is a reductive complex subalgebra of $\mathfrak{g}$, with $\mathfrak{v} \subset \mathfrak{k}$. As pointed out in § 1, $\mathfrak{v}$ has the same rank as $\mathfrak{g}$. Hence $\mathfrak{k} \cap \mathfrak{b}$ contains a Cartan subalgebra of $\mathfrak{g}$, and one can conclude that $\mathfrak{k} \cap \mathfrak{b}$ is a parabolic subalgebra of $\mathfrak{k}$. Although $K_{\mathbf{C}}$, the subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{k}$, need not be semisimple, $S = K_{\mathbf{C}}/K_{\mathbf{C}} \cap B$ is a Kähler C-space, because $K_{\mathbf{C}} \cap B$ contains the center of the reductive group $K_{\mathbf{C}}$. The compact subgroup K of $K_{\mathbf{C}}$ acts transitively on S, with isotropy group $K \cap B = V$, just as M acts transitively on X. At various times, we shall view S as the quotient space K/V, as the $K_{\mathbf{C}}$-orbit of $eB \in X$, or as the K-orbit of eB. In particular, S is a compact complex submanifold of $D \subset X$. The fibres of the fibration

$$D = G/V \xrightarrow{\xi} G/K \tag{2.1}$$

are precisely the G-translates of S, and they are all complex submanifolds of D. The holomorphic tangent vectors of D which are tangent to the fibres form a C^∞ subbundle $\mathbf{T}_v(D)$ of the holomorphic tangent bundle $\mathbf{T}(D)$. As demonstrated in § 1, the holomorphic tangent bundle of X is associated to the principal bundle $B \to G_{\mathbf{C}} \to X$ by the adjoint representation of B on $\mathfrak{g}/\mathfrak{b}$. Hence $\mathbf{T}(D)$, as a C^∞ vector bundle, is associated to $V \to G \to D$ by the adjoint representation of V on $\mathfrak{g}/\mathfrak{b}$. A vector x in the fibre of $\mathbf{T}(D)$ over eB, which is to be identified with $\mathfrak{g}/\mathfrak{b}$, is tangent to S if and only if $x \in \mathfrak{k}/\mathfrak{k} \cap \mathfrak{b}$. It follows that $\mathbf{T}_v(D)$ is associated to $V \to G \to D$ by the adjoint representation of V on $\mathfrak{k}/\mathfrak{k} \cap \mathfrak{b}$.

The maximal compact subgroup K of G determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the unique $\mathrm{ad}\,\mathfrak{k}$-invariant subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{k}$. The adjoint action of V on $\mathfrak{p}/\mathfrak{p} \cap \mathfrak{b}$ associates a C^∞ vector bundle $\mathbf{T}_h(D)$ to the principal bundle $V \to G \to D$, and since $\mathfrak{g}/\mathfrak{b} = \mathfrak{k}/\mathfrak{k} \cap \mathfrak{b} \oplus \mathfrak{p}/\mathfrak{p} \cap \mathfrak{b}$,

$$\mathbf{T}(D) = \mathbf{T}_v(D) \oplus \mathbf{T}_h(D) \tag{2.2}$$

is a G-invariant splitting of the holomorphic tangent bundle into two C^∞ subbundles. Both $\mathbf{T}_v(D)$ and $\mathbf{T}_h(D)$ may be regarded as G-invariant distributions; the former, as is obvious from its definition, is integrable, the latter in general is not. The splitting (2.2) seems to depend on the particular choice of K, which was required to be a maximal compact subgroup of G containing V. However, we shall see in § 3 that only one such group K exists; hence the splitting is intrinsic.

A holomorphic mapping F from a complex manifold Y into D is said to be horizontal if the induced tangential mapping F_* takes values only in $\mathbf{T}_h(D)$. More generally, if Y is an analytic space and F: $Y \to D$ a holomorphic map, we call F horizontal whenever the restriction to the set of manifold points of Y is horizontal according to the previous definition.

The following example, which has arisen in the first-named author's study of the periods of algebraic manifolds, may help to motivate and clarify the discussion above; details can be found in [11]. We fix positive integers r, s, and let Q be the matrix

$$\begin{pmatrix} I_{2r} & 0 \\ 0 & -I_s \end{pmatrix}.$$

Then $G_{\mathbf{C}} = \{g \in \mathrm{SL}(2r+s, \mathbf{C}) \mid {}^t g Q g = Q\}$ is a connected complex semisimple Lie group. The subgroup B consisting of all matrices $g \in G_{\mathbf{C}}$ which are of the block form

$$\begin{array}{c|ccc|} & r & r & s \\ r & A_{11} & A_{12} & A_{13} \\ r & A_{21} & A_{22} & A_{23} \\ s & A_{31} & A_{32} & A_{33} \end{array}$$

with $A_{12} + A_{21} = \sqrt{-1}\,(A_{11} - A_{22})$ and $A_{32} = \sqrt{-1}\,A_{31}$ is parabolic in $G_{\mathbf{C}}$. The Grassmann manifold $G(r, 2r+s; \mathbf{C})$ of r-planes in complex $(2r+s)$-space can be realized as the set of complex $(2r+s) \times r$ matrices of maximal rank, modulo the equivalence relation $\Omega_1 \sim \Omega_2$ if $\Omega_1 = \Omega_2 A$ for some nonsingular $r \times r$ matrix A. The equation ${}^t\Omega Q \Omega = 0$ defines a subvariety $X \subset G(r, 2r+s; \mathbf{C})$ which contains the point x_0 represented by $\Omega_0 = {}^t(I_r, \sqrt{-1}\,I_r, 0_{r \times s})$. By left-translation, $G_{\mathbf{C}}$ acts holomorphically and transitively on X, with stability group B at x_0. Thus we can identify X with the Kähler C-space $G_{\mathbf{C}}/B$.

The identity component G of the group of real matrices in $G_{\mathbf{C}}$ is a noncompact real form of $G_{\mathbf{C}}$, and $V = G \cap B$ is isomorphic to $U(r) \times \mathrm{SO}(s)$, hence compact. Now D, the G-orbit of x_0, can be described as the connected component containing x_0 of the set of points in X represented by matrices Ω such that ${}^t\bar{\Omega} Q \Omega$ is positive definite. Let K be the group $\mathrm{SO}(2r) \times \mathrm{SO}(s)$, embedded in G in the obvious manner; K is maximal compact and contains V. Its orbit at x_0, S, consists of all points in D whose representatives Ω have a zero bottom $s \times r$ block. The unique V-invariant, and in fact B-invariant, complement of the tangent space of S at x_0 in the tangent space of X is given by the vanishing of ${}^t\Omega_0 Q d\Omega$. Therefore the G-invariant subbundle $\mathbf{T}_h(D) \subset \mathbf{T}(D)$ is determined by the equation ${}^t\Omega Q d\Omega = 0$. In this particular case, $\mathbf{T}_h(D)$ is a holomorphic subbundle of $\mathbf{T}(D)$ and extends to all of X. A holomorphic mapping F: $Y \to D$ can locally be represented by a holomorphic matrix valued function $\Omega(y)$, $y \in Y$; F is horizontal if ${}^t\Omega(y) Q d\Omega(y) = 0$. The period mappings constructed in [11] have this property.

3. Structure theory of semisimple Lie algebras

In this section, we shall review and collect some facts about the structure of semisimple Lie algebras and their representations. Throughout, $\mathfrak{g}$ will denote a complex semisimple Lie algebra, $\mathfrak{m}_0$ a compact real form of $\mathfrak{g}$, and τ complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{m}_0$. We choose a maximal abelian subalgebra $\mathfrak{h}_0$ of $\mathfrak{m}_0$; complexifying $\mathfrak{h}_0$, one obtains a τ-invariant Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha}, \tag{3.1 a}$$

where Δ, the set of nonzero roots, is a subset of the dual space of $\mathfrak{h}$, and each rootspace

$$\mathfrak{g}^{\alpha} = \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x \quad \text{for all } h \in \mathfrak{h}\} \tag{3.1 b}$$

is one-dimensional. If α, β, $\alpha + \beta \in \Delta$,

$$[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] = \mathfrak{g}^{\alpha+\beta}. \tag{3.1 c}$$

Since $\mathfrak{m}_0$ is a compact real form, all roots assume real values on $\mathfrak{h}_{\mathbf{R}} = \sqrt{-1}\,\mathfrak{h}_0$. We shall regard Δ as a subset of $\mathfrak{h}_{\mathbf{R}}^*$, the dual space of $\mathfrak{h}_{\mathbf{R}}$. For every $\alpha \in \Delta$,

$$\tau(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{-\alpha} \tag{3.2}$$

because $\mathfrak{h}_{\mathbf{R}}$ is purely imaginary with respect to $\mathfrak{m}_0$.

The Cartan–Killing form

$$B(x, y) = \text{trace}\,(\text{ad}\, x\, \text{ad}\, y) \quad x, y \in \mathfrak{g} \tag{3.3}$$

restricts to a positive definite bilinear form on $\mathfrak{h}_{\mathbf{R}}$, and by duality determines an inner product $(\ ,\ )$ on $\mathfrak{h}_{\mathbf{R}}^*$. The hyperplanes $P_\alpha = \{\mu \in \mathfrak{h}_{\mathbf{R}}^* \mid (\mu, \alpha) = 0\}$, $\alpha \in \Delta$, divide $\mathfrak{h}_{\mathbf{R}}^*$ into a finite number of closed convex cones, the so-called Weyl chambers. The reflections about the hyperplanes P_α generate a group of linear transformations W, the Weyl group, which leaves Δ invariant and permutes the Weyl chambers simply and transitively. A system of positive roots is a subset $\Delta_+ \subset \Delta$ such that

(3.4)
a) for every $\alpha \in \Delta$, either α or $(-\alpha)$, but not both, belongs to Δ_+

b) if $\alpha, \beta \in \Delta_+$ and $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Delta_+$.

Equivalently, such a set Δ_+ can be described as the set of all elements of Δ which are positive with respect to some suitably chosen linear order of $\mathfrak{h}^*_{\mathbf{R}}$. To each system of positive roots Δ_+, there corresponds a distinguished Weyl chamber, the highest Weyl chamber,

$$C = \{\mu \in \mathfrak{h}^*_{\mathbf{R}} \mid (\alpha, \mu) \geqslant 0 \text{ for every } \alpha \in \Delta_+\}$$

This correspondence between systems of positive roots and Weyl chambers is bijective. Consequently W acts simply and transitively also on the collection of systems of positive roots.

Let M be a simply connected Lie group with Lie algebra $\mathfrak{m}_0$, H the subgroup of M determined by $\mathfrak{h}_0$. An element $\lambda \in \mathfrak{h}^*_{\mathbf{R}}$ which is the differential of a character of H is called a weight. The weights form a lattice

$$\Lambda = \{\lambda \in \mathfrak{h}^*_{\mathbf{R}} \mid 2(\lambda, \alpha)(\alpha, \alpha)^{-1} \in \mathbf{Z} \text{ for every } \alpha \in \Delta\}$$

in $\mathfrak{h}^*_{\mathbf{R}}$ which contains Δ. A weight λ is said to be singular if $(\lambda, \alpha) = 0$ for some $\alpha \in \Delta$, and nonsingular otherwise. If Δ_+ is a particular system of positive roots and C the corresponding highest Weyl chamber, then

$$\varrho = \tfrac{1}{2} \sum \alpha, \quad \alpha \in \Delta_+ \tag{3.5}$$

is a nonsingular weight and belongs to C; ϱ is minimal with respect to these two properties: a weight $\lambda \in C$ is nonsingular if and only if $\lambda - \varrho \in C$.

Next, we consider an irreducible skew-Hermitian representation π of $\mathfrak{m}_0$ on a finite-dimensional complex inner product space E. The complex extension of π to $\mathfrak{g}$ will be denoted by the same letter. Since $\mathfrak{h}$ is an abelian Lie algebra, its action on E determines a decomposition $E = \sum_{\lambda \in \Lambda} E_\lambda$, where $E_\lambda = \{v \in E \mid \pi(h) v = \langle \lambda, h \rangle v \text{ for every } h \in \mathfrak{h}\}$. We choose a particular system of positive roots Δ_+. There exists a unique weight λ, which is called the highest weight of π, with the property that $E_\lambda \neq 0$ and

$$\pi(x) E_\lambda = 0 \quad \text{for every } x \in \mathfrak{g}^\alpha,\ \alpha \in \Delta_+.$$

The subspace E_λ is then one-dimensional. The highest weight characterizes π up to unitary equivalence. It lies in the highest Weyl chamber; conversely, every weight in the highest Weyl chamber is the highest weight of some representation π. If the system of positive roots Δ_+ is replaced by another one, $w(\Delta_+)$, w being an element of the Weyl group, the new highest weight will be the w-translate of the original one. The representation π lifts to a connected group M with Lie algebra $\mathfrak{m}_0$ precisely when its highest weight lifts to a character of the torus H in M which corresponds to $\mathfrak{h}_0$. After only minor and rather obvious modifications are made, the statements above remain correct if $\mathfrak{m}_0$ is the Lie algebra of a compact, but not necessarily semisimple group, and $\mathfrak{g}$ the complexification of $\mathfrak{m}_0$.

We need to look more closely at the situation of § 2 on the Lie algebra level. Thus $\mathfrak{b}$ will be a parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{g}_0$ a noncompact real form of $\mathfrak{g}$, with a maximal compactly embedded subalgebra $\mathfrak{k}_0$ such that $\mathfrak{v}_0=\mathfrak{g}_0\cap\mathfrak{b}\subset\mathfrak{k}_0$. The complexification $\mathfrak{k}$ of $\mathfrak{k}_0$ has a unique ad$\mathfrak{k}$-invariant complement $\mathfrak{p}$; we set $\mathfrak{p}_0=\mathfrak{p}\cap\mathfrak{g}_0$. Then $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ is a Cartan decomposition, and $\mathfrak{k}_0\oplus\sqrt{-1}\,\mathfrak{p}_0$ is a compact real form of $\mathfrak{g}$ which contains $\mathfrak{v}_0$. Henceforth, $\mathfrak{m}_0$ will designate this particular compact real form. Let σ and τ be complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{m}_0$, respectively. They commute, and $\theta=\sigma\tau$ is an involutive automorphism of $\mathfrak{g}$ whose $(+1)$ and (-1) eigenspaces are $\mathfrak{k}$ and $\mathfrak{p}$. Since $\mathfrak{v}_0$ has the same rank as $\mathfrak{m}_0$, as was shown in § 1, we may assume that the Cartan subalgebra $\mathfrak{h}_0\subset\mathfrak{m}_0$ chosen at the beginning of this section lies in $\mathfrak{v}_0$, and hence in $\mathfrak{k}_0$. Then θ commutes with the adjoint action of $\mathfrak{h}$, and every rootspace $\mathfrak{g}^\alpha$ is contained either in $\mathfrak{k}$ or in $\mathfrak{p}$. The root α is said to be compact if the former is the case, and noncompact otherwise. We denote the sets of compact and noncompact roots by $\Delta_\mathfrak{k}$ and $\Delta_\mathfrak{p}$.

Since the rootspaces $\mathfrak{g}^\alpha$ are one-dimensional, every subalgebra $\mathfrak{u}_0$ of $\mathfrak{g}_0$ which contains $\mathfrak{h}_0$ is spanned over $\mathbf{R}$ by $\mathfrak{h}_0$ and $\mathfrak{g}_0\cap(\mathfrak{g}^\alpha\oplus\sigma(\mathfrak{g}^\alpha))$, with α ranging over a suitable subset Ψ of Δ. It is known that the exponential map, restricted to $\mathfrak{p}_0$, is a diffeomorphism. Moreover, $\mathfrak{g}_0\cap(\mathfrak{g}^\alpha\oplus\sigma(\mathfrak{g}^\alpha))\subset\mathfrak{p}_0$ whenever α is noncompact. Hence $\mathfrak{u}_0$ cannot be a compactly embedded subalgebra of $\mathfrak{g}_0$ unless $\Psi\subset\Delta_\mathfrak{k}$. In particular, $\mathfrak{k}_0$ is the only maximal compactly embedded subalgebra of $\mathfrak{g}_0$ which contains $\mathfrak{h}_0$. This verifies the statement, made in § 2, that the condition $K\supset V$ determines the maximal compact subgroup K of G uniquely.

For each $\alpha\in\Delta$, one can choose vectors $e_\alpha\in\mathfrak{g}^\alpha$ and $h_\alpha\in\mathfrak{h}_\mathbf{R}=\sqrt{-1}\,\mathfrak{h}_0$ such that

a) $B(e_\alpha, e_\beta)=\delta_{\alpha,-\beta}$, $[e_\alpha,e_{-\alpha}]=h_\alpha$

b) $B(h_\alpha, x)=\langle\alpha, x\rangle$ for $x\in\mathfrak{h}$

c) $[e_\alpha, e_\beta]=0$ if $\alpha\neq-\beta$ and $\alpha+\beta\notin\Delta$

(3.6) d) $[e_\alpha, e_\beta]=N_{\alpha,\beta}\,e_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta\in\Delta$. The $N_{\alpha,\beta}$ are nonzero real constants such that $N_{-\alpha,-\beta}=-N_{\alpha,\beta}$; also $N_{-\alpha,-\beta}=N_{-\beta,\alpha+\beta}=N_{\alpha+\beta,-\alpha}$

e) $\tau(e_\alpha)=-e_{-\alpha}$

f) $\sigma(e_\alpha)=\varepsilon_\alpha e_{-\alpha}$, where $\varepsilon_\alpha=-1$ if α is compact, $\varepsilon_\alpha=+1$ if α is noncompact

g) $\varepsilon_{\alpha+\beta}=-\varepsilon_\alpha\varepsilon_\beta$ whenever $\alpha, \beta, \alpha+\beta\in\Delta$.

A normalization with properties a)–e) is exhibited in [15]; e) implies f), because $\theta(e_\alpha)=-\varepsilon_\alpha e_\alpha$, and g) is a consequence of d). It will be convenient to define $N_{\alpha,\beta}=0$ and $e_{\alpha+\beta}=0$ if $\alpha+\beta\notin\Delta$ and $\alpha+\beta\neq0$.

From now on, Δ_+ will be a fixed system of positive roots such that $\mathfrak{g}^{-\alpha} \in \mathfrak{b}$ whenever α is positive. Such a system Δ_+ exists because $\mathfrak{b}$ contains an ad $\mathfrak{h}$-invariant maximal nilpotent subalgebra of $\mathfrak{g}$. Since $\mathfrak{v}$ is reductive and $\mathfrak{h} \subset \mathfrak{v} \subset \mathfrak{k}$,

$$\mathfrak{v} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}^{\alpha} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}^{-\alpha} \tag{3.7 a}$$

for some subset Φ of $\Delta_{\mathfrak{k}} \cap \Delta_+$. It now follows from (1.1) that

$$\mathfrak{n}_- = \oplus \mathfrak{g}^{-\alpha} \quad \alpha \in \Delta_+ - \Phi. \tag{3.7 b}$$

If α, β, $\beta - \alpha$ are nonzero roots, then

$$\alpha \in \Phi, \quad \beta \in \Delta_+ - \Phi \Rightarrow \beta \pm \alpha \in \Delta_+ - \Phi; \tag{3.8}$$

this just expresses the fact that $\mathfrak{v}$ normalizes $\mathfrak{n}_-$.

We conclude this section with a simple lemma, which will be used in § 5.

(3.1) LEMMA. *Let α be a fixed positive root, and $>$ a linear ordering of $\mathfrak{h}^*_{\mathbf{R}}$ which makes the elements of Δ_+ positive. Then*

$$\sum_{0<\beta<\alpha} N_{\beta,\alpha-\beta} N_{-\beta,\alpha} = (2\varrho - \alpha, \alpha).$$

Proof. Clearly

$$\sum_{0<\beta<\alpha} N_{\beta,\alpha-\beta} N_{-\beta,\alpha} = \sum_{\beta>0,\, \beta\neq\alpha} N_{\beta,\alpha-\beta} N_{-\beta,\alpha} - \sum_{\beta>\alpha} N_{\beta,\alpha-\beta} N_{-\beta,\alpha}.$$

In the second sum, the term corresponding to β makes a contribution only if $\beta - \alpha$ is a root, since otherwise $N_{-\beta,\alpha} = 0$. Thus we can replace β by $\alpha + \gamma$ and sum over all positive roots γ; $\gamma = \alpha$ can be excluded because 2α is never a root when α is a root. We use (3.6) a), b), c), d) to get

$$\begin{aligned}
\sum_{0<\beta<\alpha} N_{\beta,\alpha-\beta} N_{-\beta,\alpha} &= \sum_{\beta>0,\beta\neq\alpha} N_{\beta,\alpha-\beta} N_{-\beta,\alpha} - \sum_{\gamma>0,\gamma\neq\alpha} N_{\alpha+\gamma,-\gamma} N_{-\alpha-\gamma,\alpha} \\
&= \sum_{\beta>0,\beta\neq\alpha} (N_{\beta,\alpha-\beta} N_{-\beta,\alpha} - N_{-\beta,-\alpha} N_{\alpha,\beta}) = \sum_{\beta>0,\beta\neq\alpha} (N_{\beta,\alpha-\beta} N_{-\beta,\alpha} - N_{-\beta,\alpha+\beta} N_{\beta,\alpha}) \\
&= \sum_{\beta>0,\beta\neq\alpha} B([e_\beta,[e_{-\beta},e_\alpha]] - [e_{-\beta},[e_\beta,e_\alpha]], e_{-\alpha}) = \sum_{\beta>0,\beta\neq\alpha} B([h_\beta,e_\alpha],e_{-\alpha}) \\
&= \sum_{\beta>0,\beta\neq\alpha} (\beta,\alpha) = (2\varrho - \alpha, \alpha).
\end{aligned}$$

4. Curvature of homogeneous vector bundles

Let $X=M/V$ be a Kähler C-space (cf. § 1) and $D=G/V$ a noncompact dual (cf. § 2). If π: $V\to \mathrm{GL}(E)$ is an irreducible unitary representation of V on a complex vector space E, then there are defined homogeneous vector bundles $M\times_V E\to X$, $G\times_V E\to D$ which have respectively M, G invariant Hermitian metrics. We will denote both bundles by $\mathbf{E}_\pi$ and consider them as holomorphic vector bundles as follows (cf. § 1): Write $X=G_{\mathbf{C}}/B$ and extend π uniquely to an irreducible, holomorphic representation π: $B\to \mathrm{GL}(E)$. Then $G_{\mathbf{C}}\times_B E=\mathbf{E}_\pi$ is a holomorphic vector bundle over X which gives a complex structure to $M\times_V E$, and the restriction $G_{\mathbf{C}}\times_B E\,|\,D=\mathbf{E}_\pi|\,D$ gives a complex structure to $G\times_V E$.

