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LOCALLY LIPSCHITZ VECTOR OPTIMIZATION WITH INEQUALITY AND EQUALITY CONSTRAINTS

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Abstract. The present paper studies the following constrained vector optimization problem: $\min_C f(x)$, $g(x) \in -K$, $h(x) = 0$, where $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz functions, $h\colon \mathbb{R}^n \to \mathbb{R}^q$ is C^1 function, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. Two types of solutions are important for the consideration, namely w-minimizers (weakly efficient points) and i-minimizers (isolated minimizers of order 1). In terms of the Dini directional derivative first-order necessary conditions for a point x^0 to be a w-minimizer and first-order sufficient conditions for x^0 to be an i-minimizer are obtained. Their effectiveness is illustrated on an example. A comparison with some known results is done.

Keywords: vector optimization, locally Lipschitz optimization, Dini derivatives, optimality conditions

MSC 2010: 90C29, 90C30, 90C46, 49J52

1. Introduction

In this paper we deal with the local solutions of the constrained vector optimization problem

$$\min_C f(x), \quad g(x) \in -K, \quad h(x) = 0, \tag{1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $g\colon \mathbb{R}^n \to \mathbb{R}^p$ and $h\colon \mathbb{R}^n \to \mathbb{R}^q$ are given functions, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. It is supposed that f and g are locally Lipschitz and h is C^1 function. The inclusion $g(x) \in -K$ can be represented as a set of inequalities $\langle \eta, g(x)\rangle \leqslant 0$, $\eta \in K'$, where K' is the positive polar cone of K. For this reason the problem is referred as one with inequality and equality constraints. Two types of solutions are important for the considerations, namely w-minimizers (weakly efficient points) and i-minimizers (isolated minimizers of order 1). In terms of the Dini directional derivative we obtain first-order necessary conditions for a point x^0 to be a w-minimizer and first-order sufficient conditions for x^0 to be an i-minimizer. The paper generalizes the results from [9], where problems with only inequality constraints are considered.

There is a growing interest toward optimality conditions for nonsmooth vector problems, though less papers study problems with equality constraints. In the smooth case the Fritz John optimality criterion is generalized in [16] and [13]. Unified first and second-order theory based on parabolic derivatives is proposed in [6]. Nonsmooth problems within Clarke subdifferentials are treated in [7] and [8]. Recently this problem is studied with the help of scalarization [2] or by second-order technique [15], [1]. Second-order technique based on Dini derivatives for problems without equality constraints and $C^{1,1}$ data (that is differentiable with locally Lipschitz derivatives) initiates in [14] (for problems with polyhedral cones) and goes on (for arbitrary cones) in [11] and [10]. A further generalization (toward relaxing the smoothness of the problem data) for (unconstrained) problems with ℓ-stable data can be found in [5]. In [12] using suitable elimination procedure this technique is extended to problems with equality constraints (with $C^{1,1}$ objective function and inequality constraints and C^2 equality constraints). The present paper using similar elimination establishes first-order conditions for problems with locally Lipschitz objective function and inequality constraints and C^1 equality constraints. An example demonstrates the effectiveness of the obtained conditions and shows that they can work in some cases when the conditions from [7] and [8] fail.

2. Preliminaries

For the norm and the dual pairing in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. From the context it should be clear to what spaces exactly these notations are applied.

For a cone $M \subset \mathbb{R}^k$ its positive polar cone is $M' = \{\zeta \in \mathbb{R}^k\colon\ \langle\zeta,\varphi\rangle \geqslant 0$ for all $\varphi \in M\}$. If $\varphi \in \operatorname{cl}\operatorname{conv} M$ we set $M'[\varphi] = \{\zeta \in M'\colon\ \langle\zeta,\varphi\rangle = 0\}$. Then $M'[\varphi]$ is a closed convex cone and $M'[\varphi] \subset M'$. Consequently its positive polar cone $M[\varphi] := (M'[\varphi])'$ is a closed convex cone, $M \subset M[\varphi]$ and $(M[\varphi])' = M'[\varphi]$. In this paper we apply the notation $M[\varphi]$ for $M = K$ and $\varphi = -g(x^0)$.

