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LOCALLY PROJECTIVE-PLANAR LATTICES WHICH
SATISFY THE BUNDLE THEOREM.

THE OHIO STATE UNIVERSITY, PH.D., 1979

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LOCALLY PROJECTIVE-PLANAR LATTICES
WHICH SATISFY THE BUNDLE THEOREM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Jeffry Ned Kahn, B.S.

* * * * *

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1979

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ACKNOWLEDGEMENTS

While I have been at The Ohio State University, a large number of people have contributed to my mathematical growth, and I will not try to mention all of them. I do especially want to thank my advisor, Professor Dijen K. Ray-Chaudhuri, for his help and guidance throughout the time that I have spent here.

Other people to whom I am indebted for assistance in the present work are Professors Jill Yaqub and Bill Johnson of The Ohio State University and Professor Alan Hoffman of IBM. My thanks are due also to the reading committee for their efforts.

I would also like to take this opportunity to acknowledge the support of the College of Mathematical and Physical Sciences in the form of a grant in aid during part of the time that I was engaged in this research.

Finally, I want to thank jno. Moore for typing this dissertation expertly, and usually on short notice.

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3. Inversive planes satisfying the bundle theorem, to appear in *J. Combinatorial Theory (A)*.
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CHAPTER I

INTRODUCTION

1. Some Background

This investigation originated with a question about inversive planes, which was answered in [25] and [26]. The present work includes those results, as well as their analogues for Laguerre and Minkowski planes, but deals with a more general class of objects - here called "locally projective-planar lattices" - in which some properties particular to one or more of the above mentioned planes are dispensed with. Our main theorem will be introduced in section 2; we begin here with some discussion of the classical examples.

An inversive plane is an incidence structure (see chapter II) $\mathcal{I} = (\mathcal{O}, \mathcal{C})$ (elements of \mathcal{C} are called circles) which satisfies:

(0) Circles are nonempty.

(1) For each $P \in \mathcal{O}$, \mathcal{I}_P is an affine plane (where \mathcal{I}_P is the "internal" structure whose points are the points of \mathcal{O} other than P , whose blocks are the circles of \mathcal{C} which contain P , and whose incidence is that inherited from \mathcal{I}).

The order of \mathcal{I} is the (common) order of the affine planes \mathcal{I}_P .

Inversive planes arise quite naturally in geometry in the following way. Let K be a skewfield and \mathcal{O} an ovoid in $PG(3, K)$. (I.e., \mathcal{O} is a set of points satisfying: (1) no three points of \mathcal{O} are

collinear, and (2) if $P \in \mathcal{O}$, then the union of all lines meeting \mathcal{O} only in P is a plane. $\text{PG}(3, K)$ denotes three-dimensional projective space over K .) Then the following incidence structure, $\mathcal{A}(\mathcal{O})$, is an inversive plane:

The points of $\mathcal{A}(\mathcal{O})$ are the points of \mathcal{O} .

The circles of $\mathcal{A}(\mathcal{O})$ are those planes which meet \mathcal{O} in more than one point.

Incidence is inclusion.

An inversive plane is said to be egglike¹ if it is isomorphic to some $\mathcal{A}(\mathcal{O})$.

Laguerre and Minkowski planes satisfy slightly more complicated, but similar, sets of axioms; their orders are defined as for inversive planes; and we may define in each instance the egglike planes to be those arising from a construction analogous to that given above for $\mathcal{A}(\mathcal{O})$. For simplicity of exposition, all of this discussion is postponed until chapter III. (Actually, in chapter III we adopt the terminology of Benz [3], according to which the inversive planes defined above are inversive planes "in the narrow sense". For purposes of the present discussion, however, we will use the more traditional definitions: inversive planes will be as defined above, and Laguerre planes will be those objects which in chapter III are called Laguerre planes "in the narrow sense".)

We will speak of inversive, Laguerre and Minkowski planes collectively as circle-planes (although this term is used elsewhere in a much

¹ This term is due to Dembowski and Hughes [5].

broader sense, e.g., in [3], page 2).

The role which the egglike planes play in the study of circle-planes is similar, especially in the inversive and Laguerre cases, to the role of the Desarguesian planes in the theory of projective planes. In fact, this analogy - pointed out by both Hesselbach [22] and van der Waerden and Smid [33] - goes much further, and since it provided much of the initial motivation for the present work, I will dwell on it a little here.

For inversive and Laguerre planes the correspondence runs as follows. The role of the Pappian planes is taken by the subclass of egglike planes for which K (as above) is commutative, and \mathcal{Q} a non-ruled quadric (inversive case), or a cone with vertex deleted (Laguerre case). Van der Waerden and Smid [33] gave substance to this analogy by proving that the inversive and Laguerre planes belonging to these subclasses are precisely those which satisfy the "theorem of Miquel" (again see [33], or in the case of inversive planes see [14], p.255). This configurational proposition, then, corresponds to the theorem of Pappus.

In the same paper, the authors discuss a second axiom - the "Büschelsatz", or bundle theorem (see section III) - which holds in all egglike planes, and which is accordingly a candidate for the role of Desargues' theorem.

The bundle theorem does not hold in all inversive planes (see [20], for example), but it does hold in all inversive geometries of dimension greater than two ([23], [28]). Maurer's theorem [28], which

implies that all such geometries are egglike², relies heavily on this fact. All of this is, of course, precisely analogous to the situation for projective planes and higher dimensional projective spaces. (I do not know to what extent analogues of these statements have been investigated for Laguerre planes.

The principal missing link in the above analogy is supplied by the following two theorems, which we obtain in chapter III as corollaries of our main result.

THEOREM 1. All inversive planes which satisfy the bundle theorem are egglike.

THEOREM 2. All Laguerre planes which satisfy the bundle theorem are egglike.

In an early attack on this problem, Hesselbach [22] was able to prove theorem 1, and also van der Waerden - Smid's result, for the special case of "topological" inversive planes. (His paper, which predates [31], contains the earliest mention that I have seen of the bundle theorem, although for some reason van der Waerden and Smid seem generally to be given credit for first formulating the axiom.) Hesselbach also showed (for inversive planes) that the bundle theorem

² I.e., the spheres of the inversion geometry are the intersections of an ovoid with flats in $PG(d,K)$.

can be derived directly from Miquel's theorem, which corresponds to the fact that Desargues' theorem can be derived directly from Pappus'.

One other feature of the analogy deserves mention. A famous result of Segre [30] states that any oval in $PG(2,q)$, q odd, is a conic; and Barlotti [1] used Segre's theorem to show that any ovoid in $PG(3,q)$, q odd, is a quadric. These results imply respectively that a Laguerre or inversive plane of finite odd order which satisfies the bundle theorem must also satisfy Miquel's theorem, a fact which may be seen as a partial analogue of Wedderburn's theorem (a finite Desarguesian projective plane is Pappian). (The full analogue is false: examples are known of egglike inversive and Laguerre planes of finite even order which do not satisfy Miquel's theorem. They correspond respectively to the ovoids discovered by Tits [32] in $PG(3,2^{2k+1})$, $k \geq 1$ ³, and to any of the oval cones (section III) based on ovals which are not conics (see [14], page 51, and accompanying references).)

The situation for Minkowski planes is a sort of homomorphic (as opposed to isomorphic) image of the projective-inversive-Laguerre situation. Kaerlein [24] defines Miquel's theorem in Minkowski planes and proves an analogue of van der Waerden - Smid's results (with a hyperbolic quadric taking the place of the non-ruled quadric or cone). Also, as in the previous cases, egglike planes are characterized by the bundle theorem:

³ The smallest of these had been constructed earlier by Segre [31].

THEOREM 3. All Minkowski planes which satisfy the bundle theorem are egglike.

The difference in the Minkowski case is that the classes of egglike and Miquelian planes turn out to coincide, a consequence of the fact that hyperbolic quadrics are actually characterized by those of their properties which are used in the construction of Minkowski planes.

2. The Main Theorem

As will be shown in chapter III, each circle-plane \mathbb{H} determines a lattice $L = L(\mathbb{H})$ with the following properties:

(L) L is a semimodular rank 4 lattice. (Semimodular means that if $x, y \in L$ cover $x \wedge y$, then they are covered by $x \vee y$. For lattice definitions see [7]. The rank 0, 1, 2, 3 and 4 elements of L are called "0", "points", "lines", "planes" and "1" respectively.)

(LPP) For each point P of L , the interval $[P, 1]$ is a projective plane.

Lattices which satisfy (L) and (LPP) are called locally projective-planar or LPP-lattices.

If L is an LPP-lattice, and if P and Q are distinct points of L , then $P \vee Q$ is a line (by (L)), hence a point of both $[P, 1]$ and $[Q, 1]$. The order of $[P, 1]$ ($[Q, 1]$) being one less than (so equal to if infinite) the number of planes on $P \vee Q$, it follows that all of the projective planes $[P, 1]$ have the same order. This common

order is also defined to be the order of L .

If Π is an egglike circle-plane, then $L = L(\Pi)$ also satisfies the "bundle theorem":

(BT) If l_1, l_2, l_3, l_4 are lines, no three on a common plane, and no two on a common point, and if five of the six pairs $\{l_i, l_j\}$, $1 \leq i \neq j \leq 4$, are coplanar, then the sixth pair is also.

More generally, let K be a skewfield, and $S = PG(3, K)$, thought of as the subspace lattice of a four-dimensional vector space over K . For any nonempty subset \mathcal{O} of the point set of S , let $L = L(\mathcal{O})$ be the lattice whose elements are $\{0\}$ and those elements of S which contain points of \mathcal{O} , ordered as in S . Then L is an LPP-lattice which satisfies the bundle theorem.

It is not true in general that an LPP-lattice satisfying (BT) must be isomorphic to an $L(\mathcal{O})$ (see section 10), but we are able to establish such an isomorphism if we assume that \mathcal{O} is (in some sense) fairly large. For $n \in \{2, 3, 4, \dots\}$, let

$$f(n) = \max\{1, \lceil \frac{\sqrt{n-1}}{2} \rceil\}.$$

Our main result is the following.

THEOREM 4. Let L be an LPP-lattice which satisfies (BT) and
(BE) For each point P of L there is a subset $T(P)$ of the planes
on P with

$$|T(P)| < \infty \quad \text{if } L \text{ is infinite}$$

$$\leq f(n) \quad \text{if } L \text{ has finite order } n,$$

and such that any line whose only point is P lies on some plane of $T(P)$.

Then L is isomorphic to some $L(\mathcal{O})$ (with \mathcal{O} a subset of the point set of some $PG(3,K)$).

Remark. As mentioned earlier, I obtained the proof of theorem 1 some time ago. My principal aim in formulating theorem 4 was to see whether the same result could be proved without relying upon the specific structures of the circle planes. In the finite case, the function f has been chosen to be at least one in all cases, so that theorem 4 will include theorems 1-3; actually, I think that it should be possible in general to prove the theorem with a much larger f , but I have not attempted to do so.

The proof of theorem 4 is achieved by constructing a three-dimensional projective space in which the lattice L is suitably embedded. The points of the space (which we call S) are defined by means of the bundle theorem in section 3. (This is the only role which the bundle theorem plays in the proof.) In section 4, we derive some of the easier properties of S .

The two lemmas of section 5, especially 5.5, exhibit what is perhaps the underlying idea of the proof. This involves exploiting connections in the "local" structure of lines and planes containing some given point, and corresponding configurations in the structure of S itself.

The existence of lines (called "near-lines") is proved for the infinite case in section 6, and in section 7 we supply those additional details needed to obtain near-lines in the finite case. This organization is intended to allow the reader to see the main line of argument uncluttered by the more precise arguments which the finite case occasionally requires.

Finally, in section 8, we prove that the system of points and near-lines satisfies "Pasch's axiom", and the theorem follows.

The new ideas in this proof are to be found mostly in sections 6,7 and 8. The material of section 3 is essentially the same as that in section 2 of [25], though given here in a more general context, and much of sections 4 and 5 (and some of 8) is taken with only slight modification from [26].

CHAPTER II

PROOF OF THE MAIN THEOREM

An incidence structure (see [14], p.1) is usually defined as a triple $(\mathcal{P}, \mathcal{B}, I)$, with \mathcal{P} a set of "points", \mathcal{B} a set of "blocks", and $I \subseteq \mathcal{P} \times \mathcal{B}$ (or I a symmetric subset of $(\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$). In what follows, we will frequently think of a block as a subset of the point set, and when $(P, b) \in I$, we will say " $P \in b$ ", " b contains P ", etc. Incidence structures will be denoted simply $(\mathcal{P}, \mathcal{B})$.

In a poset, " $<$ " will always denote the order relation. If $x < y$ then we will say " x is on y ", " y is on x ", etc. In a lattice, we use " \vee " and " \wedge " for join and meet respectively (see [7], p.6).

Two sets are called tangent if they have intersection size one. For sets A and B , $A \setminus B$ is the set of elements of A which are not in B .

Throughout chapter II, L is an LPP-lattice satisfying (BF) and (BE). The sets of points, lines and planes of L are denoted \mathcal{O} , \mathcal{L} and \mathcal{P} respectively. The minimum element of L is 0 , and the maximum element is 1 .

3. Points

DEFINITION. A bundle complex (BC) is a subset Σ of $\mathcal{L} \cup \mathcal{P}$ satisfying

(BC1) Each $P \in \mathcal{O}$ is on exactly one line of Σ . (This line is denoted $\Sigma(P)$.)

(BC2) Any two lines of Σ are on a plane.

(BC3) A plane is in Σ iff it contains a line of Σ .

(Notice that by (BC3) the plane of (BC2) belongs to Σ .)

The idea here is that a BC should consist of all those lines and planes of L which contain some fixed point outside Θ in the reconstructed projective space.

We point out one consequence of the above definition:

If $P \in \Theta$, $\pi \in \mathcal{G}$, $P < \pi$, and if Σ is a BC containing π , then $\Sigma(P) < \pi$. (3.1)

(For there is, by (BC3), some $\ell \in \Sigma \cap \mathcal{L}$ with $\ell < \pi$. If $P < \ell$, then by (BC1) $\Sigma(P) = \ell < \pi$. Otherwise, $\pi' = \Sigma(P) \vee \ell$ is a plane by (BC2), and we have $\pi = P \vee \ell \leq \Sigma(P) \vee \ell = \pi'$. Thus $\pi' = \pi$.)

Remark 3.2. A set of lines Σ' will be $\Sigma \cap \mathcal{L}$ for some BC Σ iff it satisfies (BC1) and (BC2). When this happens, the BC Σ is unique and is given by

$$\Sigma = \Sigma' \cup \{\pi \in \mathcal{G} : \exists \ell \in \Sigma' \text{ with } \ell < \pi\}.$$

(In [25] and [26], a bundle complex was defined as a set of lines satisfying (BC1) and (BC2). As indicated by 3.2, there is not much difference between the two definitions, and the present one is adopted solely for expository purposes.)

The bundle theorem (BT) is used to guarantee the existence of BC's:

LEMMA 3.3. Let $l_1, l_2 \in \mathcal{L}$, $l_1 \vee l_2 \in \mathcal{C}$, and $l_1 \wedge l_2 = 0$.
Then there is a unique BC containing l_1 and l_2 .

Proof. By 3.2, it is sufficient to show that there is a unique set of lines Σ' containing l_1 and l_2 and satisfying (BC1) and (BC2). Notice that for l and m distinct lines, and A a point $\not\prec l \vee m$, the planes $l \vee A$ and $m \vee A$ are distinct, and so meet in a line by (LPP). If in addition, $l, m \in \Sigma'$, and Σ' satisfies (BC1) and (BC2), then $A < \Sigma'(A)$ and $\Sigma'(A) \vee l, \Sigma'(A) \vee m \in \mathcal{L}$ force

$$\Sigma'(A) = (l \vee A) \wedge (m \vee A). \quad (*)$$

This remark (eventually) proves uniqueness, since, as we will see, each of the lines $\Sigma'(A)$ in the set Σ' constructed below is defined according to (*) (using lines l and m already known to be in Σ').

Let $l_1 \vee l_2 = \pi$. If $\pi \not\prec P \in \mathcal{C}$, let $\Sigma'(P) := (l_1 \vee P) \wedge (l_2 \vee P)$. We remark that if $Q < \Sigma'(P)$, then $Q \not\prec \pi$ and $\Sigma'(Q) = \Sigma'(P)$. (For

$$\begin{aligned} \Sigma'(P) \wedge \pi &= (l_1 \vee P) \wedge (l_2 \vee P) \wedge \pi \\ &= [(l_1 \vee P) \wedge \pi] \wedge [(l_2 \vee P) \wedge \pi] \\ &= l_1 \wedge l_2 \\ &= 0. \end{aligned}$$

Thus, $Q < (l_i \vee P)$ implies $l_i \vee Q = l_i \vee P$, $i = 1, 2$, so that $\Sigma'(Q) = \Sigma'(P)$.) Moreover, if $\pi \not\prec P$, $R \in \mathcal{C}$ and $\Sigma'(P) \neq \Sigma'(R)$, then (BT) applied to l_1, l_2 , $\Sigma'(P)$ and $\Sigma'(R)$ shows that $\Sigma'(P)$ and $\Sigma'(R)$ are coplanar.

Now fix $P \not\prec \pi$. (Such a P exists: Suppose $P < \pi$ for every $P \in \mathcal{C}$. Fix $P \in \mathcal{C}$. If l is any line with $P < l \not\prec \pi$, then P is the only point on l , and there is some $\pi' \in T(P)$ with $l < \pi'$.)