Recall now that whenever we have a holomorphic, Hermitian vector bundle $\mathbf{F}\to Y$ over a complex manifold Y, there is canonically associated a connection D: $A^0(\mathbf{F})\to A^1(\mathbf{F})$ ($A^q(\mathbf{F})$ is the space of C^∞ $\mathbf{F}$-valued q-forms over Y) satisfying: (i) $D''=\bar\partial$ where $D=D'+D''$ is the decomposition of D into type; and (ii) $d(f,f')=(Df,f')+(f,Df')$ where f,f' are C^∞ sections of $\mathbf{F}$ (cf. [13]). The curvature Θ is a $\mathrm{Hom}(\mathbf{F},\mathbf{F})$-valued (1,1) form which is important in the study of the geometry of $\mathbf{F}$ as well as the sheaf cohomology (cf. [13]). We want to compute the M-invariant curvature $\Theta_X(\pi)$ in $\mathbf{E}_\pi\to X$ and the G-invariant curvature $\Theta_D(\pi)$ in $\mathbf{E}_\pi\to D$. The results we will find are these (cf. Theorem (4.13)):

(a) Both curvatures $\Theta_X(\pi)$ and $\Theta_D(\pi)$ have a canonical expression $A\wedge {}^t\bar A-B\wedge {}^t\bar B$ where A, B are matrices of $(1,0)$ forms. In other words, the curvatures will have natural expressions as a *difference of positive forms*.

(b) If $\lambda\in\mathfrak{h}^*$ is the highest weight of π, then $\Theta_X(\pi)$ will equal $A\wedge {}^t\bar A-B\wedge {}^t\bar B$ where A involves $(1,0)$ forms ω^α where $\alpha\in\Delta_+-\Phi$ satisfies $(\lambda,\alpha)>0$ and B involves the ω^β where $\beta\in\Delta_+-\Phi$ and $(\lambda,\beta)<0$. If λ is non-singular, then the *curvature forms* $\Theta_X(\pi)(\xi)$ and $\Theta_D(\pi)(\xi)$ (cf. (4.14) below) will be non-singular and $\Theta_X(\pi)(\xi)$ will have *signature* equal to the *index* $\iota(\pi)$ of λ.

(c) The curvature $\Theta_D(\pi)$ is obtained from $\Theta_X(\pi)$ by reversing the signs corresponding to the non-compact roots $\beta\in\Delta_+-\Delta_{\mathfrak{k}}$.

To begin with, we consider a pair of connected Lie groups A, B such that $B\subset A$ is a closed, reductive subgroup. Thus there is an Ad B-invariant splitting $\mathfrak{a}_0=\mathfrak{b}_0\oplus\mathfrak{t}_0$. We have in mind the pairs M, V and G, V; the reductive splittings $\mathfrak{m}_0=\mathfrak{v}_0\oplus\mathfrak{t}_0$ and $\mathfrak{g}_0=\mathfrak{v}_0\oplus\mathfrak{s}_0$ are given by the Cartan-Killing forms on $\mathfrak{m}_0$ and $\mathfrak{g}_0$; both of these forms are non-singular and are negative definite on $\mathfrak{v}_0$.

Such a reductive splitting gives an A-invariant connection in the principal bundle $B\to A\to A/B$: We will think of $\mathfrak{a}_0$ as left-invariant vector fields, or equivalently, as infinitesimal right translations, on A. The tangent space $T_a(A)$ to A at a is then $(L_a)_*\mathfrak{a}_0\cong\mathfrak{a}_0$, and the tangent space to the fibres of $A\to A/B$ is $(L_a)_*\mathfrak{t}_0\cong\mathfrak{t}_0$ (L_a is left translation by a).

Thus $(L_a)_* \mathfrak{t}_0 \cong \mathfrak{t}_0$ gives a complement to the vertical space of $A \to A/B$ at a, and since $[\mathfrak{b}_0, \mathfrak{t}_0] \subseteq \mathfrak{t}_0$, the splitting $T_a(A) \cong \mathfrak{b}_0 \oplus \mathfrak{t}_0$ is invariant by B acting on the right. This gives the horizontal space for our invariant connection in $A \to A/B$.

To find the connection form we choose a basis $a_1, ..., a_m$ for $\mathfrak{a}_0$ such that $a_1, ..., a_r$ is a basis for $\mathfrak{b}_0$ and $a_{r+1}, ..., a_m$ lie in $\mathfrak{t}_0$. Let $\varphi^1, ..., \varphi^m$ be a dual basis for the left-invariant Maurer–Cartan forms on A and set $\theta = \sum_{\varrho=1}^r a_\varrho \otimes \varphi^\varrho$. Then θ is independent of bases and is an A-invariant, $\mathfrak{b}_0$-valued differential form on A which gives the connection form of the above connection. To find the curvature form Θ, we use the *Maurer–Cartan equations*:

$$d\varphi^i = -\tfrac{1}{2} \sum_{j,k=1}^{m} c^i_{jk} \varphi^j \wedge \varphi^k \quad \left([a_j, a_k] = \sum_{i=1}^{m} c^i_{jk} a_i\right),$$

and the *Cartan structure equation:*

$$\Theta = d\theta + \tfrac{1}{2} [\theta, \theta].$$

Since $d\theta = \sum_{\varrho=1}^r a_\varrho \otimes d\varphi^\varrho$ and $[\theta, \theta] = \sum_{\varrho,\sigma=1}^r [a_\varrho, a_\sigma] \otimes \varphi^\varrho \wedge \varphi^\sigma$, we have

$$d\theta + \tfrac{1}{2} [\theta, \theta] = -\tfrac{1}{2} \sum_{\varrho=1}^{r} \sum_{\mu,\nu=r+1}^{m} c^\varrho_{\mu\nu} a_\varrho \otimes \varphi^\mu \wedge \varphi^\nu$$

because of $c^\sigma_{\varrho\mu} = 0$. This gives:

$$\Theta = -\tfrac{1}{2} \sum_{\mu,\nu=r+1}^{m} [a_\mu, a_\nu]_{\mathfrak{t}_0} \otimes \varphi^\mu \wedge \varphi^\nu, \tag{4.1}$$

where $[a]_{\mathfrak{t}_0}$ denotes the projection of $a \in \mathfrak{a}_0$ on $\mathfrak{t}_0$ relative to the splitting $\mathfrak{a}_0 = \mathfrak{b}_0 \oplus \mathfrak{t}_0$. This equation remains true if $a_1, ..., a_m$ is a basis of the complexification $\mathfrak{a}$ of $\mathfrak{a}_0$.

Suppose now that $A = M$, $B = V$ and we make the identification of $\mathfrak{m}$ with $\mathfrak{g}$ (cf. § 1). Thus we have the decomposition (3.7a) and

$$\mathfrak{v} = \mathfrak{h} \oplus \sum_{\pm\alpha \in \Phi} \mathfrak{g}^\alpha, \quad \mathfrak{t} = \sum_{\pm\beta \in \Delta_+ - \Phi} \mathfrak{g}^\beta$$

where $\Phi \subset \Delta_+$ is the set of positive roots for $\mathfrak{v}$. Since $[e_\alpha, e_\beta]_\mathfrak{v} = 0$ for $\alpha, \beta \in \Delta_+ - \Phi$, we see from (4.1) that the curvature Θ_X for the natural connection in $V \to M \to X$ is

$$\Theta_X = - \sum_{\alpha,\beta \in \Delta_+ - \Phi} [e_\alpha, e_{-\beta}]_\mathfrak{v} \otimes \omega^\alpha \wedge \omega^{-\beta}, \tag{4.2$_X$}$$

where $\omega^\alpha \in \mathfrak{g}^*$ is dual to $e_\alpha \in \mathfrak{g}^\alpha \subset \mathfrak{g}$. Using that $\omega^{-\beta} = -\bar{\omega}^\beta$ (cf. (3.6e)), we may rewrite $(4.2)_X$ as

$$\Theta_X = \sum_{\alpha,\beta \in \Delta_+ - \Phi} [e_\alpha, e_{-\beta}]_\mathfrak{v} \otimes \omega^\alpha \wedge \bar{\omega}^\beta. \tag{4.3$_X$}$$

In particular, Θ_X is a $\mathfrak{v}$-valued form of type (1, 1).

If now $A=G$, $B=V$ and we identify $\mathfrak{g}_0\oplus_{\mathbf{R}}\mathbf{C}$ with $\mathfrak{g}$ (cf. § 2), then we find that the curvature for the natural connection in $V\to G\to D$ is

$$(4.2)_D \qquad \Theta_D=-\sum_{\alpha,\beta\in\Delta_+-\Phi}[e_\alpha,e_{-\beta}]_{\mathfrak{v}}\otimes\omega^\alpha\wedge\omega^{-\beta},$$

which is formally the same as $(4.2)_X$. However, from (3.6f) we see that $\omega^{-\alpha}=-\bar{\omega}^\alpha$ for $\alpha\in\Delta_{\mathfrak{k}}$, whereas $\omega^{-\beta}=\bar{\omega}^\beta$ for $\beta\in\Delta_+-\Delta_{\mathfrak{k}}$. Thus:

$$(4.3)_D \qquad \Theta_D=-\sum_{\alpha,\beta\in\Delta_+-\Delta_{\mathfrak{k}}}[e_\alpha,e_{-\beta}]_{\mathfrak{v}}\otimes\omega^\alpha\wedge\bar{\omega}^\beta+\sum_{\alpha,\beta\in\Delta_{\mathfrak{k}}-\Phi}[e_\alpha,{}_{-\beta}]_{\mathfrak{v}}\otimes\omega^\alpha\wedge\bar{\omega}^\beta;$$

here we have used that $[e_\alpha,e_{-\beta}]_{\mathfrak{v}}=0$ for $\alpha\in\Delta_+-\Delta_{\mathfrak{k}}$ noncompact and $\beta\in\Delta_{\mathfrak{k}}-\Phi$ a positive compact root. Comparing $(4.3)_X$ and $(4.3)_D$ we see the sign reversal for the curvature in the noncompact dual D of the Kähler C-space X.

Return now to a general reductive pair A, B and let π: $B\to \mathrm{GL}(E)$ be a linear representation. Then we form the homogeneous bundle $\mathbf{E}_\pi=A\times_B E$ whose sections are the C^∞-functions f: $A\to E$ which satisfy $bf+\pi(b)f=0$ for $b\in\mathfrak{b}$. The connection in $B\to A\to A/B$ induces one in the associated bundle $\mathbf{E}_\pi$; the differential of a section f of $\mathbf{E}_\pi\to A/B$ is $Df=\sum_{\mu=r+1}^m a_\mu f\otimes\varphi^\mu$. Note that

$$Df=df-\sum_{\varrho=1}^r a_\varrho f\otimes\varphi^\varrho=df+\sum_{\varrho=1}^r\pi(a_\varrho)f\otimes\varphi^\varrho=df+\pi(\theta)f$$

where $\pi(\theta)$ is the connection form in $\mathbf{E}_\pi$.

If π is a unitary representation, then $\mathbf{E}_\pi\to A/B$ has an invariant metric. Letting f, f' be sections of $\mathbf{E}_\pi$, $(Df,f')+(f,Df')=(df,f')+(f,df')+(\pi(\theta)f,f')+(f,\pi(\theta)f')=d(f,f')$ since ${}^t\overline{\pi(\theta)}=\sum_{\varrho=1}^r{}^t\overline{\pi(a_\varrho)}\otimes\varphi^\varrho=-\sum_{\varrho=1}^r\pi(a_\varrho)\otimes\varphi^\varrho=-\pi(\theta)$. Thus D is compatible with the metric.

This discussion applies to $\mathbf{E}_\pi\to X=M/V$. We denote the connection by $D_X(\pi)$ so that $D_X(\pi)=\sum_{\pm\alpha\in\Delta_+-\Phi}e_\alpha\cdot f\otimes\omega^\alpha$. Thus the (0, 1) part $D_X(\pi)''=-\sum_{\alpha\in\Delta_+-\Phi}e_{-\alpha}\cdot f\otimes\bar{\omega}^\alpha=\bar{\partial}_{\mathbf{E}_\pi}\cdot f$ by (1.6b). It follows that $D_X(\pi)$ is the metric connection and the curvature

$$(4.4)_X \qquad \Theta_X(\pi)=\sum_{\alpha,\beta\in\Delta_+-\Phi}\pi([e_\alpha,e_{-\beta}]_{\mathfrak{v}})\otimes\omega^\alpha\wedge\bar{\omega}^\beta.$$

Similarly, the connection in $\mathbf{E}_\pi\to D$ induced from the natural one in $V\to G\to D$ is the metric connection and the curvature

$$(4.4)_D \qquad \Theta_D(\pi)=-\sum_{\alpha,\beta\in\Delta_+-\Delta_{\mathfrak{k}}}\pi([e_\alpha,e_{-\beta}]_{\mathfrak{v}})\otimes\omega^\alpha\wedge\bar{\omega}^\beta+\sum_{\alpha,\beta\in\Delta_{\mathfrak{k}}-\Phi}\pi([e_\alpha,e_{-\beta}]_{\mathfrak{v}})\otimes\omega^\alpha\wedge\bar{\omega}^\beta.$$

In summary we have:

(4.5) An irreducible unitary representation π: $V\to\mathrm{Aut}\,(E)$ gives a holomorphic vector bundle $\mathbf{E}_\pi=G_{\mathbf{C}}\times_B E$ over X which restricts to a holomorphic bundle $\mathbf{E}_\pi\to D$. The bundle

$\mathbf{E}_\pi \to X$ has an M-invariant Hermitian structure and the metric connection is the one induced from the natural connection in $V \to M \to X$. Similarly, $\mathbf{E}_\pi \to D$ has a G-invariant Hermitian structure (which is *not* the restriction of the M-invariant structure on $\mathbf{E}_\pi \to X$), and the metric connection is the one induced from the natural connection in $V \to G \to D$. The curvatures are given by $(4.4)_X$ and $(4.4)_D$ respectively.

Remark. Write $X = G_{\mathbf{C}}/B$ and let π: $B \to \mathrm{GL}(E)$ be an arbitrary holomorphic representation, $\mathbf{E}_\pi = G_{\mathbf{C}} \times_B E \to X$ the resulting holomorphic vector bundle. As a C^∞ bundle, $\mathbf{E}_\pi = M \times_V E$, and the natural connection in $V \to M \to X$ induces a connection D_π in $\mathbf{E}_\pi \to X$ such that $D''_\pi f = -\sum_{\alpha \in \Delta_+ - \Phi} e_{-\alpha} \cdot f \otimes \bar{\omega}^\alpha$. By (1.6b), $\bar{\partial}_{\mathbf{E}_\pi} f = -\sum_{\alpha \in \Delta_+ - \Phi} (e_{-\alpha} \cdot f + \pi(e_{-\alpha}) f) \otimes \bar{\omega}^\alpha$, so that $D''_\pi = \bar{\partial}_{\mathbf{E}_\pi}$ if, and only if, $\pi(\mathfrak{n}_-) = 0$.

Our program for computing curvatures had three parts (a), (b), (c) given at the beginning of this section. From $(4.4)_X$ and $(4.4)_D$ we have completed (c), and now we turn to (a) and (b).

Let π: $V \to \mathrm{GL}(E)$ be an irreducible unitary representation of V. Relative to a Cartan decomposition (3.1a) of $\mathfrak{g}$, we let $\mathfrak{v} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta(\mathfrak{v})} \mathfrak{g}^\alpha$ $(\Delta(\mathfrak{v}) = \Phi \cup \{-\Phi\})$ be the decomposition of $\mathfrak{v}$ and $\Lambda \subset \mathfrak{h}^*$ be the lattice of differentials of characters of the maximal torus $H \subset V$. We have a weight space decomposition $E = \sum_{\mu \in \Lambda(\pi)} E_\mu$ where $\Lambda(\pi) \subset \Lambda$ is the set of weights for π. A weight λ is *extremal* if, for some set of positive simple roots $\Pi \subset \Delta(\mathfrak{v})$ for V, we have $e_\alpha \cdot E_\lambda = 0$ for all $\alpha \in \Pi$. In particular, the complex structure on M/V determines a set of positive roots $\Delta_+ \subset \Delta$ and we may take for Π the simple roots for V which lie in Δ_+. Then λ is a highest weight for V, and everything is well-determined up to the actions of the Weyl group of V.

Now we assume that λ is non-singular; i.e. that $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Delta$. Then λ lies in the positive Weyl chamber (relative to Π) for V, and it lies in some Weyl chamber C_π for M. It may not be that C_π is the positive Weyl chamber (relative to Δ_+) C for M, but in any case C_π determines a set $\Delta_+(\pi)$ of positive roots for M such that C_π is the positive Weyl chamber for $\Delta_+(\pi)$. Then $\Delta_+ \cap \Delta_-(\pi)$ are those positive roots $\alpha \in \Delta_+$ with $(\alpha, \lambda) < 0$; the number of $\alpha \in \Delta_+ \cap \Delta_-(\pi)$ is called the *index* of π, denoted by $\iota(\pi)$, and this is the number of root planes through which we must reflect to get from C_π to C.

We let $\Delta_+(\pi, \mathfrak{v}) = \Delta_+(\pi) - \Delta_+(\pi) \cap \Delta(\mathfrak{v})$ and $\mathfrak{l}_+ = \sum_{\alpha \in \Delta_+(\pi, \mathfrak{v})} \mathfrak{g}^\alpha$, $\mathfrak{l}_- = \sum_{-\alpha \in \Delta_+(\pi, \mathfrak{v})} \mathfrak{g}^\alpha$; then $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{l}_+ \oplus \mathfrak{l}_-$, and $[\mathfrak{v}, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$, $[\mathfrak{v}, \mathfrak{l}_-] \subseteq \mathfrak{l}_-$. Note that $\dim(\mathfrak{n}_+ \cap \mathfrak{l}_-) = \dim(\mathfrak{n}_- \cap \mathfrak{l}_+) = \iota(\pi)$. The same argument as used to prove $(4.4)_X$ gives:

$$\Theta_X(\pi) = -\sum_{\alpha, \beta \in \Delta_+(\pi, \mathfrak{v})} \pi([e_\alpha, e_{-\beta}]_{\mathfrak{v}}) \otimes \omega^\alpha \wedge \omega^{-\beta}. \tag{4.6}$$

Using $\Delta_+(\pi)$ as a set of positive roots for M, we let F be the unitary M-module with highest weight $\lambda \in C_\pi$. Restricting to V we get a unitary V-module F_V, and we assert that:

(4.7) $$F_V = E \oplus S, \text{ where } S \text{ contains no } V\text{-module equivalent to } E;$$

i.e. the multiplicity $\mu(E, F)$ of E in F_V is one.

Proof. The highest weight space W_λ, relative to $\Delta_+(\pi)$, of F occurs with multiplicity one; thus, $\mu(E, F) \leq 1$. Since $e_\alpha \cdot W_\lambda = 0$ for $\alpha \in \Delta_+(\pi) \cap \Delta(\mathfrak{v})$ we see that $\mu(E, F) \neq 0$.

We now show that:

(4.8) $$\mathfrak{l}_+ \cdot E = 0.$$

Proof. A basis of E consists of vectors $e_{\alpha_1} \dots e_{\alpha_r} \cdot w (= \pi(e_{\alpha_1}) \dots \pi(e_{\alpha_r}) \cdot w)$ where $w \in W_\lambda = E_\lambda$ is a highest weight vector and the $\alpha_j \in \Delta(\mathfrak{v})$. If $e \in \mathfrak{l}_+$, then $e \cdot e_{\alpha_1} \dots e_{\alpha_r} = e_{\alpha_1} \cdot e \cdot e_{\alpha_2} \dots e_{\alpha_r} + [e, e_{\alpha_1}] \cdot e_{\alpha_2} \dots e_{\alpha_r}$. Since $[\mathfrak{v}, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$ and $\mathfrak{l}_+ \cdot w = 0$, we may use induction on r to conclude that $e \cdot e_{\alpha_2} \dots e_{\alpha_r} w = 0 = [e, e_{\alpha_1}] \cdot e_{\alpha_2} \dots e_{\alpha_r} w$, which proves (4.8).

From (4.8) we deduce:

(4.9) $$\mathfrak{l}_- E \subseteq S.$$

Proof. For $\alpha \in \Delta_+(\pi, \mathfrak{v})$, we write $-e_{-\alpha} = \xi_\alpha + i\eta_\alpha$ where $\xi_\alpha = 1/2(e_\alpha - e_{-\alpha})$, $\eta_\alpha = i/2(e_\alpha - e_{-\alpha})$ and ξ_α, η_α lie in the compact form $\mathfrak{m}_0$. If $w, w' \in E$, then $(e_{-\alpha} w, w') = -(\xi_\alpha w, w') - i(\eta_\alpha w, w') = (w, \xi_\alpha w') + i(w, \eta_\alpha w') = (w, (\xi_\alpha - i\eta_\alpha) w') = (w, e_\alpha w') = 0$ by (4.8). Thus $(\mathfrak{l}_- E, E) = 0$, which gives (4.9).

