LOCALLY NOETHERIAN COMMUTATIVE RINGS

BY

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Abstract. This paper centers around the theorem that a commutative ring R is noetherian if every R_P, P prime, is noetherian and every finitely generated ideal of R has only finitely many weak-Bourbaki associated primes. A more precise local version of this theorem is also given, and examples are presented to show that the assumption on the weak-Bourbaki primes cannot be deleted nor replaced by the assumption that Spec (R) is noetherian. Moreover, an alternative statement of the theorem using a variation of the weak-Bourbaki associated primes is investigated. The proof of the theorem involves a knowledge of the behavior of associated primes of an ideal under quotient ring extension, and the paper concludes with some remarks on this behavior in the more general setting of flat ring extensions.

Our terminology is that of Zariski-Samuel [17] and Bourbaki [3]. We use R to denote a commutative ring with identity and A to denote an ideal of R. (All rings of this paper are assumed to be commutative with identity.) By "ideal" we shall always mean an ideal $\neq R$. A prime ideal P of R is called an MPD (minimal prime divisor) of A if P is minimal among the prime ideals containing A. A is said to be a *radical ideal* if $A = \sqrt{A}$. If S and T are sets, we use $S \backslash T$ to denote $\{s \in S \mid s \notin T\}$; and $\subseteq$ denotes containment and $<$ strict containment.

1. **The main theorem.** We shall discuss in detail in §3 the concept of associated primes of an ideal, but for the present it will suffice to give the following definition. A prime ideal P of R is called a B_w-*prime of* A (weak-Bourbaki associated prime, [3b, p. 165, Exercise 17]) if there exists $x \in R$ such that P is an MPD of $A : x$.

1.1. LEMMA. *If A has a unique B_w-prime P, then A is P-primary (and conversely).*

Proof. Suppose $xy \in A$, $x \notin P$. If $y \notin A$, then $A : y < R$. But $x \in A : y$ implies $A : y \notin P$; so if Q is an MPD of $A : y$, then Q is a B_w-prime of A different from P. Thus, $y \in A$.

1.2. PROPOSITION. *Let R' be a quotient ring of R (with respect to a multiplicative system), let A be an ideal of R and $A' = AR'$, and let P be a prime ideal of R such that $P' = PR' \neq R'$. Then P is a B_w-prime of A if and only if P' is a B_w-prime of A'.*

Proof. $\Rightarrow$. P is an MPD of $A:x$ for some $x \in R$. Since R' is a quotient ring of R, $(A:x)R' = AR':x$; and there exists a 1-1 correspondence between primes of R' and primes of R which do not blow up, effected by extension and contraction. Therefore P' is an MPD of $AR':x$.

$\Leftarrow$. P' is a B_w-prime of A' implies there exists $y \in R'$ such that P' is an MPD of $A':y$. Then $P'R'_{P'}$ is an MPD of $(A':y)R'_{P'} = A'R'_{P'}:y$. But $R'_{P'}$ is canonically isomorphic to R_P, so if φ is the canonical homomorphism of R into $R'_{P'}$, then there exists $s \in R\backslash P$ and $r \in R$ such that $\varphi(s)$ is a unit in $R'_{P'}$ and $\varphi(s)y = \varphi(r)$. Therefore $A'R'_{P'}:y = A'R'_{P'} : \varphi(s)y = (A:r)R'_{P'}$, the last equality being a consequence of the fact that $R'_{P'}$ is a quotient ring of R. It follows that PR_P is an MPD of $(A:r)R_P$, and hence P is an MPD of $A:r$.   Q.E.D.

The above proposition is an exercise in [3b, p. 166]. We include a proof because we are interested in §4 in the degree to which Proposition 1.2 remains true when we replace the hypothesis that R' is a quotient ring of R by the weaker assumption that R' is merely a flat R-algebra.

In the proofs of the next theorem and its corollary we use the following lemma:

LEMMA. *Let $T_i = \{P_{ij}\}_{j=1}^{n_i}$, $i = 1, 2, \ldots$, be a collection of finite sets (possibly empty), and let $T = \bigcup T_i$. Suppose that T is partially ordered with respect to a relation $\leq$ and that T with respect to this relation satisfies the ascending chain condition and has the further property that if $P_{i+1j} \in T_{i+1}$, then there exists $P_{ik} \in T_i$ such that $P_{i+1j} > P_{ik}$. Then there exists a natural number n such that T_i is empty for all $i \geq n$.*

Proof. Suppose that for each natural number n there exists $i \geq n$ such that T_i is nonempty. It follows from our hypotheses that T_i is nonempty for all i. Let $T_1^n = \{P_{1j_1} \mid$ there exists a chain $P_{1j_1} < P_{2j_2} < \cdots < P_{nj_n}\}$. Since T_n is nonempty, it follows that T_1^n is also nonempty. Moreover, $T_1^n \supset T_1^m$ for $m \geq n$; and T_1^n is finite. Therefore $\bigcap_{n=1}^{\infty} T_1^n$ is nonempty. Choose $P_1 \in \bigcap_{n=1}^{\infty} T_1^n$.