Thus, $T(P) \cup \{\pi\}$ is a set of lines of the projective plane $[P, l]$ whose union contains all points of $[P, l]$. It follows that $T(P) \cup \{\pi\}$ is infinite if $[P, l]$ (and hence L) is infinite, and $|T(P) \cup \{\pi\}| \geq n + 1$ if $[P, l]$ (and hence L) has finite order n . But according to (BE), we have

$$\begin{aligned} |T(A) \cup \{\pi\}| &< \infty && \text{if } L \text{ is infinite} \\ &\leq f(n) + 1 < n + 1 && \text{if } L \text{ has finite order } n. \end{aligned}$$

This is a contradiction in either case, and proves the existence of $P \not< \pi$.) For $X < \pi$, define $\Sigma'(X) = (\Sigma'(P) \vee X) \wedge \pi$. (Since $l_1 \wedge l_2 = 0$, we must have $X \not< l_1$ or $X \not< l_2$. Say $X \not< l_1$. Then (1) $(\Sigma'(P) \vee l_1) \wedge \pi = l_1$ implies $X \not< \Sigma'(P) \vee l_1$, and (2) $l_1 \vee X = \pi$, so that our definition of $\Sigma'(X)$ is just a special case of (*). It will turn out, incidentally, that this definition does not depend on the choice of P .)

Suppose $X < \pi$ and $Y < \Sigma'(X)$. Then, $Y < \pi$ implies $Y \not< \Sigma'(P)$ (as noted earlier), so that $Y < \Sigma'(X) < \Sigma'(P) \vee X \Rightarrow \Sigma'(P) \vee Y = \Sigma'(P) \vee X \Rightarrow \Sigma'(Y) = \Sigma'(X)$. Moreover, if $X < l_i$, then $\Sigma'(P)$, $X < l_i \vee P \Rightarrow \Sigma'(P) \vee X = l_i \vee P \Rightarrow l_i \leq (\Sigma'(P) \vee X) \wedge \pi = \Sigma'(X) \Rightarrow l_i = \Sigma'(X)$ ($i = 1, 2$).

Let Σ' consist of all the lines $\Sigma'(A)$ defined above. Then, $l_1, l_2 \in \Sigma'$, and Σ' satisfies (BC1) (since we have shown $A < \Sigma'(B)$ implies $\Sigma'(A) = \Sigma'(B)$). We have also seen that $\Sigma'(A)$ and $\Sigma'(B)$

are (equal or) coplanar when $A, B \not\leq \pi$. To prove that Σ' satisfies (BC2), it thus remains to show that $\Sigma'(A)$ and $\Sigma'(B)$ are coplanar when $A < \pi$, $B \not\leq \pi$ (the case $A, B < \pi$ being trivial). We may assume that $\Sigma'(B) \neq \Sigma'(P)$. If $A < \ell_1$, then $\Sigma'(A) = \ell_1$ and $\Sigma'(A) \vee \Sigma'(B) = \ell_1 \vee B$. If $A \not\leq \ell_1$, then we may apply (BT) to the lines ℓ_1 , $\Sigma'(A)$, $\Sigma'(B)$, $\Sigma'(P)$ to show that $\Sigma'(A)$ and $\Sigma'(B)$ are coplanar (since $\ell_1 \vee \Sigma'(A) = \pi$, $\ell_1 \vee \Sigma'(B) = \ell_1 \vee B$, $\ell_1 \vee \Sigma'(P) = \ell_1 \vee P$, $\Sigma'(A) \vee \Sigma'(P) = \Sigma'(P) \vee A$, and we have already shown that $\Sigma'(B)$ and $\Sigma'(P)$ are coplanar).

This completes the proof of 3.3. □

COROLLARY 3.4. If $\ell \in \mathcal{L}$, $\pi \in \mathcal{G}$, and $\ell \wedge \pi = 0$, then there is a unique BC containing ℓ and π .

Proof. Let $X < \pi$ and let Σ be a BC with $\ell, \pi \in \Sigma$. Then (1) $X \not\leq \ell$ and $\ell \vee X \leq \ell \vee \Sigma(X) \in \mathcal{G}$ imply $\ell \vee X = \ell \vee \Sigma(X) > \Sigma(X)$ and (2) $\Sigma(X) < \pi$ (by (3.1)), so that $\Sigma(X) = (\ell \vee X) \wedge \pi$. Σ must therefore be the unique BC containing ℓ and $(\ell \vee X) \wedge \pi$ (which is a BC containing ℓ and π). □

We now extend the lattice L to a poset S by adding the set of elements $\{\Sigma: \Sigma \text{ is a BC}\}$ and the relations $0 < \Sigma < 1$ and $\Sigma < x$ whenever Σ is a BC and x is (a line or plane) in Σ . (This gives a partial ordering by (BC3).) We set $\mathcal{P} = \mathcal{G} \cup \{\Sigma: \Sigma \text{ is a BC}\}$. All elements of \mathcal{P} are now called points and denoted by capital letters.

The order of L is also taken to be the order of S . Whenever this order is finite, it will be denoted by n .

4. Basic Properties of S .

From now on, we will usually think of lines and planes as subsets of \mathcal{P} , although we will still have some occasion to use the order relation, $<$, of S , and the lattice operations, \vee and \wedge , of L . (It is not hard to see that distinct elements of $\mathcal{L} \cup \mathcal{S}$ are distinct as subsets of \mathcal{P} . They are not, in general, distinct as subsets of \mathcal{O} .) A line $\ell \in \mathcal{L}$ is called a tangent if $|\ell \cap \mathcal{O}| = 1$, and a secant if $|\ell \cap \mathcal{O}| \geq 2$.

DEFINITION. A Δ -system is a family of sets having pairwise the same intersection. This common intersection is called the kernel of the Δ -system.

As might be expected, our main task is to show that if $P \neq Q$ are in \mathcal{P} , then the set of planes containing P and Q is a Δ -system.

Remark 4.1. (a) If $P \in \mathcal{O}$ and $P \neq Q \in \mathcal{P}$, then there is a unique line (denoted PQ) which contains P and Q .

(b) If $P, Q \in \mathcal{P}$, $P \neq Q$, then there is at most one line containing P and Q .

((a) follows from the semimodularity of L if $Q \in \mathcal{O}$, and from (BC1) if $Q \in \mathcal{P} \setminus \mathcal{O}$. When $P, Q \in \mathcal{P} \setminus \mathcal{O}$, (b) follows from the uniqueness in lemma 3.3.)

PROPOSITION 4.2. Let $\ell \in \mathcal{L}$ and $\pi \in \mathcal{G}$.

- (a) $|\ell| \geq 2$.
- (b) $\ell < \pi \Leftrightarrow \ell \subseteq \pi \Leftrightarrow |\ell \cap \pi| \geq 2$.
- (c) If $\ell \not\leq \pi$, then $|\ell \cap \pi| = 1$.

Proof. We prove (c) first. If $\ell \cap \pi \cap \mathcal{O} = \emptyset$, then (c) follows from 3.4. It is therefore sufficient to show that if there are distinct $P, Q \in \ell \cap \pi$, with $P \in \mathcal{O}$, then $\ell < \pi$. But if $Q \in \mathcal{P} \setminus \mathcal{O}$ (i.e., Q is a BC), this follows from (3.1); while if $Q \in \mathcal{O}$, then P, Q, ℓ, π are elements of L , and we have $\ell = P \vee Q < \pi$. This proves (c).

Let $P \in \mathcal{O}$, $P < \ell$. Since there is a plane $\pi \not\leq P$ (there exist $Q \in \mathcal{O} \setminus \{P\}$ by (BE), and $\pi \in \mathcal{G}$ with $Q < \pi \not\leq Q \vee P$ by (LPP)), (c) implies (a). Finally, to prove (b), we note that $\ell < \pi \Rightarrow \ell \subseteq \pi$ is obvious, that $\ell \subseteq \pi \Rightarrow |\ell \cap \pi| \geq 2$ follows from (a), and that $|\ell \cap \pi| \geq 2 \Rightarrow \ell < \pi$ follows from (c). \square

PROPOSITION 4.3. If P, Q, R are distinct points with $P \in \mathcal{O}$ and $PQ \neq PR$, then there is a unique plane (denoted PQR) which contains P, Q and R .

Proof. By 4.2(b), a plane π contains P, Q, R iff $PQ, PR < \pi$. Since $P < PQ, PR$, and $PQ \neq PR$, there is exactly one such plane (by (L)). \square

COROLLARY 4.4. (a) If l is a line and $Q \in \mathcal{P} \setminus l$, then there is a unique plane (denoted Ql) which contains $\{Q\} \cup l$.

(b) If l, m are distinct, intersecting lines, then there is a unique plane (denoted lm) which contains l and m .

Proof. (a) Let $P \in l \cap \mathcal{O}$. A plane π contains $\{Q\} \cup l$ iff $\pi > l, PQ$ (by 4.2(b)), and there is exactly one such plane (by (L)).

(b) Let $l \cap m = \{P\}$ and $Q \in m \setminus \{P\}$. (Q exists by 4.2(a).) Then, a plane contains l and m iff it contains $\{Q\} \cup l$ (by 4.2(b)), and there is exactly one such plane by (a). \square

DEFINITION. The intersection of two distinct planes is called a 2-line.

PROPOSITION 4.5. The lines are precisely the 2-lines which meet \mathcal{O} .

Proof. If π_1 and π_2 are distinct planes, and there exists $P \in \pi_1 \cap \pi_2 \cap \mathcal{O}$, then $\pi_1 \wedge \pi_2$ is a line l by (LPP). If there exists $Q \in (\pi_1 \cap \pi_2) \setminus l$, then $\{Q\} \cup l \subseteq \pi_1, \pi_2 \Rightarrow \pi_1 = Ql = \pi_2$ (by 4.4(a)), a contradiction. Thus, $l = \pi_1 \cap \pi_2$.

On the other hand, if l is a line, then by (LPP), there exist distinct planes π_1, π_2 with $l = \pi_1 \wedge \pi_2$, from which it follows (as above) that $l = \pi_1 \cap \pi_2$. \square

PROPOSITION 4.6. The number of points on any 2-line is one more than (so equal to if infinite) the order of S .

Proof. Let m be a 2-line. If π, π' are distinct planes with $m = \pi \cap \pi'$, and if there exists $P \in (\pi \setminus m) \cap \Theta$, then by 4.1(a) and 4.2(b),(c), the map $Q \rightarrow PQ$ defines a bijection of the points of m and the lines on P in π . The number of points on m is thus equal to the number of points on the line π of $[P, 1]$ (which is the number we want). But there always exist such π, π', P . For, if $m = \pi_1 \cap \pi_2$, and $\pi_1 \cap \Theta \subseteq m$, then m is a line. In this case, we may choose any $P \in \Theta \setminus m$, and take $\pi = Pm$ (see 4.4(a)), and π' any other plane on m (and then $m = \pi \cap \pi'$ by 4.5). \square

PROPOSITION 4.7. If π_1, π_2, π_3 are distinct planes, then
 $|\pi_1 \cap \pi_2 \setminus \pi_3| = |\pi_1 \cap \pi_3 \setminus \pi_2|$.

Proof. If $|\pi_1 \cap \pi_2 \cap \pi_3| \leq 1$, this follows from 4.6, so assume $|\pi_1 \cap \pi_2 \cap \pi_3| \geq 2$.

If there exists $P \in \pi_1 \cap \Theta \cap \pi_2$ (or π_3), then $\ell := \pi_1 \cap \pi_2$ is a line (4.5), so $|\ell \cap \pi_3| \geq 2 \Rightarrow \ell \subseteq \pi_3$ (4.2(b)) $\Rightarrow \pi_1 \cap \pi_3 = \ell$. (4.5). Thus, $|\pi_1 \cap \pi_2 \setminus \pi_3| = |\pi_1 \cap \pi_3 \setminus \pi_2| = 0$.

We may therefore assume the existence of $P \in \pi_1 \cap \Theta$ with $P \notin \pi_2, \pi_3$. Then for $i = 2, 3$, the map $s_i: \pi_1 \cap \pi_i \rightarrow \{m: m \text{ is a line through } P \text{ in } \pi_1\}$, defined by $s_i(Q) = PQ$, is a bijection (by 4.1(a) and 4.2(b),(c)). In particular, $s_3^{-1}s_2$ induces a bijection of $(\pi_1 \cap \pi_2) \setminus \pi_3$ and $(\pi_1 \cap \pi_3) \setminus \pi_2$. \square

COROLLARY 4.8. If π_1, π_2, π_3 are distinct planes with
 $\pi_1 \cap \pi_2 \subseteq \pi_3$, then $\{\pi_1, \pi_2, \pi_3\}$ is a Δ -system.

COROLLARY 4.9. If ℓ and m are respectively a line and a 2-line contained in a plane π , then $\ell \cap m \neq \emptyset$.

Proof. Let $m = \pi_1 \cap \pi_2 \subseteq \pi$, and assume $\pi_1 \neq \pi$. Then $\emptyset \neq \ell \cap \pi_1$ (by 4.2(c)) $= \ell \cap \pi \cap \pi_1 = \ell \cap \pi_1 \cap \pi_2$ (by 4.8 if $\pi \neq \pi_2$) $= \ell \cap m$. □

DEFINITION. A near-line is a 2-line (m) which satisfies the following two equivalent conditions:

(N1) Any plane containing at least two points of m contains m .

(N2) For all $P \in \mathcal{G} \setminus m$, there is a (necessarily unique) plane, π_P , which contains $\{P\} \cup m$.

(That (N1) \Rightarrow (N2) is immediate from 4.3. Conversely, suppose m satisfies (N2), and let π be a plane containing points Q, R of m . Let $P \in \pi \cap \mathcal{G}$. If $P \in m$, then m is a line by 4.5, and satisfies (N1) by 4.2(b). If $P \notin m$, then both π and the plane of (N2) contain P, Q, R , and they must therefore be equal (4.3).)

Remark 4.10. Every line is a near-line (by 4.5 and 4.2(b)).

Remark 4.11. If ℓ is a near-line, then the set of planes meeting ℓ at least twice (i.e., containing ℓ) is a Δ -system with kernel ℓ . (For let $\ell = \pi_1 \cap \pi_2$, and suppose π_3 and π_4 are distinct planes containing ℓ . Then, if we assume $\pi_3 \neq \pi_1$, we may apply 4.8 (twice) to show $\ell = \pi_1 \cap \pi_2 = \pi_1 \cap \pi_3 = \pi_3 \cap \pi_4$.)

In particular,

there is at most one near-line containing a pair
of distinct points X and Y . (4.12)

If this near-line exists, we denote it XY .

PROPOSITION 4.13. If m is a near-line and π a plane, then either $|m \cap \pi| = 1$ or $m \subseteq \pi$.

Proof. We just have to show $m \cap \pi \neq \emptyset$. Suppose $P \in \pi \cap \Theta$ (and $P \notin m$). Then by 4.9, $\emptyset \neq (Pm \cap \pi) \cap m \subseteq \pi \cap m$. □

COROLLARY 4.14. If m and t are respectively a near-line and a 2-line contained in a plane π , then $m \cap t \neq \emptyset$.

Proof. Same as proof of 4.9. □

PROPOSITION 4.15. Let π_1, π_2, π_3 be distinct planes with $|\pi_1 \cap \pi_2 \cap \pi_3| \geq 2$. Let $X \in (\pi_1 \cap \pi_2) \setminus \pi_3$. Then

(a) Each near-line through X in π_1 meets $(\pi_1 \cap \pi_3) \setminus \pi_2$.

(b) The number of near-lines through X in π_1 is at most $|(\pi_1 \cap \pi_2) \setminus \pi_3|$.

Proof. (a) Let m be a near-line on X in π_1 . By 4.13, m contains a point Y of $\pi_1 \cap \pi_3$. If $Y \in \pi_2$, then $m = \pi_1 \cap \pi_2$ by 4.11. But then, since π_3 contains at least two points of m , (N1) implies $\pi_1 \cap \pi_2 = m \subseteq \pi_3$, contrary to assumption. Thus, $Y \in (\pi_1 \cap \pi_3) \setminus \pi_2$.

(b) By (a) and 4.12, distinct lines through X in π_1 meet $(\pi_1 \cap \pi_3) \setminus \pi_2$ in distinct points. The number of such near-lines is thus at most $|(\pi_1 \cap \pi_3) \setminus \pi_2|$, which by 4.7 is equal to $|(\pi_1 \cap \pi_2) \setminus \pi_3|$. \square

PROPOSITION 4.16. Let π_1, π_2, π_3 be distinct planes, and suppose that for some $i \in \{1, 2, 3\}$, there exists $P \in \pi_i \cap \mathcal{O}$ with $\pi_i \notin T(P)$.

Assume

$$\begin{aligned} |(\pi_1 \cap \pi_2) \setminus \pi_3| &< \infty && \text{if } S \text{ is infinite} \\ &< \sqrt{n+2-f(n)} && \text{if } S \text{ has finite order } n. \end{aligned}$$

Then, $\{\pi_1, \pi_2, \pi_3\}$ is a Δ -system.

Proof. Since $|(\pi_i \cap \pi_j) \setminus \pi_k| := c$ is constant for $\{i, j, k\} = \{1, 2, 3\}$ (by 4.7), we may assume that there exists $P \in \pi_1 \cap \mathcal{O}$ with $\pi_1 \notin T(P)$.