Choose an orthonormal basis $w_1, \dots, w_m$ for F such that $w_1, \dots, w_r$ is a basis for $E \subset F$; we shall write v_j for w_j when we think of E as a V-module, and we agree on the range of indices $1 \leq \varrho, \sigma \leq r$; $1 \leq i, j \leq m$; and $r+1 \leq \mu, \nu \leq m$. We define $A \in \operatorname{Hom}(S, E) \otimes \mathfrak{l}_+^*$ by

(4.10) $$A = \sum_{\left\{\substack{\varrho \\ \alpha \in \Delta_+(\pi, \mathfrak{v})}\right.} v_\varrho \otimes e_\alpha w_\varrho^* \otimes \omega^\alpha$$

where $w_j^* \in F^*$ is dual to $w_j \in F$ and $e_\alpha w_j^*$ is the contragredient representation. We interpret (4.10) as follows: For $w \in F$, $A \cdot w = \sum_{\varrho, \alpha} \langle e_\alpha w_\varrho^*, w \rangle v_\varrho \otimes \omega^\alpha \in E \otimes \mathfrak{l}_+^*$. Since $A w_\sigma = \sum_{\alpha, \varrho} \langle e_\alpha w_\varrho^*, w_\sigma \rangle v_\varrho \otimes \omega^\alpha = -\sum_{\alpha, \varrho} \langle w_\varrho^*, e_\alpha w_\sigma \rangle v_\varrho \otimes \omega^\alpha = 0$ by (4.8), we see that $A(E) = 0$ so that $A \in \operatorname{Hom}(S, E) \otimes \mathfrak{l}_+^*$.

Write $e_\alpha w_\mu \equiv \sum_\varrho B_{\alpha\mu}^\varrho w_\varrho$ modulo S, so that $e_\alpha w_\varrho^* = -\sum_\mu B_{\alpha\mu}^\varrho w_\mu^*$ and

(4.11) $$A = -\sum_{\alpha, \varrho, \mu} B_{\alpha\mu}^\varrho v_\varrho \otimes w_\mu^* \otimes \omega^\alpha.$$

The transposed mapping ${}^tA \in \operatorname{Hom}(E^*, S^*) \otimes \mathfrak{l}_+^*$ is given by ${}^tA = -\sum_{\varrho, \alpha, \mu} B_{\alpha\mu}^\varrho w_\mu^* \otimes v_\sigma^{**} \otimes \omega^\alpha$. Using the conjugate linear isomorphism $F \cong F^*$ given by the metric, we have ${}^t\bar{A} \in \operatorname{Hom}(E, S) \otimes \bar{\mathfrak{l}}_+^*$ where ${}^t\bar{A} = -\sum_{\varrho, \alpha, \mu} \bar{B}_{\alpha\mu}^\varrho w_\mu \otimes v_\varrho^* \otimes \bar{\omega}^\alpha$. From $B_{\alpha\mu}^\varrho = (e_\alpha w_\mu, w_\varrho) = (w_\mu, e_{-\alpha} w_\varrho) =$

$\bar{B}^{\mu}_{-\alpha\varrho}$, we have $-\sum_{\mu}\bar{B}^{\varrho}_{\alpha\mu}w_{\mu} = -\sum_{\mu}B^{\mu}_{-\alpha\varrho}w_{\mu} = -e_{-\alpha}w_{\varrho}$ so that ${}^t\bar{A} = -\sum_{\alpha,\varrho}e_{-\alpha}w_{\varrho}\otimes v_{\varrho}^*\otimes\bar{\omega}^{\alpha}$. Thus $A\wedge{}^t\bar{A}\in \operatorname{Hom}(E,E)\otimes\bar{\mathfrak{l}}_+^*\wedge\mathfrak{l}_+^*$ is given by:

$$A\wedge{}^t\bar{A} = -\sum_{\begin{cases}\alpha,\beta\in\Delta_+(\pi,\mathfrak{v})\\ \varrho,\sigma\end{cases}}\langle e_{\alpha}w_{\varrho}^*, e_{-\beta}w_{\sigma}\rangle\, v_{\varrho}\otimes v_{\sigma}^*\otimes\omega^{\alpha}\wedge\bar{\omega}^{\beta}.$$

Now $-\langle e_{\alpha}w_{\varrho}^*, e_{-\beta}w_{\sigma}\rangle = \langle w_{\varrho}^*, e_{\alpha}e_{-\beta}w_{\sigma}\rangle = \langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]\, w_{\sigma}\rangle$ (since $e_{\alpha}w_{\sigma}=0$ by (4.8)) $=$ $\langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]_{\mathfrak{v}}w_{\sigma}\rangle + \langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]_{\mathfrak{l}_+}w_{\sigma}\rangle + \langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]_{\mathfrak{l}_-}w_{\sigma}\rangle = \langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]_{\mathfrak{v}}w_{\sigma}\rangle$ by (4.8) and (4.9). Thus $A\wedge{}^t\bar{A} = \sum\langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]_{\mathfrak{v}}w_{\sigma}\rangle\, v_{\varrho}\otimes v_{\sigma}^*\otimes\omega^{\alpha}\wedge\bar{\omega}^{\beta} = -\sum_{\alpha,\beta}\pi([e_{\alpha}, e_{-\beta}]_{\mathfrak{v}})\otimes\omega^{\alpha}\wedge\bar{\omega}^{\beta}$ $=\Theta_X(\pi)$ by (4.6), i.e.

(4.12) $A\wedge{}^t\bar{A} = \Theta_X(\pi)$ where A is given by (4.10).

Write $A = A' + A''$ where

$$A' = \sum_{\begin{cases}\varrho\\ \alpha\in\Delta_+(\pi,\mathfrak{v})\cap\Delta_+\end{cases}} v_{\varrho}\otimes e_{\alpha}w_{\varrho}^*\otimes\omega^{\alpha}$$

and
$$A'' = \sum_{\begin{cases}\varrho\\ \alpha\in\Delta_+(\pi,\mathfrak{v})\cap\Delta_-\end{cases}} v_{\varrho}\otimes e_{\alpha}w_{\varrho}^*\otimes\omega^{\alpha} = -\sum_{\begin{cases}\varrho\\ \alpha\in\Delta_+(\pi,\mathfrak{v})\cap\Delta_-\end{cases}} v_{\varrho}\otimes e_{\alpha}w_{\varrho}^*\otimes\bar{\omega}^{-\alpha};$$

then A' is of type (1,0) and involves the forms ω^{α} with $\alpha\in\Delta_+-\Phi$, $(\lambda,\alpha)>0$; and A'' is of type (0,1) and involves the $\bar{\omega}^{\beta}$ with $\beta\in\Delta_+-\Phi$, $(\lambda,\beta)<0$. We claim that $A'\wedge{}^t\bar{A}'' = 0 = A''\wedge{}^t\bar{A}'$. It will suffice to show that $\langle e_{\alpha}w_{\varrho}^*, e_{-\beta}w_{\sigma}\rangle = 0$ where $\alpha\in\Delta_+\cap\Delta_+(\pi,\mathfrak{v})$, $\beta\in\Delta_-\cap\Delta_+(\pi,\mathfrak{v})$. Now $\langle e_{\alpha}w_{\varrho}^*, e_{-\beta}w_{\sigma}\rangle = -\langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]\, w_{\sigma}\rangle = -\langle w_{\varrho}^*, [e_{\alpha}, e_{-\beta}]_{\mathfrak{v}}w_{\sigma}\rangle = 0$ since α, $-\beta\in\Delta_+-\Phi$ and $[\mathfrak{n}_+, \mathfrak{n}_+]\subseteq\mathfrak{n}_+$. This gives our main results:

$(4.13)_X$ THEOREM. *The curvature* $\Theta_X(\pi) = A'\wedge{}^t\bar{A}' + A''\wedge{}^t\bar{A}''$ *where*

$$A' = \sum_{\begin{cases}\alpha\in\Delta_+-\Phi\\ (\alpha,\lambda)>0\end{cases}} v_{\varrho}\otimes e_{\alpha}w_{\varrho}^*\otimes\omega^{\alpha}$$

is a (1,0) *form with values in* Hom (S, E) *and*

$$A'' = \sum_{\begin{cases}\beta\in\Delta_+-\Phi\\ (\beta,\lambda)<0\end{cases}} v_{\sigma}\otimes e_{-\beta}w_{\sigma}^*\otimes\bar{\omega}^{\beta}$$

is a (0,1) *form with values in* Hom (S, E).

$(4.13)_D$ THEOREM. *In the noncompact case, the curvature* $\Theta_D(\pi) = P + N$ *where* $P = B'\wedge{}^t\bar{B}' - C''\wedge{}^t\bar{C}''$ *is positive,* $N = -C'\wedge{}^t\bar{C}' + B''\wedge{}^t\bar{B}''$ *is a negative term, and where*

$$B' = \sum^{*}_{\substack{\alpha\in\Delta_{\mathfrak{k}}-\Phi \\ (\lambda,\alpha)>0}} v_\varrho \otimes e_\alpha w^*_\varrho \otimes \omega^\alpha, \quad C' = \sum_{\substack{\beta\in\Delta_+-\Delta_{\mathfrak{k}} \\ (\lambda,\beta)>0}} v_\sigma \otimes e_\beta w^*_\sigma \otimes \omega^\beta,$$

$$B'' = \sum_{\substack{\alpha\in\Delta_{\mathfrak{k}}-\Phi \\ (\lambda,\alpha)<0}} v_\varrho \otimes e_{-\alpha} w^*_\varrho \otimes \bar\omega^\alpha, \quad C'' = \sum_{\substack{\beta\in\Delta_+-\Delta_{\mathfrak{k}} \\ (\lambda,\beta)<0}} v_\sigma \otimes e_{-\beta} w^*_\sigma \otimes \bar\omega^\beta.$$

We want to make some deductions from the curvature formulae. In general, if we have a holomorphic, Hermitian vector bundle $\mathbf{F} \xrightarrow{\tilde\omega} Y$ over a complex manifold Y, we choose a local unitary frame $f_1, \dots, f_r$ for $\mathbf{F}$ and write the curvature $\Theta = \sum_{\varrho,\sigma=1}^{r} \Theta^\varrho_\sigma f_\varrho \otimes f^*_\sigma$ where $\Theta^\varrho_\sigma = \sum_{i,j} \Theta^\varrho_{\sigma i\bar j}\, dz^i \wedge d\bar z^j$ is a differential form of type (1,1) on Y, and where $\Theta^\varrho_\sigma + \bar\Theta^\sigma_\varrho = 0$. For $\xi = \sum_\varrho \xi^\varrho f_\varrho$ a vector in $\mathbf{F}$ we consider the *curvature form* (cf. [1], [13]):

$$\Theta(\xi) = i \sum_{\varrho,\sigma} \Theta^\varrho_\sigma \xi^\sigma \bar\xi^\varrho, \tag{4.14}$$

which is a real (1,1) form on Y.

One way in which this form arises is as follows: Let $\hat{\mathbf{F}} = \mathbf{F} - \{\text{zero-cross-section}\}$ and let ξ be the canonical non-vanishing holomorphic section of the pull-back $\tilde\omega^*\mathbf{F} \to \hat{\mathbf{F}}$. Then the real, positive function $\varphi = (\xi, \xi)$ on $\hat{\mathbf{F}}$ defines the unit tubular neighborhood N of Y in $\mathbf{F}$ by $N = \{(y, \xi) \in \hat{\mathbf{F}}\ (y \in Y,\ \xi \in \mathbf{F}_y)$ satisfying $(\xi, \xi)_y < 1\}$. The bundle $\hat{\mathbf{F}} \to Y$ has a natural connection $T_p(\hat{\mathbf{F}}) = V_p \oplus H_p$ where the horizontal space H_p consists of those complex tangent vectors $\tau \in T_p(\hat{\mathbf{F}})$ which satisfy $\langle D\xi, \tau\rangle = 0$ in $\hat{\mathbf{F}}_p = \mathbf{F}_{\tilde\omega(p)}$. Since the tangent space $T_p(N)$ is defined by $T_p(N) = \{$all real tangent vectors τ with $\langle d\varphi, \tau\rangle = 0\}$, and since $d\varphi = (D\xi, \xi) + (\xi, D\xi)$, we see that $H_p \oplus \bar H_p \subset T_p(N)$.

Now the *E. E. Levi form* $L(\varphi)$ (cf. [1]) is defined by $L(\varphi) = i\partial\bar\partial\varphi = i\partial((\xi, D'\xi)) = i((D'\xi, D'\xi) + (\xi, D''D'\xi)) = i(D'\xi,\ D'\xi) - \Theta(\xi)$; i.e.

$$L(\varphi) = i(D'\xi, D'\xi) - \Theta(\xi) \quad \text{and} \quad L(\varphi)\,|\,H_p = -\Theta(\xi). \tag{4.15}$$

In other words, (4.15) says that, under the natural isomorphism $H_p \cong T_{\tilde\omega(p)}(Y)$, the Levi-form $L(\varphi)$ is just the negative of the curvature form (4.14). In particular, if the curvature form $\Theta(\xi)$ is everywhere non-singular, then the same will be true of $L(\varphi)$ and the signature of $L(\varphi)$ will be determined by that of $\Theta(\xi)$. If, for each non-zero $\xi \in \mathbf{F}$, we have

$$\Theta(\xi) = i \left\{ \sum_{\alpha=1}^{n-q} \omega^\alpha \wedge \bar\omega^\alpha - \sum_{\mu=n-q+1}^{n} \omega^\mu \wedge \bar\omega^\mu \right\}$$

where the $\omega^j = \sum_k A^j_k\, dz^k$ give a basis for the (1,0) forms on Y, then we will say that $\Theta(\xi)$ has *signature* q, and from [1] we have:

(4.16) If Y is compact and $\Theta(\xi)$ has everywhere signature q, then the cohomology groups $H^j(Y, O(\mathbf{F}^{(k)}))=0$ for $j \neq q$, $k > k_0$.

Here $\mathbf{F}^{(k)}$ is the kth-symmetric power of $\mathbf{F}$; in [13], a differential-geometric proof of (4.16) with a reasonably precise estimate of k_0 is given.

Returning to our case of a homogeneous bundle $\mathbf{E}_\pi \to X$ defined by an irreducible unitary representation π: $V \to \mathrm{GL}(E)$, we have:

$(4.17)_X$ THEOREM. *If the highest weight λ of π is non-singular, then the curvature form $\Theta_X(\pi)(\xi)$ is everywhere non-singular of index $\iota(\pi)$.*

Proof. From $(4.13)_X$ it will suffice to show that $\Theta_X(\xi)$ is non-singular. It will make no essential difference and will simplify notation to assume that $\iota(\pi)=0$. Referring again to $(4.13)_X$, the curvature $\Theta_X(\pi) = A \wedge {}^t\bar{A}$ where

$$A = \sum_{\begin{cases}\alpha \in \Delta_+ - \Phi \\ \varrho\end{cases}} v_\varrho \otimes e_\alpha w_\varrho^* \otimes \omega^\alpha$$

belongs to $\mathrm{Hom}(S, E) \otimes \mathfrak{l}_+^*$. Then

$$\Theta_X(\pi)(\xi) = i \sum_{\alpha, \beta, \varrho, \sigma} \langle e_\alpha w_\varrho^*, e_{-\beta} w_\sigma \rangle \bar{\xi}^\varrho \xi^\sigma \omega^\alpha \wedge \bar{\omega}^\beta = i \sum_{\alpha, \beta} (e_{-\beta}\xi, e_{-\alpha}\xi)\, \omega^\alpha \wedge \bar{\omega}^\beta$$

where $\xi = \sum_\varrho \xi^\varrho w_\varrho$ lies in E. Thus, if $\bar{\eta} = \sum_{\alpha \in \Delta_+ - \Phi} \bar{\eta}_\alpha e_\alpha$ is a (1,0) tangent vector, $\langle \Theta(\xi), \bar{\eta} \wedge \eta \rangle = \| \eta \xi \|^2$ where $\eta = \sum_{\alpha \in \Delta_+ - \Phi} \eta_\alpha e_{-\alpha}$ lies in $\mathfrak{n}_-$ and $\eta\xi \in S$. To prove (4.17) we must show:

$$(4.18) \qquad \eta : E \to S \text{ is injective for all } \eta \in \mathfrak{n}_-,\ \eta \neq 0.$$

For a subset $\Psi \subset \Phi$, we let $\langle \Psi \rangle = \sum_{\alpha \in \Psi} \alpha$. The highest weight λ of $\pi: V \to \mathrm{GL}(E)$ satisfies $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta_+$, and all weights of π have the form $\lambda - \langle \Psi \rangle$. Thus E has a weight space decomposition $E = \sum_\Psi W_{\lambda - \langle \Psi \rangle}$ where $h\,w = \langle \lambda - \langle \Psi \rangle, h \rangle\, w$ for all $w \in W_{\lambda - \langle \Psi \rangle}$. Clearly $e_{-\alpha}: W_{\lambda - \langle \Psi \rangle} \to W_{\lambda - \langle \Psi + \alpha \rangle}$, and we claim that (4.14) will be proved if we can show the following special case:

$$(4.19) \qquad e_{-\alpha}: W_{\lambda - \langle \Psi \rangle} \to W_{\lambda - \langle \Psi + \alpha \rangle} \text{ is injective for } \alpha \in \Delta_+ - \Phi.$$

Proof of (4.18) *from* (4.19). Let $\alpha_1, \dots, \alpha_l$ be a set of positive, simple roots for $\mathfrak{g}$. Every linear form $\xi \in \Lambda$ has a unique expression $\xi = \sum_{i=1}^l \xi^i \alpha_i$ where the ξ^i are rational numbers, and we say that $\xi > 0$ if the first non-zero ξ^i is positive. In this manner we give Λ the usual lexicographic ordering.

Let $w = \sum w_{\lambda-\langle\Psi\rangle}$ be a non-zero vector in E and $\eta = \sum_{\alpha\in\Delta_+-\Phi}\eta_\alpha e_{-\alpha}$ lie in $\mathfrak{n}_-$. We let $\langle\Psi_0\rangle$ be the largest linear form such that $w_{\lambda-\langle\Psi_0\rangle} \neq 0$ and α_0 the largest root with $\eta_{\alpha_0} \neq 0$. Then

$$\eta w = \eta_{\alpha_0} e_{-\alpha_0} w_{\lambda-\langle\Psi_0\rangle} + \sum_{\left\{\substack{\alpha\neq\alpha_0\\ \Psi\neq\Psi_0}\right.} \eta_\alpha e_{-\alpha} w_{\lambda-\langle\Psi\rangle}.$$

Since $\alpha_0 + \langle\Psi_0\rangle > \alpha + \langle\Psi\rangle$ for all other α, Ψ, $\lambda - \langle\Psi_0 + \alpha_0\rangle < \lambda - \langle\Psi + \alpha\rangle$ and so $\eta w \neq 0$ if $e_{-\alpha_0} w_{\lambda-\langle\Psi_0\rangle} \neq 0$.

Thus it will suffice to prove (4.19). Now $(e_{-\alpha} w_{\lambda-\langle\Psi\rangle}, e_{-\alpha} w_{\lambda-\langle\Psi\rangle}) = (w_{\lambda-\langle\Psi\rangle}, e_\alpha e_{-\alpha} w_{\lambda-\langle\Psi\rangle}) = (w_{\lambda-\langle\Psi\rangle}, [e_\alpha, e_{-\alpha}] w_{\lambda-\langle\Psi\rangle})$ (by (4.8)) $= (\lambda - \langle\Psi\rangle, \alpha)\,\|w_{\lambda-\langle\Psi\rangle}\|^2$, so that (4.19) will follow from:

(4.20) $\quad (\alpha, \xi) > 0$ for $\alpha \in \Delta_+ - \Phi$ and ξ a weight of the V-module E.

For example, if E is one-dimensional (i.e. $\mathbf{E}_\pi$ is a line bundle), then the only weight is λ and (4.20) is clear.

In general, we distinguish cases:

Case 1. Suppose that $\alpha \in \Delta_+ - \Phi$ is a simple root and that $\beta_1, \ldots, \beta_s \in \Phi$ are the positive simple roots of V. Then $\langle\Psi\rangle = \sum_{j=1}^s n_j\beta_j$, $n_j \geqslant 0$, and $(\alpha, \beta_j) \leqslant 0$. Consequently $(\alpha, \lambda - \langle\Psi\rangle) \geqslant (\alpha, \lambda) > 0$ since λ is non-singular.

Case 2. Suppose that $\alpha \in \Delta_+ - \Phi$ is arbitrary and $(\lambda - \langle\Psi\rangle, \beta) \geqslant 0$ for all $\beta \in \Phi$. Then $\alpha = \sum_{i=1}^{l-s} m_i\alpha_i + \sum_{j=1}^s n_j\beta_j$ where $\alpha_1, \ldots, \alpha_{l-s} \in \Delta_+ - \Phi$ are simple roots, all $m_i \geqslant 0$, $n_j \geqslant 0$, and some $m_i > 0$. Now

$$(\lambda - \langle\Psi\rangle, \alpha) \geqslant \sum_i m_i(\lambda - \langle\Psi\rangle, \alpha_i) > 0$$

by Case 1.

Case 3. Let $\alpha \in \Delta_+ - \Phi$ and $\mu = \lambda - \langle\Psi\rangle$ be an arbitrary weight of E. The Weyl group $W(V)$ of V acts simply transitively on the Weyl chambers of V, and so we can find $w \in W(V)$ with $(w\cdot\mu, \beta) \geqslant 0$ for all $\beta \in \Phi$. Since $\mathfrak{n}_+$ is Ad-V invariant, $w\cdot\alpha \in \Delta_+ - \Phi$ and $(\mu, \alpha) = (w\cdot\mu, w\cdot\alpha) > 0$ by Case 2.

In the noncompact case, we let $\pi\colon V \to \mathrm{GL}(E)$ be an irreducible representation and define:

$$\alpha(\pi) = \#\{\alpha \in \Delta_+ - \Delta_{\mathfrak{k}} \mid (\lambda, \alpha) > 0\} + \#\{\beta \in \Delta_{\mathfrak{k}} - \Phi \mid (\lambda, \beta) < 0\},$$

where $\#\{\ldots\}$ is the number of elements in $\{\ldots\}$.

