

Locally supersymmetric $D = 3$ non-linear sigma models

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We study non-linear sigma models with N local supersymmetries in three space-time dimensions. For $N = 1$ and 2 the target space of these models is riemannian or Kähler, respectively. All $N > 2$ theories are associated with Einstein spaces. For $N = 3$ the target space is quaternionic, while for $N = 4$ it generally decomposes into two separate quaternionic spaces, associated with inequivalent supermultiplets. For $N = 5, 6, 8$ there is a unique (symmetric) space for any given number of supermultiplets. Beyond that there are only theories based on a single supermultiplet for $N = 9, 10, 12$ and 16 , associated with coset spaces with the exceptional isometry groups $E_{4(-20)}$, $E_{6(-14)}$, $E_{7(-5)}$ and $E_{8(+8)}$, respectively. For $N = 3$ and $N \geq 5$ the $D = 2$ theories obtained by dimensional reduction are two-loop finite.

1. Introduction

For space-time dimensions $D \geq 4$ a large variety of locally supersymmetric theories has been explored, both with and without conformal invariance [1]. For $D = 2$ conformally invariant theories have been studied extensively. In contrast, only very few models have been worked out for $D = 3$. Nevertheless, gravity and supergravity in three dimensions are of interest in their own right. As is well known, three-dimensional field theories have a number of unique features. For instance, massless states do not carry helicity, so that the associated degrees of freedom can generally be described by scalar fields. Pure gravity and supergravity are topological theories and do not give rise to physical (i.e. propagating) degrees of freedom. Apart from conical singularities at the location of matter sources, space-time is flat. Notwithstanding this fact, classical gravity in three dimensions exhibits many intriguing properties [2]. More recently, pure quantum gravity in three dimensions has been reformulated as a Chern–Simons gauge theory and

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shown to be solvable in the sense that the quantum constraints (i.e. the Wheeler–DeWitt equation, in particular) can be solved exactly [3]. In addition, genuine observables (à la Dirac) can be constructed, in contrast to four-dimensional canonical gravity, where the construction of observables remains an unsolved problem even at the classical level. Moreover, the three-dimensional theory is especially amenable to a reformulation in terms of the new canonical variables proposed in ref. [4] (see also ref. [5] for a clear discussion); the exact solvability of pure quantum gravity in this approach has been demonstrated in ref. [6]. A recent treatment of pure and matter-coupled supergravity in this framework can be found in ref. [7]. Although many open questions remain, it should be clear from these remarks that three-dimensional gravity and supergravity can teach us a lot about quantum gravity in general, and that the models considered here, at the very least, can serve as non-trivial toy models.

A further motivation for studying three-dimensional supergravity is the important role it plays in the construction of two-dimensional supergravity theories via dimensional reduction. These dimensionally reduced theories have a number of remarkable properties; in particular, they possess infinite-dimensional symmetries acting on the space of solutions of the non-linear field equations [8–10]. For supergravity, these symmetries merge with the so-called “hidden symmetries” of supergravity. All these models are classically integrable in the sense that they admit linear systems for their non-linear field equations [9,10]. The belief that this classical symmetry structure should play an important role for the quantum theory was one of the main motivations for a recent investigation of the quantum divergences of these two-dimensional supergravity theories [11], which showed that for sufficiently high N (the number of independent supersymmetries) these models were two-loop finite. In order to appreciate the relevance of this result, it is important to understand the uniqueness of these theories. In ref. [11] the calculations were based on the conjectured structure of non-linear sigma models coupled to $D = 3$ supergravity with homogeneous target spaces, as they were known or expected to arise by dimensional reduction from extended supergravity in four space-time dimensions, but to date only a few of these models have been worked out explicitly [12,7].

The present paper aims at filling this gap and gives a complete classification of non-linear sigma models coupled to extended supergravity in three space-time dimensions. For rigidly supersymmetric non-linear sigma models, this analysis is almost identical to the $D = 2$ case [13]. There it was established that N -extended supersymmetric sigma models require the presence of $N - 1$ complex structures in the target space. It turns out that non-linear sigma models based on irreducible target spaces can have at most $N = 4$ supersymmetries. Extensions of this result were studied in ref. [14], where it was found that the bound on N is not affected by the presence of torsion, while for local supersymmetry the restriction $N \leq 4$ remains intact for conformally invariant theories. Without conformal invariance

there are certainly theories with $N > 4$ [15,16], but those were never studied systematically. Because three-dimensional supergravity has no conformal invariance, one expects no restriction to $N \leq 4$ (although the $N = 4$ models remain somewhat special as we shall see). On the other hand, extended supergravities in four dimensions are known to be restricted to $N \leq 8$ in view of the non-existence of consistent interacting theories describing massless particles with spin $s > 2$ (we note, however, that this bound can possibly be circumvented in certain theories which are not of the conventional type [17]). The fact that three-dimensional supergravities with even N correspond to four-dimensional theories with $N/2$ local supersymmetries and can therefore be constructed by dimensional reduction, suggests the bound $N \leq 16$ in three dimensions. Indeed, a central result of this paper is that extended theories *do* satisfy this restriction, and this fact in turn constitutes an alternative proof of the four-dimensional result. However, the result now hinges on the geometric properties of target spaces with restricted holonomy groups, a subject which has been studied in considerable depth in the mathematical literature [18].

Because the geometrical arguments leading to these restrictions are at the heart of this paper, we now briefly summarize them. The general analysis of the lagrangian and transformation rules given in sect. 3 enables us to derive the constraints on the Riemann curvature tensor, and hence on the holonomy group of the target manifold, that are imposed by local supersymmetry (see eq. (4.19), which is the crucial formula). These conditions become more and more restrictive with increasing N ; for $N > 4$, they completely determine the target manifolds, whereas they are not strong enough to determine them for $N \leq 4$. In particular, for $N = 1$, there are no restrictions at all, and the target space may be an arbitrary riemannian manifold. For $N = 2$, there is one complex structure, and the target manifold is Kähler. For $N = 3$ and 4, there are three almost-complex structures. For $N = 3$ the space is quaternionic, while for $N = 4$ the target space is locally a product of two quaternionic manifolds, associated with inequivalent supermultiplets. Nonetheless, there remains a great variety of possibilities for $N \leq 4$, as the manifolds are not homogeneous in general. For $N \geq 5$, on the other hand, eq. (4.19) implies that the holonomy group becomes “too small” in a sense to be made precise in sect. 5. We first show that all manifolds are Einstein spaces and then we derive how d (the dimension of the target space) and N are restricted: we find that an arbitrary number of supermultiplets is permitted for $N = 5, 6, 8$, while only one is allowed for $N = 9, 10, 12$ and 16. For other values of d and N no theories can exist! We can then appeal to a powerful mathematical theorem [19] and use our knowledge of the holonomy group for $N \geq 5$ to conclude that all the corresponding target manifolds must be symmetric spaces; their determination is thus simply a matter of matching the allowed values of N and d with a list of symmetric spaces. In this way, we identify a unique symmetric space for each of these values of N and d . The isometry groups of the target spaces corresponding to $N = 5, 6, 8$ are

equal to $\text{Sp}(2, k)$, $\text{SU}(4, k)$ and $\text{SO}(8, k)$, respectively, where k is the number of supermultiplets. For $N = 9, 10, 12$ and 16 the corresponding target spaces possess the exceptional isometry groups $F_{4(-20)}$, $E_{6(-14)}$, $E_{7(-5)}$ and $E_{8(+8)}$, respectively; remarkably, they can be interpreted as projective spaces over the octonions [18]. In view of our previous remarks and the fact that the maximally extended $N = 16$ theory is invariant under the ‘‘maximally extended’’ exceptional Lie group E_8 [8,12], we are intrigued by the fact that the apparent non-existence of massless particles of spin $s > 2$ in four dimensions may be related to the non-existence of exceptional groups beyond E_8 .

A characteristic feature of the non-linear sigma models with local supersymmetry is that the target-space connection for the fermions is no longer the usual Christoffel connection, but it contains extra terms proportional to the almost-complex structures associated with the extra supersymmetries (see eq. (3.27)). This aspect is crucial for the two-loop finiteness of the dimensionally reduced models, which hinges on the fact that the contraction $R_{iklm}R_j^{klm}$ of the corresponding curvature tensors remains independent of the modification of the fermionic connection [11]. From the formulae derived later (in particular (3.30) and (4.11)) it follows that this is always the case for $N = 3$ and $N > 4$. For $N = 4$ the situation is somewhat more subtle, as one is in general dealing with two separate quaternionic subspaces. Nevertheless upon using (3.30) and (4.38) one can easily establish that this property holds whenever the two subspaces are of equal dimension. In contrast the $N = 1, 2$ theories fail to be finite at one loop if the target space is not Einstein. We will not return to this topic here and leave it to the reader to verify these results.

This paper is organized as follows. In sect. 2 we review the construction of $D = 3$ supermultiplets. Sect. 3 contains the results for the invariant lagrangian and the supersymmetry transformation rules. The geometrical implications of the presence of N local supersymmetries for the target space are then worked out in sect. 4. In sect. 5 we identify the possible target spaces for $N \geq 5$. As those are all symmetric we include a discussion of the conventional formulation of extended supergravity coupled to non-linear sigma models with homogeneous target spaces and elucidate the connection with the target-space approach used in the previous sections. Some material relevant for the exceptional cosets is relegated to appendix A.