Suppose first that $c \geq 2$. Let $X, Y \in (\pi_1 \cap \pi_3) \setminus \pi_2$. If X, Y are on a near-line m , then $m = \pi_1 \cap \pi_3$ (by 4.11). But then $|\pi_2 \cap m| \geq 2$ implies $\pi_1 \cap \pi_3 = m \subseteq \pi_2$ (by (NL)), a contradiction. The near-lines containing X are thus distinct from those containing Y . Each $Q \in \pi_1 \cap \mathcal{O}$ is on lines (near-lines) QX and QY in π_1 , and since these lines are not equal, they meet only in Q (by 4.1(a)). It follows that $|\pi_1 \cap \mathcal{O}| \leq xy \leq c^2$, where x (resp. y) is the number of near-lines on X (resp. Y) in π_1 . (Note $x, y \leq c$ by 4.15(b).) On the other hand, all but at most $|T(P)|$ of the lines through P in π_1 are secants (by (BE)). So in the infinite case, $\pi_1 \cap \mathcal{O}$ must be infinite, which contradicts $|\pi_1 \cap \mathcal{O}| \leq c^2$; and in the finite case, we

have $(n + 1 - f(n)) + 1 \leq |\pi_1 \cap \Theta| \leq c^2 < n + 2 - f(n)$, which is again a contradiction.

We are reduced to the case $c = 1$. Let $X \in (\pi_1 \cap \pi_3) \setminus \pi_2$. By 4.15(b), there can be no near-line other than PX on X in π_1 . But then $\pi_1 \cap \Theta \subseteq PX$, which surely contradicts $\pi_1 \notin T(P)$. \square

5. Two Lemmas

DEFINITION. $W \in \mathcal{P}$ is called a near-point if for each $Y \in \mathcal{P} \setminus \{W\}$, W and Y lie on a near-line.

Notice that, by 4.1(a) and 4.10,

$$\text{points of } \Theta \text{ are near-points.} \quad (5.1)$$

(Near-points are weaker than points of Θ in that 4.3 does not hold if P is only assumed to be a near-point. We return to this difficulty in section 8.)

DEFINITION. In a projective plane, π , let the 6-tuple (C, U, V, W, Z, Z_1) satisfy

$$\begin{aligned} C, U, V, W, Z, Z_1 &\text{ are distinct points,} \\ C, Z, Z_1 &\text{ are on a line } k, \\ U, V, W &\text{ are on a line } \ell, \\ Z, Z_1 &\notin \ell, \\ V, W &\notin k. \end{aligned} \quad (5.2)$$

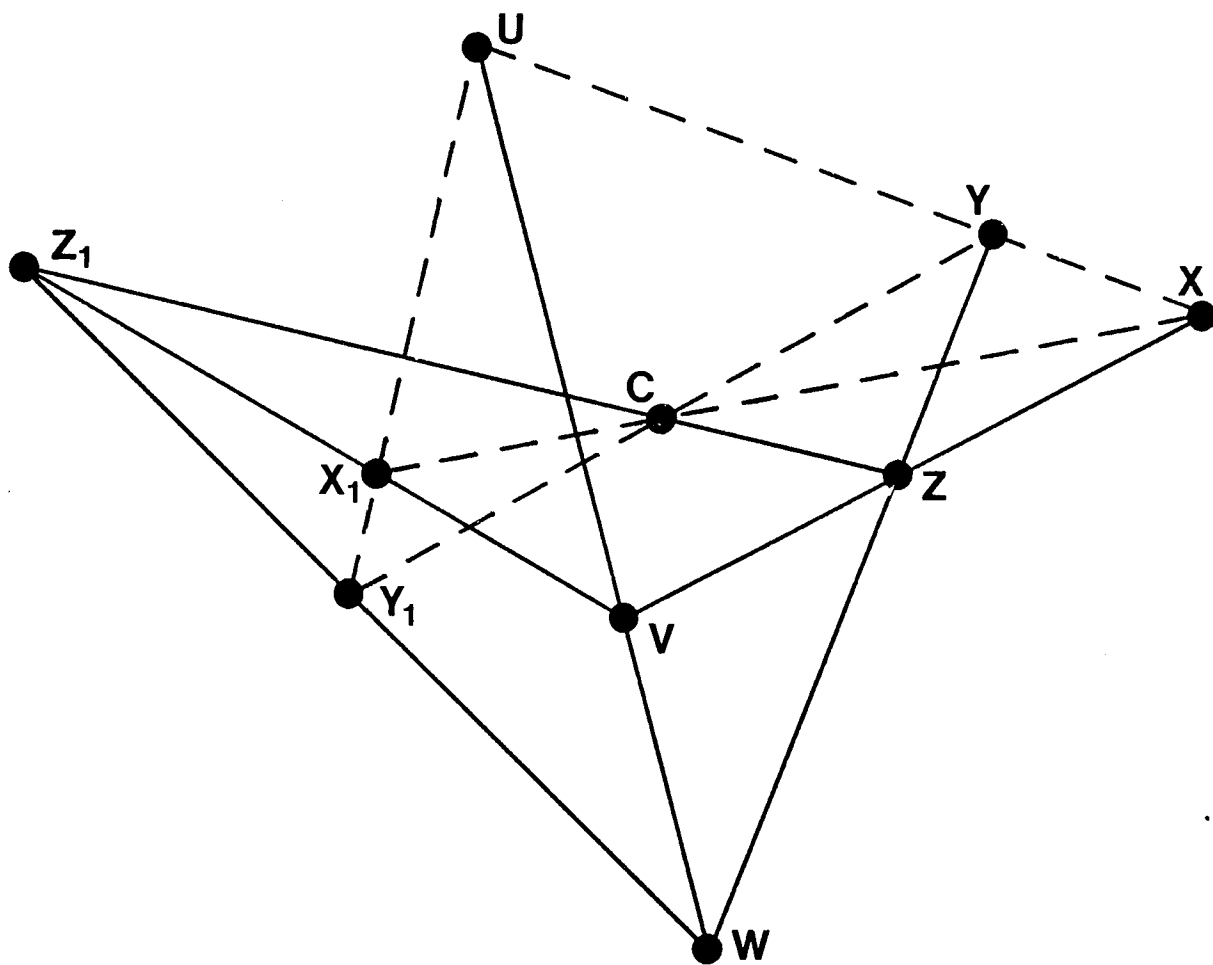


Figure 1. Desargues' Configuration

We say that π is Desarguesian with respect to (C, U, V, W, Z, Z_1) if for each $Y \in WZ \setminus \{W, Z\}$, $Y \notin CU$, we have $Y_1 \in WZ_1$, where Y_1 is defined by:

$$\begin{aligned} X &:= UY \cap VZ, \\ X_1 &:= CX \cap VZ_1, \\ Y_1 &:= UX_1 \cap CY. \end{aligned} \tag{5.3}$$

(See Figure 1. Notice that X , X_1 and Y_1 represent points, not sets. This particular notational abuse will be common from now on.)

DEFINITION. For $P \in \mathcal{O}$, we denote by S/P the incidence structure whose points and lines are respectively the lines and planes of S which contain P , incidence being inclusion. S/P is easily seen to be a projective plane with order equal to the order of S . (In fact it is essentially the same as the projective plane $[P, 1]$.) Elements of S/P are denoted by lower bars, using capital letters for points and lower case letters for lines. (We allow some ambiguity: \underline{X} is used both for a point of S/P and for a line through P (in S), and similarly for $\underline{\ell}$. The intended meaning should usually be clear from the context.)

Let the 7-tuple (P, C, U, V, W, Z, Z_1) satisfy (see figure 2):

- (i) P, C, U, V, W, Z, Z_1 are distinct points of S .
- (ii) $P, C, U \in \mathcal{O}$.
- (iii) V is a near-point. (5.4)
- (iv) C, Z, Z_1 lie on a line k .

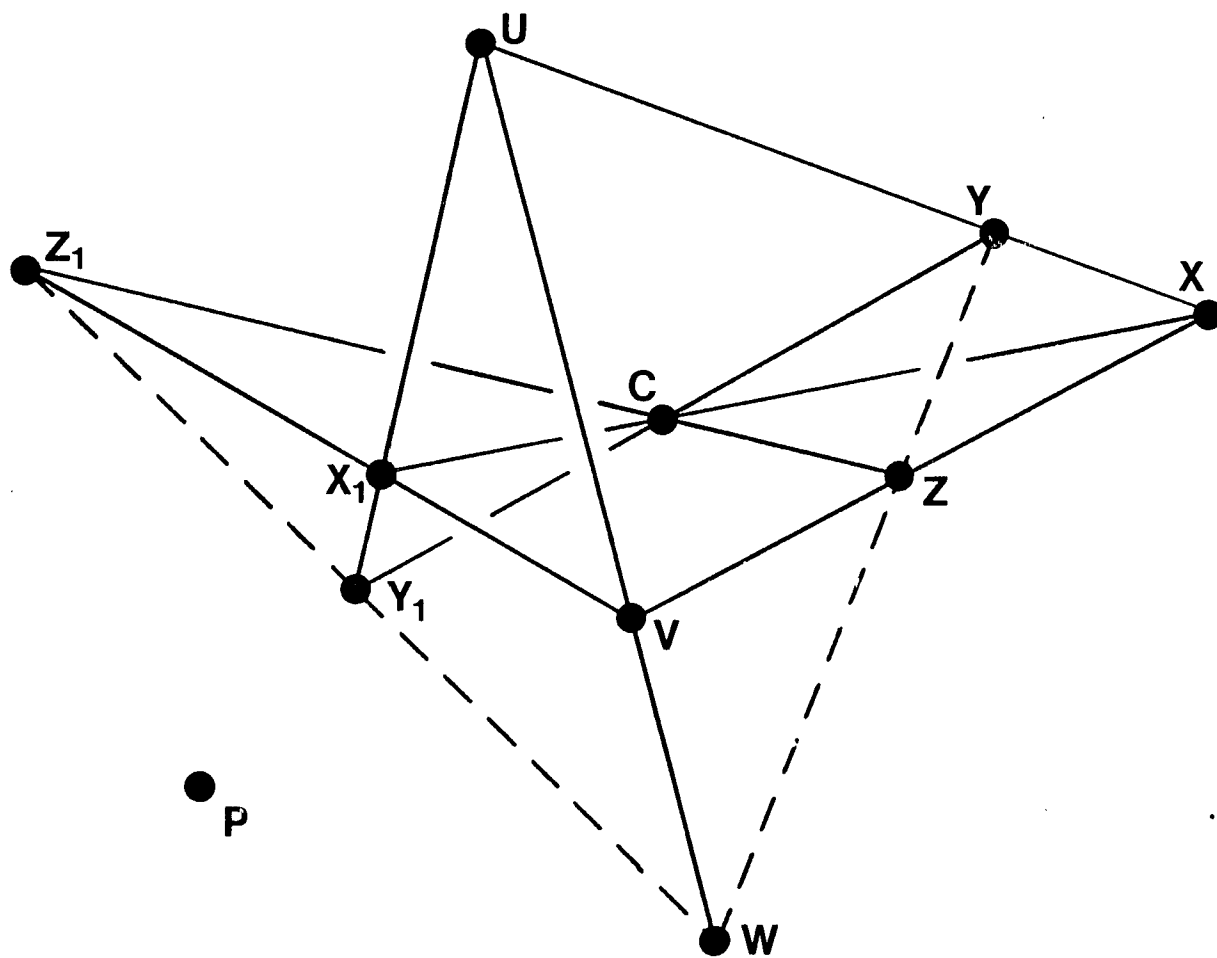


Figure 2. Construction for Lemma 5.5

- (v) U, V, W lie on a line ℓ .
- (vi) $k \cap \ell = \emptyset$.
- (vii) P does not lie on any of the planes
 UWZ, UWZ_1, CZV, CZW .
- (viii) P, C, U are not collinear.
- (ix) $\{CWZ, UWZ, FWZ\}$ is a Δ -system.

(5.4 cont.)

(Note: Both of the designations CZW and CWZ have been used above, although they represent the same plane. Similar abuses will, unfortunately, be common from now on.) Define points $\underline{C} = PC, \underline{U} = PU, \dots, \underline{Z}_1 = PZ_1$ and lines $\underline{k} = Pk, \underline{\ell} = P\ell$ of S/P . It is straightforward to check that $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$ satisfies conditions (5.2).

($\underline{C} \neq \underline{U}$ by (viii)). The second and third conditions of (5.2) come from (iv) and (v) respectively. The last two conditions and the remaining inequalities between the points are consequences of (vii).)

LEMMA 5.5. (With notation as above) S/P is Desarguesian with respect to $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$ iff $\{CWZ_1, UWZ_1, FWZ_1\}$ is a Δ -system.

Proof. Let $m = CWZ \cap UWZ$ ($\subseteq \underline{WZ}$ by (ix)). If we associate with each $Y \in m$ the point $\underline{Y} := PY$ of S/P , then by 4.9 we have a bijection of m and (the line) \underline{WZ} . Denote by Y_0 the point of m associated with the intersection point, \underline{Y}_0 , of \underline{WZ} and \underline{CU} .

If for each $Y \in m$ we let Y_1 be the unique point of $CY \cap UWZ_1$, then $Y \leftrightarrow Y_1$ is, according to 4.9, a bijection of m and $CWZ_1 \cap UWZ_1$. Notice that $W \leftrightarrow W$ and $Z \leftrightarrow Z_1$.

We can obtain the restriction of this bijection to $m \setminus \{W, Z\}$ in another way: For $Y \in m \setminus \{W, Z\}$, let X be the intersection point of the near-lines UY and VZ (which meet by 4.14, since both lie in the plane UWZ). Let X_1 be the intersection of the near-lines CX and VZ_1 (which lie in the plane CZV). Finally, the lines UX_1 and CY lie in the plane CUY , so they meet in a point which, since it lies on $UX_1 \subseteq UWZ_1$, can only be Y_1 .

Now, starting with $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$ and $\underline{Y} := PY$, $\underline{Y} \neq \underline{Y}_0$, define the points \underline{X} , \underline{X}_1 and \underline{Y}_1 of S/P according to (5.3). Then we have

$$X = UY \cap VZ \subseteq \underline{U}\underline{Y} \cap \underline{V}\underline{Z} := \underline{X} ;$$

$$X_1 = CX \cap VZ_1 \subseteq \underline{C}\underline{X} \cap \underline{V}\underline{Z}_1 := \underline{X}_1 ;$$

$$Y_1 = UX_1 \cap CY \subseteq \underline{U}\underline{X}_1 \cap \underline{C}\underline{Y} := \underline{Y}_1 .$$

In particular, $Y_1 \in \underline{Y}_1$, so that $\underline{Y}_1 \in \underline{W}\underline{Z}_1$ is equivalent to $Y_1 \in PWZ$ (by 4.2(b)). Thus

S/P is Desarguesian with respect to $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$

$$\Leftrightarrow \forall \underline{Y} \in \underline{W}\underline{Z} \setminus \{\underline{W}, \underline{Z}, \underline{Y}_0\}, \quad \underline{Y}_1 \in \underline{W}\underline{Z}_1$$

$$\Leftrightarrow \forall Y \in m \setminus \{W, Z, Y_0\}, \quad Y_1 \in PWZ_1$$

$$\Leftrightarrow PWZ_1 \supseteq (CWZ_1 \cap UWZ_1) \setminus \{(Y_0)_1\} .$$

To complete the proof, it is now sufficient to show

$$PWZ_1 \supseteq (CWZ_1 \cap UWZ_1) \setminus \{(Y_0)_1\} \Leftrightarrow \{CWZ_1, UWZ_1, PWZ_1\} \text{ is a } \Delta\text{-system.}$$

The latter statement trivially implies the former. To prove the reverse implication, notice that $U(Y_0)_1$ and $V(Y_0)_1$ are distinct near-lines on $(Y_0)_1$ in UWZ_1 (since if $U(Y_0)_1 = V(Y_0)_1$, then $(Y_0)_1 = UV \cap CWZ_1 = W$, which is false). Thus by 4.15(b),

$(Y_0)_1 \in (CWZ_1 \cap UWZ_1) \setminus PWZ_1$ is incompatible with
 $|(CWZ_1 \cap UWZ_1) \setminus PWZ_1| \leq 1$. In particular, $PWZ_1 \supseteq$
 $(CWZ_1 \cap UWZ_1) \setminus \{(Y_0)_1\}$ implies $PWZ_1 \supseteq CWZ_1 \cap UWZ_1$, which by 4.8
implies that $\{CWZ_1, UWZ_1, PWZ_1\}$ is a Δ -system. \square

LEMMA 5.6. Let $(B_1, B_2, B_3, B_4, B_5, B_6, B_7; h)$ satisfy (see figure 3):

- (a) B_1, B_2, \dots, B_7 are distinct points and h is a secant;
- (b) $B_1, B_2 \in \mathcal{O}$;
- (c) Each of the triples $\{B_1, B_4, B_6\}$, $\{B_1, B_5, B_7\}$,
 $\{B_2, B_4, B_5\}$, $\{B_2, B_6, B_7\}$ is collinear;
- (d) $B_2 \notin B_1B_4$ (i.e., the only collinear triples contained
in $\{B_1, B_2, B_4, B_5, B_6, B_7\}$ are those in (c));
- (e) $B_3 \notin B_1B_2B_4$;
- (f) $B_3 \in h$, h meets none of the lines B_1B_4 , B_1B_7 ,
 B_2B_4 , B_2B_7 ;
- (g) At least two points of $h \cap \mathcal{O}$ do not lie on the
line B_1B_2 ;
- (h) $\{B_2B_3B_4, B_1B_3B_4, B_4h\}$ and $\{B_2B_3B_6, B_1B_3B_6, B_6h\}$ are
 Δ -systems.

Then $\{B_2B_3B_5, B_1B_3B_5, B_5h\}$ is a Δ -system iff $\{B_2B_3B_7, B_1B_3B_7, B_7h\}$ is
a Δ -system.