$(4.17)_D$ THEOREM. *If the highest weight λ of π is non-singular, then the curvature form $\Theta_D(\pi)(\xi)$ is non-singular and has index $\alpha(\pi)$.*

We conclude with some applications of Theorems $(4.17)_X$ and $(4.17)_D$. First, if $\pi: V \to \mathrm{GL}(E)$ is an irreducible representation with highest weight λ, then the symmetric product $E^{(k)}$ contains the irreducible V-module E_k with highest weight $k\lambda$; we denote this representation by π_k: $V \to \mathrm{GL}(E_k)$. From (4.16) we have:

$$(4.21)_X \qquad H^j(X, \mathcal{O}(\mathbf{E}_{\pi_k})) = 0 \quad \text{for} \quad j \neq \iota(\pi), k \geqslant k_0(\lambda);$$

$$(4.21)_D \qquad H^j(Y, \mathcal{O}(\mathbf{F}_{\pi_k})) = 0 \quad \text{for} \quad j \neq \alpha(\pi), k \geqslant k_0(\lambda),$$

where $Y = \Gamma \backslash D$ is a compact quotient manifold of D by a discrete group $\Gamma \subset G$ and $\mathbf{F}_k = \Gamma \backslash \mathbf{E}_{\pi_k}$.

Remark. The vanishing theorems $(4.21)_X$ and $(4.21)_D$ are fairly crude, but they do indicate a general pattern and originally suggested the more precise vanishing theorems given in §§ 6 and 7 below.

A holomorphic vector bundle $\mathbf{F} \to Y$ over a complex manifold Y is *positive* if there exists an Hermitian metric in $\mathbf{F}$ such that the curvature form $\Theta(\xi)$ (cf. (4.14)) is positive; i.e. $\Theta(\xi)$ is non-singular and has index zero. From Theorem $(4.17)_X$ it is clear that there are plenty of positive homogeneous bundles $\mathbf{E}_\pi \to X$. The noncompact case is quite different:
(4.22) Let $Y = \Gamma \backslash D$ be a compact quotient manifold of D. Then there exists a positive homogeneous bundle $\mathbf{F}_\pi \to Y$ if, and only if, the Riemannian symmetric space G/K is Hermitian symmetric and the fibering $G/V \to G/K$ is holomorphic.

Proof. We will see in § 7 below that, in the notation of $(4.21)_D$, $H^{\alpha(\pi)}(Y, \mathcal{O}(\mathbf{F}_{\pi_k})) \neq 0$ for $k \geqslant k_0(\lambda)$. Taking (4.16) into account, it will suffice to prove: There exists π: $V \to \mathrm{GL}(E)$ with $\alpha(\pi) = 0$ if, and only if, G/K is Hermitian symmetric and $G/V \to G/K$ is holomorphic.

If we have π with $\alpha(\pi) = 0$, then the highest weight λ satisfies $(\lambda, \beta) < 0$ for $\beta \in \Delta_+ - \Delta_{\mathfrak{k}}$ and $(\lambda, \alpha) > 0$ for $\alpha \in \Delta_{\mathfrak{k}} - \Phi$. Let $\mathfrak{l}_+ = \sum_{\beta \in \Delta_+ - \Delta_{\mathfrak{k}}} \mathfrak{g}^\beta$, $\mathfrak{l}_- = \sum_{\beta \in \Delta_+ - \Delta_{\mathfrak{k}}} \mathfrak{g}^{-\beta}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$, and we claim that $[\mathfrak{k}, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$, $[\mathfrak{k}, \mathfrak{l}_-] \subseteq \mathfrak{l}_-$. (*Proof.* Write $\mathfrak{k} = \mathfrak{v} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_+ = \sum_{\alpha \in {\mathfrak{k}} - \Phi} \mathfrak{g}^\alpha$, $\mathfrak{g}_- = \sum_{\alpha \in \Delta_{\mathfrak{k}} - \Phi} \mathfrak{g}^{-\alpha}$. Since $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{v}, \mathfrak{q}_+ \oplus \mathfrak{l}_+] \subseteq \mathfrak{q}_+ \oplus \mathfrak{l}_+$, we have $[\mathfrak{v}, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$, $[\mathfrak{q}_+, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$. Thus we need that $[\mathfrak{q}_-, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$; which is the same as $-\alpha + \beta \in \Delta_+ - \Delta_{\mathfrak{k}}$ if $\alpha \in \Delta_{\mathfrak{k}} - \Phi$, $\beta \in \Delta_+ - \Delta_{\mathfrak{k}}$, and $-\alpha + \beta$ is a root. This follows from $(\lambda, -\alpha + \beta) = -(\lambda, \alpha) + (\lambda, \beta) < 0$).

The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}_+ \oplus \mathfrak{l}_-$ $(\mathfrak{l}_- = \bar{\mathfrak{l}}_+)$ gives an invariant almost-complex structure to G/K, and the integrability condition is just $[\mathfrak{k}, \mathfrak{l}_+] \subseteq \mathfrak{l}_+$. In other words, if $\alpha(\pi) = 0$, then G/K is Hermitian symmetric and $G/V \to G/K$ is holomor phic.

The converse is easy to prove and will be omitted.

Remark. This result shows that *only under very special conditions will we be able to construct automorphic forms, relative to* Γ, *on* D.

As a final application, we consider the *canonical bundle* $\mathbf{K}\to X$.

(4.23) $\mathbf{K}\to X$ is negative and $\mathbf{K}\to D$ has signature $q=\dim_{\mathbf{C}} K/V$

More precisely, the curvature form $\Theta_D(\mathbf{K})(\xi)$ is positive on the horizontal space $\mathbf{T}_h(D)$ and negative on the vertical space $\mathbf{T}_v(D)$ (cf. (2.2)).

Proof. $\mathbf{K}$ is the homogeneous line bundle obtained from the representation $\Lambda^m\,\mathrm{Ad}: V\to \mathrm{GL}(\Lambda^m\mathfrak{n}_+^*)$ where $m=\dim X=\#\{\text{roots } \alpha\in\Delta_+-\Phi\}$. Thus $\mathbf{K}$ is given by a character $\lambda\in\Lambda\subset\mathfrak{h}^*$, and, by $(4.4)_X$ and $(4.4)_D$, the curvatures are:

$$(4.24)_X \qquad \Theta_X(\mathbf{K})=\sum_{\alpha\in\Delta_+-\Phi}(\lambda,\alpha)\,\omega^\alpha\wedge\bar{\omega}^\alpha;$$

$$(4.24)_D \qquad \Theta_D(\mathbf{K})=-\sum_{\beta\in\Delta_+-\Delta_{\mathfrak{k}}}(\lambda,\beta)\,\omega^\beta\wedge\bar{\omega}^\beta+\sum_{\alpha\in\Delta_{\mathfrak{k}}-\Phi}(\lambda,\alpha)\,\omega^\alpha\wedge\bar{\omega}^\alpha.$$

In the case at hand, $\lambda=-(\sum_{\alpha\in\Delta_+-\Phi}\alpha)$ and to prove (4.23) we need to show that $(\lambda,\beta)<0$ for all $\beta\in\Delta_+-\Phi$. This is due to Borel–Hirzebruch (cf. [6], p. 512) and goes as follows: Let $\gamma\in\Delta_+-\Phi$. Then $(\gamma,\beta)\geqslant 0$ or $(\gamma,\beta)<0$ and $\gamma,\gamma+\beta,\dots,\gamma+k\beta$ is a string of roots in $\Delta_+-\Phi$ where $k=-2(\gamma,\beta)/(\beta,\beta)$. Now $(\beta,\gamma+k\beta)=-(\beta,\gamma)>0$ and

$$(\beta,\gamma+\gamma+\beta+\dots+\gamma+k\beta)=(k+1)(\beta,\gamma)+\frac{k(k+1)}{2}(\beta,\beta)=0.$$

From this it follows that

$$\sum_{\begin{cases}\alpha\in\Delta_+-\Phi\\ \alpha\neq\beta\end{cases}}(\alpha,\beta)\geqslant 0 \text{ so that } -(\lambda,\beta)=\sum_{\begin{cases}\alpha\in\Delta_+-\Phi\\ \alpha\neq\beta\end{cases}}(\alpha,\beta)+(\beta,\beta)>0.$$

5. Computation of the Laplace–Beltrami operator

Continuing with the notation of §§ 1–3, we consider a Kähler C-space $X=G_{\mathbf{C}}/B$ where B is a Borel subgroup; in this case, the set Φ of (3.7) is empty. Therefore, if G is a real form of $G_{\mathbf{C}}$ such that $G\cap B$ is compact, $\mathfrak{v}_0=\mathfrak{g}_0\cap\mathfrak{b}$ coincides with $\mathfrak{h}_0$, and in order to emphasize this fact, we shall now refer to $G\cap B$ as H. Let then $D=G/H$ be dual to X. We do not exclude the case $G=M$, i.e. G may be compact and D may coincide with X. For a compact group G, the computation below turns out to be similar to Kostant's computation of a certain algebraic Laplace operator in [21].

We keep fixed a homogeneous holomorphic line bundle $\mathbf{E}_\lambda \to D$ determined by a character λ of H; λ will tacitly be identified with its differential, which is a weight in $\mathfrak{h}^*_{\mathbf{R}}$. We recall the basis $\{e_{-\alpha} \mid \alpha \in \Delta_+\}$ of $\mathfrak{n}_-$ from § 3, and we denote the elements of the dual basis by $\omega^{-\alpha}$. If $A=(\alpha_1, \ldots, \alpha_k)$ is an ordered k-tuple of positive roots, we use the abbreviated notation ω^{-A} for the exterior product $\omega^{-\alpha_1} \wedge \ldots \wedge \omega^{-\alpha_k}$; $|A|$ will stand for $\alpha_1 + \ldots + \alpha_k$. According to remarks in § 1 and § 2, we may think of $A^k(\mathbf{E}_\lambda)$, the space of $(0, k)$-forms with values in $\mathbf{E}_\lambda$, as the subspace of $C^\infty(G) \otimes \Lambda^k \mathfrak{n}^*_-$ spanned by monomials $f\omega^{-A}$ with $f \in C^\infty(G)$ satisfying

$$xf = -\langle |A| + \lambda, h \rangle f \quad \text{for all } x \in \mathfrak{h}. \tag{5.1}$$

The equations

$$\begin{aligned} \hat{\partial}(f\omega^{-A}) &= \sum_\alpha e_{-\alpha} f \omega^{-\alpha} \wedge \omega^{-A} \\ T(f\omega^{-A}) &= \tfrac{1}{2} \sum_\alpha f \omega^{-\alpha} \wedge e_{-\alpha} \omega^{-A} \end{aligned} \tag{5.2}$$

define operators $\hat{\partial}$, T: $A^k(\mathbf{E}_\lambda) \to A^{k+1}(\mathbf{E}_\lambda)$. In these summations, α runs over Δ_+; $e_{-\alpha}$ acts on f as a left-invariant complex tangent vector field on G, and on $\omega^{-A} \in \Lambda \mathfrak{n}^*_-$ by the action contragredient to that on $\mathfrak{n}_-$. Then $\bar{\partial} = \hat{\partial} + T$, as can be read off from (1.6).

The inner product on $\mathfrak{n}^*_-$ described by $(\omega^{-\alpha}, \omega^{-\beta}) = \delta^{\alpha\beta}$ is invariant under the adjoint action of H, and thus gives rise to a G-invariant Hermitian metric on D. The line bundle $\mathbf{E}_\lambda$ has an essentially unique G-invariant metric. With respect to these choices of metrics, $\bar{\partial}$, $\hat{\partial}$, and T have formal adjoints $\bar{\partial}^*$, $\hat{\partial}^*$, T^*, and

$$\Box = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = (\hat{\partial} + T)(\hat{\partial}^* + T^*) + (\hat{\partial}^* + T^*)(\hat{\partial} + T) \tag{5.3}$$

is the Laplace–Beltrami operator.

Since $\sigma(e_{-\alpha}) = \varepsilon_\alpha e_\alpha$ is the complex conjugate of $e_{-\alpha}$ relative to $\mathfrak{g}_0$, the formal adjoint of $e_{-\alpha}$ acting on $C^\infty(G)$ is $-\varepsilon_\alpha e_\alpha$. We embed $\mathfrak{n}^*_-$ in $\mathfrak{g}^*$ by letting $\mathfrak{n}^*_-$ act trivially on $\mathfrak{h}$ and $\sigma(\mathfrak{n}_-)$. Then $\mathfrak{n}^*_-$ becomes a $\sigma(\mathfrak{n}_-)$-invariant subspace; in fact, for $\alpha, \beta, \gamma \in \Delta_+$,

$$\langle e_\alpha \omega^{-\beta}, e_{-\gamma} \rangle = -\langle \omega^{-\beta}, [e_\alpha, e_{-\gamma}] \rangle = -N_{\alpha,-\gamma}\, \delta^\beta_{\gamma-\alpha} = -N_{\alpha,-\alpha-\beta}\, \delta^{\alpha+\beta}_\gamma = N_{\alpha,\beta} \langle \omega^{-\alpha-\beta}, e_{-\gamma} \rangle,$$

here we have used (3.6 d). Thus $\mathfrak{n}^*_-$ becomes also a $\sigma(\mathfrak{n}_-)$-module, and

$$e_\alpha \omega^{-\beta} = N_{\alpha,\beta}\, \omega^{-\alpha-\beta} \quad \alpha, \beta \in \Delta_+. \tag{5.4}$$

Similarly one finds that the action of $e_{-\alpha}$ on $\mathfrak{n}^*_-$ is given by

$$e_{-\alpha}\omega^{-\beta}=\begin{cases}N_{-\alpha,\beta}\,\omega^{\alpha-\beta} & \text{if } \beta-\alpha\in\Delta_+\\ 0 & \text{otherwise}\end{cases}. \tag{5.5}$$

Together with the identity $N_{\alpha,\beta}=-N_{-\alpha,\alpha+\beta}$, (5.4) and (5.5) imply that the adjoint of $e_{-\alpha}$ acting on $\mathfrak{n}_-^*$ is e_α.

We let $e(\omega^{-\alpha})$: $\Lambda\mathfrak{n}_-^*\to\Lambda\mathfrak{n}_-^*$ denote exterior multiplication by $\omega^{-\alpha}$, and define $i(\omega^{-\alpha})$ as the adjoint operation. Explicitly,

$$\begin{aligned} i(\omega^{-\alpha})\,\omega^{-A} &= 0\\ i(\omega^{-\alpha})\,(\omega^{-\alpha}\wedge\omega^{-A}) &= \omega^{-A}, \end{aligned} \tag{5.6}$$

provided the multi-index A does not involve α. In terms of these maps,

$$\begin{aligned} \hat\partial^*(f\omega^{-A}) &= \sum_\alpha \varepsilon_\alpha e_\alpha f\, i(\omega^{-\alpha})\,\omega^{-A}\\ T^*(f\omega^{-A}) &= \tfrac12\sum_\alpha f e_\alpha i(\omega^{-\alpha})\,\omega^{-A}=\tfrac12\sum_\alpha f\, i(\omega^{-\alpha})\,e_\alpha\omega^{-A}. \end{aligned} \tag{5.7}$$

Again, the summations extend over all $\alpha\in\Delta_+$.

For $\alpha,\beta\in\Delta_+$, one gets the identities

$$\begin{aligned} &\text{a)}\quad e(\omega^{-\alpha})\,i(\omega^{-\beta})+i(\omega^{-\beta})\,e(\omega^{-\alpha})=\delta^{\alpha\beta}\\ &\text{b)}\quad e_\alpha\omega^{-A}=\sum_\beta e(e_\alpha\omega^{-\beta})\,i(\omega^{-\beta})\,\omega^{-A}\\ &\text{c)}\quad e_\alpha e(\omega^{-\beta})=e(e_\alpha\omega^{-\beta})+e(\omega^{-\beta})\,e_\alpha\\ &\text{d)}\quad e_\alpha i(\omega^{-\beta})=i(\omega^{-\beta})\,e_\alpha-i(e_{-\alpha}\omega^{-\beta}) \end{aligned} \tag{5.8}$$

which can be deduced from (5.6); in b), c), d), α may also be a negative root.

Let us compute $\hat\partial\hat\partial^*+\hat\partial^*\hat\partial$. For $f\omega^{-A}\in A^k(\mathbf{E}_\lambda)$,

$$\begin{aligned} (\hat\partial\hat\partial^*+\hat\partial^*\hat\partial)\,(f\omega^{-A}) &= -\sum_{\alpha,\beta}\varepsilon_\alpha e_\alpha e_{-\beta} f\, i(\omega^{-\alpha})\,e(\omega^{-\beta})\,\omega^{-A}-\sum_{\alpha,\beta}\varepsilon_\alpha e_{-\beta}e_\alpha f\, e(\omega^{-\beta})\,i(\omega^{-\alpha})\,\omega^{-A}\\ &= -\sum_{\alpha,\beta}\varepsilon_\alpha e_\alpha e_{-\beta} f\big(e(\omega^{-\beta})\,i(\omega^{-\alpha})+i(\omega^{-\alpha})\,e(\omega^{-\beta})\big)\,\omega^{-A}-\sum_{\alpha,\beta}\varepsilon_\alpha[e_{-\beta},e_\alpha]f\,e(\omega^{-\beta})\,i(\omega^{-\alpha})\,\omega^{-A}\\ &= \sum_\alpha\big(-\varepsilon_\alpha e_\alpha e_{-\alpha} f\omega^{-A}+\varepsilon_\alpha h_\alpha f\, e(\omega^{-\alpha})\,i(\omega^{-\alpha})\,\omega^{-A}\big)-\sum_{\alpha\neq\beta}\varepsilon_\alpha N_{-\beta,\alpha}e_{\alpha-\beta} f\, e(\omega^{-\beta})\,i(\omega^{-\alpha})\,\omega^{-A}. \end{aligned}$$

We split up the last term on the right according to the two possibilities $\alpha>\beta$ and $\beta>\alpha$, where $>$ is a linear ordering of $\mathfrak{h}_{\mathbf{R}}^*$ making the elements of Δ_+ positive. Whenever $\alpha-\beta$ is not a root, there is no contribution; hence, using (3.6 d), we get

$$\begin{aligned} \sum_{\alpha>\beta}\varepsilon_\alpha N_{-\beta,\alpha}e_{\alpha-\beta} f\, e(\omega^{-\beta})\,i(\omega^{-\alpha})\,\omega^{-A} &= \sum_{\alpha>\gamma}\varepsilon_\alpha N_{\gamma-\alpha,\alpha}e_\gamma f\, e(\omega^{\gamma-\alpha})\,i(\omega^{-\alpha})\,\omega^{-A}\\ &= -\sum_{\alpha>\gamma}\varepsilon_\alpha N_{-\gamma,\alpha}e_\gamma f\, e(\omega^{\gamma-\alpha})\,i(\omega^{-\alpha})\,\omega^{-A}=-\sum_{\alpha,\gamma}\varepsilon_\alpha e_\gamma f\, e(e_{-\gamma}\omega^{-\alpha})\,i(\omega^{-\alpha})\,\omega^{-A}. \end{aligned}$$

Similarly,

$$\sum_{\alpha>\beta} \varepsilon_\alpha N_{-\beta,\alpha} e_{\alpha-\beta} f e(\omega^{-\beta})\, i(\omega^{-\alpha})\, \omega^{-A} = \sum_{\alpha,\gamma} \varepsilon_\alpha N_{-\alpha-\gamma,\alpha} e_{-\gamma} f e(\omega^{-\alpha-\gamma})\, i(\omega^{-\alpha})\, \omega^{-A}$$
$$= -\sum_{\alpha,\gamma} \varepsilon_\alpha e_{-\gamma} f e(e_\gamma \omega^{-\alpha})\, i(\omega^{-\alpha})\, \omega^{-A},$$

and hence

$$(5.9)\qquad (\hat\partial\hat\partial^* + \hat\partial^*\hat\partial)(f\omega^{-A}) = \sum_\alpha (-\varepsilon_\alpha e_\alpha e_{-\alpha} f \omega^{-A} + \varepsilon_\alpha h_\alpha f e(\omega^{-\alpha})\, i(\omega^{-\alpha})\, \omega^{-A})$$
$$+ \sum_{\alpha,\gamma} \varepsilon_\alpha e_\gamma f e(e_{-\gamma}\omega^{-\alpha})\, i(\omega^{-\alpha})\, \omega^{-A} + \sum_{\alpha,\gamma} \varepsilon_\alpha e_{-\gamma} f e(e_\gamma \omega^{-\alpha})\, i(\omega^{-\alpha})\, \omega^{-A}.$$

Using the identities (5.8), one finds that

$$(\hat\partial T^* + T^*\hat\partial + \hat\partial^* T + T\hat\partial^*)(f\omega^{-A}) = \tfrac{1}{2}\sum_{\alpha,\beta} e_{-\alpha} f e(\omega^{-\alpha})\, i(\omega^{-\beta})\, e_\beta \omega^{-A}$$
$$+ \tfrac{1}{2}\sum_{\alpha,\beta} e_{-\alpha} f i(\omega^{-\beta})\, e_\beta e(\omega^{-\alpha})\, \omega^{-A} - \tfrac{1}{2}\sum_{\alpha,\beta} \varepsilon_\alpha e_\alpha f i(\omega^{-\alpha})\, e(\omega^{-\beta})\, e_{-\beta}\omega^{-A}$$
$$- \tfrac{1}{2}\sum_{\alpha,\beta} \varepsilon_\alpha e_\alpha f e(\omega^{-\beta})\, e_{-\beta}\, i(\omega^{-\alpha})\, \omega^{-A} = \tfrac{1}{2}\sum_{\alpha,\beta} e_{-\alpha} f i(\omega^{-\beta})\, e(e_\beta\omega^{-\alpha})\, \omega^{-A}$$
$$+ \tfrac{1}{2}\sum_{\alpha,\beta} \varepsilon_\alpha e_\alpha f e(\omega^{-\beta})\, i(e_\beta\omega^{-\alpha})\, \omega^{-A} + \tfrac{1}{2}\sum_\alpha (e_{-\alpha} f e_\alpha \omega^{-A} - \varepsilon_\alpha e_\alpha f e_{-\alpha}\omega^{-A}).$$

Since $e_\beta\omega^{-\alpha} = -e_\alpha\omega^{-\beta}$ (cf. (5.4)), and in view of (5.8a), the expression above equals

$$\tfrac{1}{2}\sum_\alpha (e_{-\alpha} f e_\alpha \omega^{-A} - \varepsilon_\alpha e_\alpha f e_{-\alpha}\omega^{-A}) + \tfrac{1}{2}\sum_{\alpha,\beta} e_{-\alpha} f e(e_\alpha\omega^{-\beta})\, i(\omega^{-\beta})\, \omega^{-A}$$
$$- \tfrac{1}{2}\sum_{\alpha,\beta} \varepsilon_\alpha e_\alpha f e(\omega^{-\beta})\, i(e_\alpha\omega^{-\beta})\, \omega^{-A}.$$