The solutions of (1) (and similarly for the problem (2) considered further) are understood in a local sense. In any case a solution is a feasible point x^0, that is a point satisfying the constraints. The feasible point x^0 is said to be a w-minimizer (weakly efficient point) for the problem (1) if there exists a neighbourhood U of x^0, such that $f(x) \notin f(x^0) - \operatorname{int} C$ for all feasible points $x \in U$.

To define an i-minimizer we need the concept of an oriented distance. Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is $d(y,A) = \inf\{\|a-y\|\colon\ a \in A\}$. The oriented distance from y to A is defined by $D(y,A) = d(y,A) - d(y,\mathbb{R}^k \setminus A)$. When $A = -C$, where C is a convex cone, then $D(y,-C) = \sup\{\langle \xi, y\rangle \colon\ \xi \in C',\ \|\xi\| = 1\}$ (here $\|\xi\|$ means the dual norm to the one given in $\mathbb{R}^k$).

We say that the feasible point x^0 is an i-minimizer (isolated minimizer of order 1) for the problem (1) (and similarly for (2)) if there exists a neighbourhood U of x^0 and a constant $A > 0$ such that

$$D\big(f(x) - f(x^0), -C\big) \geqslant A\|x - x^0\| \quad \text{for all feasible } x \in U.$$

The above definition generalizes to vector optimization problems the definition of an isolated minimizer for scalar problems from [4]. Some authors (e. g. [3]) use to say strict minimizers instead of isolated minimizers. The definition of an i-minimizer involves the norm. However, since any two norms in a finite dimensional real space are equivalent, the concept of an i-minimizer is actually norm-independent. Obviously, each i-minimizer is a w-minimizer.

For a given locally Lipschitz function $\Phi\colon\ \mathbb{R}^n \to \mathbb{R}^k$ the Dini derivative $\Phi'_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$ is defined as the set-valued Kuratowski limit

$$\Phi'_u(x^0) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t}\big(\Phi(x^0 + tu) - \Phi(x^0)\big).$$

If Φ is Fréchet differentiable at x^0 then the Dini derivative is a singleton and can be expressed in terms of the Jacobian $\Phi'_u(x^0) = \Phi'(x^0)u$. We will deal with the Dini derivative of the function $\Phi\colon\ \mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x), g(x))$. Then we use the notation $\Phi'_u(x^0) = \big(f(x^0), g(x^0)\big)'_u$. Let us note that always $\big(f(x^0), g(x^0)\big)'_u \subset f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide.

3. Problems with only inequality constraints

In this section following [9] we recall some necessary and sufficient optimality conditions for the problem with only inequality constraints

$$\min_C f(x), \quad g(x) \in -K. \tag{2}$$

The following constraint qualification of Kuhn-Tucker type appears in the Sufficient Conditions part of Theorem 1:

$$\mathbb{Q}_{0,1}(x^0) \qquad \begin{cases} \text{if } g(x^0) \in -K \text{ and } \dfrac{1}{t_k}\big(g(x^0 + t_k u^0) - g(x^0)\big) \to z^0 \in -K[-g(x^0)], \\ \text{then } \exists\, u^k \to u^0\ \exists\, k_0 \in \mathbb{N}\ \forall\, k > k_0\colon\ g(x^0 + t_k u^k) \in -K. \end{cases}$$

Theorem 1 ([9])**.** *Let f, g be locally Lipschitz functions and consider the problem* (2)*.*
(Necessary Conditions) *Let x^0 be a w-minimizer of the problem* (2)*. Then for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{N}'_{0,1} \qquad \begin{cases} \forall\,(y^0,z^0) \in \big(f(x^0),g(x^0)\big)'_u \ \exists\,(\xi^0,\eta^0) \in C' \times K'[-g(x^0)]\colon \\ (\xi^0,\eta^0) \neq (0,0) \text{ and } \langle \xi^0,y^0\rangle + \langle \eta^0,z^0\rangle \geqslant 0. \end{cases}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{S}'_{0,1} \qquad \begin{cases} \forall\,(y^0,z^0) \in \big(f(x^0),g(x^0)\big)'_u \ \exists\,(\xi^0,\eta^0) \in C' \times K'[-g(x^0)]\colon \\ (\xi^0,\eta^0) \neq (0,0) \text{ and } \langle \xi^0,y^0\rangle + \langle \eta^0,z^0\rangle > 0. \end{cases}$$

Then x^0 is an i-minimizer of order one for the problem (2)*.*

Conversely, if x^0 is an i-minimizer of order one for the problem (2) *and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ holds, then the condition $\mathbb{S}'_{0,1}$ is satisfied.*

4. Problems with inequality and equality constraints

In this section we generalize Theorem 1 to problems with both inequality and equality constraints. We prove our result under the assumption that at the feasible point x^0 the vectors $h'_1(x^0),\dots,h'_q(x^0)$, which are the components of $h'(x^0)$, are linearly independent. Under this assumption the considered problem (1) can be reduced to an equivalent problem with only inequality constraints to which Theorem 1 can be applied. Here we explain this reduction.