Proof. If $B_3 \in \mathcal{O}$, the result follows from 4.11. (In this case,
both triples are Δ -systems.) Suppose, then, that $U, V \notin B_3$ are

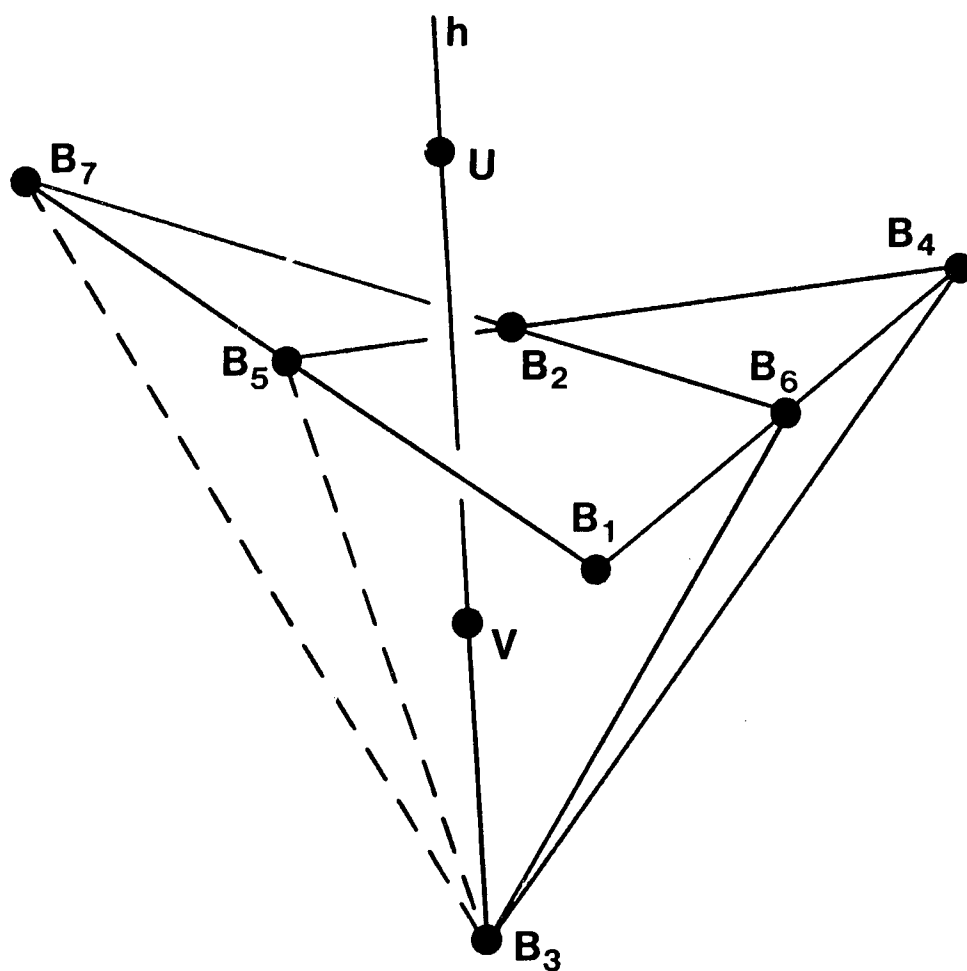


Figure 3. Hypotheses for Lemma 5.6

distinct points of $h \cap \Theta$ not lying on $B_1 B_2$. Since $|h \cap B_1 B_2 B_4| = 1$ (by (e),(f)), we may assume $V \notin B_1 B_2 B_4$. Then, it is straightforward to verify that each of the 7-tuples $(B_1, B_2, U, V, B_3, B_4, B_5)$ and $(B_1, B_2, U, V, B_3, B_6, B_7)$ satisfies the hypotheses (5.4). We observe that if we define $\underline{B}_4 = B_1 B_4$, $\underline{B}_5 = B_1 B_5$, $\underline{B}_6 = B_1 B_6$ and $\underline{B}_7 = B_1 B_7$, then $\underline{B}_4 = \underline{B}_6$ and $\underline{B}_5 = \underline{B}_7$. Thus, a double application of 5.5 yields:

$$\{B_2 B_3 B_5, UB_3 B_5 = B_5 h, B_1 B_3 B_5\} \text{ is a } \Delta\text{-system}$$

$\Leftrightarrow S/B_1$ is Desarguesian with respect to

$$(\underline{B}_2, \underline{U}, \underline{V}, \underline{B}_3, \underline{B}_4, \underline{B}_5) = (\underline{B}_2, \underline{U}, \underline{V}, \underline{B}_3, \underline{B}_6, \underline{B}_7)$$

$\Leftrightarrow \{B_2 B_3 B_7, UB_3 B_7 = B_7 h, B_1 B_3 B_7\}$ is a Δ -system. \square

6. Lines in the Infinite Case

Suppose π_1 and π_2 are affine planes on the same point set, X . We call a subset of X a constant line if it is a line of both π_1 and π_2 , and a π_i -line if it is a line of π_i , but not of π_{3-i} ($i = 1, 2$). Similarly, for $i = 1, 2$, a parallel class of π_i is called a π_i -class if it contains π_i -lines, and a constant class otherwise.

LEMMA 6.1. Let π_1 and π_2 be affine planes on the same point set, X . Let b be the number of π_1 -classes (in π_1), and suppose that

$b < \infty$ if π_1 is infinite
 $< \sqrt{n} + 1$ if π_1 is of finite order n .
then $b = 0$ (i.e., $\pi_1 = \pi_2$).⁵

Proof. For $P, Q \in X$, $P \neq Q$, let $(PQ)_i$ denote the line joining P and Q in π_i ($i = 1, 2$). Let b_2 be the number of π_2 -classes (in π_2).

If P, Q are distinct points, then $(PQ)_1$ is in a π_1 -class iff $(PQ)_2$ is in a π_2 -class. (Notice this does not preclude $(PQ)_1 = (PQ)_2$.) So if ℓ is a (constant) line of a constant class, and $P \notin \ell$, then

$$\begin{aligned}
 b &= |\{Q \in \ell: (PQ)_1 \text{ is in a } \pi_1\text{-class}\}| \\
 &= |\{Q \in \ell: (PQ)_2 \text{ is in a } \pi_2\text{-class}\}| \\
 &= b_2.
 \end{aligned}$$

Let ℓ_1 and ℓ_2 be a π_1 -line and a π_2 -line respectively. We may suppose without loss of generality (because $b = b_2$) that there is some $P \in \ell_2 \setminus \ell_1$. Then, the lines $(PA)_1$ with $A \in \ell_1 \cap \ell_2$ are all π_1 -lines (since P, A are on the π_2 -line ℓ_2), and are all distinct (since any two such A are on the π_1 -line ℓ_1 , which does not contain P). Moreover, these lines are in distinct π_1 -classes (any two intersect), and not in the π_1 -class containing ℓ_1 . It follows that $|\ell_1 \cap \ell_2| \leq b - 1$.

Again, let ℓ_1 be any π_1 -line, and let $Y \in \ell_1$. Any point of $\ell_1 \setminus \{Y\}$ is joined to Y by a π_2 -line, which meets ℓ_1 in at most $b - 2$ points other than Y . Since the number of π_2 -lines on Y is at most b , we have

⁵ J. Yaquub has pointed out that in the finite case this is due to Bruck [8].

$$|\ell_1| \leq 1 + b(b-2) < \infty \quad \text{if } \pi_1 \text{ is infinite}$$

$$< n \quad \text{if } \pi_1 \text{ has finite order } n .$$

In either case, this is a contradiction, and we conclude that π_1 -lines (and π_2 -lines) do not exist. \square

Lemma 6.1 can sometimes be used to prove the equality of projective planes of a type which we now describe.

Let $\pi \in \mathcal{S}$, and $P \in \mathcal{O} \setminus \pi$. If for each line l and plane π' on P , we map

$$\alpha: \begin{cases} l & \rightarrow l \cap \pi \\ \pi' & \rightarrow \pi' \cap \pi, \end{cases}$$

then by 4.2(b),(c), α is an isomorphism of S/P and the projective plane - called π^P - whose points are the points of π , and whose lines are the 2-lines $\pi' \cap \pi$ with $P \in \pi' \in \mathcal{S}$.

If l is a line of a projective plane π , then π^l denotes the affine plane obtained from π by deleting l and all of its points. .

LEMMA 6.2. Suppose $l \in \mathcal{L}$, $l \subseteq \pi \in \mathcal{S}$, and $E, H \in \mathcal{O} \setminus \pi$.

Suppose, moreover, that for all but at most b points W of l we have

$$EWZ \cap \pi = HWZ \cap \pi \quad \forall Z \in \pi \setminus \{W\}, \quad (6.3)$$

where

$$b < \infty \quad \text{if } S \text{ is infinite}$$

$$< \sqrt{n} + 1 \quad \text{if } S \text{ has finite order } n .$$

Then, $\pi^E = \pi^H$.

Proof. By lemma 6.1, the affine planes $(\pi^E)^\ell$ and $(\pi^H)^\ell$ are equal, since each $W \in \ell$ which satisfies (6.3) corresponds to a constant class of $(\pi^E)^\ell$. Choose $Y \in \pi \setminus \ell$ such that $Y \notin \pi'$ $\forall \pi' \in T(E)$. (Such a Y surely exists by (BE).) For any $Z \in \pi \setminus \ell \setminus \{Y\}$, we have $EYZ \cap (\pi \setminus \ell) = HYZ \cap (\pi \setminus \ell) \Rightarrow |(EYZ \cap \pi) \setminus HYZ| \leq 1$. So $EYZ \notin T(E)$ implies, by 4.16, that $EYZ \cap \pi = HYZ \cap \pi$. It follows that $EYW \cap \pi = HYW \cap \pi$ also holds for all $W \in \ell$. Finally, for $W \in \ell$ and $X \in \pi \setminus \ell$, let m be the (affine) line of $(\pi^E)^\ell = (\pi^H)^\ell$ which contains X and is parallel or equal to the line $(EWY \cap \pi) \setminus \{W\} = (HWY \cap \pi) \setminus \{W\}$. Then, $EWX \cap \pi = m \cup \{W\} = HWX \cap \pi$. \square

COROLLARY 6.4. Let ℓ be a line and suppose that at most b points of ℓ are not near-points, where

$$\begin{aligned} b < \infty & \quad \text{if } S \text{ is infinite} \\ < \sqrt{n} + 1 & \quad \text{if } S \text{ has finite order } n. \end{aligned}$$

Then, all points of ℓ are near-points.

Proof. Let $W \in \ell$, $Y \in \mathcal{P} \setminus \ell$, and $\pi = Y\ell$. If $E \in \mathcal{O} \setminus \pi$, then $EWY \cap \pi$ satisfies (N2). So given $W \in \ell$, there is, for each $Y \in \mathcal{P} \setminus \ell$, a near-line containing W and Y . Since ℓ is a near-line (4.5) containing W and any $Y \in \ell \setminus \{W\}$, W is a near-point. \square

For the remainder of section 6, S is assumed to be infinite.

Let l be a line (of S) and W a point of l , $W \notin \Theta$. Let $E, H \in \Theta$, $E, H \notin l$, $H \notin El$, and suppose EW and HW are secants. Set

$$M = \{\pi \in \mathcal{S}: E, H \notin \pi \ni l\}.$$

For $\pi \in M$, a point X of $\pi \setminus \{W\}$ is called good if $EWX \cap \pi = HWX \cap \pi$, and bad otherwise.

Fix $Q \in l \cap \Theta$, and for each $\pi \in M$, let $m(\pi)$ be the line $\pi \cap QE$. (Notice $Q \in m(\pi) \neq l$.) Define

$$M^* = \{\pi \in M: m(\pi) \text{ contains infinitely many good points}\},$$

$$M^{**} = \{\pi \in M: m(\pi) \text{ contains only finitely many bad points}\}.$$

Clearly, $M^{**} \subseteq M^*$ (see 4.6).

LEMMA 6.5. With notation as above, assume

(i) M^* is infinite,

(ii) $M^{**} \neq \emptyset$.

Then if $\pi \in M^*$, π contains no bad points.

Proof. Observe first that if $\pi \in M$, $X, Y \in \pi \setminus \{W\}$, and $Y \in EWX$, then X is good iff Y is good. It follows (from 4.9) that to prove 6.5 we need only prove

$$\text{if } \pi \in M^*, \text{ then } m(\pi) \text{ contains no bad points.} \quad (*)$$

We first show that

if $\pi \in M^{**}$, $\pi' \in M^*$, and X is a good point of $m(\pi)$, then both $EX \cap m(\pi')$ and $HX \cap m(\pi')$ are good (**)
points.

We may assume $X \neq Q$, $X \notin EH$, and $\pi \neq \pi'$, since otherwise (**) is trivial. For each $Y \in m(\pi)$, let $Y' = EY \cap m(\pi')$. Then $Y \leftrightarrow Y'$ is a bijection of $m(\pi)$ and $m(\pi')$ (by 4.9, 4.1, 4.2(b)). Choose $Y \in m(\pi)$ satisfying

$$\begin{aligned} Y &\neq Q, X, \\ Y &\notin EH, \\ Y &\text{ is good,} \\ Y' &\text{ is good.} \end{aligned}$$

(Such a Y exists: of the infinitely many $Y \in m(\pi)$ for which Y' is good, three are ruled out by the first two conditions, and since $\pi \in M^{**}$, only finitely many are ruled out by the third.) It is then straightforward to verify that $(Q, E, W, X, X', Y, Y'; HW)$ satisfies the hypotheses of lemma 5.6, and that $\{EWY', QWY' = \pi', HWY'\}$ is a Δ -system. (E.g., (a) HW secant was assumed; (g) $HW \cap QE = \emptyset$ because $H \notin E\ell$; (h) X, Y are assumed to be good.) Thus, lemma 5.6 implies that $\{EWX', \pi', HWX'\}$ is a Δ -system, i.e., $X' = EX \cap m(\pi')$ is good. That $HX \cap m(\pi')$ is good is, of course, shown in the same way. So we have (**).

Now, fix $\pi \in M^{**}$ and let π' be any plane of M^* . By (**) (and again 4.9, 4.1, 4.2(b)), the bad points of $m(\pi')$ are contained in the set $\{EX \cap m(\pi') : X \text{ is a bad point of } m(\pi)\}$. It follows that M^* is contained in, hence equal to, M^{**} . Thus (again using (**)):

If $\pi, \pi' \in M^*$ ($= M^{**}$), and if X is a bad point of $m(\pi)$, then $EX \cap m(\pi')$ and $HX \cap m(\pi')$ are also bad. (***)

Suppose that for some $\pi \in M^*$, there is a bad point, X , on $m(\pi)$, and let $k (< \infty)$ be the number of bad points on $m(\pi)$. Let $\pi = \pi_0, \pi_1, \dots, \pi_k$ be distinct planes of M^* . (This is possible by (i).) For $i = 0, 1, \dots, k$, define $X_i = EX \cap m(\pi_i)$, and $X'_i = HX_i \cap m(\pi)$. By (***) , we have $X \text{ bad} \Rightarrow X_i \text{ bad} \Rightarrow X'_i \text{ bad}$, $i = 0, 1, \dots, k$. Since $X \text{ bad}$ implies $X \neq Q$, we have $X = X_0, X_1, \dots, X_k$ are distinct points of EX . Suppose that for some $i \neq j$, $X'_i = X'_j$. Then $HX_i = HX'_i = HX'_j = HX_j$ is a line containing X_i and X_j , and must therefore be equal to EX (by 4.1(b)). But this implies $X \in EH$, which is impossible if X is bad. Thus, X'_0, X'_1, \dots, X'_k must be $k+1$ distinct bad points of $m(\pi)$, a contradiction which completes the proof of (*) and of the lemma. \square

Let ℓ be a line, $P \in \ell \cap \mathcal{O}$, and E, H distinct points of \mathcal{O} satisfying

$$EH \cap \ell = \emptyset, \quad (6.6)$$

$$\ell \not\subseteq \pi \quad \forall \pi \in T(P) \cup T(E) \cup T(H). \quad (6.7)$$

Let $P \neq Q \in \ell \cap \mathcal{O}$. All of these objects will be fixed until the end of lemma 6.17.

Suppose $W \in \ell \setminus \mathcal{O}$ satisfies

$$W \not\subseteq \pi \quad \forall \pi \in T(E) \cup T(H). \quad (6.8)$$

We remark that because of (6.7) and (BE)

$$\text{only finitely many } W \in \ell \text{ fail to satisfy (6.8)}. \quad (6.9)$$

Let $E \neq F \in EW \cap \mathcal{O}$, $H \neq G \in HW \cap \mathcal{O}$, and $A = EH \cap FG$. (Notice that $W \not\subseteq \mathcal{O}$ implies $W \neq P, Q, F, G$.)

LEMMA 6.10. (With notation as above:) For all but finitely many
 $Z \in QA$, $\{EWZ, HWZ, PAW\}$ is a Δ -system.

Proof. Let $Z \in QA$ satisfy

$$\begin{aligned} Z &\neq A, Q , \\ Z &\notin PEG , \end{aligned} \tag{6.11}$$

PZ is a secant.

Then, $(G, E, Z, A, H, F, W; PZ)$ is easily (tediously) seen to satisfy conditions (a)-(g) of lemma 5.6. (E.G., (e) $Z \notin GEA$, since otherwise $Z, A \in GEA \Rightarrow Q \in GEA$, and then $W \in GEA \Rightarrow \ell \subseteq GEA \Rightarrow EH \cap \ell \neq \emptyset$; (g) $PZ \cap GE = \emptyset$ follows from $Z \notin PEG$.) In fact, (h) also holds: $ZA (= QA)$ and ZF being lines (since $Q, F \in \mathcal{C}$), it follows from 4.11 that $\{EZA, GZA, PZA\}$ and $\{EZF, GZF, PZF\}$ are Δ -systems. Moreover, $\{EZH, GZH, PZH\}$ is a Δ -system (ZH is a line). Thus 5.6 implies that $\{EWZ, GWZ = HWZ, PWZ = PAW\}$ is a Δ -system whenever Z satisfies (6.11). But since $|PEG \cap QA| = 1$, and $PAW \notin T(P)$, there are at most finitely many $Z \in QA$ for which (6.11) does not hold. \square .