2. Massless $D = 3$ supermultiplets

Consider the extended supersymmetry algebra, with the anti-commutation relation

$$\{Q_\alpha^I, \bar{Q}_\beta^J\} = -2i\delta^{IJ}\gamma_{\alpha\beta}^\mu P_\mu, \quad (I, J = 1, \dots, N) \quad (2.1)$$

where the Q_α^I are N independent Majorana spinor charges and P_μ is the

energy-momentum operator. For states with light-like momentum, say in a frame where $P^0 = P^1 = \omega$ and $P^2 = 0$, eq. (2.1) takes the following form ^{*}:

$$\{Q_\alpha^I, Q_\beta^J\} = 2\omega\delta^{IJ}(\mathbf{1} + \sigma_3)_{\alpha\beta}. \quad (2.2)$$

In a positive-definite Hilbert space of states, Q_2^I must therefore vanish and we are left with the real charges Q_1^I , which generate an N -dimensional Clifford algebra ^{**}. In addition a fermion-number operator \mathbf{F} must exist satisfying $\mathbf{F}^2 = \mathbf{1}$, which anti-commutes with the supercharges Q_α^I . Therefore massless supermultiplets are representations of a real $(N + 1)$ -dimensional Clifford algebra of positive signature. In the basis where \mathbf{F} is diagonal we denote the bosonic indices by $A, B, \dots = 1, \dots, d$ and the fermionic indices by $\dot{A}, \dot{B}, \dots = 1, \dots, d$. The supercharges then take the form of off-diagonal gamma matrices

$$\Gamma^I = \begin{pmatrix} 0 & \Gamma_{\dot{A}\dot{B}}^I \\ \Gamma_{BC}^I & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (2.3)$$

As one can always choose a basis where the gamma matrices are symmetric, the two submatrices of Γ^I are each others transpose; in terms of the upper-right $d \times d$ matrices $\Gamma_{\dot{A}\dot{B}}^I$, which themselves have no special symmetry properties, the defining relation of the Clifford algebra reads

$$\Gamma_{\dot{A}\dot{C}}^I \Gamma_{\dot{B}\dot{C}}^J + \Gamma_{\dot{A}\dot{C}}^J \Gamma_{\dot{B}\dot{C}}^I = 2\delta^{IJ}\delta_{\dot{A}\dot{B}}. \quad (2.4)$$

The irreducible supermultiplets are listed in table 1, together with their centralizers [20].

For odd values of N the supermultiplet is unique up to a similarity transformation. For even values of N the product of the $N + 1$ generators of the algebra,

$$\tilde{F} \equiv \mathbf{F}\Gamma^1 \dots \Gamma^N \quad (2.5)$$

commutes with \mathbf{F} and Γ^I . For $N = 4 \bmod 4$ it satisfies $\tilde{F}^2 = \mathbf{1}$, so that the Clifford algebra can be decomposed into two simple ideals, associated with the projection operators $\frac{1}{2}(\mathbf{1} \pm \tilde{F})$. Inequivalent irreducible representations of the Clifford algebra correspond to one of these ideals and are characterized by $\tilde{F} = \pm \mathbf{1}$. For $N = 2 \bmod 4$ we have $\tilde{F}^2 = -\mathbf{1}$ and the representation is again unique; it cannot be decomposed into irreducible representations unless one introduces *complex* projection operators. The existence of inequivalent supermultiplets is a special feature of supersymmetry in low space-time dimensions. In higher dimensions the spinor character of the supercharges ensures that inequivalent supermultiplets have a

^{*} We use $\gamma_0 = -i\sigma_2, \gamma_1 = \sigma_1, \gamma_2 = \sigma_3$, with charge-conjugation matrix $C = \sigma_2$.

^{**} Strictly speaking the charges are hermitian; we insist on reality in view of field-theoretic applications.

TABLE 1

Irreducible massless supermultiplets with d_N the number of bosonic states. The centralizer, which constitutes a division algebra, contains the operators that commute with the supercharges *and* with fermion number.

N	d_N	Centralizer
1	1	R
2	2	C
3	4	H
4	4	H
5	8	H
6	8	C
7	8	R
8	8	R
$n + 8$	$16d_n$	as for n

different spin content, so that there is no need for making a further distinction. From table 1 we infer that the multiplets with $N = 3$ and $N = 4$ are the same; likewise $N = 5, 6, 7, 8$ have identical multiplets (this result holds again modulo 8, so that also $N = 11, 12$ have identical multiplets, and so on). However, the situation is different in the case of local supersymmetry, because the number of gravitini is not the same for different values of N .

Observe that fermions and bosons in an irreducible multiplet transform according to irreducible spinor representations of $SO(N)$. Here we recall the well-known result that the spinor representations of $SO(N)$ are real for $N = 1, 7, 8 \pmod{8}$, complex for $N = 2, 6 \pmod{8}$ and pseudo-real for $N = 3, 4, 5 \pmod{8}$ (see e.g. ref. [21]). From table 1 it is obvious that these cases correspond to the centralizers **R**, **C** and **H**, respectively. For $N = 2, \dots, 6 \pmod{8}$, the centralizer contains (at least) the identity and a real antisymmetric matrix e with $e^2 = -\mathbf{1}$, acting within the bosonic and fermionic subspaces. Clearly, e can be traded for the imaginary unit i by complexifying the representation. By use of the complex projection operators $\frac{1}{2}(\mathbf{1} \pm ie)$ the real d -dimensional $SO(N)$ representations become $(d/2)$ -dimensional *complex* representations, and the matrices Γ_{AA}^I can be replaced by complex $d/2 \times d/2$ matrices. This observation will be important for the derivation of the completeness relations and Fierz rearrangement formulas used in appendix A. For $N = 3, 4, 5 \pmod{8}$, there are two additional complex structures that anticommute with e . Either one of them can be used to show that the representation is actually pseudo-real.

In the remainder of this section we present the explicit construction of the supercharges for $N = 1, 2, 4, 8 \pmod{8}$, to facilitate the discussion in the subsequent sections (for further explicit details, see ref. [22]). The representations for intermediate values of N have the same dimensionality as one of the $N = 1, 2, 4, 8 \pmod{8}$ representations and can conveniently be studied by embedding them in the higher- N representation; the centralizer can be explicitly constructed from the

centralizer of the higher- N representation, possible extended with some of the extra gamma matrices.

We start by defining a basis of the 2×2 real matrices, consisting of the identity $\mathbf{1}$, σ_1 , σ_3 and $\epsilon \equiv -i\sigma_2$. Hence we have

$$\epsilon = \sigma_1 \sigma_3, \quad \epsilon \sigma_1 = -\sigma_3, \quad \epsilon \sigma_3 = \sigma_1. \quad (2.6)$$

For $N = 1$ we choose ($d_1 = 1$)

$$\mathbf{F}(2) = \sigma_3, \quad \Gamma^1(2) = \sigma_1, \quad (2.7)$$

where the number in parentheses indicates the dimension of the matrix. Hence, for $N = 1$ one has $\Gamma_{AA}^1 = 1$. We note the properties

$$\epsilon^2 = -1, \quad \{\epsilon, \Gamma^1\} = \{\epsilon, \mathbf{F}\} = 0. \quad (2.8)$$

For $N = 2$ a representation of the Clifford algebra is constructed by taking direct products of 2×2 matrices times the previous lower-dimensional algebra (so that $d_2 = 2$):

$$\begin{aligned} \mathbf{F}(4) &= \sigma_3 \otimes \mathbf{1}(2), \\ \Gamma^1(4) &= \sigma_1 \otimes \Gamma^1(2), \quad \text{with} \quad \Gamma^{12} = \mathbf{1} \otimes \epsilon \end{aligned} \quad (2.9)$$

$$\Gamma^2(4) = \sigma_1 \otimes \mathbf{F}(2),$$

so that

$$\Gamma_{AA}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_{AA}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.10)$$

In addition we note the existence of the following three complex structures:

$$\begin{aligned} e_1(4) &= \sigma_3 \otimes \epsilon, \\ e_2(4) &= -\epsilon \otimes \mathbf{1}(2), \quad \text{satisfying} \quad e_i e_j = -\delta_{ij} \mathbf{1} + \epsilon_{ijk} e_k \\ e_3(4) &= \sigma_1 \otimes \epsilon, \end{aligned} \quad (2.11)$$

Note that $\mathbf{F}\Gamma^1\Gamma^2 = e_1$, and

$$\begin{aligned} [e_1, \Gamma^1] &= [e_1, \Gamma^2] = [e_1, \mathbf{F}] = 0, \\ \{e_2, \Gamma^1\} &= \{e_2, \Gamma^2\} = \{e_2, \mathbf{F}\} = 0, \\ \{e_3, \Gamma^1\} &= \{e_3, \Gamma^2\} = \{e_3, \mathbf{F}\} = 0. \end{aligned} \quad (2.12)$$

The centralizer of the Clifford algebra is based on $e_0 \equiv \mathbf{1}$ and e_1 , so that the associated symmetry group is $U(1)$. Note, however, that in the bosonic or the fermionic subspace e_1 and Γ^{12} are degenerate.

For future use note the identities

$$e_1 \Gamma^1 = \Gamma^2 \mathbf{F}, \quad e_1 \Gamma^2 = \mathbf{F} \Gamma^1, \quad e_1 \mathbf{F} = \Gamma^{12}. \quad (2.13)$$

For $N = 4$ we take again direct products of 2×2 matrices times the matrices of the previous algebra (so that $d_4 = 4$):

$$\begin{aligned} \Gamma^1(8) &= \sigma_1 \otimes \Gamma^1(4), \\ \mathbf{F}(8) &= \sigma_3 \otimes \mathbf{1}(4), \quad \Gamma^2(8) = \sigma_1 \otimes \Gamma^2(4), \quad \Gamma^4(8) = \epsilon \otimes e_1(4), \\ \Gamma^3(8) &= \sigma_1 \otimes \mathbf{F}(4), \end{aligned} \quad (2.14)$$

with the complex structures

$$\begin{aligned} e_1(8) &= \mathbf{1} \otimes e_1(4), \\ e_2(8) &= \sigma_3 \otimes e_2(4), \quad \text{satisfying} \quad e_i e_j = -\delta_{ij} \mathbf{1} + \epsilon_{ijk} e_k \\ e_3(8) &= \sigma_3 \otimes e_3(4), \end{aligned} \quad (2.15)$$

Observe that $\mathbf{F} \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 = -\mathbf{1}$. As explained previously there are two inequivalent representations. A second one is, for instance, found by changing the sign of $\Gamma^1, \Gamma^2, \Gamma^3$.