Let the vectors $\bar u^j \in \mathbb{R}^n$, $j = 1,\dots,q$, be determined by

$$(3) \qquad h'_k(x^0)\bar u^j = 0 \ \text{ for } \ k \neq j, \quad \text{and} \quad h'_j(x^0)\bar u^j = 1.$$

For each $j = 1,\dots,q$, the equalities (3) constitute a system of linear equations with respect to the components of $\bar u^j$, which due to the linear independence of $h'_1(x^0),\dots,h'_q(x^0)$ has a unique solution. Moreover, the vectors $\bar u^1,\dots,\bar u^q$ solving this system are linearly independent and $\mathbb{R}^n$ is decomposed into a direct sum $\mathbb{R}^n = L \oplus L'$, where $L = \ker h'(x^0)$ and $L' = \operatorname{lin}\{\bar u^1,\dots,\bar u^q\}$. Let $u^1,\dots,u^{n-q}$ be $n-q$ linearly independent vectors in $L = \ker h'(x^0)$. We consider the system of equations

$$(4) \qquad h_k\Bigg(x^0 + \sum_{i=1}^{n-q}\tau_i u^i + \sum_{j=1}^{q}\sigma_j \bar u^j\Bigg) = 0, \quad k = 1,\dots,q.$$

Taking $\tau_1, \dots, \tau_{n-q}$ as independent variables and $\sigma_1, \dots, \sigma_q$ as dependent variables, we see that this system satisfies the requirements of the implicit function theorem at the point $\tau_1 = \dots = \tau_{n-q} = 0$, $\sigma_1 = \dots = \sigma_q = 0$ (at this point h_k take the values $h_k(x^0) = 0$ because x^0 is feasible, and the Jacobian $\partial h/\partial\sigma$ is the unit matrix and hence is non degenerate). The implicit function theorem gives that in a neighbourhood of x^0 given by $|\tau_i| < \overline{\tau}$, $i = 1, \dots, n-q$, $|\sigma_j| < \overline{\sigma}$, $j = 1, \dots, q$, this system possesses a unique solution $\sigma_j = \sigma_j(\tau_1, \dots, \tau_{n-q})$, $j = 1, \dots, q$. The functions $\sigma_j = \sigma_j(\tau_1, \dots, \tau_{n-q})$ are C^1, and

$$\sigma_j\Big|_{\tau^0} = \sigma_j(0, \dots, 0) = 0, \quad j = 1 \dots, q, \tag{5}$$

$$\frac{\partial \sigma_j}{\partial \tau_i}\Big|_{\tau^0} = 0, \quad j = 1, \dots, q, \quad i = 1, \dots, n-q, \tag{6}$$

where $\tau^0 = (0, \dots, 0)$. To show the latter we differentiate (4) with respect to τ_i obtaining

$$h_k'\left(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j\right)\left(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i} \bar{u}^j\right) = 0.$$

For $\tau = \tau^0 = 0$ we get

$$h_k'(x^0)\left(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i}\Big|_{\tau^0} \bar{u}^j\right) = 0,$$

whence on account of $u^i \in \ker h'(x^0)$ and (3) we obtain (6).

The equivalence of the problem (1) with a problem with only inequality constraints is given in the next lemma.

Lemma 1 ([12]). *Consider the problem* (1) *with* $h \in C^1$, *for which* $h_1'(x^0), \dots, h_q'(x^0)$, *are linearly independent, and* C *and* K *are closed convex cones. Then* x^0 *is a* w*-minimizer or* i*-minimizer for* (1) *if and only if* $\tau^0 = 0$ *is respectively a* w*-minimizer or* i*-minimizer for the problem*

$$\min_C \overline{f}(\tau_1, \dots, \tau_{n-q}), \quad \overline{g}(\tau_1, \dots, \tau_{n-q}) \in -K, \tag{7}$$

where

$$\overline{f}(\tau_1, \dots, \tau_{n-q}) = f\left(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \dots, \tau_{n-q}) \bar{u}^j\right),$$

$$\overline{g}(\tau_1, \dots, \tau_{n-q}) = g\left(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \dots, \tau_{n-q}) \bar{u}^j\right).$$

The next theorem is our main result.