COROLLARY 6.12. If $W \in \ell$ satisfies (6.8) and

$$\begin{aligned} &\text{the number of planes on } \ell \text{ which contain infinitely} \\ &\text{many lines through } W \text{ is infinite,} \end{aligned} \tag{6.13}$$

then, for any plane π such that

$$E, H \notin \pi \supseteq \ell , \tag{6.14}$$

π contains infinitely many lines through W ,

$\{EWZ, HWZ, \pi\}$ is a Δ -system $\forall Z \in \pi \setminus \{W\}$.

Proof. For $W \in \mathcal{O}$, this follows from 4.1(a), 4.10 and 4.11.

For $W \notin \mathcal{O}$, we obtain the present result as a special case of lemma 6.5: Any plane which satisfies conditions (6.14) is in M^* , so that (6.13) gives condition (i) of 6.5. And since W satisfies (6.8), lemma (6.10) implies $M^{**} \neq \emptyset$, which is condition 6.5(ii). (Notice that, with A as in 6.10, $QA = m(\text{PAW})$ and $\text{PAW} \in M^{**}$.) So if π satisfies (6.14) (i.e., $\pi \in M^*$), then all points of $\pi \setminus \{W\}$ are good, which is the desired result. \square

Remark 6.15. If π is a plane containing ℓ , then there is at most one point of ℓ which does not lie on infinitely many lines in π .

Proof. Suppose X, Y are distinct points of ℓ , each lying on only finitely many lines in π . If x and y are the numbers of lines of π other than ℓ on X and Y respectively, then $|(\pi \setminus \ell) \cap \mathcal{O}| \leq xy < \infty$. Thus there are infinitely many tangents on P in π , a contradiction since by (6.7), $\pi \notin T(P)$. \square

We note that, in view of 6.15,

there is at most one $W \in \ell$ which fails to satisfy (6.13). (6.16)

LEMMA 6.17. If π is any plane with $E, H \notin \pi \supseteq \ell$, then
 $\pi^E = \pi^H$.

Proof. By 6.12, if $W \in \ell$ satisfies (6.8) and (6.13), and if W is on infinitely many lines in π , then $EWZ \cap \pi = HWZ \cap \pi$ $\forall Z \in \pi \setminus \{W\}$. But according to (6.9), (6.16), and 6.15, this is true of all but finitely many $W \in \ell$. 6.17 now follows from lemma 6.2.

□

We now allow P, Q, E, H and ℓ to vary.

COROLLARY 6.18. If π is a plane, $P \in \pi \cap \mathcal{O}$, $\pi \notin T(P)$, and if $E, H \in \mathcal{O}$, $\notin \pi$, then $\pi^E = \pi^H$.

Proof. First suppose $P \notin EH$. Since only finitely many of the lines on P in π are on planes of $T(P) \cup T(E) \cup T(H)$, there is some line ℓ satisfying $P \in \ell \subseteq \pi$, $\ell \cap EH = \emptyset$, and $\ell \notin \pi'$ $\forall \pi' \in T(P) \cup T(E) \cup T(H)$. The result then follows from 6.17.

Now, suppose $P \in EH$. By (BE) (applied to P), we may choose $F \in \mathcal{O}$, $F \notin \pi \cup EH$. Then, the preceding argument gives $\pi^E = \pi^F = \pi^H$.

□

COROLLARY 6.19. Let P, W, Y be distinct points with $P \in \mathcal{O}$, $Y \notin PW$, and $PWY \notin T(P)$. Then W and Y are on a near-line.

Proof. Let $E \in \mathcal{O}$, $\notin PWY$. Then by 6.18, $EWY \cap PWY$ satisfies (N2).

□

COROLLARY 6.20. If $W \in \mathcal{P}$, and there is some $P \in \mathcal{O}$ such that $W \notin \pi \forall \pi \in T(P)$, then W is a near-point.

Proof. If $Y \notin PW$, then W, Y are on a near-line by 6.19. If $Y \in PW \setminus \{W\}$, then W, Y are on the near-line PW . \square

COROLLARY 6.21. All points are near-points. (Any two points lie on a near-line.)

Proof. Given $W \in \mathcal{P}$, choose $P, Q \in \mathcal{O}$ with $W \notin PQ$ and $Q \notin \pi \forall \pi \in T(P)$. (Choose P first and use (BE).) Then, any $\pi \in T(P)$ contains just one point of QW . Thus, all but finitely many points of QW are near-points (by 6.20). So all points of QW (and W in particular) are near-points (by 6.4). \square

7. Lines in the Finite Case

In this section, we assume that S has finite order n . Until we reach corollary 7.16, we will assume that n is at least 4, in order to have enough elements to work with. As will be shown, 7.16 is obtained quite easily when $n = 2, 3$, without the aid of any of the earlier results of this section.

As in the discussion preceding lemma 6.5, let ℓ be a line, $W \in \ell$, $W \notin \mathcal{O}$. Assume that $E, H \in \mathcal{O}$, $E, H \notin \ell$, $H \notin El$, and that EW and HW are secants. Assume further that there exists $P \in \ell \cap \mathcal{O}$ with $\ell \not\subseteq \pi \forall \pi \in T(P)$. Set

$$M = \{\pi \in \mathcal{S}: E, H \notin \pi \supseteq \ell\},$$

and for $\pi \in M$, let a point X of $\pi \setminus \{W\}$ be called good if

$EWX \cap \pi = HWX \cap \pi$, and bad otherwise. Let $Q \in \ell \cap \Theta$, and for each $\pi \in M$, let $m(\pi) = \pi \cap QE_H$.

LEMMA 7.1. (With notation as above) Let t be a nonnegative integer, and suppose

there is some $\pi_0 \in M$ for which the number of bad points on $m(\pi_0)$ is at most t . (7.2)

(a) Suppose M contains at least $t + 1$ planes π (one of which may be π_0) for which

the number of good points on $m(\pi) \setminus \{Q\}$ is at least $t + 2$. (7.3)

Then if $\pi \in M$ satisfies (7.3), π contains no bad points.

(b) If $t = 0$, then no $\pi \in M$ contains any bad points.

Proof. The proof of (a) is the same as the proof of lemma 6.5.

Proof of (b): We may assume that W is not a near-point, since otherwise the result follows from 4.11. We first remark that if π is any plane containing ℓ , then π contains at least two lines on W other than ℓ . For if $(\pi \setminus \ell) \cap \Theta \subseteq k$ for some line k on W , then $\pi \notin T(P)$ implies that there are at least $n - f(n)$ points of Θ on k , so that (by (5.1)) at most $f(n) + 1 < \sqrt{n} + 1$ points of k are not near-points. But then by 6.4, all points of k (including W) are near-points, contrary to assumption. Since for any $\pi \in M$, each line of π on W meets $m(\pi)$ in a good point, (b) now follows from (a). □

COROLLARY 7.4. Under the hypotheses of 7.1(a), no $\pi \in M$ contains any bad points.

Proof. 7.1(a) guarantees the existence of a π_0 which satisfies (7.2) with $t = 0$, so the result follows from 7.1(b). \square

As in the discussion preceding lemma 6.10, let l be a line, $P \in l \cap \mathcal{O}$, and E, H distinct points of \mathcal{O} satisfying (6.6) and (6.7). Let $P \neq Q \in l \cap \mathcal{O}$. All of these elements will be fixed until the end of lemma 7.13.

Suppose $W \in l \setminus \mathcal{O}$ satisfies (6.8). (Notice that (6.7) and (BE) imply that

$$\text{at most } 2f(n) \text{ } W \in l \text{ fail to satisfy (6.8).} \quad (7.5)$$

Let $E \neq F \in EW \cap \mathcal{O}$, $H \neq G \in HW \cap \mathcal{O}$, and define $A = EH \cap FG$, $A' = EG \cap FH$.

LEMMA 7.6. With notation as above, for all but at most $f(n)$ $Z \in QA$, $\{EWZ, HWZ, PAW\}$ is a Δ -system.

Proof. Let $Z_0 = QA \cap PEG$, and suppose that PZ_0 is a secant. (Notice that $Z \neq Q, A$.) Each of the systems $(G, E, Z_0, A, H, F, W; PZ_0)$ and $(H, F, Z_0, A, G, E, W; PZ_0)$ is easily seen to satisfy all of the hypotheses of 5.6, with the possible exception of (g). (This is as in lemma 6.10. Z_0 satisfies (6.11) except for the condition $Z_0 \notin PEG$, which in 6.10 was used to obtain 5.6(g).)

Suppose first that 5.6(g) is satisfied by (at least) one of these systems. Then $\{EWZ_{\circ}, GWZ_{\circ}, PAW\}$ is shown to be a Δ -system as in the proof of 6.10. (Notice that $EWZ_{\circ} = FWZ_{\circ}$ and $GWZ_{\circ} = HWZ_{\circ}$.) 7.6 now follows: The proof of lemma 6.10 shows that $\{EWZ, HWZ, PAW\}$ is a Δ -system whenever $Z \in QA$ satisfies conditions (6.11). But the same result holds trivially when $Z = A$, and by 4.11 when $Z = Q$; and as indicated above, it also holds when $Z \in PEG$, provided PZ is a secant. Thus, the only points Z of QA for which $\{EWZ, HWZ, PAW\}$ may fail to be a Δ -system are those for which PZ is a tangent. This gives 7.6.

Now suppose that neither system satisfies 5.6(g). That (g) fails for the system $(G, E, Z_{\circ}, A, H, F, W; PZ_{\circ})$ implies that the only two points of $PZ_{\circ} \cap \Theta$ are P and $PZ_{\circ} \cap GE$. Similarly, since (g) fails for the second system, PZ_{\circ} and HF must intersect in the only point of $PZ_{\circ} \cap \Theta$ other than P . Since this point lies on GE and HF , it must be A' , and in particular we have $A' \in \Theta$ and $PA'W = PAW$ (see figure 4).

Since $n \geq 4$, we may choose in $PA'W$ a secant m on P with $m \neq \ell, PA, PA'$ (using (BE)). Then, whenever $Z \in m \setminus \{P\}$ and $Z \neq m \cap EHW$, the system $(G, F, Z, A', H, E, W; PZ)$ satisfies the hypotheses of lemma 5.6. (The advantage of having $A' \in \Theta$ is that ZA' is automatically a line, whereas previously we needed $Z \in QA$ to force ZA to be a line.) Again, as in the proof of 6.10, $\{FWZ = EWZ, HWZ, PA'W = PAW\}$ is a Δ -system. Of course, this is also true if $Z = P$ or $Z = m \cap EHW$ (trivially in the latter case). But as

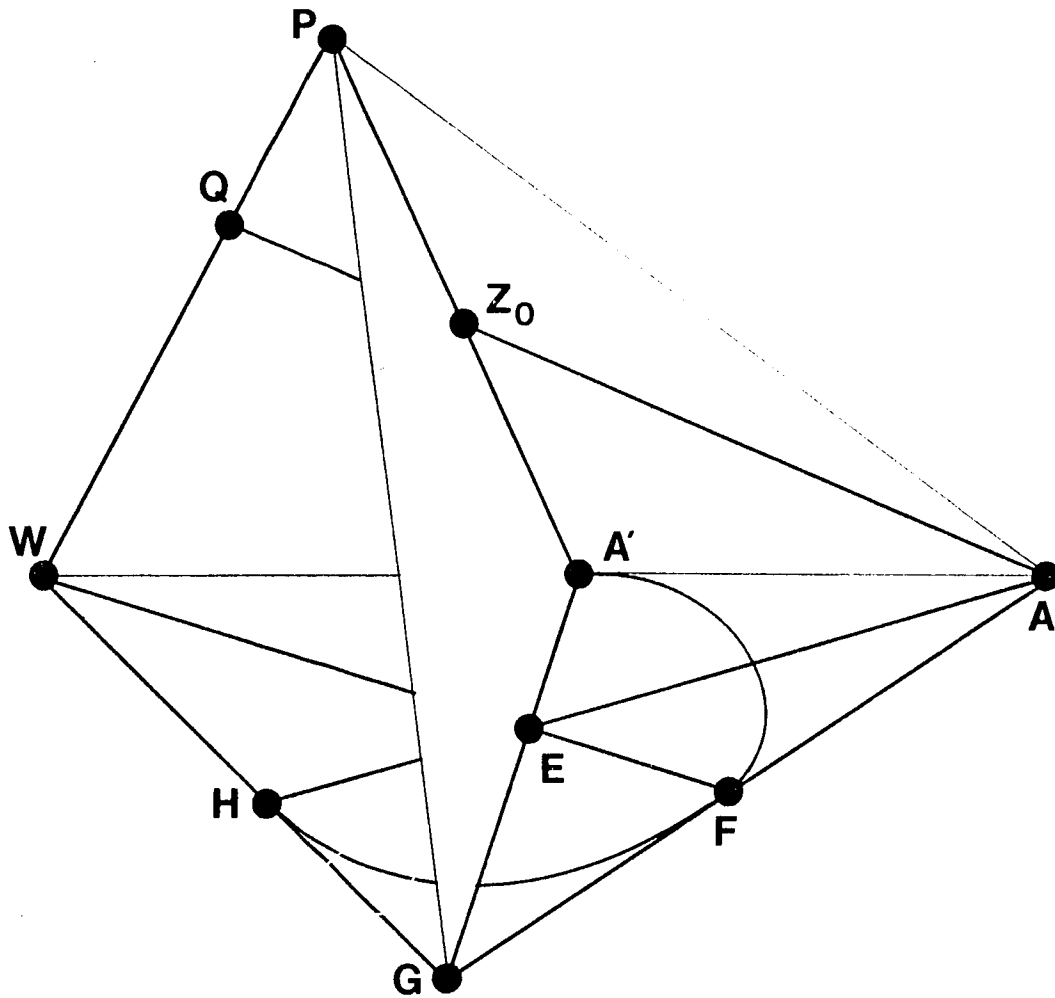


Figure 4. Case Involving A' in the Proof of Lemma 7.6

observed earlier (see the first paragraph of the proof of 6.5),
 $\{EWZ, HWZ, PAW\}$ is a Δ -system for each $Z \in m$ iff $\{EWZ, HWZ, PAW\}$ is
 a Δ -system for each $Z \in PAW \setminus \{W\}$.

This completes the proof of 7.6. \square

COROLLARY 7.7. With notation as in lemma 7.6, if $f(n) = 1$ then
 $\{EWZ, HWZ, PAW\}$ is a Δ -system $\forall Z \in PAW \setminus \{W\}$.

Proof. By 7.6, there is at most one ($= f(n)$) $Z \in QA$ for which
 $\{EWZ, HWZ, PAW\}$ is not a Δ -system. If such a Z (say Z_0) exists,
 then by 4.9, we have for $X \in PAW \setminus \{W\}$:

$$\begin{aligned} X \in EWZ_0 &\Leftrightarrow X \notin EWZ \quad \forall Z \in QA \setminus \{Z_0\} \\ &\Leftrightarrow X \notin HWZ \quad \forall Z \in QA \setminus \{Z_0\} \\ &\Leftrightarrow X \in HWZ_0. \end{aligned}$$

That is, $EWZ_0 \cap PAW = HWZ_0 \cap PAW$. This completes the proof. \square

The next two corollaries are derived from 7.4, 7.6 and 7.1(b), 7.7
 respectively in the same way as 6.12 was derived from 6.5, 6.10.

COROLLARY 7.8. Let $f(n) \geq 2$. If $W \in \ell$ satisfies (6.8) and
the number of planes on ℓ which contain at least
 $f(n) + 2$ lines $\neq \ell$ through W is at least (7.9)
 $f(n) + 1$,

then for any plane π with $E, H \notin \pi \ni \ell$, $\{EWZ, HWZ, \pi\}$ is a Δ -system
 $\forall Z \in \pi \setminus \{W\}$.

COROLLARY 7.10. Let $f(n) = 1$. If $W \in \ell$ satisfies (6.8),
then for any plane π with $E, H \notin \pi \ni \ell$, $\{EWZ, HWZ, \pi\}$ is a Δ -system
 $\forall Z \in \pi \setminus \{W\}$.

Remark 7.11. Let $f(n) \geq 2$. If π is a plane containing ℓ , then there is at most one point of ℓ which lies on less than $f(n) + 2$ lines other than ℓ in π .

Proof. Since P lies on at least $n - f(n)$ secants other than ℓ in π , there is at most one $W \in \ell$ for which the number of lines $\neq \ell$ on W in π is $< \sqrt{n - f(n)}$ (see proof of 6.15). But

$$\begin{aligned} \sqrt{n - f(n)} > f(n) + 1 &\Leftrightarrow n - f(n) > (f(n))^2 + 2f(n) + 1 \\ &\Leftrightarrow (f(n))^2 + 3f(n) + 1 < n. \end{aligned}$$

Since the last inequality is always true if $f(n) \geq 2$, this proves 7.11. □

From 7.11, it follows that

$$\text{there is at most one } W \in \ell \text{ which fails to satisfy 7.9.} \quad (7.12)$$

(here we are using (1) the number of planes on ℓ is $n+1$, and

$$(2) \quad f(n) < \frac{1}{2}(n + 1) .)$$

LEMMA 7.13. If π is any plane with $E, H \notin \pi \ni \ell$, then $\pi^E = \pi^H$.