Now, for each $n \geq 2$, let $T_2^n = \{P_{2j_2} \mid$ there exists a chain $P_1 < P_{2j_2} < \cdots < P_{nj_n}\}$. Since $P_1 \in T_1^n$, T_2^n is nonempty. Also, $T_2^n \supset T_2^m$ for $m \geq n$; and T_2^n is finite. Therefore $\bigcap_{n=2}^{\infty} T_2^n$ is nonempty. Choose $P_2 \in \bigcap_{n=2}^{\infty} T_2^n$.

Thus, proceeding inductively, we construct a chain $P_1 < P_2 < \cdots$, which contradicts the ascending chain condition.

1.3. THEOREM. *Let P be a prime ideal of R, and suppose the following hold:*

(i) *PR_M is finitely generated for every maximal ideal $M \supset P$ (and hence also for every prime ideal containing P),*

(ii) *there exists a finitely generated ideal B_0 such that $\sqrt{B_0} = P$ and such that for every finitely generated ideal B with $B_0 \subset B \subset P$, B has only finitely many B_w-primes,*

(iii) *the set of B_w-primes of the ideals B of (ii) satisfies the a.c.c. (ascending chain condition).*

Then P is finitely generated.

Proof. Let B be a finitely generated ideal of R such that $B_0 \subset B \subset P$ and such that $BR_P = PR_P$. By (ii), B has only finitely many B_w-primes different from P, say

$P_1, \ldots, P_n$. Then $\sqrt{B} = P$ implies $P \subset P_i$. Since the P_i are finite in number and each PR_{P_i} is finitely generated (by (i)), there exists a finitely generated ideal A such that $B \subset A \subset P$ and $AR_{P_i} = PR_{P_i}$, $i = 1, \ldots, n$. Note that each B_w-prime of A contains $P = \sqrt{A}$.

Claim. If $Q_1, \ldots, Q_m$ is the set of B_w-primes of A different from P, then each Q_i properly contains some P_j. To see this, it suffices to show that if Q is a prime ideal of R which contains P but does not properly contain some P_j, then $AR_Q = PR_Q$; for AR_{Q_i} has $Q_i R_{Q_i}$ as a B_w-prime by Proposition 1.2, and hence $AR_{Q_i} \neq PR_{Q_i}$. If $Q = P_j$ for some j, then by the choice of A, $AR_Q = PR_Q$. On the other hand, if $Q \not\supset P_j$ for all j, then BR_Q has PR_Q as its unique B_w-prime by 1.2. But then BR_Q is PR_Q-primary by Lemma 1.1. Since R_P is a quotient ring of R_Q and $BR_P = PR_P$, we therefore conclude that $BR_Q = PR_Q$, and a fortiori $AR_Q = PR_Q$.

We proceed to define inductively a chain $\{A_i\}$ of finitely generated ideals containing B_0 and contained in P and such that $A_i R_P = PR_P$. We take A_0 to be the ideal generated by the ideal B_0 of (ii) and a finite set of elements of R which generate PR_P (by (i) such a set exists). Then $B_0 \subset A_0 \subset P$ and $A_0 R_P = PR_P$. If now $N_1, \ldots, N_r$ are the B_w-primes of A_n distinct from P, then we choose A_{n+1} to be a finitely generated ideal such that $A_n \subset A_{n+1} \subset P$ and $A_{n+1} R_{N_i} = PR_{N_i}$, $i = 1, \ldots, r$. By our previous observations and the a.c.c. on the B_w-primes of the $\{A_i\}$, we can apply the above lemma to conclude that there exists m such that A_m has P as its unique B_w-prime. Then A_m is P-primary by Lemma 1.1. Since $A_m R_P = PR_P$, we therefore conclude that $A_m = P$. Thus, P is finitely generated.   Q.E.D.

Note that the converse to Theorem 1.3 is trivially true. The following corollary gives a criterion for R to be noetherian when every localization of R is. In §§2 and 3 we shall justify by examples the hypotheses used here.

1.4. COROLLARY. *If every R_P, P a prime ideal of R, is noetherian and every finitely generated ideal of R has only finitely many B_w-primes, then R is noetherian.*

Proof. By Cohen's theorem, it is sufficient to check that every prime ideal P of R is finitely generated. Any chain of primes is contained in a maximal prime M, so R_M is noetherian implies the chain must be finite; thus, (iii) of Theorem 1.3 is satisfied. We must yet check that $P = \sqrt{B_0}$ for some finitely generated ideal B_0. Since PR_P is finitely generated, there exists a finitely generated ideal A of R such that $AR_P = PR_P$. Since A has only finitely many B_w-primes by hypothesis, A has a fortiori only finitely many MPD's, one of them being P. Therefore we can enlarge A to a finitely generated ideal A_1 such that $A \subset A_1 \subset P$ and such that no MPD of A_1 different from P is an MPD of A. We similarly enlarge A_1 and thus construct a chain $A \subset A_1 \subset A_2 \subset \cdots \subset P$ such that no MPD of A_{i+1} different from P is an MPD of A_i. By applying the above lemma to the MPD'S different from P of the A_i, we conclude that there exists an n such that P is the only MPD of A_n. Therefore $A_n = B_0$ is the required ideal.   Q.E.D.