Now $\omega^{-A} \to \sum_\beta e(\omega^{-\beta})\, i(e_\alpha\omega^{-\beta})\, \omega^{-A}$ is dual to the mapping $\omega^{-A} \to \sum_\beta e(e_\alpha\omega^{-\beta})\, i(\omega^{-\beta})\, \omega^{-A} = e_\alpha\omega^{-A}$. Hence

$$(5.10)\qquad (\hat\partial T^* + T^*\hat\partial + \hat\partial^* T + T\hat\partial^*)(f\omega^{-A}) = \sum_\alpha (e_{-\alpha} f e_\alpha \omega^{-A} - \varepsilon_\alpha e_\alpha f e_{-\alpha}\omega^{-A}).$$

In order to complete the computation of $\square$, we must attack $TT^* + T^*T$. First of all, we observe that T and T^* ought to be regarded as endomorphisms of $\wedge\mathfrak{n}_-^*$. If α is a positive root,

$$T^* e(\omega^{-\alpha}) = \tfrac{1}{2}\sum_\beta e_\beta i(\omega^{-\beta})\, e(\omega^{-\alpha}) = \tfrac{1}{2} e_\alpha - \tfrac{1}{2}\sum_\beta e_\beta e(\omega^{-\alpha})\, i(\omega^{-\beta})$$
$$(5.11)\qquad = \tfrac{1}{2} e_\alpha - \tfrac{1}{2}\sum_\beta e(e_\beta\omega^{-\alpha})\, i(\omega^{-\beta}) - e(w^{-\alpha})\, T^*$$
$$= \tfrac{1}{2} e_\alpha + \tfrac{1}{2}\sum_\beta e(e_\alpha\omega^{-\beta})\, i(\omega^{-\beta}) - e(\omega^{-\alpha})\, T^* = e_\alpha - e(\omega^{-\alpha})\, T^*.$$

Similarly,

$$T e(\omega^{-\alpha}) = \tfrac{1}{2}\sum_{\beta} e(\omega^{-\beta})\, e_{-\beta} e(\omega^{-\alpha}) = \tfrac{1}{2}\sum_{\beta<\alpha} N_{-\beta,\alpha} e(\omega^{-\beta})\, e(\omega^{\beta-\alpha}) - e(\omega^{-\alpha})\, T. \tag{5.12}$$

Repeated application of (5.11) and (5.12) yields

$$\begin{aligned}(TT^* + T^*T)\, e(\omega^{-\alpha}) &= \tfrac{1}{2}\sum_{\beta<\alpha} N_{-\beta,\alpha} T^* e(\omega^{-\beta})\, e(\omega^{\beta-\alpha}) - T^* e(\omega^{-\alpha})\, T + T e_\alpha - T e(\omega^{-\alpha})\, T^* \\ &= e(\omega^{-\alpha})\,(TT^* + T^*T) + T e_\alpha - e_\alpha T + \tfrac{1}{2}\sum_{\beta<\alpha} N_{-\beta,\alpha}(T^* e(\omega^{-\beta})\, e(\omega^{\beta-\alpha}) - e(\omega^{-\beta})\, e(\omega^{\beta-\alpha}) T^*) \\ &= e(\omega^{-\alpha})\,(TT^* + T^*T) + T e_\alpha - e_\alpha T + \tfrac{1}{2}\sum_{\beta<\alpha} N_{-\beta,\alpha}(e_\beta e(\omega^{\beta-\alpha}) - e(\omega^{-\beta})\, e_{\alpha-\beta}).\end{aligned} \tag{5.13}$$

Replacing $\alpha-\beta$ by β in the second half of the last term and noticing that $N_{\beta-\alpha,\alpha} = -N_{-\beta,\alpha}$, one obtains

$$\begin{aligned}\sum_{\beta<\alpha} N_{-\beta,\alpha}(e_\beta e(\omega^{\beta-\alpha}) - e(\omega^{-\beta})\, e_{\alpha-\beta}) &= \sum_{\beta}(e(e_\beta e_{-\beta}\omega^{-\alpha}) + 2\, e(e_{-\beta}\omega^{-\alpha})\, e_\beta) \\ &= (2\varrho - \alpha, \alpha)\, e(\omega^{-\alpha}) + 2\sum_{\beta} e(e_{-\beta}\omega^{-\alpha})\, e_\beta.\end{aligned} \tag{5.14}$$

In the last step, Lemma 3.1 was used; ϱ is one-half of the sum of the positive roots.

If α, β, γ are positive roots,

$$(e_{-\beta}e_\alpha - e_\alpha e_{-\beta})\,\omega^{-\gamma} - [e_{-\beta}, e_\alpha]\,\omega^{-\gamma} = \begin{cases} 0 & \text{if } \gamma > \beta \\ (\gamma, \alpha)\,\omega^{-\alpha} & \text{if } \gamma = \beta \\ N_{\alpha,\gamma-\beta} N_{-\beta,\gamma}\,\omega^{\beta-\gamma-\alpha} & \text{if } \gamma < \beta. \end{cases}$$

One can verify this by observing that the isomorphism

$$\varphi : \mathfrak{n}_-^* \to \sigma(\mathfrak{n}_-) \subset \mathfrak{g}$$

determined by $\varphi(\omega^{-\alpha}) = e_\alpha$ commutes with the action of $\sigma(\mathfrak{n}_-)$, and $e_{-\beta}\varphi(\omega^{-\alpha}) = \varphi(e_{-\beta}\omega^{-\alpha})$ when $\alpha > \beta$. Hence

$$\begin{aligned}T e_\alpha - e_\alpha T &= \tfrac{1}{2}\sum_{\beta}(e(\omega^{-\beta})\, e_{-\beta} e_\alpha - e_\alpha e(\omega^{-\beta})\, e_{-\beta}) \\ &= -\tfrac{1}{2}\sum_{\beta} e(e_\alpha\omega^{-\beta})\, e_{-\beta} + \tfrac{1}{2}\sum_{\beta} e(\omega^{-\beta})\,(e_{-\beta}e_\alpha - e_\alpha e_{-\beta}) \\ &= \tfrac{1}{2}\sum_{\beta} e(e_\beta\omega^{-\alpha})\, e_{-\beta} + \tfrac{1}{2}\sum_{\beta,\gamma} e(\omega^{-\beta})\, e((e_{-\beta}e_\alpha - e_\alpha e_{-\beta})\,\omega^{-\gamma})\, i(\omega^{-\gamma})) \\ &= \tfrac{1}{2}\sum_{\beta} e(e_\beta\omega^{-\alpha})\, e_{-\beta} + \tfrac{1}{2}\sum_{\beta<\alpha,\gamma} N_{-\beta,\alpha} e(\omega^{-\beta})\, e(e_{\alpha-\beta}\omega^{-\gamma})\, i(\omega^{-\gamma}) \\ &\quad + \tfrac{1}{2}\sum_{\gamma+\alpha>\beta>\alpha} N_{-\beta,\alpha} e(\omega^{-\beta})\, e(e_{\alpha-\beta}\omega^{-\gamma})\, i(\omega^{-\gamma}) \\ &\quad + \tfrac{1}{2}\sum_{\gamma+\alpha>\beta>\gamma} N_{\alpha,\gamma-\beta} N_{-\beta,\gamma}\, e(\omega^{-\beta})\, e(\omega^{\beta-\gamma-\alpha})\, i(\omega^{-\gamma}) \\ &\quad - \tfrac{1}{2}\sum_{\gamma}(\alpha,\gamma)\, e(\omega^{-\alpha})\, e(\omega^{-\gamma})\, i(\omega^{-\gamma}) + \tfrac{1}{2}\sum_{\beta}(\alpha,\beta)\, e(\omega^{-\beta})\, e(\omega^{-\alpha})\, i(\omega^{-\beta}).\end{aligned} \tag{5.15}$$

We transform the second, third, and fourth terms on the right of (5.15) by changing the indices of summation and applying (3.6 d):

$$\sum_{\beta<\alpha,\gamma} N_{-\beta,\alpha}\, e(\omega^{-\beta})\, e(e_{\alpha-\beta}\,\omega^{-\gamma})\, i(\omega^{-\gamma}) = \sum_{\beta<\alpha,\gamma} N_{\beta-\alpha,\alpha}\, e(\omega^{\beta-\alpha})\, e(e_{\beta}\,\omega^{-\gamma})\, i(\omega^{-\gamma})$$

(5.16 a)

$$= -\sum_{\beta} e(e_{-\beta}\,\omega^{-\alpha})\, e_{\beta}$$

$$\sum_{\gamma+\alpha>\beta>\alpha} N_{-\beta,\alpha}\, e(\omega^{-\beta})\, e(e_{\alpha-\beta}\,\omega^{-\gamma})\, i(\omega^{-\gamma}) = \sum_{\gamma>\beta} N_{-\alpha-\beta,\alpha}\, e(\omega^{-\alpha-\beta})\, e(e_{-\beta}\,\omega^{-\gamma})\, i(\omega^{-\gamma})$$

(5.16 b)

$$= -\sum_{\beta} e(e_{\beta}\,\omega^{-\alpha})\, e_{-\beta}$$

$$\sum_{\gamma+\alpha>\beta>\gamma} N_{\alpha,\gamma-\beta}\, N_{-\beta,\gamma}\, e(\omega^{-\beta})\, e(\omega^{\beta-\alpha-\gamma})\, i(\omega^{-\gamma})$$

(5.16 c)

$$= -\sum_{\alpha>\beta,\gamma} N_{\alpha,-\beta}\, N_{-\beta-\gamma,\gamma}\, e(\omega^{\beta-\alpha})\, e(\omega^{-\beta-\gamma})\, i(\omega^{-\gamma}) = -\sum_{\beta} e(e_{-\beta}\,\omega^{-\alpha})\, e_{\beta}.$$

Together, (5.13)–(5.16) lead to the identity

$$(TT^* + T^*T)\, e(\omega^{-\alpha}) = e(\omega^{-\alpha})\, (TT^* + T^*T) + \tfrac{1}{2}\,(2\varrho-\alpha,\alpha) - \sum_{\beta}(\alpha,\beta)\, e(\omega^{-\alpha})\, e(\omega^{-\beta})\, i(\omega^{-\beta}).$$

For every k-tuple of positive roots A,

$$\sum_{\beta}(\alpha,\beta)\, e(\omega^{-\beta})\, i(\omega^{-\beta})\, \omega^{-A} = (\alpha, |A|)\, \omega^{-A}.$$

Thus, by induction on k,

$$(T^*T + TT^*)\,\omega^{-A} = \tfrac{1}{2}\,(|A|, 2\varrho - |A|)\,\omega^{-A}. \tag{5.17}$$

The computations of this section can be summed up as follows:

PROPOSITION 5.1. *For every monomial $f\omega^{-A} \in A^k(\mathbf{E}_\gamma)$,*

$$\Box\,(f\omega^{-A}) = \sum_{\alpha} \varepsilon_{\alpha}\,(h_{\alpha} f\, e(\omega^{-\alpha})\, i(\omega^{-\alpha})\,\omega^{-A} - e_{\alpha} e_{-\alpha} f\,\omega^{-A}) + \sum_{\alpha,\beta}(\varepsilon_{\beta}-\varepsilon_{\alpha})\, e_{\alpha} f\, e(e_{-\alpha}\,\omega^{-\beta})\, i(\omega^{-\beta})\,\omega^{-A}$$
$$+ \sum_{\alpha,\beta}(\varepsilon_{\beta}+1)\, e_{-\alpha} f\, e(e_{\alpha}\,\omega^{-\beta})\, i(\omega^{-\beta})\,\omega^{-A} + \tfrac{1}{2}\,(|A|, 2\varrho-|A|)\, f\omega^{-A}.$$

For the preceding computations, the assumption that we were dealing with a homogeneous complex manifold $D = G/H$, where H is a torus, was not really crucial, and similar expressions for $\Box$ can be deduced more generally for manifolds $D = G/V$, when V is not necessarily a torus. In one case, the formula even simplifies considerably when one works with a non-toral isotropy group, and it is perhaps of interest to mention this special formula, although we shall not use it here.

Let $D=G/V$ be a noncompact Hermitian symmetric space, i.e. $V=K$ is a maximal compact subgroup of G and $\mathfrak{n}_-$ is an Ad K-invariant abelian subalgebra of $\mathfrak{g}$, and let $\mathbf{E}_\pi \to D$ be the homogeneous holomorphic vector bundle determined by an irreducible representation $\pi: V \to \mathrm{GL}(E)$, whose highest weight we shall denote by λ. The $\mathbf{E}_\pi$-valued $(0, k)$-forms on D can be identified with the V-invariant elements

$$\sum f_{i,A} \otimes v_i \otimes \omega^{-A} \in C^\infty(G) \otimes E \otimes \Lambda^k \mathfrak{n}_-^*,$$

when V is made to act on $C^\infty(G)$ by right translation and on E and $\Lambda^k \mathfrak{n}_-^*$ in the obvious manner. We construct the Laplace–Beltrami operator $\square$ after choosing essentially unique Hermitian metrics on D and $\mathbf{E}_\pi$. Then, as Okamoto and Ozeki [27] have shown,

$$\square\left(\sum f_{i,A} \otimes v_i \otimes \omega^{-A}\right) = \tfrac{1}{2} \sum \left((\lambda + 2\varrho, \lambda) - \Omega\right) f_{i,A}) \otimes v_i \otimes \omega^{-A}. \tag{5.18}$$

Here Ω is the Casimir operator of $\mathfrak{g}$, and ϱ is one-half of the sum of the positive roots. This formula, as well as its derivation, is identical to the formula for $\square$ in the corresponding compact case, except for a switch in sign. Its simplicity stems from two facts: $\mathfrak{n}_-$ is an abelian subalgebra of $\mathfrak{g}$, and the Casimir operator of K acts on $\Lambda^k \mathfrak{n}_-^*$ as a constant (Lemma 4.1 of [26]).

6. The generalized Borel–Weil theorem

Let $X = G_{\mathbf{C}}/B = M/V$ be a Kähler C-space, and $\mathbf{E}_\pi \to X$ be the homogeneous holomorphic vector bundle arising from some irreducible representation π of V. We choose systems of positive roots Δ_+ for $(\mathfrak{g}, \mathfrak{h})$ and Φ for $(\mathfrak{v}, \mathfrak{h})$, as described by (3.7), and we denote the highest weight of π with respect to Φ by λ; as before, ϱ is one-half of the sum of the elements of Δ_+. Since the action of M on X lifts to $\mathcal{O}(\mathbf{E}_\pi)$, the sheaf of germs of holomorphic sections of $\mathbf{E}_\pi$, the cohomology groups $H^k(X, \mathcal{O}(\mathbf{E}_\pi))$ become M-modules. These cohomology groups are the subject of the generalized Borel–Weil theorem of Bott [7]:

(6.1) THEOREM. *If $\lambda+\varrho$ is singular, i.e. $(\lambda+\varrho, \alpha)=0$ for some $\alpha \in \Delta$, then $H^k(X, \mathcal{O}(\mathbf{E}_\pi)) = 0$ for every k. If $\lambda+\varrho$ is nonsingular, let w be that element of the Weyl group which carries $\lambda+\varrho$ into the highest Weyl chamber, and l the number of $\alpha \in \Delta_+$ such that $w(\alpha)$ is negative; then $H^k(X, \mathcal{O}(\mathbf{E}_\pi))$ vanishes if $k \neq l$, and is the irreducible M-module of highest weight $w(\lambda+\varrho)-\varrho$ if $k=l$.*

Proof. First, we assume that B is a Borel subgroup of $G_{\mathbf{C}}$, as we did in § 5; M will take on the role which G played there. Then $V = M \cap B$ is a torus, and π therefore must be a one-dimensional representation. In order to be consistent with the notation of § 5, we shall refer to V as H and to π as λ. Again we identify the space of $\mathbf{E}_\lambda$-valued $(0, k)$-forms, $A^k(\mathbf{E}_\lambda)$, with a subspace of $C^\infty(M)\otimes\Lambda^k\mathfrak{n}_-^*$. It is known that the cohomology groups of the sheaf $O(\mathbf{E}_\lambda)$ can be computed from the complex $\{A^k(\mathbf{E}_\lambda),\bar\partial\}$, and that the cohomology groups of this complex in turn are isomorphic to the spaces of harmonic forms $H^k(\mathbf{E}_\lambda)=\{\psi\in A^k(\mathbf{E}_\lambda)\mid \Box\psi=0\}$ (cf. [16]). Since the Hermitian metrics used to define $\Box$ were M-invariant, the subspaces $H^k(\mathbf{E}_\lambda)\subset A^k(\mathbf{E}_\lambda)$ are preserved by M, and $H^k(\mathbf{E}_\lambda)\simeq H^k(X, O(\mathbf{E}_\lambda))$ is an isomorphism of M-modules.

We consider a particular monomial $f\omega^{-A}\in A^k(\mathbf{E}_\lambda)$, with the property that the function $f\in C^\infty(M)$ is an eigenfunction of the Casimir operator $\Omega=\sum_{\alpha>0}(e_\alpha e_{-\alpha}+e_{-\alpha}e_\alpha)+\sum h_i h_i=\sum_{\alpha>0}(2e_\alpha e_{-\alpha}-h_\alpha)+\sum h_i h_i$; here $\{h_1, h_2, \ldots, h_r\}$ is an orthonormal basis of $\mathfrak{h}$. Since $\sum_{\alpha>0}e_\alpha e_{-\alpha}$ differs from $\frac{1}{2}\Omega$ by an operator in the universal enveloping algebra of $\mathfrak{h}$, and since f satisfies (5.1), $\sum_{\alpha>0}e_\alpha e_{-\alpha}f$ is a multiple of f. Now M is compact, so $\varepsilon_\alpha=-1$ for every $\alpha\in\Delta_+$. Hence only the first and last terms on the right side of the identity in proposition 5.1 remain, and $\Box(f\omega^{-A})$ is a multiple of $f\omega^{-A}$. As a consequence of the Peter–Weyl theorem, for example, $C^\infty(M)$ decomposes discretely into eigenspaces of Ω. In the corresponding decomposition of $A^k(\mathbf{E}_\lambda)$, each subspace is $\Box$-invariant, because Ω commutes with every left-invariant differential operator on M. We conclude that $H^k(\mathbf{E}_\lambda)$ has a basis of harmonic monomials.

Since $\Box$ is the sum of the two semidefinite operators $\bar\partial\bar\partial^*$ and $\bar\partial^*\bar\partial$, a monomial $f\omega^{-A}\in A^k(\mathbf{E}_\lambda)$ is harmonic if and only if $\bar\partial(f\omega^{-A})=0=\bar\partial^*(f\omega^{-A})$. When $\bar\partial(f\omega^{-A})=\hat\partial(f\omega^{-A})+T(f\omega^{-A})$ (cf. (5.2)) is expressed as a sum of monomials, ω^{-A} can be factored out from each term of $\hat\partial(f\omega^{-A})$, and from no term of $T(f\omega^{-A})$. Thus $\bar\partial(f\omega^{-A})=0$ if and only if $\hat\partial(f\omega^{-A})=0=T(f\omega^{-A})$; similarly, $\bar\partial^*(f\omega^{-A})=0$ is equivalent to $\hat\partial^*(f\omega^{-A})=0=T^*(f\omega^{-A})$. Hence $f\omega^{-A}$ is harmonic precisely when $\hat\partial(fw^{-A})$, $\hat\partial^*(fw^{-A})$, $T(fw^{-A})$, and $T^*(f\omega^{-A})$ all vanish.

As was pointed out already, T and T^* may be viewed as endomorphisms of $\Lambda\mathfrak{n}_-^*$. Since

$$\begin{aligned}T\omega^{-A}&=\tfrac{1}{2}\sum_\beta e(\omega^{-\beta})\,e_{-\beta}\omega^{-A}=\tfrac{1}{2}\sum_{\beta,\alpha}e(\omega^{-\beta})\,e(e_{-\beta}\omega^{-\alpha})\,i(\omega^{-\alpha})\,\omega^{-A}\\&=\tfrac{1}{2}\sum_{\beta<\alpha}N_{-\beta,\alpha}\,e(\omega^{-\beta})\,e(\omega^{\beta-\alpha})\;i(\omega^{-\alpha})\,\omega^{-A}=\tfrac{1}{2}\sum_{\beta+\gamma=\alpha}N_{\beta,\gamma}\,e(\omega^{-\beta})\,e(\omega^{-\gamma})\,i(\omega^{-\alpha})\,\omega^{-A},\end{aligned}$$

$T\omega^{-A}=0$ if and only if for every pair β, $\gamma\in\Delta_+$ such that $\beta+\gamma\in\Delta$ and such that neither β nor γ occur in A, $\beta+\gamma$ also does not occur in A. Analogously, $T^*\omega^{-A}=0$ if and only if for every pair of roots β, γ belonging to A and whose sum is a root, $\beta+\gamma$ occurs in A. If w is an element of the Weyl group, and if A consists precisely of the roots common to Δ_+ and $w^{-1}(-\Delta_+)$, then A has the two properties equivalent to $T\omega^{-A}=0=T^*\omega^{-A}$. Conversely, if A has these two properties, the set

$$\tilde\Delta_+=\{\alpha\in\Delta\mid -\alpha\in A\}\cup\{\alpha\in\Delta_+\mid\alpha\notin A\}$$

contains $\beta+\gamma$ whenever $\beta,\gamma\in\tilde{\Delta}_+$ and $\beta+\gamma\in\Delta$; moreover, Δ is the disjoint union of $\tilde{\Delta}_+$ and $-\tilde{\Delta}_+$. Thus $\tilde{\Delta}_+$ is a system of positive roots and is of the form $w^{-1}(\Delta_+)$, for some w in the Weyl group; A is the set $\Delta_+\cap w^{-1}(-\Delta_+)$, ordered in some way.