Proof. First let $f(n) \geq 2$. By 7.8, we have $EWZ \cap \pi = HWZ \cap \pi \forall Z \in \pi \setminus \{W\}$, provided $W \in \ell$ satisfies (6.8) and (7.9). By (7.5) and (7.12), this holds for all but at most $2f(n) + 1 < \sqrt{n} + 1$ $W \in \ell$, and the result follows from 6.2.

If $f(n) = 1$, then by 7.10, $W \in \ell$ need only satisfy (6.8) to insure that $EWZ \cap \pi = HWZ \cap \pi \quad \forall Z \in \pi \setminus \{W\}$. By 7.5, there are at most $2 < \sqrt{n} + 1$ $W \in \ell$ which fail (6.8), and again we may apply 6.2. \square

We now allow P, Q, E, H and ℓ to vary.

COROLLARY 7.14. If π is a plane, $P \in \pi \cap \mathcal{O}$, $\pi \notin T(P)$, and if $E, H \in \mathcal{O}$, $\notin \pi$, then $\pi^E = \pi^H$.

Proof. Same as 6.18. We make use of the fact that $3f(n) + 1 < n + 1$, and in particular the assumption $n \geq 4$. \square

COROLLARY 7.15. Let P, W, Y be distinct points with $P \in \mathcal{O}$, $Y \notin PW$, and $PWY \notin T(P)$. Then W, Y are on a near-line.

Proof. Same as 6.19.

We now drop the assumption $n \geq 4$.

COROLLARY 7.16. If $W \in \mathcal{P}$ and there is some $P \in \mathcal{O}$ such that $W \notin \pi \quad \forall \pi \in T(P)$, then W is a near-point.

Proof. Same as 6.20 if $n \geq 4$. If $n = 2$ or 3 , then PW a secant implies that there are at most two points on PW which are not near-points. The result then follows from 6.4. \square

COROLLARY 7.17. All points are near-points. (Any two points lie on a near-line.)

Proof. Same as 6.21, using the fact that $f(n) < \sqrt{n} + 1$. \square

8. Planes

In this section, we place no restriction on the order of S . For any two points X, Y , we may now write XY for the near-line containing X and Y .

Remark 8.1. If π is a plane and $P, Q \in \mathcal{O}$, $\notin \pi$, then $\pi^P = \pi^Q$. (The lines of this projective plane are the near-lines contained in π .)

LEMMA 8.2. Let $P \in \mathcal{O}$, and suppose that in S/P the 6-tuple $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$ satisfies (5.2). If each of (the lines) $\underline{C}, \underline{U}$ is a secant, then S/P is Desarguesian with respect to $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$.

Proof. This will follow from lemma 5.5 if we can find $C, U, V, W, Z, Z_1 \in \mathcal{P}$ such that $\underline{C} = PC, \dots, \underline{Z}_1 = PZ_1$, and (P, C, U, V, W, Z, Z_1) satisfies conditions (i), (ii) and (iv)-(viii) of (5.4). (Conditions (iii) and (ix) and the fact that $\{CWZ_1, UWZ_1, PWZ_1\}$ is a Δ -system are guaranteed by 6.21, 7.17.) To do this, choose $C \in (\underline{C} \cap \mathcal{O}) \setminus \{P\}$, and $U \in (\underline{U} \cap \mathcal{O}) \setminus \{P\}$. Let ℓ be a line on U in \underline{UV} with $P, C \notin \ell$, and let $V = \underline{V} \cap \ell$, $W = \underline{W} \cap \ell$. Let k be a line on C in \underline{CZ} with $P \notin k$, $k \cap \ell = \emptyset$, and let $Z = \underline{Z} \cap k$, $Z_1 = \underline{Z}_1 \cap k$. Then conditions (5.4)(i), (ii), (iv)-(vi) and (viii) are

obvious, and condition (vii) follows from the last two conditions of 5.2 (applied to $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$). \square

LEMMA 8.3. S/P is Desarguesian for all $P \in \mathcal{O}$.

Proof. If the order of S is 2 (actually if it's ≤ 8), there is nothing to prove, so assume that the order of S is at least 3. By 8.1, it is sufficient to prove that S/P is Desarguesian for some $P \in \mathcal{O}$. (For if S/P is Desarguesian, and $Q \in \mathcal{O} \setminus \{P\}$, then we can choose a plane π not containing P, Q , and we have $S/P \cong \pi^P = \pi^Q \cong S/Q$.)

Fix $P, Q \in \mathcal{O}$ with $P \notin \pi \ \forall \pi \in T(Q)$. We show that S/P is Desarguesian by showing that it is Desarguesian with respect to $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$ whenever this 6-tuple satisfies conditions (5.2).

Given such a 6-tuple, we first show that we can choose points $C \in \underline{C} \setminus \{P\}$, $U \in \underline{U} \setminus \{P\}$ and a plane π such that $C, U \notin \pi' \ \forall \pi' \in T(Q)$, $U \notin QC$, $C, U \in \pi$, and $P, Q \notin \pi$. Notice that by our choice of P and Q , any $\pi' \in T(Q)$ meets each line through P in just one point (this point being Q if the line in question is PQ). It follows that there exist points $C_1, C_2 \in \underline{C} \setminus \{P\}$ and $U_1, U_2 \in \underline{U} \setminus \{P\}$ which do not lie on any plane of $T(Q)$. (In the finite case, the assumption $n \geq 3$ implies $n - f(n) \geq 2$.) At least two of the near-lines $C_1U_1, C_1U_2, C_2U_1, C_2U_2$ (say C_hU_i and C_jU_k) do not contain Q . It is possible to find some $R \in \mathcal{O} \setminus PCU$ which does not lie on at least one of QC_hU_i, QC_jU_k (since otherwise

$\mathcal{O} \subseteq PCU \cup (QC_h U_i \cap QC_j U_k)$ violates (BE) applied to P). Say $R \notin QC_h U_i$.
Then $C = C_h$, $U = U_i$ and $\pi = RCU$ have the desired properties.

With C, U, π as above, set $V = \underline{V} \cap \pi$, $W = \underline{W} \cap \pi$, $Z = \underline{Z} \cap \pi$
and $Z_1 = \underline{Z}_1 \cap \pi$. Then

S/P is Desarguesian with respect to $(\underline{C}, \underline{U}, \underline{V}, \underline{W}, \underline{Z}, \underline{Z}_1)$

$\Leftrightarrow \pi^P = \pi^Q$ is Desarguesian with respect to (C, U, V, W, Z, Z_1)

$\Leftrightarrow S/Q$ is Desarguesian with respect to $(QC, QU, QV, QW, QZ, QZ_1)$.

But the last statement is, because of our choice of C and U , a
consequence of 8.2. This completes the proof of the lemma. \square

DEFINITION. A point A will be called planar if whenever the
5-tuple (A, B, C, D, E) satisfies

$$\begin{aligned} &A, B, C, D, E \text{ are distinct points,} \\ &C \in AB, \\ &E \in AD, \\ &AB \neq AD, \end{aligned} \tag{8.4}$$

the lines BD and CE intersect.

Remark 8.5. All points of \mathcal{O} are planar.

If $A, B, C \in \mathcal{P}$ do not lie on a near-line, then they are contained
in at most one plane, since the intersection of two distinct planes

containing A and B is AB . It will be convenient to let ABC denote this common plane when it exists, and the empty set when there is no plane containing A , B and C .

LEMMA 8.6. If $U \in \mathcal{P}$, and if for some $X \in \mathcal{O}$ ($X \neq U$), the line XU contains some planar point $Y \neq X$, then U is planar.

Proof. Of course, we may assume $Y \neq U$. Suppose (U, V, W, X_1, Y_1) satisfies (8.4). We must show that $VX_1 \cap WY_1 \neq \emptyset$. We may assume that

$$\begin{aligned} & \text{the only coplanar triples from } \{U, V, W, X_1, Y_1\} \\ & \text{are } \{U, V, W\} \text{ and } \{U, X_1, Y_1\}, \end{aligned} \tag{8.7}$$

since otherwise, the whole configuration is easily seen to lie in a plane, and the result follows from 4.14. At this point, the reader may find it helpful to refer to figure 5.

Let $Z = XV \cap YW$ and $C = XX_1 \cap YY_1$. ($XV, YW \subseteq XUV$, and $XX_1, YY_1 \subseteq XUX_1$, so Z and C exist.) By the planarity of X and Y respectively, there exist points $Z_1 = ZC \cap VX_1$ and $Z_2 = ZC \cap WY_1$. Notice at this point that if S has order 2, then $Z_1, Z_2 \in ZC \setminus \{Z, C\}$ implies $Z_1 = Z_2 \in VX_1 \cap WY_1$ (by 4.6), which is the desired conclusion. For the remainder of the proof, we assume that the order of S is at least 3.

Suppose there exists $P \in \mathcal{O}$ such that $P \notin XUV \cup XUX_1 \cup XVX_1 \cup YWY_1$. (Recall that $YWY_1 := \emptyset$ if y, W, Y_1 do not lie on a plane.) Notice that according to (8.7), the planes PUV , PUX_1 , PVX_1 and PWY_1 are

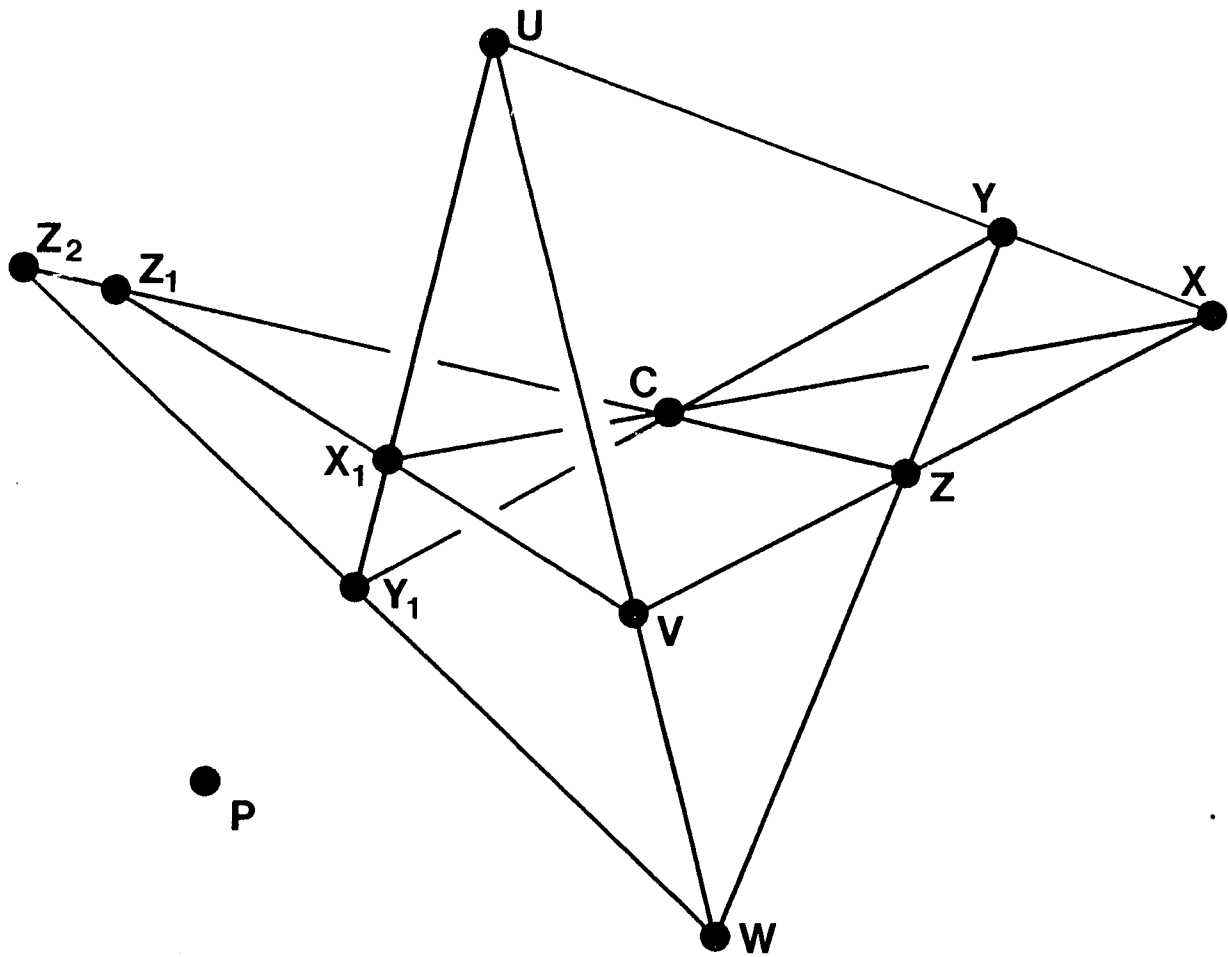


Figure 5. Construction for Lemma 8.6

distinct. Now if in S/P we set $\underline{U} = PU$, $\underline{V} = PV$, etc., then in view of lemma 8.3, the lines \underline{ZC} , \underline{VX}_1 and \underline{WY}_1 must be concurrent. But this implies

$$PZ_1 = \underline{ZC} \cap \underline{VX}_1 = \underline{ZC} \cap \underline{WY}_1 = PZ_2 .$$

If $Z_1 \neq Z_2$, then $P \in Z_1Z_2 = ZC \subseteq XVX_1$, contrary to assumption.

We conclude that $Z_1 = Z_2$ is the desired intersection point of VX_1 and WY_1 .

Thus $VX_1 \cap WY_1 = \emptyset$ forces

$$\mathcal{O} \subseteq XUV \cup XUX_1 \cup XVX_1 \cup WY_1 ; \quad (8.8)$$

and by interchanging V and X_1 with W and Y_1 respectively and repeating the above argument, we see that the nonintersection of VX_1 and WY_1 implies

$$\mathcal{O} \subseteq XUV \cup XUX_1 \cup XWY_1 \cup YVX_1 . \quad (8.9)$$

Assume $VX_1 \cap WY_1 = \emptyset$. Then the combination of (8.8) and (8.9) yields

$$\begin{aligned} \mathcal{O} \subseteq XUV \cup XUX_1 \cup (XVX_1 \cap XWY_1) \cup (YVX_1 \cap YWY_1) \\ \cup (XVX_1 \cap YVX_1 = VX_1) \cup (XWY_1 \cap YWY_1 = WY_1) . \end{aligned}$$

Observing that $\mathcal{O} \cap VX_1 = \mathcal{O} \cap WY_1 = \emptyset$ (by 8.7), that $XVX_1 \cap XWY_1 = XZ_2$, and that $YVX_1 \cap YWY_1 = YZ_1$ or \emptyset (depending on whether at least one of YVX_1 , YWY_1 is empty), we find that

$$\mathcal{O} \subseteq XUV \cup XUX_1 \cup XZ_2 \cup YZ_1 . \quad (8.10)$$

In particular, $J = T(X) \cup \{XUV, XUX_1, XYZ_1\}$ is a set of planes whose

union contains all lines through X except possibly XZ_2 , and furthermore,

$$\begin{aligned} |J| &< \infty && \text{if } S \text{ is infinite} \\ &\leq f(n) + 3 && \text{if } S \text{ has finite order } n. \end{aligned}$$

In the infinite case, this is clearly impossible. For the finite case, let $N = |\{(\ell, \pi) : X \in \ell \subseteq \pi \in J\}|$. Noting that some line on X must lie in more than one $\pi \in J$, we have $(f(n) + 3)(n + 1) \geq N > n^2 + n \Rightarrow f(n) > n - 3$, a contradiction if n is at least 4.

When $n = 3$, we get a similar contradiction with a little more work: First, observe that XZ_1 is a tangent (since $XZ_1 \cap \Theta \subseteq (XZ_1 \cap XUV) \cup (XZ_1 \cap XUX_1) \cup (XZ_1 \cap XZ_2) \cup (XZ_1 \cap YZ_1) = \{X, Z_1\}$, and $Z_1 \notin \Theta$ by (8.7)). Let $UZ_2 \setminus \{U, Z_2\} = \{A, B\}$. The lines XA , XB are tangents by (8.10) (since $XUZ_2 \cap XUV = XUZ_2 \cap XUX_1 = XUZ_2 \cap XYZ_1 = XU$). But then each of the three noncoplanar lines XZ_1 , XA , XB lies on a plane of $T(X)$, which is impossible since $|T(X)| \leq f(3) = 1$.

We conclude that $VX_1 \cap WY_1 \neq \emptyset$, which proves the lemma. \square

COROLLARY 8.11. All points are planar.

Proof. Give $W \in \mathcal{P}$, choose $P, Q \in \Theta$ with $W \notin PQ$ and $Q \notin \pi \forall \pi \in T(P)$. Then QW contains some point $Z \neq Q$ which does not lie on any $\pi \in T(P)$. Since PZ contains a point of Θ other than P , Z is planar by 8.6. But then a second application of 8.6 shows that, since QW contains the planar point $Z \neq Q$, W must be planar.

\square

It is well known (see [14], pp. 27-28) that an incidence structure $(\mathcal{X}, \mathcal{R})$ with $|\mathcal{R}| \geq 2$ either is a projective plane or is isomorphic to the point-line incidence structure of a projective geometry over some skewfield K , provided it satisfies the following three axioms:

Any two distinct $X, Y \in \mathcal{X}$ are on exactly
one $\ell \in \mathcal{R}$. (8.12)

$|\ell| \geq 3$ for each $\ell \in \mathcal{R}$. (8.13)

If a, b, c, d, e are distinct elements
of \mathcal{R} , no three containing a common $X \in \mathcal{X}$,
if a meets b , and if each of a, b meets
each of c, d , then c meets d . (8.14)

Let \mathcal{M} be the set of near-lines of S . We have already seen that $(\mathcal{P}, \mathcal{M})$ satisfies (8.12)-(8.14) (see 6.21 and 7.17, 4.6, and 8.11). We may therefore take \mathcal{P} and \mathcal{M} to be the point and line sets of a projective geometry S^* of dimension at least two. If $\pi \in \mathcal{G}$, then π contains any near-line which it meets in at least two points, and any two near-lines in π intersect (4.14), so π is a plane of S^* . Then $\mathcal{P} \not\subseteq \pi$ and the fact that π meets every near-line (see 4.13) implies that the dimension of S^* is 3.