2. **Two examples.** We shall give in this section two examples to show that some obvious conjectures in connection with Corollary 1.4 must be ruled out. First, the most obvious conjecture is that every R_P is noetherian implies R is noetherian. The customary counterexample is due to Nakano [13, p. 426] and is quite involved. (See also [2, §4] where a generalized form is worked out.) It is possible to give a simple example, which we do in 2.2.

Thus, the assumption in 1.4 that every finitely generated ideal of R has only finitely many B_w-primes, or some alternative, is certainly necessary for the validity of the corollary. A pleasant alternative would be the assumption that Spec (R) is noetherian, i.e. that radical ideals of R have a.c.c. Some equivalent statements are the following [15]:

(i) Spec (R) is noetherian.

(ii) For any ideal A of R, there exists a finitely generated ideal A_0 such that $\sqrt{A} = \sqrt{A_0}$.

(iii) For any prime ideal P of R, there exists a finitely generated ideal P_0 such that $P = \sqrt{P_0}$.

(iv) Every ideal of R has only finitely many MPD's and R has a.c.c. on prime ideals.

Our example 2.3 is considerably more complicated than 2.2 and is a finite-dimensional domain R which satisfies (i)–(iv) and has every R_P noetherian but is not itself noetherian. Thus, this condition is not suitable either. Incidentally, the example also shows then that the assumption in Theorem 1.3 that every finitely generated ideal B such that $B_0 \subset B \subset P$ have only finitely many B_w-primes cannot be omitted.

2.2. Example of a domain D such that every D_P is a noetherian valuation ring but D is not itself noetherian.

Since this example uses the technique of constructing a domain by means of an appropriate lattice ordered group, we shall review first a few of the facts related to this construction. If D is a domain with quotient field K, let $U(D)$ denote the group of units of D, and let * denote nonzero elements. Let $\varphi: K^* \to K^*/U(D) = G$ be the canonical homomorphism of the (multiplicative) group K^* onto the group G. It is customary to switch to additive notation for G, and we do so. An ordering is defined on G by taking $G^+ = \varphi(D^*)$, and the resulting ordered group is called the *group of divisibility* of D (where here $G^+ = \{g \in G \mid g \geq 0\}$). We are particularly interested in the case when G is lattice ordered, since we then have

THEOREM (JAFFARD [8, p. 78]). *If G is a lattice ordered group, then there exists a domain D having G as group of divisibility. We henceforth assume that G denotes a lattice ordered group (although much of what follows can be done in greater generality; see [14]).*

We now carry over the ideal-theoretic concepts from D to G. A *segment* of a lattice ordered group G is a subset $A \neq G^+$ of G^+ such that $a \in A$ and $b \geq a$ implies

$b \in A$, and $a, b \in A$ implies inf $\{a, b\} \in A$. A is called a *prime segment* if $a, b \in G^+ \backslash A$ implies $a+b \notin A$. A is a *principal segment* if there exists $a \in A$ such that $A = \{b \in G^+ \mid b \geq a\}$. G is *noetherian* if the a.c.c. holds for the segments. dim G (or rank G)$= n$ if there exists a chain $P_1 < P_2 < \cdots < P_n$ of n prime segments and no such longer chain, etc. Finally, let us sketch yet how the concept of localization at a prime carries over to G (see [8, p. 36]). If P is a prime segment of G, let H be the subgroup of G generated by $G^+ \backslash P$. Let φ_P be the canonical homomorphism $G \to G/H = G_P$, and define an order on G_P by taking $G_P^+ = \varphi_P(G^+)$. G_P is the *localization of G at P*. If $P' = \varphi^{-1}(P) \cup 0$ then P' is a prime ideal of D; and G_P is then the group of divisibility of $D_{P'}$, with $\varphi_P \cdot \varphi : K^* \to G_P$ being the associated group of divisibility homomorphism. Note in particular that when G_P is totally ordered, then $\varphi_P \cdot \varphi$ is a valuation having ring $D_{P'}$.