This time all e_i commute with Γ^I and \mathbf{F} ,

$$[e_i, \Gamma^I] = [e_i, \mathbf{F}] = 0. \quad (2.16)$$

so that the centralizer of the algebra consists of $e_0 \equiv \mathbf{1}$ and e_i associated with the group $SU(2)$.

The $SO(4)$ generators are

$$\begin{aligned} \Gamma^{12} &= \mathbf{1} \otimes \Gamma^{12}, & \Gamma^{34} &= \sigma_3 \otimes \mathbf{F} e_1 = \sigma_3 \otimes \Gamma^{12}, \\ \Gamma^{23} &= \mathbf{1} \otimes \Gamma^2 \mathbf{F} = \mathbf{1} \otimes e_1 \Gamma^1, & \Gamma^{14} &= \sigma_3 \otimes e_1 \Gamma^1, \\ \Gamma^{31} &= \mathbf{1} \otimes \mathbf{F} \Gamma^1 = \mathbf{1} \otimes e_1 \Gamma^2, & \Gamma^{24} &= \sigma_3 \otimes e_1 \Gamma^2, \end{aligned} \quad (2.17)$$

where we made use of the identities derived previously for $N = 2$. This shows that

$$\mathbf{F} \Gamma^{IJ} = \frac{1}{2} \epsilon^{IJKL} \Gamma^{KL}. \quad (2.18)$$

Therefore the $SO(4)$ group factors into two $SO(3)$ groups, one acting on the bosons (the selfdual component) and one on the fermions (the anti-selfdual component). This feature will play an important role in the discussion of $N = 4$ theories in sects. 4 and 5.

For $N = 8$, we have $d_g = 8$ from table 1. The gamma matrices are then explicitly given by

$$\begin{aligned}
 \Gamma^1(16) &= \sigma_1 \otimes \Gamma^1(8), \\
 \Gamma^2(16) &= \sigma_1 \otimes \Gamma^2(8), \quad \Gamma^6(16) = \epsilon \otimes e_1(8), \\
 \mathbf{F}(16) &= \sigma_3 \otimes \mathbf{1}(8), \quad \Gamma^3(16) = \sigma_1 \otimes \Gamma^3(8), \quad \Gamma^7(16) = \epsilon \otimes e_2(8), \\
 \Gamma^4(16) &= \sigma_1 \otimes \Gamma^4(8), \quad \Gamma^8(16) = \epsilon \otimes e_3(8), \\
 \Gamma^5(16) &= \sigma_1 \otimes \mathbf{F}(8),
 \end{aligned} \tag{2.19}$$

Just as for $N = 4$ this representation is not unique; a second inequivalent representation exists, and may, for instance, be obtained by changing the sign of Γ^6 , Γ^7 and Γ^8 .

For $N > 8$ the pattern repeats itself; for $N = n + 8$, the dimensionality of the gamma matrices equals $16d_n$ and we put ($n \leq 8$)

$$\begin{aligned}
 \mathbf{F} = \mathbf{F}(2d_n) \otimes \mathbf{1}(16), \quad \Gamma^I = \Gamma^I(2d_n) \otimes \Gamma^I(16), \quad \Gamma^{8+a} = \Gamma^a(2d_n) \otimes \mathbf{1}(16), \\
 \Gamma^9 = \Gamma^1(2d_n) \otimes \mathbf{F}(16),
 \end{aligned} \tag{2.20}$$

where $I = 1, \dots, 8$ and $a = 2, \dots, n$, while $\Gamma^I(2d_n)$ and $\Gamma^a(2d_n)$ are the $(2d_n \times 2d_n)$ gamma matrices corresponding to the irreducible representation of the n -dimensional Clifford algebra. The centralizer is of the form $\mathbf{Z}(2d_n) \otimes \mathbf{1}(16)$, where $\mathbf{Z}(2d_n)$ is the centralizer of the n -dimensional Clifford algebra.

Finally, let us add that for *reducible* representations, the centralizer generates the group $SO(k)$, $U(k)$ or $Sp(k)$, depending on whether the centralizer for an irreducible representation corresponds to \mathbf{R} , \mathbf{C} or \mathbf{H} , respectively. Here k denotes the number of irreducible representations. The case of $N = 4 \pmod 4$ is again exceptional because one is dealing with inequivalent representations [23]. For k_1 and k_2 inequivalent representations, the corresponding groups are $SO(k_1) \otimes SO(k_2)$ (for $N = 8 \pmod 8$) and $Sp(k_1) \otimes Sp(k_2)$ (for $N = 4 \pmod 8$).

3. Lagrangian and transformation rules

In this section we present the full lagrangian and transformation rules for a non-linear sigma model coupled to N -extended supergravity. Let us first introduce

the separate lagrangians for pure supergravity and the non-linear sigma model. The supergravity lagrangian can be written as follows ^{*}:

$$\mathcal{L}_{s.g.} = -\frac{1}{2}i\epsilon^{\mu\nu\rho}\left\{e_{\mu}^a R_{\nu\rho a}(\omega) + \bar{\psi}_{\mu}^I D_{\nu}(\omega)\psi_{\rho}^I\right\}, \quad (3.1)$$

with the SO(2, 1) covariant derivative acting on a spinor as

$$D_{\mu}(\omega)\psi = \left(\partial_{\mu} + \frac{1}{2}\omega_{\mu}^a\gamma_a\right)\psi. \quad (3.2)$$

The spin-connection field ω_{μ}^a will be regarded as an independent field (first-order formalism). Its field equation implies that the supercovariant torsion tensor vanishes, i.e.

$$D_{[\mu}(\omega)e_{\nu]}^a - \frac{1}{4}\bar{\psi}_{\mu}^I\gamma^a\psi_{\nu}^I = 0, \quad (3.3)$$

where

$$D_{\mu}(\omega)e_{\nu}^a = \partial_{\mu}e_{\nu}^a + i\epsilon^{abc}\omega_{\mu b}e_{\nu c}. \quad (3.4)$$

From (3.4) one determines the spin connection; substituting the result into the field strength

$$R_{\mu\nu}^a(\omega) = \partial_{\mu}\omega_{\nu}^a - \partial_{\nu}\omega_{\mu}^a + i\epsilon^{abc}\omega_{\mu b}\omega_{\nu c} \quad (3.5)$$

yields the Riemann tensor (up to gravitino-dependent terms). The lagrangian (3.1) is locally supersymmetric under N independent supersymmetries. There is no restriction on the number of independent local supersymmetries and the theory is topological [3].

The rigidly supersymmetric non-linear sigma model is described by the lagrangian

$$\mathcal{L}_{\text{matter}} = -\frac{1}{2}g_{ij}(\phi)\left\{\partial_{\mu}\phi^i\partial^{\mu}\phi^j + \bar{\chi}^i\mathcal{D}(\overset{\circ}{\Gamma})\chi^j\right\} + \mathcal{L}_{\chi^i}, \quad (3.6)$$

where the target-space connection $\overset{\circ}{\Gamma}$ equals the Christoffel symbol and the covariant derivative is defined by (for arbitrary connection Γ)

$$D_{\mu}(\Gamma)\chi^i \equiv \partial_{\mu}\chi^i + \Gamma_{jk}^i(\phi)\partial_{\mu}\phi^j\chi^k. \quad (3.7)$$

We denote the dimension of the target space by d , so that $i, j, \dots = 1, \dots, d$. The

^{*} We use the Pauli-Källén metric with $\gamma_{[a}\gamma_b\gamma_{c]} = i\epsilon_{abc}$, $\gamma_{ab} \equiv \gamma_{[a}\gamma_{b]} = i\epsilon_{abc}\gamma^c$. Readers who prefer the $(-, +, +)$ metric multiply Dirac conjugate spinors and ϵ_{abc} by i , and ϵ^{abc} by $-i$.

χ^4 terms are proportional to the Riemann tensor of the sigma-model target space,

$$\mathcal{L}_{\chi^4} = -\frac{1}{24}R_{ijkl}(\phi) \bar{\chi}^i \gamma_a \chi^j \bar{\chi}^k \gamma^a \chi^l. \quad (3.8)$$

Ignoring the extra space-time coordinate, the lagrangian (3.6) is identical to the one in two dimensions; the χ^4 terms can be rewritten in a form where we sum over only two independent gamma matrices, by using the cyclicity of the Riemann tensor.

The non-linear sigma models have $N = 1, 2$ or 4 independent rigid supersymmetries. The extra supersymmetries are associated with complex structures f_{Pj}^i , labeled by $P = 2, \dots, N$, which are hermitian,

$$g_{ij}f_{Pk}^j + g_{kj}f_{Pi}^j = 0, \quad (3.9)$$

and satisfy the Clifford property

$$f_{Pk}^i f_{Qj}^k + f_{Qk}^i f_{Pj}^k = -2\delta_{PQ}\delta_j^i. \quad (3.10)$$

Furthermore they are covariantly constant (with respect to the Christoffel connection),

$$D_i(\overset{\circ}{\Gamma})f_{Pk}^j \equiv \partial_i f_{Pk}^j + \overset{\circ}{\Gamma}_{ii}^j f_{Pk}^l - \overset{\circ}{\Gamma}_{ik}^l f_{Pl}^j = 0. \quad (3.11)$$

The upper limit on N arises because the holonomy group commutes with the complex structures. Therefore this group must either act reducibly in target space, in which case the target space becomes reducible (i.e. it decomposes into separate spaces), or, by Schur's lemma (see e.g. ref. [24]), the complex structures must generate a division algebra; the largest such algebra is the quaternionic one with three complex structures, corresponding to $N = 4$ [13]. Alternatively, one may make use of the fact that these models are invariant under $SO(N)$ rotations on the fermions (for $N = 4$ one has only $SO(3)$). Combining these transformations with supersymmetry proves that the theory must be invariant under non-uniform translations of space-time coordinates as soon as $N > 4$, which implies that the target space is reducible [14].