Theorem 2. *Consider the problem* (1) *with* f, g *being locally Lipschitz functions,* $h \in C^1$, *and* C *and* K *closed convex cones. Let* x^0 *be a feasible point and let the vectors* $h_1'(x^0), \ldots, h_q'(x^0)$, *the components of* $h'(x^0)$, *be linearly independent.*

(Necessary Conditions). *Let* x^0 *be a* w*-minimizer of the problem* (1). *Then for each* $u \in \ker h'(x^0) \setminus \{0\}$ *the following condition is satisfied:*

$$\mathbb{N}' \qquad \begin{cases} \forall\,(y^0, z^0) \in \big(f(x^0), g(x^0)\big)_u' \ \exists\,(\xi^0, \eta^0)\colon \\ (\xi^0, \eta^0) \in C' \times K'[-g(x^0)],\ (\xi^0, \eta^0) \neq (0,0) \\ \text{and } \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \geqslant 0. \end{cases}$$

(Sufficient Conditions). *Suppose that for each* $u \in \ker h'(x^0) \setminus \{0\}$ *the following condition is satisfied:*

$$\mathbb{S}' \qquad \begin{cases} \forall\,(y^0, z^0) \in \big(f(x^0), g(x^0)\big)_u' \ \exists\,(\xi^0, \eta^0)\colon \\ (\xi^0, \eta^0) \in C' \times K'[-g(x^0)],\ (\xi^0, \eta^0) \neq (0,0) \\ \text{and } \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle > 0. \end{cases}$$

Then x^0 *is an* i*-minimizer of the problem* (1).

Proof. According to Lemma 1 the feasible point x^0 is a w-minimizer or i-minimizer of the problem (1) if and only if $\tau^0 = (0, \ldots, 0)$ is respectively a w-minimizer or i-minimizer of the problem with only inequality constraints (7). It remains to apply Theorem 1 to (7) and to express the necessary and sufficient conditions through the data of the problem (1).

We deal first with the necessary conditions. Lemma 1 gives that if τ^0 is a w-minimizer of (7), then for each $\tau = (\tau_1, \ldots, \tau_{n-q}) \in \mathbb{R}^{n-q} \setminus \{0\}$ it holds

$$\begin{aligned} &\forall\,(y^0, z^0) \in \big(\overline{f}(\tau^0), \overline{g}(\tau^0)\big)_\tau' \ \exists\,(\xi^0, \eta^0) \in C' \times K'[-\overline{g}(\tau^0)]\colon \\ &\qquad (\xi^0, \eta^0) \neq (0,0) \quad \text{and} \quad \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \geqslant 0. \end{aligned} \tag{8}$$

To the fixed vector $\tau = (\tau_1, \ldots, \tau_{n-q})$ we juxtapose the vector

$$u = \sum_{i=1}^{n-q} \tau_i u^i. \tag{9}$$

Since the vectors $u^1, \ldots, u^{n-q}$ form a base in $\ker h'(x^0)$, obviously (9) gives a one-to-one correspondence between the vectors τ in $\mathbb{R}^{n-q} \setminus \{0\}$ and the vectors u in $\ker h'(x^0) \setminus \{0\}$. Now we express the condition (8) using the vector u instead of τ and x^0, f, g instead of τ^0, $\overline{f}$, $\overline{g}$.