We conclude this summary with a few elementary observations about the relationship between D and G. If A is any segment (prime segment) of G, then $\varphi^{-1}(A) \cup \{0\}$ is an ideal (prime ideal) of D. Conversely, *if one uses the K and φ given by the proof of Jaffard's theorem*, it can be seen that for any ideal $A \neq (0)$ of D, $\varphi(A^*)$ is a segment of G, and $\varphi^{-1}(\varphi(A^*)) = A^*$. Thus, if one uses Jaffard's construction, there exists a 1-1 correspondence between nonzero ideals of D and segments of G; and this correspondence preserves primeness and most of the other ideal-theoretic concepts. It can furthermore be seen [7, p. 1370] that the D given by Jaffard's construction is actually a Bezout domain (i.e. finitely generated ideals are principal).

To construct example 2.2, it is sufficient then to give a lattice ordered group G such that for every prime segment P of G, the group G_P is order isomorphic to Z, where $Z=$ additive group of integers with the usual order. The idea for the example comes from a similar example given by Underwood in [16, Example 3.10].

Let $N = \{1, 2, 3, \ldots\}$, $Z^+ = \{0, 1, 2, 3, \ldots\}$. Let $G =$ group of all functions from N to Z such that $f(i)$ is constant for all i greater than some integer, i.e. $G = \{f \in Z^N \mid f(N) \text{ is finite}\}$. G is a lattice ordered group under the ordering $f \leq g$ if and only if $f(i) \leq g(i)$ for all $i \in N$. Let

$$P_i = \{f \in G \mid f(i) > 0\}, \qquad i = 1, 2, \ldots,$$
$$P_\infty = \{f \in G \mid \text{there exists } n \text{ such that } f(i) > 0 \text{ for all } i > n\}.$$

P_i, P_∞ are clearly prime segments of G.

Claim. These are the only prime segments. For suppose Q is a prime segment $\nsubseteq P_i, P_\infty, i = 1, 2, \ldots$. Since $Q \nsubseteq P_\infty$, there exists $f_\infty \in Q$ such that $f_\infty(i) = 0$ for all $i >$ some integer n. Also, $Q \nsubseteq P_1, \ldots, P_n$ implies there exists $f_j \in Q$ such that $f_j(j) = 0$, $j = 1, \ldots, n$. Therefore, $0 = \inf\{f_1, \ldots, f_n, f_\infty\} \in Q$, a contradiction. Thus, P_i, P_∞ are the only *maximal* segments of G. Now consider the maps $\varphi_i : G \to Z$, $i = 1, 2, \ldots, \infty$, defined by

$$\varphi_j(f) = f(j), \qquad j = 1, 2, \ldots,$$
$$\varphi_\infty(f) = f(n+1),$$

where $n+1$ is chosen such that f is constant for $i \geq n+1$. The φ_i are lattice homomorphisms of G onto Z. Since Z has the unique prime segment $Z^+ \backslash 0$, it follows from the usual correspondence between prime segments under a lattice homomorphism that P_i contains no prime segment properly. Thus, P_i, $i = 1, \ldots, \infty$, are the only prime segments of G; and G_{P_i} is order isomorphic to Z. Finally, G is not noetherian since $f = 1$ has infinitely many MPD's. It is also interesting to note that each P_i, $i < \infty$, is principal but P_∞ is not.

2.3. Example of a (2-dim) domain R such that R_P is noetherian for every prime ideal P of R and Spec (R) is noetherian, but R is not noetherian. (By Corollary 1.4, a 1-dim example of this type cannot exist.) It follows from the usual theorems on integral extensions that if $R \subset R'$ and R' is integral over R, then R' satisfies (iv) of 2.1 implies R does also; so it is sufficient to exhibit a nonnoetherian domain R such that

    (a) R_P is noetherian for every prime P of R,

    (b) there exists a noetherian domain R' such that R' is integral over R.

Let k be a field of characteristic 0 and $x_1, x_2, \ldots, y, z$ be indeterminates over k, let $K = k(x_1, x_2, \ldots)$, and let $J = K[y, z]$. $P_i = (y, z-i)$, $i = 1, 2, \ldots$, are prime ideals of J; and $J_{P_i} = R_i'$ are 2-dim local domains of the form $R_i' = K + M_i$, where M_i = maximal ideal of R_i'. Consider the subring $R_i = K_i + M_i$ of R_i', where $K_i = k(x_1, x_2, \ldots, x_i^i, \ldots)$ is the subfield of K generated by $\{x_j\}_{j \neq i}$ and x_i^i over k. Note that R_i' is a finite integral extension of R_i and hence that R_i is a 2-dim local domain. (That R_i is noetherian can easily be checked directly, or else apply Eakin's theorem [4] or [11].) Let $R = \bigcap R_i$ and $R' = \bigcap R_i' = \bigcap J_{P_i}$. R' is then a Krull ring whose essential valuation rings are the localizations of J at those minimal primes which are contained in some P_i. Since the minimal primes of J are principal, this set can also be described as the set of minimal primes of J which are contained in $\bigcup P_i$. Therefore by [12, p. 116, 33.5], $R' = J_S$, where $S = J \backslash \bigcup P_i$.