So far we have put Newton's constant to unity, but in what follows we want to be a little more explicit about the dimension of the various quantities. It is convenient to choose all boson fields dimensionless, with the exception of the spin connection which has dimension [1] (in mass units); the fermion fields have dimension [$\frac{1}{2}$] and the supersymmetry transformation parameter dimension [$-\frac{1}{2}$]. In this way none of the transformation rules will contain dimensional parameters, whereas the lagrangian contains just an overall constant $1/\kappa$, where κ has dimension [-1]. Hence we write

$$\mathcal{L} = \frac{1}{\kappa} \{ \mathcal{L}_{s.g.} + \mathcal{L}_{kin} + \mathcal{L}_N + \mathcal{L}_{\chi^4} \} \quad (3.12)$$

Here $\mathcal{L}_{\text{s.g.}}$ is the supergravity lagrangian, modified by extra matter-dependent connection terms (here and henceforth we decompose the indices I into $I = 1$ and $I = P = 2, \dots, N$; the gravitino field and corresponding supersymmetry parameter with $I = 1$ are denoted by ψ_μ and ϵ , respectively),

$$\mathcal{L}_{\text{s.g.}} = -\frac{1}{2}i\epsilon^{\mu\nu\rho}\left\{e_\mu^a R_{\nu\rho a}(\omega) + \bar{\psi}_\mu D_\nu(\omega, Q)\psi_\rho + \bar{\psi}_\mu^P D_\nu(\omega, Q)\psi_\rho^P\right\}, \quad (3.13)$$

where

$$\begin{aligned} D_\mu(\omega, Q)\psi_\nu &= D_\mu(\omega)\psi_\nu - \partial_\mu\phi^i Q_i^P(\phi)\psi_\nu^P, \\ D_\mu(\omega, Q)\psi_\nu^P &= D_\mu(\omega)\psi_\nu^P + \partial_\mu\phi_i\left[Q_i^{PQ}(\phi)\psi_\nu^Q + Q_i^P(\phi)\psi_\nu\right]. \end{aligned} \quad (3.14)$$

Clearly Q_i^P and Q_i^{PQ} can be combined into an $\text{SO}(N)$ target-space connection Q_i^J .

The term \mathcal{L}_{kin} refers to the properly covariantized kinetic terms of the non-linear sigma model,

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}eg_{ij}(\phi)\left\{g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j + \bar{\chi}^i\mathcal{D}(\omega, \Gamma)\chi^j\right\}, \quad (3.15)$$

where the connection Γ is no longer the Christoffel connection but may contain extra terms. As only the anti-symmetric part of Γ appears in (3.15), we may assume without loss of generality that the metric postulate remains satisfied,

$$D_i(\Gamma)g_{jk} = 0. \quad (3.16)$$

The torsion now receives contributions from the spinor fields χ^i , so that (3.3) changes into

$$D_{[\mu}(\omega)e_{\nu]}^a - \frac{1}{4}\bar{\psi}_\mu^I\gamma^a\psi_\nu^I - \frac{1}{8}ie\epsilon_{\mu\nu\rho}e^{\rho a}g_{ij}\bar{\chi}^i\chi^j = 0. \quad (3.17)$$

Just as in the case of rigid supersymmetry, the extra supersymmetries are associated with tensors f_{Pj}^i . However, in the context of local supersymmetry these tensors are usually not complex structures, but only almost-complex structures (for definitions, see e.g. ref. [25]); indeed, as we shall see later, their Nijenhuis tensors do not vanish in general. The almost-complex structures appear in the lagrangian \mathcal{L}_{N} , which refers to the Noether terms with certain higher-order modifications to ensure the supercovariance of the χ^i field equation,

$$\begin{aligned} \mathcal{L}_{\text{N}} &= \frac{1}{4}eg_{ij}\bar{\chi}^i\gamma^\mu(\not{\partial}\phi^k + \hat{\not{\partial}}\phi^k)(\delta_k^j\psi_\mu - f_{Pk}^j\psi_\mu^P) \\ &= \frac{1}{2}eg_{ij}\bar{\chi}^i\gamma^\mu\not{\partial}\phi^k(\delta_k^j\psi_\mu - f_{Pk}^j\psi_\mu^P) \\ &\quad + \frac{1}{16}eg_{ij}\bar{\chi}^i\chi^j(\bar{\psi}_\nu\gamma^\mu\gamma^\nu\psi_\mu + \bar{\psi}_\nu^P\gamma^\mu\gamma^\nu\psi_\mu^P) \\ &\quad + \frac{1}{16}e\bar{\chi}^i\gamma_\rho\chi^j\left[(f_{[P}f_{Q]})_{ij}\bar{\psi}_\nu^P\gamma^\rho\gamma^\mu\gamma^\nu\psi_\mu^Q + f_{Pij}\bar{\psi}_\nu^P(\gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu\gamma^\rho)\psi_\mu\right], \end{aligned} \quad (3.18)$$

where we used that the supercovariant derivative of ϕ^i is equal to

$$\hat{\partial}_\mu \phi^i = \partial_\mu \phi^i - \frac{1}{2} \left(\delta_j^i \bar{\psi}_\mu + f_{Pj}^i \bar{\psi}_\mu^P \right) \chi^j. \quad (3.19)$$

Also the χ^4 terms are modified due to the local supersymmetry, and we find

$$\mathcal{L}_{\chi^4} = \frac{1}{16} e \left(g_{ij} \bar{\chi}^i \chi^j \right)^2 - \frac{1}{24} e R_{ijkl} \bar{\chi}^i \gamma_a \chi^j \bar{\chi}^k \gamma^a \chi^l. \quad (3.20)$$

The supersymmetry transformation rules are

$$\delta e_\mu^a = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu + \frac{1}{2} \bar{\epsilon}^P \gamma^a \psi_\mu^P, \quad (3.21)$$

$$\delta \psi_\mu = D_\mu(\omega, Q) \epsilon + \delta \phi^i Q_i^P \psi_\mu^P - \frac{1}{8} \bar{\chi}^i \gamma^\nu \chi^j f_{Pij} \gamma_{\mu\nu} \epsilon^P, \quad (3.22)$$

$$\begin{aligned} \delta \psi_\mu^P &= D_\mu(\omega, Q) \epsilon^P - \delta \phi^i \left[Q_i^P \psi_\mu + Q_i^{PQ} \psi_\mu^Q \right] \\ &\quad + \frac{1}{8} \bar{\chi}^i \gamma^\nu \chi^j \left[(f_{[P} f_{Q]})_{ij} \gamma_{\mu\nu} \epsilon^Q + f_{Pij} \gamma_{\mu\nu} \epsilon \right], \end{aligned} \quad (3.23)$$

$$\delta \phi^i = \frac{1}{2} \left(\delta_j^i \bar{\epsilon} + f_{Pj}^i \bar{\epsilon}^P \right) \chi^j, \quad (3.24)$$

$$\delta \chi^i = \frac{1}{2} \hat{\partial} \phi^i \left(\epsilon \delta_j^i - f_{Pj}^i \epsilon^P \right) - \Gamma_{jk}^i \delta \phi^j \chi^k. \quad (3.25)$$

Let us now briefly comment on the derivation of these results. One starts with the sum of (3.1) and (3.15) and follows the same strategy as in ref. [14] by introducing an as yet undetermined connection Γ into the lagrangian and transformation rules. The first variations are standard and quickly reveal the need for the Noether terms. At that point one has variations proportional to $\partial\phi\partial\phi\chi\epsilon$ and $\partial\phi\partial\phi\psi\epsilon$. The former can be cancelled by introducing the Q -dependent terms in the gravitino transformation rules, which at the same time requires one to add corresponding $Q\psi\psi$ terms to the action. This restricts the form of Q_i to $SO(N)$ target space connections (cf. eq. (3.14)), and leads in turn to new $\partial\phi\partial\phi\psi\epsilon$ variations. Both the $\partial\phi\partial\phi\chi\epsilon$ and $\partial\phi\partial\phi\psi\epsilon$ variations vanish provided the $SO(N)$ curvatures satisfy the condition

$$\begin{aligned} R_{ij}^P(Q) &\equiv \partial_i Q_j^P + Q_i^{PQ} Q_j^Q - (i \leftrightarrow j) = -\frac{1}{2} f_{Pij}, \\ R_{ij}^{PQ}(Q) &\equiv \partial_i Q_j^{PQ} + Q_i^{PR} Q_j^{RQ} - Q_i^P Q_j^Q - (i \leftrightarrow j) = \frac{1}{2} (f_{[P} f_{Q]})_{ij}, \end{aligned} \quad (3.26)$$

the connection Γ is given by

$$\Gamma_{ik}^j = \hat{\Gamma}_{ik}^j - Q_i^P f_{Pk}^j, \quad (3.27)$$

and the almost-complex structures are covariantly constant in the following sense:

$$D_k(\Gamma) f_{Pij} + Q_k^P Q f_{Qij} + Q_k^Q (f_{[P} f_{Q]})_{ij} = 0. \quad (3.28)$$

The latter result ensures that the Bianchi identities of the $SO(N)$ curvatures remain consistent with the constraints (3.26). It also allows the evaluation of the Nijenhuis tensors (no summation over P implied)

$$N_{Pij}^k = f_{Pi}^l f_{P[l;j]}^k - (i \leftrightarrow j), \quad (3.29)$$

which satisfy $N_{Pji}^j = 0$, but vanish only for $N = 2$ where the complex structure is covariantly constant with respect to the Christoffel connection. Let us also note that the curvature associated with the connection (3.27) is equal to

$$R_{ijkl}(\Gamma) = R_{ijkl} - \frac{1}{2} f_{ij}^P f_{kl}^P, \quad (3.30)$$

where we used eq. (3.28).