Since the near-lines and planes of S^* which meet \mathcal{O} are precisely the lines of \mathcal{L} and the planes of \mathcal{G} (by 4.5 and 4.3 respectively), it follows that our original lattice L is the lattice $L(\mathcal{O})$ defined by the subset \mathcal{O} of the point set of S^* .

This completes the proof of Theorem 4.

CHAPTER III
COROLLARIES AND EXAMPLES

9. Corollaries for Circle-planes

In this section, we define three types of "circle-planes" - inversive, Laguerre and Minkowski planes - and give canonical constructions. We then show how each circle plane Π may be embedded in an LPP-lattice $L(\Pi)$. Any circle plane has an affine plane naturally associated with each of its points. These affine planes, which may be seen as the essential link between the different types of circle-planes, are pointed out in connection with the lattices $L(\Pi)$. Since the lattices $L(\Pi)$ all satisfy (BE), it is a simple matter to obtain theorem 3 and generalizations of theorems 1 and 2 as corollaries of theorem 4. All of this is quite straightforward, and is included here mainly because, as mentioned in the introduction, the circle planes provided the initial motivation for the present study.

Throughout this section, Π will denote an incidence structure $(\mathcal{O}, \mathcal{C})$, and the blocks of Π will be called circles. The letters x and y are reserved for circles. Circles x and y are tangent at the point P if $x \cap y = \{P\}$. If x, y are distinct circles with $|x \cap y| \geq 2$, then $x \cap y$ is called a trail. The letters s and t are reserved for trails. If $P, Q \in \mathcal{O}$, and either $P = Q$, or no circle contains both P and Q , then P is parallel to Q .

(written $P \parallel Q$). Trails s and t are parallel (written $s \parallel t$) if each $S \in s$ is parallel to some $T \in t$ and vice versa. (The notion of a trail is useful in the discussion of inversive and Laguerre planes, while parallelism of points occurs in Laguerre and Minkowski planes.)

Π (as above) is called an inversive plane if it satisfies:

(I1) Any three points are on at least one circle. If $x, y, z \in \mathcal{C}$ and $|x \cap y \cap z| \geq 2$, then $x \cap y = y \cap z = z \cap x$.

(I2) If P and Q are points, and x a circle containing P but not Q , then there is a unique circle which contains Q and is tangent to x at P .

(I3) Circles are nonempty. There exist four points which do not lie on a common circle.

An inversive plane Π is an inversive plane in the narrow sense if each trail contains exactly two points. (Equivalently, Π satisfies (I2), (I3) and

(I1') Any three distinct points are on exactly one circle.

These definitions are easily seen to be equivalent to the definition of inversive plane given in section 1. For example, if we assume (I1'), (I2), (I3), let $P \in \mathcal{C}$ and define Π_P as \mathcal{L}_P in section 1, then (a) If $Q \neq R$ are points distinct from P , and x the unique

circle containing P, Q, R , then in Π_P , x is the unique line containing Q and R ; (b) if x is a circle through P , and Q a point not on x , then in Π_P , the circle given by (I2) is the unique line through Q parallel to x ; (c) the existence of a triangle in Π_P is guaranteed by (I3).)

For Π an inversive plane, we make a few elementary observations: If P and Q are distinct points, there is a unique trail, denoted (PQ) , which contains P and Q ; any circle containing P and Q contains (PQ) . If t is a trail and $P \in \mathcal{C} \setminus t$, then there is a unique circle containing P and t ; or put another way, if s and t are distinct, intersecting trails, then there is a unique circle containing s and t . For each $P \in \mathcal{C}$, tangency (at P) defines an equivalence relation on the circles through P . (For if x, y, z are distinct circles with $x \cap y = x \cap z = \{P\}$, and if $P \neq Q \in y \cap z$, then y and z are distinct circles containing Q and tangent to x at P , which violates (I2).) The equivalence classes under this relation are called pencils (at P). If t is a trail, $P \in t$, and β a pencil at P , then there is a unique $x \in \beta$ with $t \subseteq x$.

Much of the above discussion of inversive planes is drawn from [3], to which the reader is referred for further details.

Π is called a Laguerre plane if it satisfies:

(L1) a) Any two distinct, non-parallel points P, Q are on a unique trail (denoted (PQ)).

b) If P, Q, R are distinct points with $Q \parallel P \parallel R$, and if there is no circle containing P, Q and R , then $(PQ) \parallel (PR)$.

(L2) If P and Q are non-parallel points, and x a circle containing P but not Q , then there is a unique circle which contains Q and is tangent to x at P .

(L3) If $P \in \mathcal{O}$, $x \in \mathcal{C}$, and $P \notin x$, then there is a unique $Q \in x$ with $P \parallel Q$.

(L4) There exists a circle x and a point P with $|x| \geq 3$ and $P \notin x$.

A Laguerre plane Π is called a Laguerre plane in the narrow sense if each trail contains exactly two points. (Equivalently, Π satisfies (L2), (L3), (L4) and

(L1) Any three pairwise non-parallel points are on a unique circle.)

Let Π be a Laguerre plane. We note some elementary properties: Parallelism of points is an equivalence relation whose equivalence classes are called generators. For $P \in \mathcal{O}$, g_P is the generator containing P . Every circle meets every generator in a unique point. Similarly, parallelism of trails is an equivalence relation. The equivalence class containing the trail t is denoted g_t .

Also: If P and Q are distinct points, and x a circle containing P and Q ($\Rightarrow P \parallel Q$), then $(PQ) \subseteq x$. If P, Q, R are pairwise non-parallel points with $(PQ) \parallel (PR)$, then there is a unique circle containing P, Q and R (i.e., containing $(PQ) \cup (PR)$). If s and t are non-parallel trails, then there is at most one generator which meets both s and t . (Proof: let $s = (PQ)$, $t = (P_1Q_1)$ with $P \parallel P_1 \parallel Q \parallel Q_1$. Then (L1)b) implies $(PQ) \parallel (P_1Q) \parallel (P_1Q_1)$.) For each $P \in \mathcal{O}$, tangency (at P) defines an equivalence relation on the circles through P . The equivalence classes are called pencils (at P). If t is a trail, $P \in t$, and β a pencil at P , then there is a unique $x \in \beta$ with $t \subseteq x$.

Much of the above discussion and a great deal of additional information on Laguerre planes, may be found in [6].

Finally, Π is called a Minkowski plane if there exist two equivalence relations, \parallel_+ and \parallel_- , on \mathcal{O} such that

(M1) If $P \parallel Q$ then either $P \parallel_+ Q$ or $P \parallel_- Q$.

(M2) If $P, Q \in \mathcal{O}$, then there is a unique point R for which $P \parallel_+ R \parallel_- Q$.

(M3) If $P \in \mathcal{O}$, $x \in \mathcal{C}$, then there exist unique points $Q, R \in x$ with $Q \parallel_+ P \parallel_- R$.

(M4) Any three pairwise non-parallel points are on a unique circle.

(M5) If P and Q are non-parallel points, and x a circle containing P but not Q , then there is a unique circle which contains Q and is tangent to x at P .

(M6) There exist four pairwise non-parallel points which do not lie on a common circle.

Equivalence classes under \parallel_+ and \parallel_- are called (+)-generators and (-)-generators respectively (and a generator is an equivalence class of either type). The (+)- and (-)-generators containing $P \in \mathcal{O}$ are denoted g_P^+ and g_P^- respectively. Any two points of a generator are parallel, and every circle meets every generator (both by (M3)). If $P, Q \in \mathcal{O}$ satisfy $P \parallel_+ Q$ and $P \parallel_- Q$, then $P = Q$. As in the inversive and Laguerre cases, for each $P \in \mathcal{O}$, tangency (at P) defines an equivalence relation on the circles through P . The equivalence classes are again called pencils (at P).

Canonical examples of circle-planes are obtained by taking plane sections of various objects in projective 3-space. We first need to define these objects.

Let K be a skewfield and $S = PG(d, K)$ for some $d \geq 2$. A subset \mathcal{O} of the point set of S is called a semi-ovoid (see [9]) if for each $P \in \mathcal{O}$ the union of all lines meeting \mathcal{O} only in P is a hyperplane. If $d = 2$, a semi-ovoid is also called a semi-oval. A semi-ovoid (semi-oval) is called an ovoid (oval) if no three of its

points are collinear. (Ovoids in $PG(3,K)$ were defined in section 1.)

Let $d = 3$. If π is a plane of S , u a semi-oval in π , and V a point of S not lying on π , then the union of all lines which contain V and meet u is called a semi-oval cone (or an oval cone if u is an oval). V is called the vertex of the cone.

Suppose that $d = 3$ and K is commutative. \mathcal{Q} is a hyperbolic quadric if it is, with respect to some choice of coordinates, the set of points satisfying the equation $x_0x_3 = x_1x_2$. If \mathcal{Q} is a hyperbolic quadric, then the set of lines contained in \mathcal{Q} consists of two disjoint sets of pairwise skew lines, say R and R' , such that $\mathcal{Q} = \bigcup_{\ell \in R} \ell = \bigcup_{m \in R'} m$. If $\ell \in R$ and $m \in R'$, then ℓ and m meet in a unique point. R and R' are called the rulings of \mathcal{Q} . The converse of these observations is given by the following proposition. (I assume that this fact is in the literature, but have been unable to locate it.)

PROPOSITION 9.1. Let \mathcal{Q} be a subset of the point set of $PG(3,K)$, K a skewfield, and suppose that $\mathcal{Q} = \bigcup_{\ell \in R} \ell = \bigcup_{m \in R'} m$, where R and R' are distinct sets of pairwise skew lines, and every line of R meets every line of R' (in a necessarily unique point). Then K is commutative and \mathcal{Q} is a hyperbolic quadric.

Proof. Let $\ell_\infty, \ell_0, \ell_1$ be distinct lines of R , and m_∞, m_0, m_1 distinct lines of R' . We may assign homogeneous coordinates as follows: $\ell_\infty \cap m_\infty = (0,0,0,1)$, $\ell_0 \cap m_\infty = (0,0,1,0)$, $\ell_\infty \cap m_0 = (0,1,0,0)$,

$\ell_0 \cap m_0 = (1,0,0,0)$, and $\ell_1 \cap m_1 = (1,1,1,1)$. Then (it is easily checked that)

$$R = \{\ell_\infty\} \cup \{\ell_a := \langle(1,a,0,0), (0,0,1,a)\rangle : a \in K\} ,$$

$$R' = \{m_\infty\} \cup \{m_b := \langle(1,0,b,0), (0,1,0,b)\rangle : b \in K\} .$$

Let $a, b \in K \setminus \{0\}$, and let X be the intersection point of ℓ_a and m_b . Multiplying by scalars on the right, we have

$$(1, a, c, ac) = X = (1, d, b, bd)$$

for some $c, d \in K$. But this implies $c = b$, $d = a$, and $ab = ba$.

Since a and b were arbitrary in $K \setminus \{0\}$, K is commutative.

And with respect to the above choice of coordinates, \mathcal{O} is the set of points satisfying $x_0x_3 = x_1x_2$. \square

We may now construct our canonical examples. Again, let $S = \text{PG}(3, K)$, K a skewfield. Let \mathcal{O} be a subset of the point set of S .

If \mathcal{O} is a semi-ovoid, then the following incidence structure, $\Pi(\mathcal{O})$, is an inversive plane:

The points of $\Pi(\mathcal{O})$ are the points of \mathcal{O} .

The circles of $\Pi(\mathcal{O})$ are the planes which contain at least two points of \mathcal{O} .

Incidence is containment.

$\Pi(\mathcal{O})$ is an inversive plane in the narrow sense iff \mathcal{O} is an ovoid.

If \mathcal{O} is a simi-oval cone with vertex V , then the following incidence structure, $\Pi(\mathcal{O})$, is a Laguerre plane:

The points of $\Pi(\mathcal{O})$ are the points of \mathcal{O} other than V .

The circles of $\Pi(\mathcal{Q})$ are the planes which do not contain V .

Incidence is containment.

(The lines through V which lie in \mathcal{Q} turn out to be the generators of $\Pi(\mathcal{Q})$.) $\Pi(\mathcal{Q})$ is a Laguerre plane in the narrow sense iff \mathcal{Q} is an oval cone.

If \mathcal{Q} is a hyperbolic quadric, then the following incidence structure, $\Pi(\mathcal{Q})$, is a Minkowski plane:

The points of $\Pi(\mathcal{Q})$ are the points of \mathcal{Q} .

The circles of $\Pi(\mathcal{Q})$ are the planes which do not contain rulings of \mathcal{Q} .

Incidence is containment.

(The rulings of \mathcal{Q} turn out to be the generators of $\Pi(\mathcal{Q})$.)

A circle plane will be called egglike if it is isomorphic to a circle plane arising from one of the above constructions. The reader should find it helpful to keep these models in mind during the discussion of the lattices associated with the circle-planes.

Remark. At this point, we can see that proposition 9.1 justifies the statement, made at the end of section 1, that the egglike Minkowski planes are no more general than the Miquelian ones.

Before defining the lattices $L(\Pi)$, we mention one rather obvious lemma, which is sometimes useful in checking that a given poset is a semimodular lattice.

LEMMA 9.2. Let L be a poset having a lower bound, 0 . (Elements of L which cover 0 are called points.) Suppose

(i) If P and Q are distinct points, then their least upper bound exists and covers P, Q .

(ii) For each point P , the set of elements $\geq P$, ordered as in L , forms a semimodular lattice (denoted L^P).

(iii) Any element of $L \setminus \{0\}$ is in L^P for some point P .

Then L is a semimodular lattice. (All definitions may be found in [7].)

Proof. We first show the existence of least upper bounds (l.u.b.'s). Let $x, y \in L \setminus \{0\}$. If there is a point P with $x, y \geq P$, then x and y have a l.u.b. in L^P , and this is clearly a l.u.b. in L as well. If no such P exists, let X, Y be points with $x \geq X, y \geq Y$. Then $z \geq x, y \Leftrightarrow z \geq x, y, X \vee Y \Leftrightarrow z \geq x \vee (X \vee Y), y$ (noting that $x \vee (X \vee Y)$ exists since $x, X \vee Y \in L^X$) $\Leftrightarrow z \geq (x \vee (X \vee Y)) \vee y$. The latter is thus a l.u.b. for x and y .

We next show that there exist greatest lower bounds (g.l.b.'s). Let $x, y \in L$. If there is a point P with $x, y \geq P$, then x and y have a g.l.b., w , in L^P , and w is also a g.l.b. for x and y in L . (For $z \leq x, y$ implies $P \vee z \leq x, y$, which gives

$z \leq P \vee z \leq w$.) Of course if there is no such point P , then 0 is the only lower bound for x and y (so it is a g.l.b.). We have shown that L is a lattice.

Finally, let $x, y \in L$ cover $x \wedge y$. If there is a point P with $x, y \geq P$, then $x \vee y$ covers x, y by the semimodularity of L^P . But if $x \wedge y = 0$, then x and y must be points, so that $x \vee y$ covers x and y by (ii). Thus L is semimodular. \square

If $\Pi = (\mathcal{O}, \mathcal{C})$ is a circle-plane, and β a pencil of Π , then $\underline{\beta}$ denotes the point which the circles of β have in common. If $P \in \mathcal{O}$, $x \in \mathcal{C}$, and $P \in x$, then $[P, x]$ denotes the pencil at P which contains x . In what follows, an affine or projective plane may be thought of as an incidence structure, a lattice, or a poset (with $P < \ell$ if P is a point lying on the line ℓ). There is, of course, no essential difference between these points of view, and we will use them interchangeably.

We next define LPP-lattices associated with the various circle planes.

First, let Π be an inversive plane. We may regard Π as a poset consisting of the points, trails and circles of Π , ordered by containment, i.e., with

$$\begin{aligned} P < t & \quad \text{if} & \quad P \in t, \\ P < x & \quad \text{if} & \quad P \in x, \\ t < x & \quad \text{if} & \quad t \subseteq x, \end{aligned}$$

whenever P is a point, t a trail, and x a circle. For each $P \in \mathcal{O}$, the set of elements $> P$, equipped with this ordering, is an affine plane, which we denote Π_P . (See [3]. For affine plane axioms, see [14], p.116. That the Π_P are affine planes is immediate from our remarks following the definition of inversive plane. For inversive planes in the narrow sense, these affine planes are essentially the same as the affine planes which appear in the definition of inversive plane given in section 1.)

Since the parallel classes of Π_P are the pencils at P , Π may be extended to an LPP-lattice $L(\Pi)$ as follows: The elements of $L(\Pi) \setminus \Pi$ are the pencils of Π , a "tangent plane", $\langle P \rangle$, for each $P \in \mathcal{O}$, a minimum element, 0 , and a maximum element, 1 . The additional relations are:

$$\begin{aligned} P < [P,x] < x & \quad \text{if} \quad P \in x, \\ P < \langle P \rangle, \\ [P,x] < \langle P \rangle & \quad \text{if} \quad P \in x, \\ 0 \leq \alpha \leq 1, \end{aligned}$$

whenever $P \in \mathcal{O}$, $x \in \mathcal{C}$, and $\alpha \in L(\Pi)$. That $L(\Pi)$ is an LPP-lattice is easily checked using 9.2. (Notice that distinct points P, Q are covered by their l.u.b., (PQ) , and that for each point P , $[P,1]$ is the projective plane obtained by adjoining ideal elements to Π_P .) $L(\Pi)$ also satisfies (BE) with $T(P) = \{\langle P \rangle\}$, since the only lines tangent to \mathcal{O} at P are the pencils at P .