We next note that $\{P_i\}$ is exactly the set of primes of J which are maximal with respect to not meeting S. For, if $Q \neq 0$ is a prime of J such that $Q \subset \bigcup P_i$, then Q has height 1 or 2. Since J is a UFD, Q has height 1 implies Q is principal; and then $Q \subset P_i$ for some i. If height $Q = 2$, then Q is maximal in J; so $Q + (y) \subset \bigcup P_i$ implies $y \in Q$. But Q is principal mod (y), so $Q = (y, f)$, $f \in K[z]$. Then $f \in \bigcup P_i$ implies $f \in P_i$ for some i: so $Q \subset P_i$. It now follows that $\{P_i J_S\}$ is exactly the set of maximal ideals of $J_S = R'$.

Now let us check that R' is integral over R. $r' \in R'$ implies $r' \in k(x_1, \ldots, x_n)(y, z) = L_n$, for some n. $r' \in R_i' \cap L_n$ implies $r' = a_i + b_i$, $a_i \in k(x_1, \ldots, x_n)$, $b_i \in M_i$. Since $a_1, \ldots, a_n$ are integral over $k(x_1, x_2^2, \ldots, x_n^n)$, there exists a monic polynomial $f(T) \in k(x_1, x_2^2, \ldots, x_n^n)[T]$ such that $f(a_i) = 0$, $i = 1, \ldots, n$. Therefore $f(r') \in M_i \subset R_i$, $i = 1, \ldots, n$; so $f(r') \in \bigcap R_i = R$. Thus, r' is integral over R.

Let us denote the prime $P_i J_S$ of R' by N_i'. As we have observed, $\{N_i'\}$ is exactly the set of maximal ideals of R'. Since R' is integral over R, it follows that $\{N_i\}$, where $N_i = N_i' \cap R$, is exactly the set of maximal ideals of R. To show each R_P is

noetherian, it suffices to show then that each R_{N_i} is noetherian. We shall, in fact, show that $R_{N_i} = R_i$. $R_{N_i} \subseteq R_i$ is clear since N_i is contained in the maximal ideal M_i of R_i. Conversely, if $a \in R_i$, we must exhibit an $r \in R \setminus N_i$ such that $ra \in R$. Since $R' = J_S$ and $R'_i = J_{P_i}$ and $N'_i = P_i J_S$, it follows that $a \in R'_i = R'_{N'_i}$. Therefore there exists $r' \in R' \setminus N'_i$ such that $r'a \in R'$. If $L_n = k(x_1, \ldots, x_n, y, z)$, then $r'a$ and r' are in L_n for some n; so $r'a$ and r' are in $R_j, j > n$. We also have $r' = b + m$, $b \in K \cap L_n$, $m \in M_i$; and since r' is a unit of R'_i, we see that $b \neq 0$. Hence there is a monic polynomial $g(T) \in (K_i \cap L_n)[T]$ such that $g(b)$ is a nonzero element of K_i. It follows that $g(r')$ is a unit of R_i. Also, $a \cdot g(r') \in R'$; so if $r = \pi(z - j)g(r'), j = 1, \ldots, n, j \neq i$, then $r \in R \setminus N_i$ and $ra \in R$. Thus, $R_{N_i} = R_i$ and each localization of R is noetherian.

Finally, we observe that R is not itself noetherian. For, if $i \geq 2$, then $x_i y \notin yR$ but $(z - i)x_i y \in yR$. Since R' is integral over R and N'_i is the only prime of R' which contains both y and $z - i$, it follows that $N_i = \sqrt{(yR : x_i y)}, i \geq 2$. Thus, yR is an ideal of R with an infinite number of B_w-primes (in fact, an infinite number of ZS-primes; see §3), so R is not noetherian.

3. Associated prime ideals.

We devote this section to an examination of some alternative hypotheses in Theorem 1.3 and Corollary 1.4. Let us begin by considering some possible alternatives to the B_w-primes we employ there. The following four definitions of associated primes of an ideal A can be found, either as definitions or as characterizations, in [3b, p. 131], [17, Vol. 1, p. 211], [3b, p. 165, Exercise 17], and [12, §7], respectively.

3.1. DEFINITIONS. (B) P is an associated prime of A in the Bourbaki sense if $P = A : x$ for some $x \in R$.

(ZS) P is an associated prime of A in the Zariski-Samuel sense if there exists $x \in R$ such that $P = \sqrt{(A : x)}$.

(B_w) P is an associated prime of A in the weak-Bourbaki sense if there exists $x \in R$ such that P is an MPD (minimal prime divisor) of $A : x$.

(N) P is an associated prime of A in the Nagata sense if there exists a multiplicative system S of R such that $S \cap A = \varnothing$ and such that $PR_S \supset AR_S$ and PR_S is maximal with respect to being contained in the set of zero-divisors mod AR_S.