At this point all variations of the lagrangian linear in the spinor fields vanish. Subsequently one concentrates on the terms proportional to three spinors with a derivative acting on one of them. This then requires one to introduce the $\delta\phi Q\psi\epsilon$ and the $\chi^2\epsilon$ variations in (3.22) (3.23) and the $\psi\chi\epsilon$ variations contained in the supercovariant derivative in (3.25). The gravitino fields in the lagrangian and transformation rules are restricted by supercovariance arguments; therefore, in view of dimensional arguments, the only extra variations that one expects are possible $\chi^2\epsilon$ terms in (3.25). However, it turns out that those are not needed and one determines directly the χ^4 terms in the action (cf. eq. (3.20)) by making use of the integrability conditions that are derived directly from (3.28) and (3.26). We refrain from giving these conditions here, as they will be discussed in the next section (cf. eq. (4.4)). By virtue of the integrability conditions also the remaining variations, all cubic and quintic in the spinor fields, cancel after tedious but straightforward calculation!

4. Target-space geometry

In this section we study the implications of local supersymmetry on the target-space geometry. The most obvious restriction concerns the dimension of the target space. Locally it must be decomposable into a number of supermultiplets. Therefore we must have $d = kd_N$, where k is an integer denoting the number of irreducible supermultiplets and d_N is the number of bosonic states of an irreducible supermultiplet listed in table 1. For $N = 1, 2$ the remaining implications are rather straightforward. When $N = 1$ the target space is a riemannian manifold

of arbitrary dimension (as $d_1 = 1$) and no special properties are required while for $N = 2$ we are dealing with a Kähler space, as there is a complex structure that is covariantly constant with respect to the Christoffel connection (cf. eq. (3.28)). Obviously such a space must be of even dimension. It then follows that the Ricci tensor is related to the first Chern class.

The analysis for $N > 2$ is more involved. It is convenient to adopt a manifest $SO(N)$ notation. First introduce the anti-symmetric tensors f_{ij}^{IJ} (we freely raise and lower $SO(N)$ indices),

$$f^{PQ} = f^{[P} f^{Q]}, \quad f^{1P} = \pm f^P, \quad (4.1)$$

and the $SO(N)$ target-space connections Q_i^{IJ} , consisting of Q_i^{PQ} and

$$Q_i^{1P} = \mp Q_i^P. \quad (4.2)$$

With these definitions eqs. (3.26) and (3.28) can be written as

$$\begin{aligned} R_{ij}^{IJ}(Q) &\equiv \partial_i Q_j^{IJ} - \partial_j Q_i^{IJ} + 2Q_i^{K[I} Q_j^{J]K} = \frac{1}{2} f_{ij}^{IJ}, \\ D_i f_{jk}^{IJ} &\equiv D_i(\tilde{F}^{\otimes}) f_{jk}^{IJ} + 2Q_i^{K[I} f_{jk}^{J]K} = 0. \end{aligned} \quad (4.3)$$

They lead to the integrability condition

$$R_{ijmk} f^{IJm}{}_l - R_{ijml} f^{IJm}{}_k = -f_{ij}^{K[I} f_{kl}^{J]K}, \quad (4.4)$$

which, as pointed out in sect. 3, was required for the cancellation of the supersymmetry variations of the action that are cubic and quintic in the spinor fields.

Obviously the tensors f_{IJ} act as generators of $SO(N)$ in target space,

$$f_{IJ} f_{KL} - f_{KL} f_{IJ} = 4\delta_{K[I} f_{J]L} - 4\delta_{L[I} f_{J]K}. \quad (4.5)$$

In addition they satisfy

$$\begin{aligned} (f_{IJ})^2 &= -\mathbf{1} \quad (I \text{ and } J \text{ fixed}) \\ f_{IK} f_{KJ} &= (N-1)\delta_{IJ} - (N-2)f_{IJ}, \\ f_{IJij} f_{KL}{}^{ij} &= 2d\delta_{I[K} \delta_{L]J} \mp \delta_{N,4} \epsilon_{IJKL} J^k{}_k. \end{aligned} \quad (4.6)$$

The tensor J is defined by

$$(f_{[P_1} \cdots f_{P_{N-1}]})^i{}_j = J^i{}_j \epsilon_{P_1 \dots P_{N-1}}. \quad (4.7)$$

For *even* values of N it satisfies the following properties:

$$J^i_k f^k_{Pj} = f^i_{Pk} J^k_j, \quad D_i(\overset{\circ}{\Gamma}) J^j_k = 0, \quad J^2 = (-)^{N/2} \mathbf{1}, \quad J_{ij} = (-)^{N/2} J_{ji}, \quad (4.8)$$

and must be traceless, unless $N = 4$. For $N = 4$ one derives

$$f_P f_Q = -\delta_{PQ} \mathbf{1} - \epsilon_{PQR} J f_R. \quad (4.9)$$

Hence J^k_k is the trace of the product of the three almost-complex structures, which is constant so that it may be evaluated at any point in target space. As J is symmetric for $N = 4$ and its square is equal to the unit matrix (cf. eq. (4.8)), we find

$$J^k_k = d_+ - d_-, \quad (4.10)$$

where d_{\pm} are the dimensions of the subspaces for which the eigenvalue of J is equal to ± 1 . More generally, for $N = 4 \bmod 4$, the subspaces with $J = \pm 1$ correspond to the inequivalent supermultiplets discussed in sect. 2.

Let us now proceed for a general value of $N > 2$. First we note that for $N = 3$ the tensors f^{IJ} define precisely three almost-complex structures, which are covariantly constant with respect to a non-trivial $\text{SO}(3) \sim \text{Sp}(1)$ connection (cf. eq. (4.3)). Hence the target space must be quaternionic for $N = 3$. Leaving the special case of $N = 4$ until the end of this section, we now continue as generally as possible for $N > 2$. Contracting (4.4) with f^{MN}_{kl} and making use of (4.6) gives

$$R_{ijkl} f^{kl} = \frac{1}{4} df_{IJij}, \quad (4.11)$$

while contracting (4.4) with g^{jl} , using the cyclicity of the Riemann tensor and the above result (4.11), yields

$$R_{ij} \equiv R_{ikjl} g^{kl} = c g_{ij}, \quad (4.12)$$

where

$$c = N - 2 + \frac{1}{8}d > 0. \quad (4.13)$$

Hence we are dealing with an Einstein space ^{*}.

^{*} For $N = 3$ this is in accord with the fact that quaternionic spaces of dimension higher than four are always Einstein [26]. In the case at hand, the result also holds true for a four-dimensional target space. Our conventions here are such that positive curvature ($c > 0$) corresponds to non-compact manifolds; this convention is opposite to the one commonly adopted in the mathematical literature.

Now decompose the Riemann curvature as $R_{ijkl} = \hat{R}_{ijkl} + \frac{1}{8} f_{ij}^{IJ} f_{kl}^{IJ}$, so that (4.4) reads

$$\hat{R}_{ijmk} f^{IJm}_l - \hat{R}_{ijml} f^{IJm}_k = 0. \quad (4.14)$$

This motivates us to introduce the set of independent antisymmetric tensors $h_{ij}^\alpha(\phi)$, labelled by indices α defined by the requirement that they commute with the $SO(N)$ generators,

$$h_{ik}^\alpha f^{Ijk}_j - h_{jk}^\alpha f^{Ijk}_i = 0. \quad (4.15)$$

For the moment we restrict ourselves to a given point in target space, but the fact that the $SO(N)$ generators are realized everywhere on the manifold (in the spinor representation), implies that the number of independent tensors h^α and their associated Lie-bracket structure is the same everywhere. Obviously the h^α generate the subgroup H' of $SO(d)$ that commutes with $SO(N)$; it will play an important role in what follows. Because of Schur's lemma, H' must be one the groups $SO(k_1) \otimes SO(k_2)$, $U(k_1) \otimes U(k_2)$ or $Sp(k_1) \otimes Sp(k_2)$, where k_1 and k_2 denote the number of inequivalent $SO(N)$ representations of the target space, and we have $k = k_1 + k_2$, as every irreducible supermultiplet contains precisely one irreducible $SO(N)$ multiplet of scalar fields. The nature of the group is determined by the centralizer of the $SO(N)$ representation and can be read off from table 1; for $N = 7, 8, 9 \pmod 8$ the group is orthogonal, for $N = 2, 6 \pmod 8$, it is unitary, and for $N = 3, 4, 5 \pmod 8$ it is symplectic. For odd N the spinor representation is unique, so that one has $k_1 = k$ and $k_2 = 0$. The structure constants of H' , which may at this point depend on the target-space coordinates, are defined by

$$h^\alpha h^\beta - h^\beta h^\alpha = f_\gamma^{\alpha\beta} h^\gamma. \quad (4.16)$$

From the arguments given above, as well as from more general considerations, it follows that the compact group H' factorizes into a direct product of an abelian group and a number of simple groups. In what follows these factor groups will generically be denoted by H'' . By a suitable redefinition we ensure that an index α refers exclusively to one of these factor groups. Without loss of generality it is possible to impose the normalization condition

$$h_{ij}^\alpha h^{bij} = 2d_N \delta^{\alpha\beta}. \quad (4.17)$$

With this normalization it follows that $\delta^{\alpha\beta}$ is an invariant tensor under H' , which may be used to raise and lower indices. The structure constants $f^{\alpha\beta\gamma}$ are then totally antisymmetric.