We will show that (8) transforms into $\mathbb{N}'$. Observe that $K'[-\overline{g}(\tau^0)] = K'[-g(x^0)]$ due to $\overline{g}(\tau^0) = g(x^0)$. Therefore, $(\xi^0, \eta^0) \in C' \times K'[-\overline{g}(\tau^0)]$ can be written as

$(\xi^0, \eta^0) \in C' \times K'[-g(x^0)]$. It remains to show that $(y^0, z^0) \in \big(\overline{f}(\tau^0), \overline{g}(\tau^0)\big)'_\tau$ is equivalent to $(y^0, z^0) \in \big(f(x^0), g(x^0)\big)'_u$, where u and τ are related by (9). Indeed, let

$$y^0 = \lim_k \frac{1}{t_k}\big(\overline{f}(\tau^0 + t_k\tau) - \overline{f}(\tau^0)\big), \quad z^0 = \lim_k \frac{1}{t_k}\big(\overline{g}(\tau^0 + t_k\tau) - \overline{g}(\tau^0)\big),$$

with some sequence $t_k \to 0^+$. In order to prove that $(y^0, z^0) \in \big(f(x^0), g(x^0)\big)'_u$ it is enough to show that

$$y^0 = \lim_k \frac{1}{t_k}\big(f(x^0 + t_k u) - f(x^0)\big), \quad z^0 = \lim_k \frac{1}{t_k}\big(g(x^0 + t_k u) - g(x^0)\big).$$

We show only the first equality. The second one is derived similarly. Assume that f is Lipschitz with constant λ in a neighbourhood of x^0. Then

$$\begin{aligned}
&\frac{1}{t_k}\big(f(x^0 + t_k u) - f(x^0)\big) \\
&\quad = \frac{1}{t_k}\big(\overline{f}(\tau^0 + t_k\tau) - \overline{f}(\tau^0)\big) \\
&\qquad + \frac{1}{t_k}\left(f(x^0 + t_k u) - f\left(x^0 + t_k u + \sum_{j=1}^{q} \sigma_j(t_k\tau_1, \dots, t_k\tau_{n-q})\bar{u}^j\right)\right) \to y^0.
\end{aligned}$$

In the above limit the first term tends toward y^0 and the second toward 0. The latter follows by the following chain of inequalities, true for sufficiently large k:

$$\begin{aligned}
&\left|\frac{1}{t_k}\left(f(x^0 + t_k u) - f\left(x^0 + t_k u + \sum_{j=1}^{q} \sigma_j(t_k\tau_1, \dots, t_k\tau_{n-q})\bar{u}^j\right)\right)\right| \\
&\quad \leqslant \frac{\lambda}{t_k} \sum_{j=1}^{q} |\sigma_j(t_k\tau_1, \dots, t_k\tau_{n-q}) - \sigma_j(\tau^0)| \cdot \|\bar{u}^j\| \\
&\quad \leqslant \lambda \sum_{j=1}^{q} \sum_{i=1}^{n-q} \left|\frac{\partial \sigma_j}{\partial \tau_i}(\theta_k t_k \tau_1, \dots, \theta_k t_k \tau_{n-q})\right| \cdot \|\bar{u}^j\| \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Here $0 < \theta_k < 1$ is given by the mean-value theorem. We have also used the fact that $\sigma_j \in C^1$ and the equalities (5) and (6).

The above reasonings prove the Necessary Conditions of the theorem. The Sufficient Conditions are proved in a similar way. □

Let us make the following remark. Theorem 1 gives also the converse of the sufficient conditions. To obtain a similar converse for the problem (1) with both equalities and inequalities constraints we can write the constraint qualification $\mathbb{Q}_{0,1}(\tau^0)$ for the problem (7) and reformulate it in terms of the problem (1). What we get is the following constraint qualification:

$$\mathbb{Q}(x^0) \qquad \begin{cases} \text{if } g(x^0) \in -K,\ h(x^0) = 0,\ \bar{u} = \sum_{i=1}^{n-q} \overline{\tau}_i u^i \in \ker h'(x^0) \\ \text{and } \dfrac{1}{t_k}\big(g(x^0 + t_k\bar{u}) - g(x^0)\big) \to z^0 \in -K[-g(x^0)], \\ \text{then } \exists\, \bar{u}^k = \sum_{i=1}^{n-q} \overline{\tau}_i^k u^i \to \bar{u}\ \exists\, k_0 \in \mathbb{N} \\ \forall\, k > k_0\colon\ g\Big(x^0 + t_k\bar{u}^k + \sum_{j=1}^{q} \sigma_j(t_k\overline{\tau}_1^k, \ldots, t_k\overline{\tau}_{n-q}^k)\Big) \in -K. \end{cases}$$

It should be noted here that if at some feasible point x^0 the constrained qualification $\mathbb{Q}(x^0)$ holds, then the condition $\mathbb{S}'$ is implied by the fact that x^0 is an i-minimizer of the problem (1).