When R is noetherian, then the associated primes in any of the above senses coincide with the usual associated primes of A. In general, P satisfies (B) $\Rightarrow P$ satisfies (ZS) $\Rightarrow P$ satisfies (B_w) $\Rightarrow P$ satisfies (N), and none of these implications is reversible. See [16] for details([2]).

(B) and (ZS) seem to be too restrictive to be of much use in the nonnoetherian case; for example, an ideal may have no ZS-primes. On the other hand, at least every MPD of an ideal must be a B_w-prime. Moreover, as we have shown in Proposition 1.2, the B_w-primes localize well.

It is conceivable that replacing "B_w-primes" by "ZS-primes" in Corollary 1.4 would yield a better result. However, it can be seen that Nakano's example [13,

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([2]) The ZS-primes defined by Underwood in [16] are slightly different from our ZS-primes.

p. 426] gives a domain D such that every D_P, P prime, is a noetherian valuation ring and yet every finitely generated ideal of D has no ZS-primes; so 1.4 is, in fact, false with this replacement. (Such a domain can also be constructed by using Jaffard's theorem as in 2.2 but we shall omit the details. Note that 2.2 does not itself yield the required example since it is easily seen that there exist principal ideals of that example which have infinitely many ZS-primes.) We shall show in Corollary 3.4 though that if a mild condition is added to the hypothesis of 1.4, then such a change to ZS-primes can be successfully made. We first examine how to reformulate Theorem 1.3 and Corollary 1.4 in terms of ZS-primes.

3.2. LEMMA. *Let A be an ideal of R. If $\{\sqrt{(A:x)}\}$, $x \in R$, has a.c.c., then every B_w-prime of A is also a ZS-prime.*

Proof. Suppose A has a B_w-prime P which is not a ZS-prime. Among the ideals of the form $\sqrt{(A:x)}$ which have P as an MPD, choose a maximal such ideal $B = \sqrt{(A:x)}$. $P \neq B$ since P is not a ZS-prime of A, so there exists an MPD Q of B such that $Q \neq P$. Choose $y \in Q \backslash P$. y is in an MPD of B implies y is a zero-divisor mod B [3a, p. 94]. Therefore, there exists $z \notin B$ such that $zy \in B$. By raising to a power, we may assume $zy \in A:x$. If now $B' = \sqrt{(A:xy)}$, then $B \subset B'$ and $z \in B'$; so $B < B'$. Moreover, $y \notin P$ implies $A:xy = (A:x):y \subset P$; so $B' \subset P$. Thus, B' contradicts the maximality of B.   Q.E.D.

The following theorem is our version of Theorem 1.3 in terms of ZS-primes.

3.3. THEOREM. *If P is a prime ideal of R such that*
  (i) *PR_M is finitely generated for every maximal ideal $M \supset P$,*
  (ii) *there exists a finitely generated ideal B_0 such that $\sqrt{B_0} = P$ and such that for every finitely generated ideal B with $B_0 \subset B \subset P$, B has only finitely many ZS-primes,*
  (iii) *the set of ideals of the form $\sqrt{(B:x)}$, B an ideal from (ii), $x \in R$, satisfies the a.c.c.,*
then P is finitely generated.

Proof. By Lemma 3.2 the ZS- and B_w-primes of an ideal B of (ii) coincide. (iii) implies the a.c.c. on ZS-primes of ideals B from (ii) and hence also the a.c.c. on B_w-primes of such ideals. Therefore Theorem 1.3 applies.   Q.E.D.

The next corollary is the ZS-analogue of Corollary 1.4.

3.4. COROLLARY. *If R is a ring such that*
  (i) *every R_P, P prime, is noetherian,*
  (ii) *every finitely generated ideal of R has only finitely many ZS-primes,*
  (iii) *Spec (R) is noetherian,*
then R is noetherian.

Proof. Since Spec (R) is noetherian, R has a.c.c. on radical ideals. Therefore Theorem 3.3 applies; so every prime ideal of R is finitely generated, and hence by Cohen's theorem R is noetherian.   Q.E.D.

We conclude this section with some further observations on the associated primes of an ideal.

3.5. PROPOSITION. *Let A be an ideal of R. If A has only finitely many B_w-primes, then $\{ZS\text{-}primes\} = \{B_w\text{-}primes\} = \{N\text{-}primes\}$.*

Proof. Since the hypothesis of 3.2 is satisfied when A has only finitely many B_w-primes, every B_w-prime of A is also a ZS-prime. By our previous remarks, it therefore remains to see that every N-prime of A is a B_w-prime. First consider the case where P is a maximal N-prime of A, i.e. a prime which is maximal with respect to containing A and being contained in the zero-divisors mod A. Let $Q_1, \ldots, Q_n$ be the B_w-primes of A which do not contain P. Choose $x \in P$, $\notin Q_1 \cup \cdots \cup Q_n$. Since x is a zero-divisor mod A, there exists $y \notin A$ such that $xy \in A$. Then $x \in A:y$ implies $A:y \nsubseteq Q_1 \cup \cdots \cup Q_n$; and since any MPD of $A:y$ is a B_w-prime of A, it follows that P is the only MPD of $A:y$. Thus, P is a B_w-prime of A (and even a ZS-prime of A). Since B_w-primes localize well (1.2), and since every N-prime extends to a maximal N-prime in some R_S, it follows that every N-prime of A is a B_w-prime.   Q.E.D.