Taking the covariant derivative of (4.15) it follows that the covariant derivative of h^α commutes with f^{IJ} , and must therefore be proportional to the same tensors, i.e.

$$D_i(\hat{\Gamma}^\alpha)h_{jk}^\alpha(\phi) = \Omega_i^{\alpha\beta}(\phi)h_{jk}^\beta(\phi). \quad (4.18)$$

In other words, the tensors h_{ij}^α are covariantly constant with respect to the Christoffel connection and some connection $\Omega_i^{\alpha\beta}$. In view of (4.17) this connection is anti-symmetric in α and β .

The fact that \hat{R} commutes with $\text{SO}(N)$ (cf. eq. (4.14)) thus implies that locally the Riemann tensor can be written as

$$R_{ijkl} = \frac{1}{8} \{ f_{ij}^{IJ} f_{kl}^{IJ} + C_{\alpha\beta} h_{ij}^\alpha h_{kl}^\beta \}, \quad (4.19)$$

where $C_{\alpha\beta}(\phi)$ is some unknown tensor, symmetric in α and β , so that the curvature satisfies the pair-exchange property. According to (4.18), (4.19) and the second equation of (4.3), the curvature and its multiple covariant derivatives take their values in the algebra corresponding $\text{SO}(N) \otimes H'$. Therefore the target-space holonomy group must be contained in this group. Note, however, that the holonomy group could in principle be smaller than $\text{SO}(N) \otimes H'$, depending on the actual values taken by the tensor $C_{\alpha\beta}$ and the connection $\Omega_i^{\alpha\beta}$. It is known [18] that spaces with restricted holonomy groups have special properties, so we expect (4.19) to have important consequences. We shall return to this aspect in sect. 5.

The fact that we are dealing with an Einstein space implies

$$C_{\alpha\beta} h_i^{\alpha k} h_{kj}^\beta = [N(N-1) - 8c] g_{ij}. \quad (4.20)$$

Obviously, the above expression is invariant under H' , so that

$$C_{\delta(\alpha} f_{\beta)}^{\delta\gamma} h^\alpha h^\beta = 0. \quad (4.21)$$

To ensure that the Riemann curvature satisfies the cyclicity property, the tensors f^{IJ} and h^α should satisfy

$$f_{[ij}^{IJ} f_{kl]}^{IJ} + C_{\alpha\beta} h_{[ij}^\alpha h_{kl]}^\beta = 0. \quad (4.22)$$

It is not easy to solve this equation in full generality. Therefore we first consider its contraction with f_{kl}^{KL} and h_{kl}^α , using eq. (4.5) and (4.6) and

$$f_{ij}^{ij} h_{ij}^\alpha = 0. \quad (4.23)$$

The latter relation follows from the cyclicity of the trace and the fact that (for $N > 2$) every tensor f^{IJ} can be written as the commutator of two such tensors (cf.

eq. (4.5)). Note that this is also in accord with (4.11) and (4.19). For the generators h^α we used the same argument when imposing (4.17) to ensure that the trace of the product of two generators belonging to different factor groups H'' vanishes.

The contraction of (4.22) with f leads again to (4.20), while with h^α we find

$$2d_N C_\alpha^\beta + C_{\gamma\delta} f_\lambda^{\beta\gamma} f_\alpha^{\delta\lambda} - 16c \delta_\alpha^\beta = 0. \quad (4.24)$$

This result shows that $C_{\alpha\beta}$ vanishes when α and β belong to the different factor groups of H' . For that reason we may consider (4.24) and (4.21) for the simple subgroups separately. For the abelian factor (4.24) can be solved directly,

$$C_{\alpha\beta}(H'') = \frac{8c}{d_N} \delta_{\alpha\beta}, \quad \alpha, \beta \in h'' \text{ abelian}. \quad (4.25)$$

For the simple factor groups, it is more difficult to find the solution of $C_{\alpha\beta}$, but after multiplying with $h^\alpha h^\beta$, with α and β belonging to the generators of the simple factor group, and making use of (4.21), we find

$$C_{\alpha\beta}(H'') h^\alpha h^\beta = \frac{16c}{2d_N + c_2(H'')} h^\alpha h^\alpha, \quad \text{with } \alpha, \beta \in h'', \quad (4.26)$$

where

$$f_{\gamma\delta}^\alpha f^{\gamma\delta}_\beta = c_2(H'') \delta_\beta^\alpha. \quad (4.27)$$

In the last equation we used Schur's lemma. Observe that (4.26) applies also to the abelian factor, as $c_2(H'') = 0$ in that case.

Now there is one more conclusion we can draw from (4.22), namely that the group $SO(N) \otimes H'$ must act *irreducibly* on the target space. To show this, it is convenient to rewrite (4.22) with tangent-space indices. Let us then assume that there is a subspace which is left invariant by $SO(N) \otimes H'$, so that this group acts reducibly. Denote the indices of this invariant subspace by $i_\parallel, j_\parallel, \dots$, and the indices of its orthogonal complement by i_\perp, j_\perp, \dots . Subsequently consider the cyclicity equation (4.22), with indices $i_\parallel, j_\parallel, k_\perp$ and l_\perp . Because of the invariance of the subspace there are no generators with mixed indices, so that (4.22) reduces to

$$f_{i_\parallel j_\parallel}^{IJ} f_{k_\perp l_\perp}^{IJ} + C_{\alpha\beta} h_{i_\parallel j_\parallel}^\alpha h_{k_\perp l_\perp}^\beta = 0. \quad (4.28)$$

However, contracting this with $f_{k_\perp l_\perp}^{KL}$ leads to an immediate contradiction. Hence we conclude that $SO(N) \otimes H'$ acts *irreducibly* on the target space.

By Schur's lemma, this shows that the abelian factor in H' has dimension 0 or 1, with the square of its corresponding generator h equal to $h^2 = -(2/k)\mathbf{1}$. Further-

more both $C_{\alpha\beta}h^\alpha h^\beta$ and $h^\alpha h^\alpha$, with the generators restricted to one of the factor groups H'' , are proportional to the unit matrix. In this way we find

$$(h^\alpha h^\alpha)_{ij} = -\frac{2 \dim H''}{k} g_{ij},$$

$$C_{\alpha\beta}(H'')(h^\alpha h^\beta)_{ij} = -\frac{32c}{2d_N + c_2(H'')} \frac{\dim H''}{k} g_{ij}, \quad (4.29)$$

where the sum extends over the generators of each of the factor groups H'' separately. Last but not least, as $SO(N) \otimes H'$ leaves the subspace invariant constituted by equivalent $SO(N)$ representations, it follows that the target space should decompose entirely into $SO(N)$ representations that are *equivalent*. Consequently, we may put $k_1 = k$ and $k_2 = 0$.

Now we substitute (4.29) into (4.20) to obtain a relation between N and the number of supermultiplets. Using that c_2 equals $2(k-2)$, $4k$ and $8(k+1)$, for $SO(k)$, $SU(k)$ and $Sp(k)$, while the dimensions of these groups are equal to $\frac{1}{2}k(k-1)$, k^2-1 and $k(2k+1)$, respectively, leads to the following equations:

$$\frac{N(N-1)}{8c} = \begin{cases} \frac{d_N - 1}{d_N + k - 2} & \text{for } N = 7, 8, 9 \text{ mod } 8, \\ \frac{d_N^2 - 4}{d_N(d_N + 2k)} & \text{for } N = 6 \text{ mod } 4, \\ \frac{d_N + 2}{d_N + 4k + 4} & \text{for } N = 3, 5, 12 \text{ mod } 8, \end{cases} \quad (4.30)$$

where c was defined in (4.13). From these equations one may verify that $N(N-1) - d_N$ must be positive, which implies that there can be no solutions for $N > 17$. Therefore it remains to search for a finite number of explicit solutions, which are rather rare in view of the fact that the parameters N and k must be integers. The result of this search is shown in table 2.

We should stress that so far we did not determine the tensor $C_{\alpha\beta}$. An obvious solution is to choose it equal to $\delta_{\alpha\beta}$ for every factor group H'' . In that case the Riemann tensor takes its values in the algebra corresponding to $SO(N) \otimes H'$ (in the spinor representation of $SO(N)$ and the defining representation of H'), and it is also invariant under this group. However, it is possible that there are alternative solutions for $C_{\alpha\beta}$, corresponding to nontrivial solutions of (4.24). The Riemann tensor could then take its values in the algebra corresponding to a subgroup of $SO(N) \otimes H'$ (which should still act irreducibly on the target space). Let us denote this group by \hat{H}' and assume that it can be written as a product of subgroups \hat{H}'' that are abelian (because of Schur's lemma, the abelian group is at most one-di-

TABLE 2

All solutions to (4.30) with $N = 3$ or $N \geq 5$, which correspond to possible non-linear sigma models coupled to extended supergravity in terms of N and the number of supermultiplets k . The case $N = 4$ is given for comparison. There one can have two independent quaternionic subspaces corresponding to k_+ and k_- inequivalent supermultiplets

N	d_N	k	c	H'
16	128	1	30	1
12	64	1	18	Sp(1)
10	32	1	12	U(1)
9	16	1	9	1
8	8	k	$6 + k$	SO(k)
6	8	k	$4 + k$	U(k)
5	8	k	$3 + k$	Sp(k)
4	4	k_{\pm}	$2 + k_{\pm}$	Sp(k_{\pm})
3	4	k	$\frac{1}{2}(2 + k)$	Sp(k)

mensional) or simple. In addition to (4.24) also the following condition must then be satisfied:

$$\sum_{\hat{H}'' \subset \hat{H}'} \frac{\dim \hat{H}''}{2d_N + c_2(\hat{H}'')} = \sum_{H'' \subset H'} \frac{\dim H''}{2d_N + c_2(H'')}, \tag{4.31}$$

where the subgroups H'' are known from table 2. For an explicit example of this phenomenon consider $d_N = 4$ with the indices α, β taking values in the Lie algebra corresponding to Sp(k). In that case one obvious solution corresponds to $C_{\alpha\beta} \propto \delta_{\alpha\beta}$, while a second solutions is obtained by restricting $C_{\alpha\beta}$ to take only non-zero values for α, β corresponding to the generators of the obvious U(k) subgroup. We leave it to the reader to verify that in both cases one can satisfy (4.24) and (4.31). This example is relevant for $N = 3$, where indeed there exist homogeneous spaces corresponding to these solutions, namely Sp(1, k)/(Sp(1) \otimes Sp(k)) and U(2, k)/(U(2) \otimes U(k)). As we shall discuss in sect. 5, the fact that the holonomy group is reduced has important consequences for the target space.