The next example shows the effectiveness of the conditions from Theorem 2 for particular problems. This example is used in the next section to compare Theorem 2 with some results of [8] and [7]. For brevity we omit some of the calculations. Applying Theorem 2 we follow the usual procedure. First we find the set N_w of the critical points, that is, the points satisfying the Necessary Conditions, which contains all the w-minimizers. Among the critical points we distinguish the set of the i-minimizers satisfying the Sufficient Conditions. The problem considered in this example is with locally Lipschitz data, but not with C^1 data (the function g is not C^1).

Example 1. Consider the problem (1), for which $n = 2$, $m = 2$, $p = 1$, $q = 1$, the cones are $C = \mathbb{R}^2_+$ and $K = \mathbb{R}_+$, and the functions f, g, h, are given by

$$f(x_1, x_2) = (x_1, -x_2), \quad g(x_1, x_2) = \min(x_1, x_2),$$
$$h(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2 - x_1 - x_2.$$

Then the sets N_w and S_i of the feasible points satisfying respectively the Necessary Conditions $\mathbb{N}'$ and the Sufficient Conditions $\mathbb{S}'$ are given by $N_w = N_w^1 \cup N_w^2$ and $S_i = S_i^1 \cup S_i^2$, where

$$\begin{aligned} N_w^1 &= \Big\{\Big(x_1, \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1 + 1})\Big)\colon\ \frac{3}{8} \leqslant x_1 \leqslant 1\Big\}, \\ N_w^2 &= \Big\{\Big(\frac{1}{2}(2x_2 + 1 - \sqrt{8x_2 + 1}), x_2\Big)\colon\ \frac{3}{8} \leqslant x_2 \leqslant 1\Big\}, \\ S_i^1 &= \Big\{\Big(x_1, \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1 + 1})\Big)\colon\ \frac{3}{8} < x_1 \leqslant 1\Big\}, \\ S_i^2 &= \Big\{\Big(\frac{1}{2}(2x_2 + 1 - \sqrt{8x_2 + 1}), x_2\Big)\colon\ \frac{3}{8} < x_2 \leqslant 1\Big\}. \end{aligned}$$

Indeed, the set of the feasible points in this example is $F = F^1 \cup F^2$, where

$$F^1 = \Big\{\Big(x_1, \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1 + 1})\Big) \colon\ 0 \leqslant x_1 \leqslant 1\Big\},$$
$$F^2 = \Big\{\Big(\frac{1}{2}(2x_2 + 1 - \sqrt{8x_2 + 1}), x_2\Big) \colon\ 0 \leqslant x_2 \leqslant 1\Big\}.$$

We have $h_1'(x) = h'(x) = (2x_1 - 2x_2 - 1, -2x_1 + 2x_2 - 1)$. Obviously, the two components of $h_1'(x)$ cannot vanish simultaneously, which guarantees the linear independence of the single-valued set $\{h_1'(x)\}$ at any feasible point x. Clearly, if $u \in \mathbb{R}^2$, then

$$h'(x)u = (2x_1 - 2x_2 - 1)u_1 + (-2x_1 + 2x_2 - 1)u_2,$$
$$\ker h'(x) = \{(2x_1 - 2x_2 + 1, 2x_1 - 2x_2 - 1)t \colon\ t \in \mathbb{R}\}.$$

The Dini derivatives are given by

$$f_u'(x) = f'(x)u = (u_1, -u_2),$$
$$g_u'(x) = \begin{cases} u_1, & x_1 < x_2, \\ u_1, & x_1 = x_2,\ u_1 \leqslant u_2, \\ u_2, & x_1 = x_2,\ u_2 \leqslant u_1, \\ u_2, & x_2 < x_1. \end{cases}$$

Obviously $C' = C = \mathbb{R}^2_+$ and $K' = K = \mathbb{R}_+$. For $z^0 \in K'$ we have also $K'[z^0] = \{0\}$ when $z^0 < 0$, and $K'[z^0] = \mathbb{R}_+$ when $z^0 = 0$.