Underwood proves in [16, Theorem 3.4 and Proposition 3.6] that if $A = \sqrt{A}$, then the ZS- and B-primes coincide with the irredundant MPD's of A (P is an irredundant MPD of A if it cannot be deleted when A is written as the intersection of all its MPD's). Nakano's example [13, p. 426] shows that an ideal A may have no ZS-primes but infinitely many B_w-primes; and Underwood gives examples in [16, Examples 3.9–3.15] which show that ZS-primes and B_w-primes of an ideal A may be present in almost any combination.

Also, in connection with Lemma 3.2 and Proposition 3.5, it is possible to have an ideal A such that $\{\sqrt{(A:x)}\}$, $x \in R$, satisfies the a.c.c. but A still does not have a finite number of B_w-primes; this example was used in another context in [16, 3.9].

3.6. EXAMPLE. Let $R = k[x_1, x_2, \ldots]$ be a polynomial ring over the field k in the indeterminates x_i, and let $A = (\{x_i x_j\}_{i<j})$. Then $A = \bigcap P_n$, $P_n = (x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots)$; and $\{P_n\}$ is the set of MPD's of A.

If $A:r \nsubseteq P_n$, $r \in R$, then $r \in P_n$; so $x_n r \in A$, and hence $x_n \in A:r$. Let P be any MPD of $A:r$. Then $P \supset P_n$ for some n; so if $P \neq P_n$, then $A:r \nsubseteq P_n$ and hence $x_n \in A:r \subset P$. Thus, $P = (x_1, x_2, \ldots)$ if $P \neq P_n$. But $r \notin A$ implies $r \notin P_j$ for some j; so $A:r \subset P_j < P$. Thus, P is not an MPD of $A:r$ if $P \neq P_n$; so $\{P_n\}$ is exactly the set of B_w-primes of A.

Now let us check that $\sqrt{(A:r)}$, $r \in R$, has a.c.c. $A:r \neq A$ implies $r \in P_n$ for some n. Hence r has constant term equal to 0. If r involves only the variables $x_1, \ldots, x_m$, then $x_i \in A:r$ for all $i > m$. Thus, the MPD's of $A:r$ can only come from among the set $P_1, \ldots, P_m$. Hence any proper ideal of the form $\sqrt{(A:r)}$, $r \in R$, is either A or a finite intersection of ideals from $\{P_n\}$.

4. **Generalizations of 1.2.** Since Proposition 1.2 is crucial for the main result of §1, it is perhaps of interest to examine the hypothesis further. Let us first observe

that the proof remains valid if we merely assume that R' is a generalized quotient ring of R, in the sense of [1], i.e. there exists a homomorphism $\varphi \colon R \to R'$ which makes R' a flat R-module, and R' is contained in the total quotient ring of $\varphi(R)$. Let us see what happens if one tries to weaken this assumption further to merely: R' is an R-algebra which is flat as an R-module. *We make the following assumptions throughout this section*: R' is an R-algebra which is flat as an R-module, A is an ideal of R, $A' = AR'$, P is a prime ideal of R containing A, $P' = PR' \neq R'$. By replacing R by the canonical homomorphic image of R in R' we may also, without loss of generality, assume that $R \subset R'$. (The above assumptions also imply $P' \cap R = P$.)

If P' is a prime ideal, then 4.3 will show that P is a B_w-prime of A implies P' is a B_w-prime of A'. However, it can happen that P is a B_w-prime of A and yet $P' = PR'$ is not even prime. For example, take R to be a valuation ring with quotient field K having the property that there exists an algebraic extension K' of K such that the integral closure R' of R in K' is not a valuation ring. Then R' has at least two primes lying over P [17, Vol. 2, p. 27, Theorem 12], so $P' = PR'$ cannot be prime. Note that by [3a, p. 29, Proposition 3], a module is flat over a Bezout domain if and only if the module is torsion free; in particular, R' is flat over R.

We can, of course, avoid the above difficulty by merely looking at the MPD's of P'. We observe first that flatness of R' over R yields the following proposition.

4.1. Proposition. *If $z(A)$ and $z(A')$ denote, respectively, the zero divisors of R modulo A and of R' modulo A', then $z(A') \cap R \subset z(A)$. If in addition $A' \cap R = A$, then $z(A) = z(A') \cap R$.*

Proof. If $x \in z(A') \cap R$, then $(A:x)R' = A':x \neq A'$. Hence $A:x \neq A$ and $x \in z(A)$. Conversely, if $x \in z(A)$ and $A' \cap R = A$, then $A:x \neq A$ implies that $A' \neq (A:x)R' = A':x$ so that $x \in z(A')$.