Let us assume that A is of this form. Then $|A|=\varrho-w^{-1}(\varrho)$, and if $f\omega^{-A}$ is to belong to $A^k(\mathbf{E}_\lambda)$,

$$hf=\langle-\lambda-\varrho+w^{-1}(\varrho),h\rangle f \quad \text{for } h\in\mathfrak{h}. \tag{6.1a}$$

The equations $\hat{\partial}(f\omega^{-A})=0=\hat{\partial}^*(f\omega^{-A})$ are now equivalent to

$$e_{-\alpha}f=0 \quad \text{for } \alpha\in w^{-1}(\Delta_+). \tag{6.1b}$$

Hence, for the particular choice of A made above, the space of harmonic monomials $f\omega^{-A}$ is isomorphic to the subspace of $C^\infty(M)$ determined by the differential equations (6.1).

In (6.1), $e_{-\alpha}$ and h are regarded as left-invariant complex tangent vector fields, i.e. as linear combinations of infinitesimal generators of one-parameter groups acting on the right. Therefore, from the Peter–Weyl expansion

$$L^2(M)\simeq \textstyle\sum_{i\in\hat{M}} W_i\otimes W_i^*.$$

(Here $\hat{M}$ is the set of equivalence classes of irreducible representations of M, and W_i^* is the M-module contragredient to W_i), these differential equations pick out $\sum W_i\otimes U_i$, where $U_i\subset W_i^*$ is the subspace of vectors v which satisfy

$$hv=\langle-\lambda-\varrho+w^{-1}(\varrho),h\rangle v \quad \text{for } h\in\mathfrak{h}$$

$$e_{-\alpha}v=0 \quad \text{for } \alpha\in w^{-1}(\Delta_+).$$

According to the highest weight theory, $U_i=0$ unless $\varrho-w(\lambda+\varrho)$ is the lowest weight of V_i^*, i.e. unless V_i has highest weight $w(\lambda+\varrho)-\varrho$; and if $U_i\neq 0$, U_i is one-dimensional. Consequently, the M-module of harmonic monomials $f\omega^{-A}$, with A of the form $\Delta_+\cap w^{-1}(-\Delta_+)$, is irreducible and has highest weight $w(\lambda+\varrho)-\varrho$, provided that this weight belongs to the highest Weyl chamber, and is zero otherwise. We had seen already that all harmonic monomials arise in this fashion, and that $H^k(\mathbf{E}_\lambda)$ is spanned by monomials. Finally, $w(\lambda+\varrho)-\varrho$ lies in the highest Weyl chamber precisely when $\lambda+\varrho$ is non-singular and when w is the unique element of the Weyl group which carries $\lambda+\varrho$ into the interior of the highest Weyl chamber. This proves the theorem in the special case when B is a Borel subgroup.

Suppose now that B is an arbitrary parabolic subgroup of $G_{\mathbf{C}}$, and $\mathbf{E}_\pi\to X=G_{\mathbf{C}}/B$

the homogeneous holomorphic vector bundle determined by the irreducible representation $\pi: V \to \mathrm{GL}(E)$, where $V = M \cap B$. Instead of pushing through the computation of § 5 and the proof above in this situation, which is possible but bothersome, we shall sketch an argument of Bott that reduces the problem to the case already covered. In the notation of (3.7),

$$\mathfrak{b}_1 = \mathfrak{h} \oplus \sum \mathfrak{g}^{-\alpha}, \quad \alpha \in \Delta_+,$$

and

$$\mathfrak{b}_1 \cap \mathfrak{v} = \mathfrak{h} \oplus \sum \mathfrak{g}^{-\alpha}, \quad \alpha \in \Phi$$

are Borel subalgebras of $\mathfrak{g}$ and $\mathfrak{v}$. We denote the subgroups of $G_{\mathbf{C}}$ corresponding to $\mathfrak{b}_1$, $\mathfrak{v}$, $\mathfrak{h}_0$ by B_1, $V_{\mathbf{C}}$, H, respectively. Just as M acts transitive on $X = G_{\mathbf{C}}/B$ and on $G_{\mathbf{C}}/B_1$, V acts transitively on $V_{\mathbf{C}}/B_1 \cap V_{\mathbf{C}}$. The quotient spaces $X_1 = G_{\mathbf{C}}/B_1 = M/H$ and $Y = V_{\mathbf{C}}/B_1 \cap V_{\mathbf{C}} = V/H$ are Kähler C-spaces; $V_{\mathbf{C}}$, of course, is only reductive and not semisimple, but this causes no problems because $B_1 \cap V_{\mathbf{C}}$ contains the center of $V_{\mathbf{C}}$. The highest weight λ of π can be regarded as a character of H and determines a homogeneous holomorphic line bundle $\mathbf{F}_\lambda \to X_1$. The natural quotient map $q: X_1 \to X$ exhibits X_1 as a holomorphic fibre bundle over X with fibre Y. Let $R^k q(\mathbf{F}_\lambda)$ be the kth direct image sheaf of $\mathcal{O}(\mathbf{F}_\lambda)$ under q, i.e. the sheaf arising from the presheaf

$$U \to H^k(q^{-1}(U), \mathcal{O}(\mathbf{F}_\lambda))$$

for open subsets U of X. Since $q: X_1 \to X$ is locally a product, $R^k q(\mathbf{F}_\lambda)$ is the sheaf of germs of holomorphic sections of a vector bundle, whose fibre over the "origin" eV is isomorphic to $H^k(Y, \mathcal{O}_Y(\mathbf{F}_\lambda))$ as V-module. In the first half of this proof, the semisimplicity of M was never used. Thus we may let Y and V play the roles of X and M; $B_1 \cap V_{\mathbf{C}}$ is Borel in $V_{\mathbf{C}}$, we recall. Since λ lies in the highest Weyl chamber of $(\mathfrak{v}, \mathfrak{h})$ with respect to the system of positive roots Φ, $H^k(Y, \mathcal{O}_Y(\mathbf{F}_\lambda)) = 0$ for $k \neq 0$, and $H^0(Y, \mathcal{O}_Y(\mathbf{F}_\lambda))$ is isomorphic as V-module to E, the fibre of $\mathbf{E}_\pi$ over eV. Hence $R^k q(\mathbf{F}_\lambda) = 0$ for $k \neq 0$, and because V acts irreducibly on E, $R^0 q(\mathbf{F}_\lambda) = \mathcal{O}(\mathbf{E}_\pi)$. The Leray spectral sequence collapses, to give an isomorphism $H^*(X, \mathcal{O}(\mathbf{E}_\pi)) \simeq H^*(X_1, \mathcal{O}(\mathbf{F}_\lambda))$, which clearly commutes with the action of M. Now the special case of Theorem 6.1 can be applied to $H^*(X_1, \mathcal{O}(\mathbf{F}_\lambda))$ and leads to the desired description of $H^*(X, \mathcal{O}(\mathbf{E}_\pi))$.

7. Cohomology in the noncompact case

We consider a manifold $D = G/V$ dual to the Kähler C-space $X = G_{\mathbf{C}}/B = M/V$, and a discrete subgroup Γ of G which acts on D without fixed points, such that $\Gamma \backslash G$ is compact. Then $Y = \Gamma \backslash D$ is a compact complex manifold. Let $\pi: V \to \mathrm{GL}(E)$ be an irreducible representation whose highest weight, relative to the system of positive roots Φ of $(\mathfrak{v}, \mathfrak{h})$, shall be denoted by λ. The corresponding vector bundle $\mathbf{E}_\pi \to D$ is Γ-invariant and therefore drops to a holomorphic vector bundle $\mathbf{F}_\pi \to Y$. Let $\alpha(\lambda)$ be the number of positive compact roots β with $(\lambda, \beta) < 0$ plus the number of positive noncompact roots β with $(\lambda, \beta) > 0$.

(7.1) LEMMA. *There exists a constant η, depending only on D, such that whenever $|(\lambda, \alpha)| \geqslant \eta$ for every $\alpha \in \Delta$, $H^k(Y, O(\mathbf{F}_\pi)) = 0$ for $k \neq \alpha(\lambda)$.*

Proof. We choose groups B_1, $V_{\mathbf{C}}$, H as in the last part of the proof of theorem 6.1. Then $D_1 = G/H$ is dual to $X_1 = G_{\mathbf{C}}/B_1 = M/H$. We set $Y_1 = \Gamma \backslash D_1$; $Y_1 \to Y$ is a holomorphic fibre bundle with fibre $V_{\mathbf{C}}/B_1 \cap V_{\mathbf{C}}$. Just as in the proof of Theorem 6.1, the Leray spectral sequence of this fibering establishes an isomorphism between the cohomology groups of $\mathbf{F}_\pi$ on Y and those of the line bundle determined by λ on Y_1.

Thus we may as well assume that $D = G/H$ is dual to $X = G_{\mathbf{C}}/B$, where B is a Borel subgroup. Since H is a torus, π must now be a one-dimensional representation of H, and so we shall refer to it as λ. The cohomology group $H^k(Y, O(\mathbf{E}_\lambda))$ is isomorphic to the space of $\mathbf{E}_\lambda$-valued harmonic $(0, k)$-forms on Y, which in turn is naturally isomorphic to $\mathcal{H}^k_\Gamma(\mathbf{E}_\lambda)$, the space of Γ-invariant $\mathbf{E}_\lambda$-valued harmonic $(0, k)$-forms on D. If $\varphi \in \mathcal{H}^k_\Gamma(\mathbf{E}_\lambda)$, we can write, in the notation of § 5, $\varphi = \sum_i f_i \omega^{-A_i}$; here the A_i are ordered k-tuples of positive roots, and the f_i are Γ-invariant C^∞ functions on G. Let Δ_1 be the set of roots α such that $\varepsilon_\alpha(\alpha, \lambda) > 0$. Then in view of Proposition 5.1 and (5.8a),

$$\begin{aligned} 0 = \Box \varphi = & -\sum_i \sum_{\alpha \in \Delta_+ \cap \Delta_1} \varepsilon_\alpha (h_\alpha f_i i(\omega^{-\alpha}) e(\omega^{-\alpha}) \omega^{-A_i} + e_{-\alpha} e_\alpha f_i \omega^{-A_i}) \\ & + \sum_i \sum_{\alpha \in \Delta_+ - \Delta_1} \varepsilon_\alpha (h_\alpha f_i e(\omega^{-\alpha}) i(\omega^{-\alpha}) \omega^{-A_i} - e_\alpha e_{-\alpha} f_i \omega^{-A_i}) \\ & + \sum_i \sum_{\alpha, \beta \in \Delta_+} (\varepsilon_\beta - \varepsilon_\alpha) e_\alpha f_i e(e_{-\alpha} \omega^{-\beta}) i(\omega^{-\beta}) \omega^{-A_i} \\ & + \sum_i \sum_{\alpha, \beta \in \Delta_+} (\varepsilon_\beta + 1) e_{-\alpha} f_i e(e_\alpha \omega^{-\beta}) i(\omega^{-\beta}) \omega^{-A_i} + \tfrac{1}{2} \sum_i (|A_i|, 2\varrho - |A_i|) f_i \omega^{-A_i}. \end{aligned}$$

The functions f_i satisfy $h_\alpha f_i = -(\alpha, \lambda + |A_i|) f_i$ (cf. (5.1)); thus, if we set

$$c_i = \sum_{\alpha \in (\Delta_+ \cap \Delta_1) - A_i} \varepsilon_\alpha (\alpha, \lambda + |A_i|) - \sum_{\alpha \in A_i \cap (\Delta_+ - \Delta_1)} \varepsilon_\alpha (\alpha, \lambda + |A_i|) + \tfrac{1}{2} (|A_i|, 2\varrho - |A_i|),$$

the identity above becomes

$$\begin{aligned} 0 = \Box \varphi = \sum_i c_i f_i \omega^{-A_i} - \sum_i \sum_{\alpha \in \Delta_1} \varepsilon_\alpha e_{-\alpha} e_\alpha f_i \omega^{-A_i} + \sum_i \sum_{\alpha, \beta \in \Delta_+} (\varepsilon_\beta - \varepsilon_\alpha) e_\alpha f_i e(e_{-\alpha} \omega^{-\beta}) i(\omega^{-\beta}) \omega^{-A_i} \\ + \sum_i \sum_{\alpha, \beta \in \Delta_+} (\varepsilon_\beta + 1) e_{-\alpha} f_i e(e_\alpha \omega^{-\beta}) i(\omega^{-\beta}) \omega^{-A_i}. \end{aligned}$$

Integration over a fundamental domain of Γ, with respect to the Hermitian metrics introduced in § 5, determines an inner product on $\mathcal{H}^k_\Gamma(\mathbf{E}_\lambda)$. Equivalently, this inner product can be described as being induced by the natural inner product of $L^2(\Gamma\backslash G)$ and the inner product $(\omega^{-\alpha}, \omega^{-\beta}) = \delta^{\alpha\beta}$ on $\mathfrak{n}^*_-$. Let c be the smallest constant among the c_i, and b a suitably chosen positive constant. Then, because $-\varepsilon_\alpha\, e_{-\alpha}$ is the adjoint of e_α acting on $L^2(\Gamma\backslash G)$,

$$\begin{aligned}0 = (\Box\varphi, \varphi) &= \sum_i c_i\,(f_i, f_i) + \sum_i \sum_{\alpha\in\Delta_1} (e_\alpha f_i, e_\alpha f_i)\\
&+ \sum_{i,j} \sum_{\alpha,\beta\in\Delta_+} (\varepsilon_\beta - \varepsilon_\alpha)\,(e_\alpha f_i, f_j)\,(e(e_{-\alpha}\omega^{-\beta})\, i(\omega^{-\beta})\,\omega^{-A_i}, \omega^{-A_j})\\
&+ \sum_{i,j} \sum_{\alpha,\beta\in\Delta_+} (\varepsilon_\beta + 1)\,(e_{-\alpha} f_i, f_j)\,(e(e_\alpha\omega^{-\beta})\, i(\omega^{-\beta})\,\omega^{-A_i}, \omega^{-A_j})\\
&\geqslant c(\varphi, \varphi) + \sum_i \sum_{\alpha\in\Delta_1} (e_\alpha f_i, e_\alpha f_i) - b\sum_{i,j} \sum_{\alpha\in\Delta_+} \left|(e_\alpha f_i, f_j)\right| - b\sum_{i,j} \sum_{\alpha\in\Delta_+} \left|(e_{-\alpha} f_i, f_j)\right|.\end{aligned}$$

For every root α, positive or negative, $|(e_\alpha f_i, f_j)| = |(e_{-\alpha} f_j, f_i)|$, again because $-\varepsilon_\alpha e_{-\alpha}$ is the adjoint of e_α. Hence

$$0 \geqslant c(\varphi, \varphi) + \sum_i \sum_{\alpha\in\Delta_1} (e_\alpha f_i, e_\alpha f_i) - 2b\sum_{i,j} \sum_{\alpha\in\Delta_1} (e_\alpha f_i, e_\alpha f_i)^{\frac{1}{2}}\,(f_j, f_j)^{\frac{1}{2}}.$$

We now use the inequality $2bxy \leqslant x^2 + b^2y^2$, to get

$$0 \geqslant c(\varphi, \varphi) + \sum_i \sum_{\alpha\in\Delta_1} (e_\alpha f_i, e_\alpha f_i) - \sum_i \sum_{\alpha\in\Delta_1} (e_\alpha f_i, e_\alpha f_i) - b^2(\varphi, \varphi) = (c - b^2)\,(\varphi, \varphi).$$

If the integer k is different from $\alpha(\lambda)$, every k-tuple of distinct positive roots either must have a nonempty intersection with $\Delta_+ - \Delta_1$, or cannot contain all of $\Delta_+ \cap \Delta_1$, or both. In this case, by choosing the constant η in the statement of the lemma sufficiently large, we can make all of the c_i greater than b^2. Then the last inequality above will imply the vanishing of every $\varphi \in \mathcal{H}^k_\Gamma(\mathbf{E}_\lambda)$, as desired.

(7.2) Theorem *Let $\mathbf{F}_\pi \to Y$ be as above and assume that* $|(\lambda, \alpha)| \geqslant \eta$ *for all* $\alpha \in \Delta$ (*cf. Lemma* 7.1). *Then* $H^k(Y, O(\mathbf{F}_\pi)) = 0$ *for* $k \neq \alpha(\pi)$ *and*

$$\dim H^{\alpha(\pi)}(Y, O(\mathbf{F}_\pi)) = c(D)\mu(Y) \dim W_\lambda$$

where $c(D) > 0$ *depends only on* D, $\mu(Y)$ *is the volume of* Y, *and* W_λ *is the irreducible* M*-module with highest weight* $w(\lambda + \varrho) - \varrho$ (*cf. Theorem* 6.1).

Proof. As in the proof of Theorem 6.1, we may assume that $Y = \Gamma\backslash G/H$ where $H \subset G$ is a maximal torus, and that the bundle in question is a line bundle $\mathbf{F}_\lambda \to Y$. Also, the vanishing part of Theorem (7.2) is included in Lemma (7.1), and so it will suffice to prove that:

$$\chi(Y, O(\mathbf{F}_\lambda)) = (-1)^{\alpha(\lambda)}\, c(D)\, \mu(Y) \dim W_\lambda, \tag{7.3}$$

where $\chi(Y, O(\mathbf{F}_\lambda)) = \sum_{k=0}^{m} (-1)^k \dim H^k(Y, O(\mathbf{F}_\lambda))$ is the *sheaf Euler characteristic* of $O(\mathbf{F}_\lambda)$.

For this we will use the Hirzebruch–Riemann–Roch theorem for the nonalgebraic complex manifold Y, which has been proved by Atiyah–Singer (cf. [3]). Taking into account the Riemann–Roch formula given in [16] and (7.3), we must show:

$$T(Y, \mathbf{F}_\lambda) = (-1)^{\alpha(\lambda)}\, c(D)\, \mu(Y) \dim W_\lambda, \tag{7.4}$$

where $T(Y, \mathbf{F}_\lambda)$ is the *Todd genus* of the line bundle $\mathbf{F}_\lambda \to Y$.

Now the holomorphic tangent bundle $\mathbf{T}(Y)$ arises from the restriction to D of the homogeneous vector bundle $G_{\mathbf{C}} \times_B \mathfrak{n}_+^*$ (cf. § 1); thus $\mathbf{T}(Y) \to Y$ has a solvable structure group and $\mathbf{T}(Y)$ has a composition series with successive quotients the homogeneous line bundles $\mathbf{F}_\alpha \to Y$ where α runs through the positive roots Δ_+. Topologically, $\mathbf{T}(Y) \cong \sum_{\alpha \in \Delta_+} \mathbf{F}_\alpha$. For each $\gamma \in \Lambda \subset \mathfrak{h}^*$, we let $c(\gamma) \in H^2(Y, \mathbf{Q})$be the Chern class of the homogeneous line bundle $\mathbf{F}_\gamma \to Y$. Then c can be regarded as a homomorphism from Λ into $H^2(Y, \mathbf{Q})$, which extends uniquely to an algebra homomorphism from the symmetric algebra of $\mathfrak{h}^*$ into $H^*(Y, \mathbf{C})$. More generally, to each holomorphic function f which is defined on a neighborhood of zero in $\mathfrak{h}$, one can associate $c(f) \in H^*(Y, \mathbf{C})$. With this convention, and using Hirzebruch's notation [16], the Todd genus of $\mathbf{F}_\lambda$ can be expressed as

$$T(Y, \mathbf{F}_\lambda) = c\left\{ e^\lambda \prod_{\alpha \in \Delta_+} \frac{\alpha}{1 - e^{-\alpha}} \right\} [Y]. \tag{7.5}$$

Here λ and the α's are viewed as functions on $\mathfrak{h}$. Now

$$\prod_{\alpha \in \Delta_+} \frac{\alpha}{1 - e^{-\alpha}} = e^\varrho \prod_{\alpha \in \Delta_+} \frac{\alpha}{e^{\alpha/2} - e^{-\alpha/2}},$$

where as usual $\varrho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$, and hence

$$T(Y, \mathbf{F}_\lambda) = c\left\{ e^{\lambda + \varrho} \prod_{\alpha \in \Delta_+} \frac{\alpha}{e^{\alpha/2} - e^{-\alpha/2}} \right\} [Y].$$

The action of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ on $\mathfrak{h}^*$ induces an action on the image of c in $H^*(Y, \mathbf{C})$. The Chern class $c(\gamma)$ is represented by the curvature form

$$\frac{i}{2\pi} \Theta_D(\gamma) = \frac{i}{2\pi} \left\{ - \sum_{\alpha \in \Delta_+} (\gamma, \alpha)\, \omega^\alpha \wedge \omega^{-\alpha} \right\} \tag{7.6}$$

(cf. $(4.2)_D$ and [13]); thus every cohomology class of top degree, which can be represented by a multiple of the differential form $\prod_{\alpha\in\Delta_+}\omega^\alpha\wedge\omega^{-\alpha}$, is carried into sgn w times itself by each Weyl reflection w. Since evaluation on the fundamental cycle involves only the component of top degree, and since the expression $\prod_{\alpha\in\Delta_+}\alpha/(e^{\alpha/2}-e^{-\alpha/2})$ is invariant under the Weyl group, (7.5) may be rewritten as

$$T(Y,\mathbf{F}_\lambda)=\frac{1}{N}\,c\left\{\sum_w \operatorname{sgn} w\, e^{w(\lambda+\varrho)}\prod_{\alpha\in\Delta_+}\frac{\alpha}{e^{\alpha/2}-e^{-\alpha/2}}\right\}[Y],$$

N being the order of the Weyl group. According to Weyl's character formula [30], $\sum_w \operatorname{sgn} w\, e^{w(\lambda+\varrho)}\prod_{\alpha\in\Delta_+}(1/e^{\alpha/2}-e^{-\alpha/2})$ equals, up to sign, the character of the M-module W_λ mentioned in the statement of the theorem. Consequently, this expression defines a holomorphic function on $\mathfrak{h}$ whose value at the origin is $\pm\dim W_\lambda$. Moreover, $c\,(\prod_{\alpha\in\Delta_+}\alpha)$ is a cohomology class of top degree. Thus

$$T(Y,\mathbf{F}_\lambda)=\pm\frac{1}{N}\dim W_\lambda\, c\left(\prod_{\alpha\in\Delta_+}\alpha\right)[Y].$$

In view of (7.6), $c\,(\prod_{\alpha\in\Delta_+}\alpha)$ is represented by the differential form

$$\left(-\frac{i}{2\pi}\right)^m\prod_{\alpha\in\Delta_+}\sum_{\beta\in\Delta_+}(\alpha,\beta)\,\omega^\beta\wedge\omega^{-\beta}, \tag{7.7}$$

which is a constant multiple of the volume form of D. The proof of Theorem (7.2) will be complete as soon as the differential form (7.7) is shown not to vanish identically. The computation above carries over word-for-word to the case of a line bundle $\mathbf{F}_\lambda$ over the compact flag manifold $X=M/H$. Thus the Euler characteristic of the trivial line bundle over X, for example, which is known to be different from zero (Theorem (6.1)), is a multiple of the expression (7.7), reinterpreted as a differential form on X and integrated over the fundamental cycle. Thus the form (7.7) cannot vanish, and we are done.