At this point we have not yet attempted to solve (4.22). The easiest way to find solutions to this equation is to assume that one is dealing with a homogeneous space, in which case (4.22) is just one of the Jacobi identities for the generators of the isometry group. This will also be discussed in sect. 5. For a coset space G/H one expects the Riemann tensor to take its values in the Lie algebra of H. In the case at hand we know that H must be contained in SO(N) \otimes H'. For a given group H one knows the dimension of G, and in this way it is relatively easy to find coset spaces that satisfy all the restrictions given above.

Now we turn to a discussion of the $N = 4$ theories. An important role is played by the symmetric tensor J , whose definition and main properties were given in eq.

(4.6)–(4.10). As its eigenvalues are equal to ± 1 , we can use it to define the projection operators

$$\Pi_{\pm j}^i = \frac{1}{2}(\delta_j^i \pm J_j^i). \quad (4.32)$$

By means of these projectors one decomposes the target space into two subspaces. Because of the fact that the tensors Π_{\pm} are covariantly constant, the Riemann tensor is only nonvanishing when all its indices take values in the same subspace (to see this use the cyclicity of the curvature). Hence we decompose the curvature into two tensors $R_{ijkl}^{(\pm)}$, satisfying

$$\Pi_{\pm i}^j R_{jklm}^{(\mp)} = \Pi_{\pm i}^j D_j(\overset{\circ}{I}^{\mp}) R_{klmn}^{(\mp)} = 0, \quad (4.33)$$

where the second equation follows from the first one combined with the Bianchi identity. Under these circumstances, the space is locally a product of two separate riemannian spaces; this means that one can choose coordinates such that the metric acquires a block-diagonal form, in accordance with the projectors (4.32), where the metric of one subspace does not depend on the coordinates of the other one.

Furthermore, because the almost-complex structures commute with the tensor J , they can be decomposed into almost-complex structures belonging to the two subspaces. Hence we may introduce two tensors $f_p^{(\pm)i}$, which are only non-zero when both indices take values in the corresponding subspace, although at this stage they may still depend on the coordinates of both subspaces. Decomposing the SO(4) connections in terms of two sets of SO(3) connections,

$$Q_i^{(\pm)P} = -\frac{1}{2}\epsilon^{PQR}Q_i^{QR} \mp Q_i^P, \quad (4.34)$$

one can write (3.28) as follows:

$$D_k(\overset{\circ}{I}^{\pm})f_{ij}^{(\pm)P} + \epsilon^{PQR}Q_k^{(\pm)Q}f_{ij}^{(\pm)R} = 0, \quad (4.35)$$

while, according to (3.26), the curvatures of the two connections are equal to

$$R_{ij}^P(Q^{(\pm)}) = \partial_i Q_j^{(\pm)P} - \partial_j Q_i^{(\pm)P} + \epsilon^{PQR}Q_i^{(\pm)Q}Q_j^{(\pm)R} = \pm f_{ij}^{(\pm)P}. \quad (4.36)$$

Hence the curvatures $R^P(Q^{(\pm)})$ vanish in the subspace projected out by Π_{\mp} . Therefore by a suitable SO(3) gauge transformation, one can ensure that the connections $Q^{(\pm)P}$ vanish in this subspace. The remaining identities then ensure that the two spaces decouple completely, with separate complex structures $f^{(\pm)}$ and connections $Q^{(\pm)}$ with components in the corresponding subspace and depending only on the coordinates of that subspace. Note that the tensors $f^{(\pm)P}$ define almost-complex structures in their respective subspaces. We should perhaps

point out here that these two subspaces do not decouple in the field theory, but interact via the coupling to the dreibein and gravitino fields.

Hence we may now concentrate on one of these subspaces separately. Dropping all superscripts (\pm), the geometry in the subspace is subject to the following equations:

$$\begin{aligned} f_P f_Q &= -\delta_{PQ} \mathbf{1} \mp \epsilon_{PQR} f_R, \\ D_k(\tilde{F}^{\circ}) f_{ij}^P + \epsilon^{PQR} Q_k^Q f_{ij}^R &= 0, \\ R_{ij}^P(Q) &= \pm f_{ij}^P. \end{aligned} \quad (4.37)$$

The subspace transforms under the action of the corresponding $SO(3)$ group according to inequivalent representations. Again, as we have three almost-complex structures that are covariantly constant with respect to a non-trivial $Sp(1)$ connection, the space is quaternionic.

For reasons of comparison we repeat some of the same steps as in the more general case. Contracting the integrability condition corresponding to the second equation of (4.37) with the almost-complex structures and the metric yields the analogue of (4.11) and (4.12), but with different normalizations,

$$R_{ijkl} f_P^{kl} = \frac{1}{2} d_{\pm} f_{Pij}, \quad R_{ij} = \frac{1}{4} (8 + d_{\pm}) g_{ij}, \quad (4.38)$$

where $d_{\pm} = 4k_{\pm}$ is the dimension of the subspace and k_{\pm} the number of supermultiplets (which equals the quaternionic dimension of the subspace). Furthermore we have a similar decomposition of the curvature as in (4.19),

$$R_{ijkl} = \frac{1}{2} \{ f_{ij}^P f_{kl}^P + C_{\alpha\beta} h_{ij}^{\alpha} h_{kl}^{\beta} \}, \quad (4.39)$$

where the tensors h^{α} , together with the identity, span the centralizer of the almost-complex structures, so that they generate the group $Sp(k_{\pm})$. Together with the complex structures they generate the group $Sp(1) \otimes Sp(k_{\pm})$, which must again act irreducibly. Again one derives

$$C_{\alpha\beta} (h^{\alpha} h^{\beta})_{ij} = -\frac{1}{2} (2 + d_{\pm}) g_{ij}. \quad (4.40)$$

We should point out that the presence of the two separate quaternionic spaces can be understood from $N = 2$ supergravity in four space-time dimensions. In that case there exist two inequivalent matter multiplets. The vector multiplets, whose scalar fields parametrize a Kähler manifold [27], and the scalar (or hyper-)multiplets, whose scalar fields parametrize a quaternionic manifold [28]. Upon dimensional reduction the Kähler space of the vector multiplets is converted into a

quaternionic space (although not the most general) [29], so that one obtains two quaternionic spaces associated with inequivalent supermultiplets.

Perhaps we should explain why this phenomenon can only happen for $N = 4$, while there are inequivalent multiplets for all values $N = 4 \bmod 4$, as we showed in section 2. The reason is that the group $SO(N) \otimes H'$ must act irreducibly in the target space, so that only one type of multiplet is allowed. The situation for $N = 4$ is different, because the group $SO(4)$ factors into two separate $SO(3)$ groups, each of them acting in a different subspace of the target space.

The question that remains to be answered is what the possible spaces are corresponding to $N > 4$. As we shall argue in the next section, it turns out that these spaces are unique. After identifying each one of them it is rather straightforward to verify that all equations of this section are indeed satisfied.

5. Homogeneous spaces

A striking feature of the results derived in sect. 4 is that, except for the low values $N \leq 4$, the number of possible theories is rather limited. In particular, for $N > 8$, there remain only four theories based on a single supermultiplet corresponding to $N = 9, 10, 12$ and 16 . The bound $N \leq 16$ was obtained here solely on the basis of mathematical considerations; since there is no helicity in three dimensions, we cannot rely on “physical” arguments, unlike in four space-time dimensions, where the analogous bound $N \leq 8$ follows from requiring absence of massless states of helicity higher than 2. The arguments of sect. 4 are not yet strong enough to determine the target manifolds, since we used only a contracted version of (4.22); to find out what the possible spaces are, one must exploit the full content of these identities. Fortunately, we can now invoke a powerful mathematical theorem to prove that the target spaces are, in fact, symmetric and therefore homogeneous for sufficiently high N .

Theorem 5.1 [19]: Let \mathcal{M} be an irreducible riemannian manifold. If the holonomy group at a point $p \in \mathcal{M}$ does not act transitively on the unit sphere in the tangent space $T_p \mathcal{M}$ at p , then \mathcal{M} is a symmetric space of rank ≥ 2 .