Next let Π be a Laguerre plane. We regard Π as a poset whose elements are the points, trails, \parallel -equivalence classes of trails, and circles of Π , with

$$P < t, \mathfrak{g}_t \quad \text{if} \quad P \in t,$$

$$P < x \quad \text{if} \quad P \in x,$$

$$t < \mathfrak{g}_t,$$

$$t < x \quad \text{if} \quad t \subseteq x,$$

whenever P is a point, t a trail, and x a circle. For each $P \in \mathcal{O}$, the set of elements $> P$, with the above ordering, is an affine plane (see [6]), which we denote Π_P .

One of the parallel classes of Π_P is the set $\{\mathfrak{g}_t: t \text{ a trail, } P \in t\}$, and the remaining classes are the pencils at P . This leads to a natural extension of Π to an LPP-lattice, $L(\Pi)$, as follows: The elements of $L(\Pi) \setminus \Pi$ are the pencils and generators of Π , a "tangent plane", $\langle g \rangle$, for each generator g , a minimum element, 0 , and a maximum element, 1 . The additional relations are:

$$P < [P, x] \quad \text{if} \quad P \in x,$$

$$P < \mathfrak{g}_P, \langle \mathfrak{g}_P \rangle,$$

$$[P, x] < x, \langle \mathfrak{g}_P \rangle \quad \text{if} \quad P \in x,$$

$$\mathfrak{g}_P < \langle \mathfrak{g}_P \rangle,$$

$$\mathfrak{g}_P < \mathfrak{g}_t \quad \text{if} \quad P \in t,$$

$$0 \leq \alpha \leq 1,$$

whenever $P \in \mathcal{O}$, $x \in \mathcal{C}$, t is a trail, and $\alpha \in L(\Pi)$. Again $[P, 1]$ is a projective completion of the affine plane Π_P , and $L(\Pi)$ is easily seen (using 9.2) to be an LPP-lattice. $L(\Pi)$ also satisfies

(BE) with $T(P) = \{\langle g_P \rangle\}$, since the only lines tangent to \mathcal{O} at P are the pencils at P .

Finally, if Π is a Minkowski plane, then we define the full lattice $L(\Pi)$ as follows: Elements of $L(\Pi)$ are $\mathcal{O} \cup \mathfrak{L} \cup \mathfrak{G} \cup \{0,1\}$, with

$$\mathfrak{L} = \{\{P,Q\}: P,Q \in \mathcal{O}, P \not\parallel Q\} \cup \{\beta: \beta \text{ is a pencil of } \Pi\} \\ \cup \{g: g \text{ is a generator of } \Pi\},$$

$$\mathfrak{G} = \mathcal{C} \cup \{\langle P \rangle: P \in \mathcal{O}\}.$$

Relations in $L(\Pi)$ are:

$$P < \{P,Q\} \quad \text{if} \quad P \not\parallel Q,$$

$$P < \beta \quad \text{if} \quad \beta = \{P\},$$

$$P < g_P^+, g_P^-,$$

$$P < x \quad \text{if} \quad P \in x,$$

$$P < \langle P \rangle,$$

$$\{P,Q\} < x \quad \text{if} \quad P,Q \in x \quad (P \neq Q),$$

$$\{P,Q\} < \langle R \rangle \quad \text{if} \quad P \parallel_+ R \parallel_- Q, \quad P \not\parallel Q,$$

$$\beta < x \quad \text{if} \quad x \in \beta,$$

$$\beta < \langle P \rangle \quad \text{if} \quad \beta = \{P\},$$

$$g < \langle P \rangle \quad \text{if} \quad P \in g,$$

$$0 \leq \underline{\alpha} \leq 1,$$

whenever P, Q and R are points, x a circle, β a pencil, g a generator, and $\alpha \in L(\Pi)$. Again it is straightforward to check that $L(\Pi)$ is an LPP-lattice (see following remark). As before, $L(\Pi)$ satisfies (BE), in this case with $T(P) = \langle P \rangle$.

Remark. For each $P \in \mathcal{O}$, $[P, l]$ is a projective plane, the projective completion of an affine plane, Π_P , consisting of P , l , the pairs $\{P, Q\}$, the circles containing P , and the planes $\langle Q \rangle$ with $P \neq Q \parallel P$, ordered as in $L(\Pi)$. In other words, Π_P consists of P and all $\alpha \in L(\Pi)$ with $P < \alpha \not\subseteq \langle P \rangle$. Π_P may also be (and usually is) thought of as the incidence structure whose points are the points of Π not parallel to P , and whose lines are the circles containing P and the generators not containing P , with the natural incidence.

DEFINITION. A circle plane, Π , is said to satisfy the bundle theorem if the corresponding lattice, $L(\Pi)$, satisfies (BT).

Remark. Equivalent formulations can, of course, be made without reference to the lattices. This becomes rather cumbersome for Laguerre and Minkowski planes; for inversive planes in the narrow sense, one case of the bundle theorem is exhibited in figure 6. The hypotheses of the bundle theorem may also be weakened in various ways. For example, Benz [2], dealing with inversive and Laguerre planes in the narrow sense, confines his statement of the bundle theorem to the

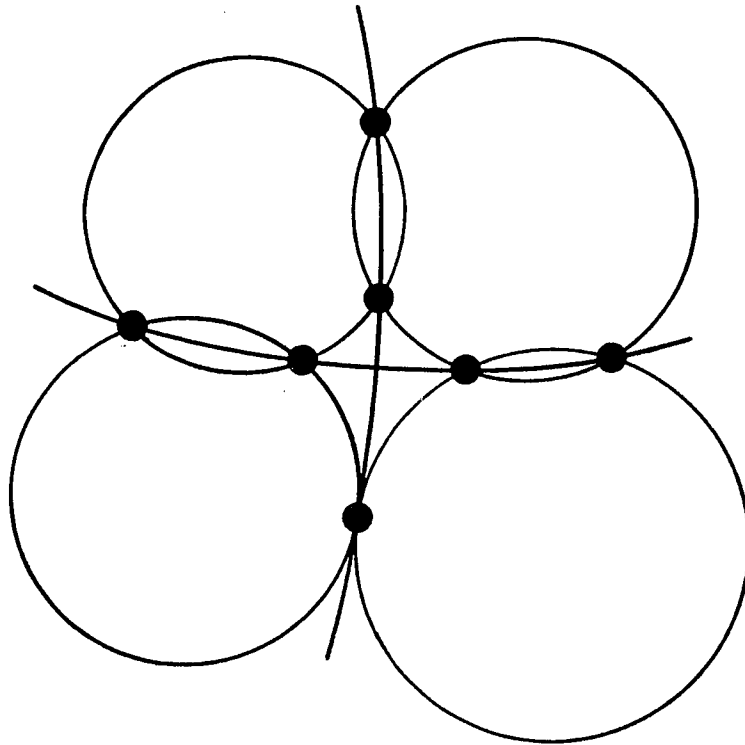


Figure 6. The Bundle Theorem

case when all of the l_i (see (BT)) are trails (i.e., point pairs). This can be shown (but not without a fair amount of tedious work) to imply the bundle theorem as we have stated it.

We can now obtain theorem 3 and the following two results as corollaries of theorem 4.

COROLLARY 9.3. All inversive planes which satisfy the bundle theorem are egglike.

COROLLARY 9.4. All Laguerre planes which satisfy the bundle theorem are egglike.

Of course theorem 1 is a special case of 9.3 (but not the same as 9.3, since theorem 1 referred to what we are now calling inversive planes in the narrow sense). Similarly, theorem 2 is a special case of 9.4.

As a consequence of theorem 4, we find that if $\Pi = (\mathcal{O}, \mathcal{C})$ is a circle-plane satisfying the bundle theorem, then we may identify \mathcal{O} with a subset of the point set of some three-dimensional projective space, S^* , and $L(\Pi)$ with the lattice $L(\mathcal{O})$. In this context, when we speak of a line or plane, α , of $L(\Pi)$ as a set of points, we mean the set of all points of S^* on α (i.e., not just the points of \mathcal{O} on α).

To prove the above results, it only remains to show that \mathcal{O} is (in S^*) a point set of the desired type (i.e., \mathcal{O} is a semi-ovoid,

a semi-oval cone, or a doubly ruled quadric if Π is an inversive, Laguerre, or Minkowski plane respectively). Notice that by the definition of $L(\Theta)$, the lines and planes of S^* which meet Θ are precisely the lines and planes of $L(\Theta) = L(\Pi)$.

First, let Π be an inversive plane, and $P \in \Theta$. By the definition of $L(\Pi)$, we see that the tangents to Θ through P are the pencils at P , and that these are also precisely the lines through P which lie in $\langle P \rangle$. Thus Θ is a semi-ovoid, which proves 9.3.

Next let Π be a Laguerre plane. First notice that any two generators are coplanar (since if $g_P \neq g_Q$ are generators, they lie on the plane $g_{(PQ)}$). It follows that there is a point $V \notin \Theta$ of S^* which lies on all of the generators. We assert that $\Theta^* := \Theta \cup \{V\}$ is a semi-oval cone.

The lines of $L(\Pi)$ are the generators, trails and pencils of Π . From the definition of $L(\Pi)$, we see that if g is a generator, then any trail or pencil coplanar with g has a point of Θ in common with g . Thus any BC containing g consists entirely of generators, i.e., is equal to the point V . But the points of S^* are the points of Θ and the BC's of $L(\Pi)$, so that $g \setminus \{V\} \subseteq \Theta$.

It follows that if x is any plane not containing V (i.e., x is a circle of Π), then Θ^* is the union of the lines which contain V and meet $x \cap \Theta$. Thus to prove that Θ^* is a semi-oval cone, we have only to show that $x \cap \Theta$ is a semi-oval. Let $P \in x \cap \Theta$. Then (again checking the definition of $L(\Pi)$) the lines on P in x

are the trails t with $P \in t \subseteq x$, and the pencil $[P,x]$. The latter is thus the unique tangent to $x \cap \mathcal{O}$ through P in x . This proves 9.4.

Finally, let Π be a Minkowski plane. We first observe that if g is a generator, then (checking the definition of $L(\Pi)$) any line of $L(\Pi)$ coplanar with g also has a point in common with g . It follows (by an argument similar to one given in the Laguerre case) that $g \subseteq \mathcal{O}$. But then by 9.1, \mathcal{O} is clearly a hyperbolic quadric with rulings the set of (+)-generators and the set of (-)-generators. This proves theorem 3.

10. A Class of Examples

Suppose L is an LPP-lattice satisfying (BT). (As usual \mathcal{O} , \mathcal{L} and \mathcal{S} are the point, line and plane sets of L .) Let $P \in \mathcal{O}$, and let $J(P)$ be the set of planes whose only point is P .

Let L' be a poset obtained from L by deleting the elements of $J(P)$, and adding a new set of elements, $J'(P)$, in such a way that

(1) If $\alpha \in J'(P)$, then the relations involving α are $\alpha < 1$, $0 < \alpha$, $P < \alpha$, and $l < \alpha$ for each l in some $\mathcal{L}_\alpha \subseteq \{l \in \mathcal{L}: P < l, Q \not< l \forall Q \in \mathcal{O} \setminus \{P\}\}$.

(2) The interval from P to 1 , say $[P,1]'$, in L' is a projective plane.

Then L' is again an LPP-lattice satisfying (BT). (That L' is an LPP-lattice is easily seen using 9.2. That it satisfies (BT) is a triviality, since if l_1, l_2, l_3, l_4 satisfy the hypotheses of (BT), then any two of them contain at least two distinct points, and so are coplanar in L' iff they are coplanar in L .

If \mathcal{O} is a sufficiently small set of points in $PG(3, K)$, then the above construction may be applied to $L(\mathcal{O})$ to obtain examples of LPP-lattices satisfying (BT) which are not isomorphic to any $L(\mathcal{O})$. In particular, we can do this whenever there is some $P \in \mathcal{O}$ for which the "partial plane" $[P, l] \setminus J(P)$ can be completed to a non-Desarguesian plane.

Obviously, one can weaken (BE) considerably without admitting any of the examples described in the preceding paragraph. For example, I would not be surprised if theorem 4 still holds when (BE) is replaced by the following condition.

(BE') For each $P \in \mathcal{O}$, $[P, l] \setminus J(P)$ can be extended to a projective plane by addition of lines in only one way (up to isomorphism).

Here $J(P)$ is, as above, the set of planes of L whose only point is P . I have not investigated the possibility of proving theorem 4 under this weaker assumption.

BIBLIOGRAPHY

1. Barlotti, A.: "Un'estensione del teorema di Segre-Kustaanheimo", Boll. Un. Mat. Ital. 10(1955), 248-252.
2. Benz, W.: " $(8_3, 6_4)$ -Konfigurationen in Laguerre-, Möbius- und weiteren Geometrien", Math. Z. 70(1958), 283-296.
3. Benz, W.: "Über Möbiusebenen. Ein Bericht", Jber. Deutsch. Math. Verein. 63(1960), 1-27.
4. Benz, W.: "Über die Grundlagen der Geometrie der Kreise in der pseudo-euklidischen (Minkowskischen) Geometrie", J. Reine Angew. Math. 232(1968), 41-76.
5. Benz, W.: Vorlesungen Über Geometrie der Algebren, Springer, Berlin-Heidelberg-New York, 1973.
6. Benz, W. and Mäurer, H.: "Über die Grundlagen der Laguerre-Geometrie. Ein Bericht", Jber. Deutsch. Math. Verein. 67(1965), Abt. 1, 14-42.
7. Birkhoff, G.: Lattice Theory, 3rd ed., Amer. Math. Soc., New York, 1967.
8. Bruck, R.H.: "Finite nets. II. Uniqueness and imbedding", Pacif. J. Math. 13(1963), 421-457.
9. Buekenhout, F.: "Characterizations of semi-quadrics: a survey", pp. 393-421 in Colloquio Internazionale sulle Teorie Combinatorie (Roma 1973) Tomo I, Atti dei Convegni Lincei 17(1976).
10. Cameron, P.J.: "Extensions of designs: variations on a theme", pp. 23-43 in Combinatorial Surveys, edited by P.J. Cameron, Academic Press, London-New York-San Francisco, 1977.
11. Chen, Y.: "A characterization of some geometries of chains", Can. J. Math. 26(1974), 257-272.
12. Dembowski, P.: "Inversive planes of even order", Bull. Amer. Math. Soc. 69(1963), 850-854.
13. Dembowski, P.: "Möbiusebenen gerader Ordnung", Math. Ann. 157(1964), 179-205.

14. Dembowski, P.: Finite geometries, Springer, Berlin-Heidelberg-New York, 1968.
15. Dembowski, P. and Hughes, D.H.: "On finite inversive planes", J. London Math. Soc. 40(1965), 171-182.
16. Dembowski, P. and Wagner, A.: "Some characterizations of finite projective spaces", Arch. Math. 11(1960), 465-469.
17. Denniston, R.H.F.: "Uniqueness of the inversive plane of order 5", Manuscripta Math. 8(1973), 11-19.
18. Denniston, R.H.F.: "Uniqueness of the inversive plane of order 7", Manuscripta Math. 8(1973), 21-26.
19. Doyen, J. and Hubant, X.: "Finite regular locally projective spaces", Math. Z. 119(1971), 83-88.
20. Ewald, G.: "Beispiel einer Möbiusebene mit nichtisomorphen affinen Unterebenen", Arch. Math. 11(1960), 146-150.
21. Herzer, A.: "Büschelsätze zur Charakterisierung projektiv darstellbar Zykelebenen", Math. Z. 164(1979), 215-238.
22. Hesselbach, B.: "Über zwei Vierecksätze in der Kreisgeometrie", Abh. Math. Sem. Univ. Hamburg, 9(1933), 265-271.
23. Hoffman, A.J.: "On the foundations of inversion geometry", Trans. Amer. Math. Soc. 71(1951), 218-242.
24. Kaerlein, G.: "Der Satz von Miquel in der pseudo-euklidischen (Minkowskischen) Geometrie", Ph.D. thesis, Bochum (1970).
25. Kahn, J.: "Finite inversive planes satisfying the bundle theorem", submitted.
26. Kahn, J.: "Inversive planes satisfying the bundle theorem", to appear in J. Combinatorial Theory (A).
27. Kantor, W.M.: "Dimension and embedding theorems for geometric lattices", J. Combinatorial Theory (A) 18(1975), 12-26.
28. Mäurer, H.: "Ein axiomatischer Aufbau der mindestens 3-dimensionalen Möbius-Geometrie", Math. Z. 103(1968), 282-305.
29. Qvist, B.: "Some remarks concerning curves of the second degree in a finite plane", Ann. Acad. Sci. Fenn. 134(1952), 1-27.
30. Segre, B.: "Ovals in a finite projective plane", Can. J. Math. 7(1955), 414-416.
31. Segre, B.: "On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two", Acta. Arith. 5(1959), 315-332.

32. Tits, J.: "Ovoides et groupes de Suzuki", Arch. Math. 13(1962), 187-198.
33. van der Waerden, B.L. and Smid, L.J.: "Eine Axiomatik der Kreisgeometrie und der Laguerre-Geometrie", Math. Ann. 110(1935), 753-776.
34. Witt, E.: "Über Steinersche System", Abh. Hamburg 12(1938), 265-275.