Remark. We want to speculate a little on the possible implications of theorem (7.2). The spaces $Y=\Gamma\backslash D$ give a class of compact, complex manifolds which arise quite naturally in algebraic geometry (cf. [11]) and function theory (cf. Langlands [24]), and for which the higher sheaf cohomology $H^{a(\pi)}(Y,\,\mathcal{O}(\mathbf{F}_\pi))$, instead of being as usual an obstruction or, at best, a sideshow, is now the main object of interest.

For the purposes of the harmonic analysis on G there is at least a conjectural explanation. Roughly speaking, we should first let $\Gamma\subset G$ be an arbitrary discrete subgroup and we should consider $\mathcal{H}^k_\Gamma(\mathbf{F}_\pi)$, which is by definition the space of Γ-invariant, $\mathbf{F}_\pi$-valued harmonic $(0,k)$-forms φ on $D=G/V$ satisfying $\int_{\Gamma\backslash D}\|\varphi\|^2<\infty$. If $\Gamma\backslash G$ is compact and $N(\Gamma\cap V=\{e\}$, then $\mathcal{H}^k_\Gamma(\mathbf{F}_\pi) \cong H^k(Y, O(\mathbf{F}_\pi))$. The vanishing part of Theorem (7.2) goes through for general Γ (cf. Theorem (7.8) below), and the remaining group $\mathcal{H}^{\alpha(\pi)}_\Gamma(\mathbf{F}_\pi)$ *should* be closely related with the contribution of the discrete series to $L^2(\Gamma\backslash G)$. What is missing is the general existence theorem for $\mathcal{H}^{\alpha(\pi)}_\Gamma(\mathbf{F}_\pi)$. In case $\Gamma\backslash G$ is compact, we found existence from the Riemann–Roch theorem on $\Gamma\backslash G/V$; it is fairly clear that we could have equally well used the Atiyah–Bott–Lefschetz fixed point formula on $\Gamma\backslash G$. For the opposite extreme $\Gamma = \{e\}$, there is the Langlands conjecture, which is formally stated at the end of this paragraph, and whose present status is discussed in the introduction.

The relation between the cohomology $H^{\alpha(\pi)}(D, O(\mathbf{F}_\pi))$ and algebraic geometry is presently quite obscure. Let us illustrate the problem. Suppose that $V \subset \mathbf{P}_3$ is an algebraic surface of degree n given by an equation

$$F(\xi) = \sum_{i_0+i_1+i_2+i_3=n} \lambda_{i_0 i_1 i_2 i_3} \xi_0^{i_0} \xi_1^{i_1} \xi_2^{i_2} \xi_3^{i_3} = 0$$

where $\xi = [\xi_0, \xi_1, \xi_2, \xi_3]$ are homogeneous coordinates. Then V is determined by the homogeneous vector $\lambda = [..., \lambda_{i_0 i_1 i_2 i_3}, ...] \in \mathbf{P}_N$ of coefficients in $F(\xi)$, and we write V_λ for V. The set of points $\lambda \in \mathbf{P}_N$ for which V_λ is singular forms an algebraic hypersurface $S \subset \mathbf{P}_N$, and we let $B = \mathbf{P}_N - S$. Then $\{V_\lambda\}_{\lambda \in B}$ gives the algebraic family of all non-singular surfaces of degree n in $\mathbf{P}_3$.

The parameter space B is a connected, open manifold and we fix a base point $0 \in B$. If $\lambda \in B$ and $\lambda(t)$ $(0 \leqslant t \leqslant 1, \lambda(0) = 0, \lambda(1) = \lambda)$ is a curve from 0 to λ, there is an induced diffeomorphism $f_\lambda: V_0 \to V_\lambda$. The isotopy class of f_λ depends only on the homotopy class of the path $\{\lambda(t)\}$, and the induced mappings $f_\lambda^*: H^2(V_\lambda, \mathbf{Q}) \to H^2(V_0, \mathbf{Q})$ all preserve the Chern class ω of the standard positive line bundle $\mathbf{H} \to \mathbf{P}_3$ (ω is the cohomology class of a hyperplane section). Thus, if we let $\Gamma \subset \operatorname{Aut}(H^2(V_0, \mathbf{Q}))$ be the group of all automorphisms $(g_\lambda^{-1} f_\lambda)^*$ where $f_\lambda: V_0 \to V_\lambda$ and $g_\lambda: V_0 \to V_\lambda$ arise from paths in B, then Γ is a discrete subgroup of $\mathrm{GL}(H^2(V_0, \mathbf{R}))$ and every $T \in \Gamma$ preserves both ω and the cup-product pairing $Q: H^2(V_0, \mathbf{Q}) \otimes H^2(V_0, \mathbf{Q}) \to \mathbf{Q}$.

Let $E \subset H^2(V_0, \mathbf{C})$ be the subspace of classes $\varphi \in H^2(V_0, \mathbf{C})$ with $\varphi \cdot \omega = 0$. Since $\omega \cdot \omega = n > 0$, E is defined over $\mathbf{Q}$, $Q: E \otimes E \to \mathbf{C}$ is non-singular, and Γ acts on E. We let $r = \dim H^{2,0}(V_0) = \dim H^{2,0}(V_\lambda)$ for all λ. By the *Hodge index theorem* (cf. Hodge [18]), $\dim E = 2r + s$ where Q is equivalent to $\sum_{j=1}^{2r} x_j^2 - \sum_{k=1}^{s} y_k^2$ over $\mathbf{R}$. Denote by $G(r, E)$ the Grassmann variety of all r-planes in E and $X \subset G(r, E)$ the r-planes S which satisfy $Q(S, S) = 0$. Then X is a Kähler C-space $G_{\mathbf{C}}/B$ where $G_{\mathbf{C}} \subset \mathrm{GL}(E)$ is the complex orthogonal group of Q (cf. the end of § 2). The set of r-planes $S \in X$ which satisfy $Q(S, \bar{S}) > 0$ gives a non-compact dual $D \subset X$; here $D \cong G/G \cap B$ where $G \subset G_{\mathbf{C}}$ is the real orthogonal group of Q and D is the

G-orbit of a point (cf. the end of § 2 again). As a real homogeneous space, $D \cong SO(2r, s)/U(r) \times SO(s)$, and Γ is a properly discontinuous group of analytic automorphisms of D. We let Y be the analytic space $\Gamma\backslash D$.

If $\lambda \in B$, the subspace $S_\lambda = f_\lambda^*\{H^{2,0}(V_\lambda)\}$ lies in D (cf. Hodge [18]) and is well-defined modulo Γ. This gives the period mapping Φ: $B \to Y$, and in [11] it is proved that Φ is a holomorphic, horizontal mapping and the differential Φ_* is essentially injective.

If the degree n is 1, 2, or 3, then V_λ is rational and $r=0$. Suppose that $n=4$. Then the canonical bundle of V_λ is trivial, V_λ is a *Kummer surface*, and $\dim H^{2,0}(V_\lambda)=r=1$. In this case $D \cong SO(2, s)/SO(2) \times SO(s)$ is an Hermitian symmetric domain of type IV (cf. [15]), and the horizontal map condition is vacuous. Taking the homogeneous line bundle $\mathbf{F}_\pi \to D$ to be a high positive power $\mathbf{K}^\mu$ of the canonical bundle of D, the integer $\alpha(\pi)$ equals zero and $\mathcal{H}^0_\Gamma(\mathbf{F}_\pi) \cong H^0(D, \mathcal{O}(\mathbf{K}^\mu))^\Gamma$ is the space of *automorphic forms of weight* μ (here we are using that Γ is of finite index in an arithmetic subgroup of G, cf. [12]). The quotient $f=\varphi/\psi$ of two automorphic forms of the same weight is an *automorphic function*, and it is proved in [12] that:

(i) $F=f\circ\Phi$ is a *rational* function on B; and (ii) if $\mathcal{K}$ is the field of rational functions on B and $\mathcal{K}_\Phi \subset \mathcal{K}$ is the subfield generated by the functions $F=f\circ\Phi$, then $\mathcal{K}_\Phi$ gives the same equivalence relation as Φ; i.e. $\Phi(\lambda_1)=\Phi(\lambda_2)$ if and only if, $F(\lambda_1)=F(\lambda_2)$ for all $F \in \mathcal{K}_\Phi$. In conclusion:

(7.8) *The equivalence relation* Periods of $\{V_{\lambda_1}\}$ = Periods of $\{V_{\lambda_2}\}$ *on B is an algebraic equivalence relation given by the subfield $\mathcal{K}_\Phi \subset \mathcal{K}$; the automorphic functions invert the period mapping up to rational functions.*

If $n \geqslant 5$, then $r>1$ and $D \cong SO(2r, s)/U(r) \times SO(s)$ does *not* fibre holomorphically over an Hermitian symmetric space. Thus $H^0(D, \mathcal{O}(\mathbf{F}_\pi))^\Gamma = \{0\}$ for all non-trivial homogeneous line bundles and there are *no automorphic forms.*

Now since the center of $U(r) \times SO(s)$ is a circle, the homogeneous line bundles over D are essentially the powers $\mathbf{K}^\mu$ of the canonical bundle. The condition that the period mapping Φ: $B \to \Gamma\backslash D$ be horizontal is non-vacuous if $n \geqslant 5$, and from $(4.24)_D$ we see that: For $\mu>0$, the curvature $\Theta_D(\mathbf{K}^\mu)$ is positive on $\Phi_*\{\mathrm{T}_\lambda(B)\}$. Thus we can hope to have rational, holomorphic sections of $\Phi^*(\mathbf{K}^\mu) \to B$ $(\mu >> 0)$ which will lead to the conclusion (7.8) for all $n \geqslant 4$. The problem is to construct these sections a priori and, in particular, to see if they can be obtained somehow from the automorphic cohomology

$$\mathcal{H}_\Gamma^{\frac{r^2-r}{2}}(D, \mathbf{K}^\mu) \qquad \left(\alpha(\pi) = \frac{r^2-r}{2} \text{ in this case}\right).$$

We close this section with some remarks about a conjecture of R. P. Langlands.

Let π be an irreducible unitary representation of a noncompact semisimple Lie group G. Then π can be thought of as an infinite-dimensional unitary matrix whose entries are functions on G; π is said to be *square-integrable* if one, or equivalently every, matrix entry belongs to $L^2(G)$. According to the fundamental results of Harish-Chandra [14], G has square-integrable representations if and only if it contains a compact Cartan subgroup. Suppose now that H is a particular compact Cartan subgroup, which we keep fixed from now on. Then, again according to Harish-Chandra, to each nonsingular character λ of H there corresponds, in a well-determined way, an irreducible square-integrable unitary representation π_λ; every such representation arises in this fashion; and π_λ and π_μ are equivalent if and only if the characters λ and μ of H are related by an element of the normalizer of H in G.

For simplicity, we assume that G can be embedded as a real form in a complex semisimple Lie group $G_{\mathbf{C}}$. We denote the Lie algebras of $G_{\mathbf{C}}$, G, H by $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{h}_0$, and the complexification of $\mathfrak{h}_0$ by $\mathfrak{h}$. We choose a system of positive roots Δ_+ for $(\mathfrak{g}, \mathfrak{h})$; then

$$\mathfrak{b} = \mathfrak{h} \oplus \sum \mathfrak{g}^{-\alpha} \quad \alpha \in \Delta_+$$

is a Borel subalgebra of $\mathfrak{g}$. Let B be the corresponding subgroup of $G_{\mathbf{C}}$. Since H is compact, the roots assume purely imaginary values on $\mathfrak{h}_0$, and the complex conjugate of $\mathfrak{g}^{-\alpha}$ with respect to $\mathfrak{g}_0$ is $\mathfrak{g}^{\alpha}$. It follows that $\mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{h}_0$, and that H is the identity component of $G \cap B$, which therefore normalizes H. Because the normalizer of H in G is compact, the argument at the beginning of § 2 shows that $D = G/H$ is dual to the Kähler C-space $G_{\mathbf{C}}/B$.

Let $\mathbf{L}_\lambda \to D$ be the homogeneous holomorphic line bundle determined by the character λ of H. Following Andreotti and Vesentini [2], we shall introduce certain "L^2-cohomology groups" of $\mathbf{L}_\lambda$. An equivalent, but more elaborate definition has been given in [27]. As before, $A^k(\mathbf{L}_\lambda)$ is the space of $\mathbf{L}_\lambda$-valued $(0, k)$-forms on D. The G-invariant Hermitian metrics on $\mathbf{L}_\lambda$ and D chosen in § 5, by integration over D, give rise to an inner product $(\,,\,)$ on $A^k_c(\mathbf{L}_\lambda)$, the subspace of compactly supported forms in $A^k(\mathbf{L}_\lambda)$. We denote the completion of $A^k_c(\mathbf{L}_\lambda)$ with respect to this inner product by $L^k(\mathbf{L}_\lambda)$; the elements of $L^k(\mathbf{L}_\lambda)$ will be thought of as differential forms with measurable coefficients which are square-integrable over D. The kth "L^2-cohomology group" of $\mathbf{L}_\lambda$ is defined as

$$\mathcal{H}^k(\mathbf{L}_\lambda) = \{\varphi \in A^k(\mathbf{L}_\lambda) \cap L^k(\mathbf{L}_\lambda) \mid \Box\varphi = 0\},$$

where $\Box$ is the Laplace-Beltrami operator. According to proposition 7 of [2], every $\varphi \in \mathcal{H}^k(\mathbf{L}_\lambda)$ is $\bar\partial$-closed, and hence determines an element of $H^k(D, \mathcal{O}(\mathbf{L}_\lambda))$. However, since D is noncompact, there is no reason to expect the mapping $\mathcal{H}^k(\mathbf{L}_\lambda) \to H^k(D, \mathcal{O}(\mathbf{L}_\lambda))$ to be nontrivial. If $\{\varphi_n\}$ is a sequence in $\mathcal{H}^k(\mathbf{L}_\lambda)$ which converges to a limit φ in $L^k(\mathbf{L}_\lambda)$, then φ is necessarily a weak solution of $\square\varphi=0$; but since $\square$ is an elliptic differential operator, every weak solution must be smooth and is actually a strong solution. Thus $\mathcal{H}^k(\mathbf{L}_\lambda)$, as a closed subspace of $L^k(\mathbf{L}_\lambda)$, has the structure of a Hilbert space. By translation, G acts unitarily on $\mathcal{H}^k(\mathbf{L}_\lambda)$. Recall the definition of the integer $\alpha(\lambda)$.

CONJECTURE (Langlands). *If $\lambda+\varrho$ is singular, $\mathcal{H}^k(\mathbf{L}_\lambda)=0$ for every k. If $\lambda+\varrho$ is nonsingular, $\mathcal{H}^k(\mathbf{L}_\lambda)=0$ for $k\neq\alpha(\lambda+\varrho)$; and the action of G in dimension $k=\alpha(\lambda+\varrho)$ is irreducible and equivalent to $\pi_{\lambda+\varrho}$.*

In a closely related conjecture, Langlands has postulated a connection between the dimensions of the cohomology groups considered in theorem 7.2 and the multiplicities of the representations π_μ in $L^2(\Gamma\backslash G)$; for details, the reader is referred to [24]. We offer the vanishing theorem below as a partial result in the direction of the Langlands conjecture.

(7.8) THEOREM. *There exists a constant η, which depends only on D, such that $|(\lambda,\alpha)|>\eta$ for every $\alpha\in\Delta_+$ implies $\mathcal{H}^k(\mathbf{L}_\lambda)=0$ for $k\neq\alpha(\lambda)$.*

Proof. The Hermitian metric on D, being G-invariant, is complete. Thus, in view of proposition 8 on p. 94 of [2], it suffices to establish an inequality

$$(\square\varphi,\varphi)\geqslant\delta(\varphi,\varphi),\quad \text{for } \varphi\in A^k_c(\mathbf{L}_\lambda),\ k\neq\alpha(\lambda),$$

provided that λ satisfies the hypothesis, where δ is a positive constant, independent of λ. This inequality follows from an argument which is formally identical to the proof of Lemma 7.1, except that the functions f_i are now compactly supported C^∞ functions on G.

8. The pseudoconvexity of dual manifolds of Kähler C-spaces

Every noncompact Hermitian symmetric space has the important property of being a Stein manifold. A manifold $D=G/V$ which is dual to a Kähler C-space, unless it is Hermitian symmetric, contains compact subvarieties of positive dimension and therefore cannot be a Stein manifold. However, as we shall show next, D comes as close to possessing this property as the presence of compact subvarieties will permit.

Let f be a C^∞ real-valued function on a complex manifold Y. The *Levi form* of f, $L(f)$, is the Hermitian form on the holomorphic tangent bundle of Y which, in terms of local coordinates $z^1, \dots, z^n$ is given by

$$L(f) = \sum_{i,j} \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j.$$

We say that f is an *exhaustion function* of Y if for every real number c the set $f^{-1}(\{x \leq c\})$ is compact. Recall the definition of the horizontal distribution $\mathbf{T}_h(D)$ from § 2. Since a complex manifold is a Stein manifold if and only if it admits an exhaustion function whose Levi form is positive definite at every point, the following theorem may be thought of as an extension of the assertion that the noncompact Hermitian symmetric spaces are Stein manifolds:

(8.1) THEOREM. *On every manifold D which is dual to a Kähler C-space, there exists an exhaustion function whose Levi form, restricted to $\mathbf{T}_h(D)$, is positive definite at every point.*

Andreotti–Grauert's [1] generalization of theorem B now implies

(8.2) COROLLARY. *If $\mathfrak{F}$ is a coherent analytic sheaf over D, then $H^k(D, \mathfrak{F}) = 0$ for $k > \dim_{\mathbf{C}} S = \frac{1}{2} \dim_{\mathbf{R}} K/V$.*

(8.3) COROLLARY. *A horizontal analytic mapping F: $Y \to D$ of a connected compact analytic space Y into D is constant.*

Proof. Let f be the function whose existence is guaranteed by the theorem. Since F is horizontal, F^*f is a plurisubharmonic function on the compact space Y, and hence must be constant. Thus $F^*(L(f))$, which is defined at the manifold points of Y, vanishes; and this cannot happen unless the tangential map F_* vanishes identically, i.e. unless F is constant.

Proof of Theorem 8.1. We shall use the notation of § 2 and § 3 freely. In particular, $D = G/V$ will be regarded as an open subset of $X = M/V = G_{\mathbf{C}}/B$. Let μ be the negative of the sum of the roots in $\Delta_+ - \Phi$. As was pointed out in § 4, μ determines a one-dimensional representation of V, and the corresponding line bundle $\mathbf{L}_\mu \to X$ is the canonical bundle. This line bundle can be given an M-invariant metric γ_M, and its restriction to D a G-invariant metric γ_G; both are unique up to multiplicative constants. The ratio of these two metrics is a positive C^∞ function on D. Hence $f = -\log(\gamma_G/\gamma_M)$ is well-defined on D. The Levi form of f is precisely the difference of the curvature forms of $\mathbf{L}_\mu$ corresponding to γ_G and γ_M, which were given by (4.23); it follows that $L(f)$ is positive definite on $\mathbf{T}_h(D)$.

In order to prove that f is an exhaustion function, it suffices to show that the ratio γ_G/γ_M extends to a continuous function on X whose restriction to the topological boundary ∂D of D vanishes.

The holomorphic cotangent bundle $\mathbf{T}^*(X)$ is associated to the principal bundle $B \to G_{\mathbf{C}} \to X$ by the adjoint representation of B on $\mathfrak{n}_-$. This can be verified by observing that the tangent bundle is associated to the principal bundle by the adjoint action of B on $\mathfrak{g}/\mathfrak{b}$, and that the Killing form establishes an Ad B-invariant nondegenerate bilinear pairing between $\mathfrak{g}/\mathfrak{b}$ and $\mathfrak{n}_-$. Thus we can identify the fibre of $\mathbf{T}^*(X)$ over a point $gB \in X = G_{\mathbf{C}}/B$ with $\operatorname{Ad} g(\mathfrak{n}_-)$. We define M-invariant and G-invariant Hermitian forms h_M and h_G on $\mathbf{T}^*(X)$ by setting

$$h_M(x, y) = -B(x, \tau(y)), \quad h_G(x, y) = B(x, \sigma(y))$$

for $x, y \in \operatorname{Ad} g(\mathfrak{n}_-)$. The former is positive definite; the latter nondegenerate, at least at eB, and by G-invariance then over all of D. Let $h(gB)$ be the product of the eigenvalues of h_G on $\operatorname{Ad} g(\mathfrak{n}_-)$ with respect to h_M. By construction, h is a continuous function on all of X. Because $\mathbf{L}_\mu$ and $\Lambda^n \mathbf{T}^*(X)$, $n = \dim_{\mathbf{C}} X$, coincide, the restriction of h to D is proportional to the ratio γ_G/γ_M. It only remains to be shown that h_G is degenerate on $\operatorname{Ad} g(\mathfrak{n}_-)$ whenever $gB \in \partial D$.