Further we denote for brevity

$$\mathcal{L} = \mathcal{L}(\xi^0, \eta^0; y^0, z^0) = \langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle = \xi_1^0 y_1^0 + \xi_2^0 y_2^0 + \eta^0 z^0.$$

Let x be a feasible point and $u \in \ker h'(x) \setminus \{(0,0)\}$. We can distinguish the following cases:

1. $x_1 = \frac{1}{2}(2x_2 + 1 - \sqrt{8x_2 + 1})$, $\frac{3}{8} \leqslant x_2 \leqslant 1$.
   Now $y^0 = (u_1, -u_2)$, $z^0 = u_1$, $\mathcal{L} = \xi_1^0 u_1 - \xi_2^0 u_2 + \eta^0 u_1$, where

$$u_1 = (2x_1 - 2x_2 + 1)t = (2 - \sqrt{8x_2 + 1})t,$$
$$u_2 = (2x_1 - 2x_2 - 1)t = -\sqrt{8x_2 + 1}\, t,\ \ t \neq 0.$$

   We have the possibilities:
   1a. $t > 0$. Taking $\xi^0 = (0, 1)$, $\eta^0 = 0$, we get $\mathcal{L} = \sqrt{8x_2 + 1}\, t > 0$.
   1b. $t < 0$. Taking $\xi^0 = (1, 0)$, $\eta^0 = 0$, we get $\mathcal{L} = (2 - \sqrt{8x_2 + 1})t \geqslant 0$ with strict inequality for $x_2 > \frac{3}{8}$ and equality for $x_2 = \frac{3}{8}$.
2. $x_1 = \frac{1}{2}(2x_2 + 1 - \sqrt{8x_2 + 1})$, $0 < x_2 < \frac{3}{8}$.

Now y^0, z^0, u and $\mathcal{L}$ are expressed as in the case **1**. In particular

$$\mathcal{L} = (\xi_1^0 + \eta^0)(2 - \sqrt{8x_2+1})t + \xi_2^0\sqrt{8x_2+1}\,t < 0$$

for all $t < 0$ and $(\xi^0, \eta^0) \in C' \times K'[-g(x)] = \mathbb{R}_+^2 \times \{0\}$, $(\xi^0, \eta^0) \neq (0,0,0)$, since

$$(2 - \sqrt{8x_2+1})t < 0 \quad \text{and} \quad \sqrt{8x_2+1}\,t < 0.$$

3. $x_2 = \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1+1})$, $\frac{3}{8} \leqslant x_1 \leqslant 1$.
Now $y^0 = (u_1, -u_2)$, $z^0 = u_2$, $\mathcal{L} = \xi_1^0 u_1 - \xi_2^0 u_2 + \eta^0 u_2$, where

$$u_1 = (2x_1 - 2x_2 + 1)t = \sqrt{8x_1+1}\,t,$$
$$u_2 = (2x_1 - 2x_2 - 1)t = (-2 + \sqrt{8x_1+1})t, \quad t \neq 0.$$

We have the possibilities:
3a. $t > 0$. Taking $\xi^0 = (1,0)$, $\eta^0 = 0$, we get $\mathcal{L} = \sqrt{8x_1+1}\,t > 0$.
3b. $t < 0$. Taking $\xi^0 = (0,1)$, $\eta^0 = 0$, we get $\mathcal{L} = (2 - \sqrt{8x_1+1})t \geqslant 0$ with strict inequality for $x_1 > \frac{3}{8}$ and equality for $x_1 = \frac{3}{8}$.
4. $x_2 = \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1+1})$, $0 < x_1 < \frac{3}{8}$.
Now y^0, z^0, u and $\mathcal{L}$ are expressed as in the case 3. In particular

$$\mathcal{L} = \xi_2^0\sqrt{8x_1+1}\,t + (\xi_2^0 - \eta^0)(2 - \sqrt{8x_1+1})t < 0$$

for all $t < 0$ and $(\xi^0, \eta^0) \in C' \times K'[-g(x)] = \mathbb{R}_+^2 \times \{0\} \setminus \{(0,0,0)\}$, since

$$\sqrt{8x_1+1}\,t < 0 \quad \text{and} \quad (2 - \sqrt{8x_1+1})t < 0.$$

5. $x_1 = 0$, $x_2 = 0$.
Now $y^0 = (u_1, -u_2)$, $z^0 = u_1$ when $u_1 \leqslant u_2$ and $z^0 = u_2$ when $u_2 \leqslant u_1$,

$$u_1 = (2x_1 - 2x_2 + 1)t = t,$$
$$u_2 = (2x_1 - 2x_2 - 1)t = -t, \quad t \neq 0,$$
$$\mathcal{L} = \xi_1^0 u_1 - \xi_2^0 u_2 + \eta^0 z^0 = \begin{cases} (\xi_1^0 + \xi_2^0 - \eta^0)t, & t > 0, \\ (\xi_1^0 + \xi_2^0 + \eta^0)t, & t < 0. \end{cases}$$

Obviously, when $t < 0$ we have $\mathcal{L} < 0$.