4.2. Corollary. *If A is P-primary then any associated prime of A' in R' (using any one of the definitions of associated primes) lies over P.*

Proof. This is immediate from 4.1, for if Q' is an associated prime of A', then $Q' \subset z(A')$.

In particular, 4.2 implies that any MPD of P' in R' lies over P. It follows that for an arbitrary ideal A of R any MPD of $AR' = A'$ lies over an MPD of A.

4.3. Corollary. *If P is a B_w-prime of A, then each MPD of P' is a B_w-prime of A'.*

Proof. P is an MPD of $A:x$ for some $x \in R$. $(A:x)R' = AR':x$ by flatness; so if Q' is an MPD of P', then Q' lies over P, and hence Q' is an MPD of $AR':x$. Q.E.D.

It can happen, however, in the case where R' is not a generalized quotient ring of R, that P' is a B_w-prime of A' and yet P is not a B_w-prime of A. Our next example illustrates this. (Example 4.4 also provides a negative answer to the more general question raised by D. Lazard in [9a, p. 19, Remark 3.4].)

4.4. Example. Let R be a valuation ring having maximal ideal P and a chain of primes $P_1 < P_2 < \cdots$ such that $\bigcup P_i = P$ (such a ring may be obtained by choosing the value group of R to be $Z \oplus Z \oplus \cdots$ with lexicographic order; see, for example, [6, p. 248]). Let $A = aR$ for some $a \neq 0$ in P. Then $A : y$ is principal for each $y \in R$, so P cannot be an MPD of $A : y$. Thus P is not a B_w-prime of A. Let x be an indeterminate over R and consider the ring $R' = R + P'$, where $P' = PR[x/a]$. Since $R' \subseteq R[x/a]$, we see that R' is a torsion free and hence flat R-module. Note that $P' \cap R = P$ and $R'/P' \cong R/P$. Hence P' is a maximal ideal of R'. Moreover, $P' = PR'$. It is clear that $PR' \subseteq P'$, and to show that $P' \subseteq PR'$ it will suffice to show that $b(x/a)^i \in PR'$ for any $b \in P$. Since P is not principal there exist $b_1, b_2 \in P$ such that $b = b_1 b_2$. Hence $b(x/a)^i = b_1 b_2 (x/a)^i$, and $b_1 b_2 (x/a)^i \in PR'$ since $b_2(x/a)^i \in P' \subseteq R'$. Finally we show that $P' = aR' : x$, and hence that P' is a B-prime (and a fortiori a B_w-prime) of $aR' = AR'$. That $P' \subseteq aR' : x$ follows from the fact that $P \subseteq aR' : x$, i.e. $x/a \cdot P \subseteq R'$. Since P' is maximal, to show $P' = aR' : x$, it suffices to observe that $x/a \notin R' = R + PR[x/a]$, which is clear from the transcendence of x/a over R.

Thus, in general, P' is a B_w-prime of A' does not imply P is a B_w-prime of A. We can show, however, that P must be an N-prime of A.

4.5. Proposition. *If Q' is a B_w-prime of A' and $Q' \cap R = P$, then P is an N-prime of A.*

Proof. If $S = R \backslash P$, then R'_S is flat over R_S [3a, p. 115, Proposition 13]; and by Proposition 1.2, $Q'R'_S$ is a B_w-prime of $A'R'_S = AR'_S$. Hence $Q'R'_S$ is contained in $z(AR'_S)$, the set of zero divisors of R'_S modulo AR'_S. We have $PR_S \subseteq Q'R'_S$, so Proposition 4.1 implies that PR_S consists of zero divisors modulo AR'_S. Since PR_S is a maximal ideal, it follows that PR_S is a maximal N-prime of AR_S. Hence P is an N-prime of A.   Q.E.D.

We have not been able to determine whether N-primes of R' necessarily contract to N-primes of R (even in the case where R is an integral domain and R' is contained in the quotient field of R).

Question. If Q' is an N-prime of A' must it follow that $Q = Q' \cap R$ is an N-prime of A?

A difficulty in considering this question is that N-primes are not preserved by localization, i.e. Q is an N-prime of A does not imply QR_Q is an N-prime of AR_Q. (See [10] or [5].) In connection with the above question we can make the following observations: applying 4.1 we see that $Q \subseteq z(A)$. Hence when Q is maximal we certainly have that Q is an N-prime of A. Moreover, if there exists a multiplicative system S' of R' with $S' \cap R = S$ such that QR_S is maximal and $Q'R'_{S'}$ is an N-prime of $AR'_{S'}$, then QR_S is an N-prime of AR_S and hence Q is an N-prime of A.

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