Suppose then that gB is a point of ∂D; for brevity, we set $\tilde{\mathfrak{n}}_- = \operatorname{Ad} g(\mathfrak{n}_-)$, $\tilde{\mathfrak{b}} = \operatorname{Ad} g(\mathfrak{b})$. As a consequence of Bruhat's lemma, the maximal nilpotent ideal $\tilde{\mathfrak{n}}_-$ of $\tilde{\mathfrak{b}}$ has a complement $\tilde{\mathfrak{v}}$ in $\tilde{\mathfrak{b}}$ such that

$$\tilde{\mathfrak{b}} \cap \sigma(\tilde{\mathfrak{b}}) = (\tilde{\mathfrak{v}} \cap \sigma(\tilde{\mathfrak{b}})) \oplus (\tilde{\mathfrak{n}}_- \cap \sigma(\tilde{\mathfrak{b}}));$$

indeed, if $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\mathfrak{g}$ in the intersection of the two parabolic subalgebras $\tilde{\mathfrak{b}}$ and $\sigma(\tilde{\mathfrak{b}})$, the (unique) maximal reductive subalgebra of $\tilde{\mathfrak{b}}$ which contains $\tilde{\mathfrak{h}}$ is a suitable choice for $\tilde{\mathfrak{v}}$. Notice that $\tilde{\mathfrak{v}}$ and $\mathfrak{v}$ have the same dimension. Since $gB \in \partial D$, the G-orbit of gB cannot have interior, and the Lie algebra $\mathfrak{g}_0 \cap \tilde{\mathfrak{b}}$ of the isotropy subgroup of G at gB must be of higher dimension than $\mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{v}$. Equivalently, because $\tilde{\mathfrak{b}} \cap \sigma(\tilde{\mathfrak{b}})$ is the complexification of $\mathfrak{g}_0 \cap \tilde{\mathfrak{b}}$,

$$\dim \mathfrak{v} < \dim \tilde{\mathfrak{b}} \cap \sigma(\tilde{\mathfrak{b}}) = \dim \tilde{\mathfrak{v}} \cap \sigma(\tilde{\mathfrak{b}}) + \dim \tilde{\mathfrak{n}}_- \cap \sigma(\tilde{\mathfrak{b}}) \leqslant \dim \mathfrak{v} + \dim \tilde{\mathfrak{n}}_- \cap \sigma(\tilde{\mathfrak{b}}).$$

Thus we can choose a nonzero vector $y \in \tilde{\mathfrak{n}}_- \cap \sigma(\tilde{\mathfrak{b}})$. Finally, since $B(\tilde{\mathfrak{n}}_-, \mathfrak{b}) = 0$, $h_G(x, y) = B(x, \sigma(y)) = 0$ for every $x \in \mathfrak{n}_-$, i.e. h_G is degenerate on $\tilde{\mathfrak{n}}_-$, as was to be shown.

9. Horizontal mappings are negatively curved

In recent papers ([9], [20], [22], [32], etc.) several authors have shown that negatively curved Hermitian manifolds in certain ways behave like bounded domains. In view of the presence of parabolic compact subvarieties, dual manifolds of Kähler C-spaces cannot, in general, be expected to be negatively curved. However, the analogy to the situation of § 8 might suggest that they are negatively curved in the horizontal directions, and this is indeed the case.

Let $D=G/V$ be a manifold dual to the Kähler C-space $X=G_{\mathbf{C}}/B$, as in § 2. The holomorphic tangent space of D at eH is naturally isomorphic to $\sigma(\mathfrak{n}_-)=\tau(\mathfrak{n}_-)$, and

$$(x, y) = -B(x, \tau(y)), \quad x, y \in \sigma(\mathfrak{n}_-)$$

defines an Ad V-invariant inner product on $\sigma(\mathfrak{n}_-)$. By translation, the inner product gives rise to a G-invariant Hermitian metric on D. This metric, because of its homogeneity, turns D into a complete Hermitian manifold.

(9.1) THEOREM. *The holomorphic sectional curvatures of D corresponding to directions in $\mathbf{T}_h(D)$ are negative and bounded away from zero.*

We shall deduce some corollaries before proving the theorem. As a direct consequence of Corollary 8.3 in [32], we get

(9.2) COROLLARY. *The family of all horizontal holomorphic mappings of a fixed complex manifold into D is normal.*

In his paper [20], Kobayashi has introduced an intrinsic pseudodistance which is defined for each connected complex manifold Y: given a pair of points $x, y \in Y$, we consider chains of points $x=x_0, x_1, ..., x_n=y$ in Y, holomorphic mappings $f_1, ..., f_n$ of the unit disc Δ into Y, and points $a_i, b_i \in \Delta$, such that $f_i(a_i)=x_{i-1}$, $f_i(b_i)=x_i$; denote by d_i the distance from a_i to b_i, measured with respect to the Poincaré metric on Δ; now let $\varkappa(x, y)$ be the infimum of all possible sums $\sum_1^n d_i$ obtained in this manner. Then $\varkappa$ is a pseudodistance for Y. The arguments leading to theorem 3.8 of [20], which are also implicit in [32], together with theorem 9.1 imply

(9.3) COROLLARY. *A horizontal holomorphic mapping of a complex manifold Y into D is distance decreasing with respect to the Kobayashi pseudodistance on Y and the Hermitian metric on D, suitably renormalized.*

It is not difficult to verify that the pseudodistance vanishes identically on $\mathbf{C}^m$. Hence

(9.4) COROLLARY. *Every horizontal holomorphic mapping of a complex Euclidean space into D reduces to a constant.*

(9.5) COROLLARY. *If Y is a complex manifold, $S \subset Y$ a subvariety of codimension at least two, and F: $Y-S \to D$ a horizontal holomorphic mapping, then F can be extended over S.*

Proof. The proof of Theorem 3.3 in [22] can be modified slightly to cover this case. Alternatively, one may proceed as follows. Since the set of singular points of S forms a subvariety of lower dimension, an inductive argument allows us to assume that S is a submanifold. Now, because the problem is a local one, we need to consider only the case of a polycylinder Y and a linear subspace S. For dimensional reasons, if an analytic disc in Y is perturbed by arbitrarily little, it can be made disjoint from S, while still lying in Y. Hence the Kobayashi pseudodistance of Y, which is a true distance in this instance, agrees on $Y-S$ with the pseudodistance of $Y-S$. It follows that the distance decreasing map F from $Y-S$ into the complete metric space D must extend continuously, and hence holomorphically, to all of Y.

Let Γ be a discrete subgroup of G. Then Γ acts on D as a properly discontinuous group of analytic automorphisms, and $\Gamma\backslash D$ has the structure of a normal analytic space such that the quotient map $D\to\Gamma\backslash D$ becomes holomorphic. A holomorphic mapping of an analytic space into $\Gamma\backslash D$ is said to be locally liftable if in some neighborhood of each point of the domain it can be factored through the quotient map $D\to\Gamma\backslash D$. We call such a mapping horizontal if all of the local liftings are horizontal as defined previously. The period mappings introduced in [11] have these two properties.

(9.6) LEMMA. *A locally liftable holomorphic mapping $F\colon Y\to\Gamma\backslash D$ can be lifted globally if Y is simply connected.*

Proof. If Γ were known to act freely, this would be a direct consequence of the monodromy theorem. In general, the usual proof of the monodromy theorem still applies, when it is combined with the following fact: let $U\subset Y$ be open and irreducible, and $F_1, F_2\colon U\to D$ two liftings of $F|U$; then there exists a $\gamma\in\Gamma$ such that $\gamma\circ F_1=F_2$. Indeed, because Γ acts properly discontinuously, if U is shrunk, there exist only finitely many elements $\gamma_1, \ldots, \gamma_n\in\Gamma$ such that $\gamma_i\circ F_1(U)\cap F_2(U)$ is non-empty. Thus there must exist some $\gamma\in\Gamma$ and a sequence $\{y_n\}$ which accumulates in U such that $\gamma\circ F_1(y_n)=F_2(y_n)$ for every n. Hence $\gamma\circ F_1=F_2$, as was asserted.

This lemma, together with corollaries 8.3 and 9.4 immediately implies

(9.7) COROLLARY. *Let Y be an analytic space whose universal covering space is either compact or a Euclidean space. Then every horizontal locally liftable holomorphic map $F\colon Y\to\Gamma\backslash D$ is constant.*

Extending holomorphic maps over subvarieties is a local problem, and the removal of a subvariety of codimension at least two will not increase the connectivity of a manifold. Thus we have:

(9.8) COROLLARY. *Let S be a subvariety of codimension at least two of a complex manifold Y, and F: $Y-S\to\Gamma\backslash D$ a horizontal, locally liftable holomorphic mapping. Then F can be continued to all of Y.*

Proof of Theorem 9.1. We recall the structure equations of a Hermitian manifold. Let Y be a complex manifold, with Hermitian metric $ds^2=\sum_i \omega^i\bar{\omega}^i$, where $\{\omega^i\}$ is a (local) frame of forms of type (1, 0). With respect to this frame, the connection is represented by the unique matrix of one-forms (φ^i_j) such that:

$$\begin{cases} d\omega^i+\sum_j \varphi^i_j\wedge\omega^j \quad \text{is of type } (2,0) \\ \varphi^i_j+\bar{\varphi}^j_i=0. \end{cases}$$

The curvature form is a differential two-form Ω which takes values in the bundle of endomorphisms of the holomorphic tangent bundle; if $\{e_i\}$ is the frame of vector fields dual to $\{\omega^i\}$, then

$$\Omega e_i=\sum_j \Omega^j_i e_j,$$

where

$$\Omega^j_i=d\varphi^j_i+\sum_k \varphi^j_k\wedge\varphi^k_i.$$

For a (1, 0)-vector x of unit length, the holomorphic sectional curvature in the direction of x is given by $(\Omega(x,\bar{x})\,x, x)$.

Let us now consider a manifold $D=G/V$ as in § 2. We shall also use the notation of § 3, especially (3.6). Corresponding to every $\alpha\in\Delta$, we define linear functionals ω^α and α on $\mathfrak{g}$:

$$\omega^\alpha(e_\beta)=\delta^\alpha_\beta,\quad \alpha(e_\beta)=0 \quad \text{for } \beta\in\Delta$$

$$\omega^\alpha(h)=0,\quad \alpha(h)=\langle\alpha, h\rangle \quad \text{for } h\in\mathfrak{h}.$$

We identify these linear functionals with left-invariant complex valued one-forms on G. Then

$$\bar{\omega}^\alpha=\varepsilon_\alpha\omega^{-\alpha},\quad \bar{\alpha}=-\alpha.$$

In view of (3.6), the Maurer–Cartan equations assert that

$$d\omega^\alpha=-\tfrac{1}{2}\sum_{\beta+\gamma=\alpha} N_{\beta,\gamma}\,\omega^\beta\wedge\omega^\gamma-\alpha\wedge\omega^\alpha, \tag{9.9}$$

with β, γ ranging over the set of all nonzero roots.

According to the definition of the Hermitian metric on D, $\{s^*\omega^\alpha \mid \alpha\in\Delta_+-\Phi\}$ is a local unitary frame of (1, 0)-forms on D, whenever s is a local section of the principal bundle $V\to G\to D$. The corresponding connection and curvature forms can also be expressed as the pullback via s of left-invariant forms on G; thus we can transfer our computations from D to G.

From now on, α, β, γ will always denote elements of $\Delta_+ - \Phi$, and as indices of summation, these letters will range over $\Delta_+ - \Phi$, subject to whatever other conditions are indicated. A sum of two roots cannot belong to $\Delta_+ - \Phi$ unless at least one of the two does (3.8); moreover, $N_{\beta,\gamma}$ is skewsymmetric in the indices. Hence, for $\alpha \in \Delta_+ - \Phi$, we can rewrite (9.9) as follows:

$$\begin{aligned} d\omega^\alpha &= -\tfrac{1}{2} \sum_{\beta+\gamma=\alpha} N_{\beta,\gamma}\, \omega^\beta \wedge \omega^\gamma - \sum_{\alpha-\beta\in\Phi} N_{\alpha-\beta,\beta}\, \omega^{\alpha-\beta} \wedge \omega^\beta - \sum_{\beta-\alpha\in\Delta_+} N_{\alpha-\beta,\beta}\, \omega^{\alpha-\beta} \wedge \omega^\beta - \alpha \wedge \omega^\alpha \\ &= -\sum_{\beta+\gamma=\alpha} (\tfrac{1}{2} + \varepsilon_\gamma) N_{\beta,\gamma}\, \omega^\beta \wedge \omega^\gamma + \sum_{\beta+\gamma=\alpha} \varepsilon_\gamma N_{\alpha,-\beta}\, \omega^{\alpha-\beta} \wedge \omega^\beta - \sum_{\alpha-\beta\in\Phi} N_{\alpha,-\beta}\, \omega^{\alpha-\beta} \wedge \omega^\beta \\ &\quad - \sum_{\beta-\alpha\in\Delta_+} N_{\alpha,-\beta}\, \omega^{\alpha-\beta} \wedge \omega^\beta - \alpha \wedge \omega^\alpha = -\sum_\beta \varphi^\alpha_\beta \wedge \omega^\beta + \tau^\alpha \end{aligned}$$

where

$$\varphi^\alpha_\beta = \begin{cases} N_{\alpha,-\beta}\, \omega^{\alpha-\beta} & \text{if } \beta-\alpha \in \Delta_+ \\ -\varepsilon_{\alpha-\beta} N_{\alpha,-\beta}\, \omega^{\alpha-\beta} & \text{if } \alpha-\beta \in \Delta_+ \\ \alpha & \text{if } \alpha=\beta \\ 0 & \text{otherwise} \end{cases} \tag{9.10}$$

and

$$\tau^\alpha = -\sum_{\beta+\gamma=\alpha} (\tfrac{1}{2} + \varepsilon_\gamma) N_{\beta,\gamma}\, \omega^\beta \wedge \omega^\gamma.$$

Observe that $\varphi^\beta_\alpha = -\bar{\varphi}^\alpha_\beta$, and that $s^*\tau^\alpha$ is of type (2,0) whenever s is a local section of $G \to D$. Hence (φ^α_β) is the connection form (or, to be more precise, $(s^*\varphi^\alpha_\beta)$ represents the connection relative to the frame $(s^*\omega^\alpha)$) and $(\Omega^\alpha_\beta) = (d\varphi^\alpha_\beta + \sum_\gamma \varphi^\alpha_\gamma \wedge \varphi^\gamma_\beta)$ the curvature form. When the φ^α_β are regarded as linear functionals on $\mathfrak{g}$, (9.2) becomes

$$\varphi^\alpha_\beta(x) = \begin{cases} -B(x, [e_\beta, e_{-\alpha}]) & \text{if } x \in \mathfrak{p} \cap \sigma(\mathfrak{n}_-) \\ B(x, [e_\beta, e_{-\alpha}]) & \text{if } x \in \mathfrak{k} \text{ or } x \in \mathfrak{p} \cap \mathfrak{n}_-. \end{cases}$$

Under the natural isomorphism between $\sigma(\mathfrak{n}_-)$ and the holomorphic tangent space of D at eV, $\mathfrak{p} \cap \sigma(\mathfrak{n}_-)$ corresponds to the fibre of $\mathbf{T}_h(D)$. Suppose now that $x \in \mathfrak{p} \cap \sigma(\mathfrak{n}_-)$, i.e.

$$x = \sum_{\alpha \in \Delta_+ \cap \Delta_\mathfrak{p}} a^\alpha e_\alpha, \quad a^\alpha \in \mathbf{C}.$$

It will be convenient to set $a^\alpha = 0$ if $\alpha \in \Delta_\mathfrak{k}$. Then $\bar{x} = \sigma(x) = \sum_\alpha \bar{a}^\alpha e_{-\alpha}$, and hence

$$\begin{aligned} (\Omega(x,\bar{x})\, x, x) &= \sum_{\alpha,\beta} \bar{a}^\alpha a^\beta\, \Omega^\alpha_\beta(x,\bar{x}) = -\sum_{\alpha,\beta} \bar{a}^\alpha a^\beta\, \varphi^\alpha_\beta([x,\bar{x}]) + \sum_{\alpha,\beta,\gamma} \bar{a}^\alpha a^\beta \varphi^\alpha_\gamma(x)\, \varphi^\gamma_\beta(\bar{x}) \\ &\quad - \sum_{\alpha,\beta,\gamma} \bar{a}^\alpha a^\beta\, \varphi^\alpha_\gamma(\bar{x})\, \varphi^\gamma_\beta(x) = -B([x,\bar{x}],[x,\bar{x}]) - \sum_\gamma B(x,[e_\gamma,\bar{x}])\, B(\bar{x},[x,e_{-\gamma}]) \\ &\quad + \sum_\gamma B(\bar{x},[e_\gamma,\bar{x}])\, B(x,[x,e_{-\gamma}]) = -B([x,\bar{x}],[x,\bar{x}]) - \sum_\gamma B([x,\bar{x}],e_\gamma)\, B([x,\bar{x}],e_{-\gamma}). \end{aligned}$$

Since $x \in \mathfrak{p}$, $[x, \bar{x}]$ belongs to $\sqrt{-1}\,\mathfrak{k}_0$, and we can write $[x, \bar{x}] = v + y - \sigma(y)$, with $y \in \mathfrak{k} \cap \mathfrak{n}_-$, $v \in \sqrt{-1}\,\mathfrak{v}_0$. Explicitly, $[x, \bar{x}] = \sum_\alpha |a^\alpha|^2 h_\alpha + \sum_{\alpha \neq \beta} a^\alpha \bar{a}^\beta [e_\alpha, e_{-\beta}]$; all terms of the first of these two sums lie in the cone generated by the positive roots in $\mathfrak{h}_{\mathbf{R}}$, which shows that $[x, \bar{x}] \neq 0$ unless $x = 0$. Because the Hermitian form $B(y, \sigma(y))$ is negative definite on $\mathfrak{k}$,

$$(\Omega(x, \bar{x})\, x, x) = B([x, \bar{x}], \sigma[x, \bar{x}]) + B(y, \sigma(y)) < 0,$$

provided $x \neq 0$. Thus the holomorphic sectional curvatures of D in the horizontal directions are negative and bounded away from zero, at least at eV, and because of the homogeneity of the metric, everywhere.

References

[1]. Andreotti, A. & Grauert, H., Théorèmes de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90 (1962), 193–259.

[2]. Andreotti, A. & Vesentini, E., Carleman estimates for the Laplace-Beltrami operator on complex manifolds. *Inst. Hautes Études Sci. Publ. Math.*, 25 (1965), 81–130.

[3]. Atiyah, M. F. & Singer, I. M., The index of elliptic operators I, III. *Ann. of Math.*, 87 (1968), 484–530 and 546–604.

[4]. Borel, A., Kählerian coset spaces of semi-simple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 40 (1954), 1140–1151.

[5]. Borel, A. & Weil, A., Representations linéaires et espaces homogènes kähleriennes des groupes de Lie compacts. *Séminaire Bourbaki*, (May 1954) exp. 100.

[6]. Borel, A. & Hirzebruch, F., Characteristic classes and homogeneous spaces I, II. *Amer. J. Math.*, 80 (1958), 458–438 and 81 (1959), 315–382.

[7]. Bott, R., Homogeneous vector bundles. *Ann. of Math.*, 66 (1957), 203–248.

[8]. Calabi, E. & Vesentini, E., On compact, locally symmetric Kähler manifolds. *Ann. of Math.*, 71 (1960), 472–507.

[9]. Grauert, H. & Reckziegel, H., Hermitesche Metriken und normale Familien holomopher Abbildungen. *Math. Z.*, 89 (1965), 108–125.

[10]. Griffiths, P. A., Some results on locally homogeneous complex manifolds. *Proc. Nat. Acad. Sci. U.S.A.*, 56 (1966), 413–416.

[11]. —— Periods of integrals on algebraic manifolds I, II. *Amer. J. Math.*, 90 (1968), 568–626.

[12]. —— Periods of integrals on algebraic manifolds III. To appear.

[13]. —— Hermitian differential geometry and positive vector bundles. To appear.

[14]. Harish-Chandra, Discrete series for semisimple Lie groups II. *Acta Math.*, 116 (1966), 1–111.

[15]. Helgason, S., *Differential geometry and symmetric spaces*. Academic Press, New York 1962.

[16]. Hirzebruch, F., *Neue topologische Methoden in der algebraischen Geometrie*. Ergebnisse der Mathematik, 9 (1956).

[17]. —— Automorphe Formen und der Satz von Reimann-Roch. *Symposium International de Topologica Algebraica, Universidad de Mexico* (1958), 129–143.

[18]. Hodge, W. V. D., *The theory and applications of harmonic integrals*. 2nd ed., Cambridge University Press (1959).

[19]. Ise, M., Generalized automorphic forms and certain holomorphic vector bundles. *Amer. J. Math.*, 86 (1964), 70–108.

[20]. Kobayashi, S., Invariant distances on complex manifolds and holomorphic mappings. *J. Math. Soc. Japan*, 19 (1967), 460–480.

[21]. Kostant, B., Lie algebra cohomology and the generalized Borel–Weil theorem. *Ann. of Math.*, 74 (1961), 329–387.

[22]. Kwack, M. H., *Generalizations of the big Picard theorem*. Thesis, Berkeley, 1968.

[23]. Langlands, R., The dimensions of spaces of automorphic forms. *Amer. J. Math.* 85 (1963), 99–125.

[24]. —— Dimensions of spaces of automorphic forms. Algebraic groups and discontinuous subgroups. *Proc. of Symposia in Pure Mathematics*, vol. IX, Amer. Math. Soc., Providence, R. I. (1966), 253–257.

[25]. Matsushima, Y. & Murakami, S., On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds. *Ann. of Math.*, 78 (1963), 365–416.

[26]. —— On certain cohomology groups attached to Hermitian symmetric spaces. *Osaka J. Math.*, 2 (1965), 1–35.

[27]. Okamoto, K. & Ozeki, H., On square-integrable $\bar{\partial}$-cohomology spaces attached to Hermitian symmetric spaces. *Osaka J. Math.*, 4 (1967), 95–110.

[28]. Schmid, W., *Homogeneous complex manifolds and representations of semisimple Lie groups.* Thesis, Berkeley, 1967.

[29]. —— Homogeneous complex manifolds and representations of semisimple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 59 (1968), 56–59.

[30]. Stiefel, E., Kristallographische Bestimmung der Charactere der geschlossenen Lie'schen Gruppen. *Comm. Math. Helv.*, 17 (1944), 165–200.

[31]. Wang, H. C., Closed manifolds with homogeneous complex structure. *Amer. J. Math.*, 76 (1954), 1–32.

[32]. Wu, H. H., Normal families of holomorphic mappings. *Acta Math.*, 119 (1967), 193–233

Received April 3, 1969.