Thus, on the basis of Theorem 2 we see that the points which do not belong to the set N_w determined above are not w-minimizers, and the points from the set S_i are i-minimizers. The efficiency for points in the set $N_w \backslash S_i = \{(-1/8, 3/8), (3/8, -1/8)\}$ needs a separate investigation. It can be shown directly from the definition that the point $(-1/8, 3/8)$ is a w-minimizer but not an i-minimizer (actually it is an isolated minimizer of order 2, a concept defined in [10]), while the point $(3/8, -1/8)$ is not a w-minimizer.

5. Some comparison

First-order optimality conditions for the problem (1) with locally Lipschitz functions are well-known from the classical monograph of Clarke [7] (see Theorem 6.3.1 therein), where the particular case $C = K = \mathbb{R}^n_+$ is treated. A generalization to problems with arbitrary cones C and K is presented in [8] and involves Clarke's generalized Jacobians. Recall that Clarke's generalized Jacobian for the vector function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ at a point x^0, denoted by $\partial f(x^0)$, is defined as the convex hull of all limits of sequences $f'(x^k)$, where $x^k \to x^0$ and the gradient $f'(x^k)$ exists. The following result is a particular case of Theorem 2 in [8].

Theorem 3. *Consider the problem* (1) *with f, g being locally Lipschitz functions, $h \in C^1$, and C and K closed convex cones. Let x^0 be a feasible point and assume it is a w-minimizer of the problem* (1). *Then there exist vectors $\tau \in C'$, $\lambda \in K'[-g(x^0)]$, $\mu \in \mathbb{R}^q$, not all zero, such that*

$$0 \in \partial(\tau f + \lambda g + \mu h)(x^0). \tag{10}$$

The following observation gives some comparison between Theorems 3 and 2.

Observation. Consider the problem (1) with f, g and h as defined in Example 1 and let N_w be the set described there. Then the set of points satisfying the condition (10) is $N_w^C = N_w \cup \{(0,0)\}$. Therefore, Theorem 3 does not reject the point $(0,0)$ as a w-minimizer, while Theorem 2 does (because $(0,\,0) \notin N_w$).

Indeed, it is easy to check that all the points in the set N_w satisfy the necessary conditions of Theorem 3. This is easily seen, since the functions f, g, and h are continuously differentiable at the points $x \in N_w$. Let $\operatorname{conv} A$ denote the convex hull of the set A. At the point $(0,0)$, which clearly is not a w-minimizer, we have $\partial g(0,0) = \operatorname{conv}\{(1,0),(0,1)\}$, while $g_1(x) = x_1$ and $g_2(x) = x_2$ are continuously differentiable at $(0,0)$ and their generalized Jacobian coincides with their gradient. Straightforward calculations show that the condition (10) is satisfied choosing $\tau = (0,1)$, $\lambda = 1$, and $\mu = 0$. Hence, the necessary conditions of Theorem 3 are satisfied at $(0,0)$, although $(0,0)$ is not a w-minimizer.

Similarly, one can show that also the necessary optimality conditions given in Clarke [7, Theorem 6.3.1] hold at the point $(0,0)$. On the contrary, the necessary conditions of Theorem 2 do not hold at $(0,0)$ and on this basis it follows that this point is not a w-minimizer.

This observation is significant, since in fact $(0,0)$ is the only point requiring special attention. Indeed, Clarke's generalized Jacobian is introduced to treat nonsmooth problems. But $(0,0)$ is the only point among those satisfying the equality constraints at which the problem fails to be C^1.

It is also worth recalling that neither Theorem 3 nor Theorem 6.3.1 in [7] give sufficient optimality conditions, while Theorem 2 does. Moreover, Theorem 2 allows to distinguish the i-minimizers, which as Example 1 shows are rather typical type of solutions for vector optimization problems.

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