The content of this theorem can be rephrased as follows: if the holonomy group of \mathcal{M} is sufficiently “small” with respect to the generic holonomy group (i.e. $SO(d)$ for an arbitrary d -dimensional riemannian manifold), then the manifold is completely determined; if, on the other hand, it is “large”, then little can be said, and there is a greater variety of spaces. We note, however, that the possibly holonomy groups for irreducible non-symmetric riemannian manifolds cannot be arbitrary subgroups of $SO(d)$, but are strongly restricted; a complete list is given in corollary 10.92 of ref. [18]. In the case at hand, all the necessary information is encoded in the explicit formula (4.19) for the curvature tensor, which tells us that the

TABLE 3

Complete list of target spaces for $N \geq 5$ supergravity theories. The coefficient c , defined in (4.13), coincides with the dual Coxeter number of the group G

N	d_N	k	c	G/H	Rank
16	128	1	30	$E_{8(+8)}/SO(16)$	8
12	64	1	18	$E_{7(-5)}/(SO(12) \otimes SO(3))$	4
10	32	1	12	$E_{6(-14)}/(SO(10) \otimes SO(2))$	2
9	16	1	9	$F_{4(-20)}/SO(9)$	1
8	8	k	$6+k$	$SO(8, k)/(SO(8) \otimes SO(k))$	$\max(8, k)$
6	8	k	$4+k$	$SU(4, k)/S(U(4) \times U(k))$	$\max(4, k)$
5	8	k	$3+k$	$Sp(2, k)/(Sp(2) \times Sp(k))$	$\max(2, k)$

holonomy group is contained in $SO(N) \otimes H'$, where the centralizer subgroup H' can be read off from table 2. As the dimension of the target space is $d = kd_N$, we must therefore check whether or not the group $SO(N) \otimes H'$ acts transitively on the unit sphere S^{d-1} . When it does not, then the holonomy group $SO(N) \otimes \hat{H}$, which is contained in it, does not act transitively either and we can apply the theorem. This allows us to understand the limitations on the number of possible theories from a slightly different point of view: extended supergravity theories are scarce because the mismatch between the actual holonomy group $SO(N) \otimes \hat{H}$ and the generic holonomy group $SO(d) = SO(kd_N)$ becomes too big for $N > 4$. For $N \leq 4$, the information provided by (4.19) is not sufficient to completely determine the manifold. In particular, for $N = 1$, there are no restrictions at all, and the target space is an arbitrary riemannian manifold. For $N = 2$, the holonomy group has a $U(1)$ factor; since there is one complex structure, the manifold must be Kähler, and the holonomy group is contained in $U(k)$ with $d = 2k$. As this group acts transitively on the sphere S^{2k-1} , we get no further restrictions from the theorem. For $N = 3$ and $N = 4$, the target spaces are quaternionic manifolds of dimension $d = 4k$ and $d_{\pm} = 4k_{\pm}$, respectively, and the holonomy group is contained in $Sp(1) \otimes Sp(k)$. Since the group $Sp(1) \otimes Sp(k)$ acts transitively on the sphere S^{4k-1} , the theorem imposes no immediate restrictions on the manifold. For all higher values of N with the exception of $N = 9$, the group $SO(N) \otimes H'$, and therefore the holonomy group does *not* act transitively. According to the theorem we can then uniquely determine the possible target manifolds by matching the values of N and d with the list of symmetric spaces. This identification leads to the list of spaces shown in table 3, which forms a central result of this paper*. All non-linear sigma models coupled to $N \geq 5$ supergravity are thus uniquely deter-

* By some abuse of notation we wrote orthogonal groups for the cosets where possible. It should be clear from the text in sect. 4 what the representations are in which the isotropy group acts. As $SO(N)$ acts in the spinor representation it would be appropriate to denote it as $Spin(N)$, whereas the $SO(3)$ group for $N = 12$ is actually $Sp(1)$. Observe the importance of triality for the $N = 8$ coset space, which can be used to interchange vector and spinor representations of $SO(8)$.

mined. The maximal number of supersymmetries is $N = 16$, which corresponds to the theory constructed quite some time ago in the first paper of ref. [12]. The case $N = 9$ may seem special, as $\text{Spin}(9)$ does act transitively on S^{15} , but it can be shown that the coset space $F_4/\text{Spin}(9)$ (which is of rank 1) is the only solution [30] *.

We expect that the theories with even N in table 3 can be obtained by dimensional reduction of the corresponding $N/2$ theories in four space-time dimensions. To obtain the theories with odd N , one would have to further truncate the dimensionally reduced theories, but, evidently, neither the target spaces nor the fact that there are no theories for certain odd values of N below $N = 16$ and none at all above $N = 16$ could have been reliably predicted on the basis of such arguments. We should perhaps point out that exceptional groups (including G_2) also appear for symmetric quaternionic spaces. All homogeneous quaternionic spaces are known and were given in ref. [31] (see also ref. [23]).

Having established that the target spaces are symmetric for sufficiently high N , we devote the remainder of this section to elucidating some features of the relation between the target-space formulation of locally supersymmetric theories as given in sect. 3 and the formulation of extended supergravity theories as G/H coset space theories (see, for instance, refs. [12,32]). In particular we shall indicate how some of the results of our work arise in the context of the latter formulation. We assume, in accord with the spaces listed in table 3, that G is a non-compact group and H its maximal compact subgroup, so that the space is symmetric. For $N \geq 5$ the possible choices for G and H can be gleaned from table 3, but our results can be applied for other cases as well. Together with the results derived in sect. 3, this information then gives an explicit representation of the lagrangian and supersymmetry transformations of the theory.

Let us first discuss the group-theoretical aspects in a little more detail. From sect. 4 we know that the group H always factorizes according to $\text{SO}(N) \otimes \hat{H}$, where $\hat{H} \subset H'$ (for the spaces listed in table 3, \hat{H} and H' do actually coincide). The generators of the group \hat{H} will be denoted by h^α where the indices α now take their values in the Lie algebra of \hat{H} : $\alpha = 1, \dots, \dim \hat{H}$. They commute with fermion number and with the matrices Γ_{AA}^I ,

$$h_{AC}^\alpha \Gamma_{CB}^I + h_{BC}^\alpha \Gamma_{AC}^I = 0. \quad (5.1)$$

Denoting the $\text{SO}(N)$ generators by $X^{IJ} = -X^{JI}$, where $I, J, \dots = 1, \dots, N$, and the remaining (coset) generators by Y^A , where the bosonic indices A, B, \dots (or

* In ref. [18] the reader may find the list of subgroups of $\text{SO}(d)$ which act transitively on S^{d-1} . Besides the regular groups, there are three exceptional cases, namely G_2 acting on S^6 , $\text{Spin}(7)$ on S^7 and $\text{Spin}(9)$ on S^{15} . The first two of these play no role in our analysis, because the associated manifolds are Ricci flat [18], which would lead to a contradiction with (4.12) and (4.13).

the fermionic ones $\hat{A}, \hat{B}, \dots = 1, \dots, d$ were already introduced in sect. 2, the Lie algebra of G is characterized by the commutation relations

$$\begin{aligned} [X^{IJ}, X^{KL}] &= \delta^{JK}X^{IL} - \delta^{IK}X^{JL} - \delta^{JL}X^{IK} + \delta^{IL}X^{JK}, \\ [X^\alpha, X^\beta] &= f^{\alpha\beta}{}_\gamma X^\gamma, \quad [X^{IJ}, X^\alpha] = 0, \\ [X^{IJ}, Y^A] &= -\frac{1}{2}\Gamma_{AB}^{IJ}Y^B, \quad [X^\alpha, Y^A] = -h_{AB}^\alpha Y^B, \\ [Y^A, Y^B] &= \frac{1}{4}\Gamma_{AB}^{IJ}X^{IJ} + \frac{1}{8}C_{\alpha\beta}h_{AB}^\alpha X^\beta, \end{aligned} \quad (5.2)$$

where $\Gamma_{AB}^{IJ} \equiv \Gamma_{AA}^I \Gamma_{BB}^J$, so that $\frac{1}{2}\Gamma_{AB}^{IJ}$ generates the spinor representation of $SO(N)$. Likewise

$$h_{AC}^\alpha h_{CB}^\beta - h_{AC}^\beta h_{CB}^\alpha = f^{\alpha\beta}{}_\gamma h_{AB}^\gamma. \quad (5.3)$$

The tensor $C_{\alpha\beta}$ coincides with the tensor introduced in (4.19). Most of the Jacobi identities implied by the algebra (5.2) are trivially satisfied once we assume that $C_{\alpha\beta}$ is \hat{H} -invariant. The remaining identity, and the one that leads to the most stringent constraints on G , arises from the commutator $[[Y^A, Y^B], Y^C]$; it reads

$$\Gamma_{[AB}^{IJ} \Gamma_{CD]}^{IJ} + C_{\alpha\beta} h_{[AB}^\alpha h_{CD]}^\beta = 0. \quad (5.4)$$

This equation is just (4.19), except that $C_{\alpha\beta}$ is now assumed to be \hat{H} -invariant. From sect. 3 we can therefore deduce its values for the spaces listed in table 3, using the normalization (4.17). For $N=16$ and 9 , $C_{\alpha\beta}$ obviously vanishes; for $N=12, 10, 8$ and 5 , \hat{H} is simple, so that $C_{\alpha\beta}$ is proportional to the identity, and its eigenvalues are equal to $2, 3, 8$ and 2 , respectively. The case $N=6$ is slightly more complicated. For the $SU(k)$ subgroup $C_{\alpha\beta}$ is proportional to the identity with eigenvalue equal to 4 , whereas for the $U(1)$ subgroup, we have the eigenvalue $4+k$. In appendix A we will give an explicit proof of the Jacobi identity (5.4) for the groups E_8, E_7, E_6 and F_4 .

In the coset space formulation the scalar fields that parametrize the coset space are characterized by a matrix $\mathcal{V}(x) \in G/H$, on which G acts as a rigid symmetry group from the left, while H is realized as a local symmetry acting from the right. To understand that this description is equivalent to the one in terms of the target-space coordinate fields $\phi^i(x)$, we note that the matrix \mathcal{V} represents $d = \dim(G/H) = \dim G - \dim H$ physical degrees of freedom. The spurious (gauge) degrees of freedom associated with the subgroup H can be eliminated by choosing a special (“unitary”) gauge where the matrix \mathcal{V} is directly parametrized through the target-space coordinate $\phi^i(x)$ used before, i.e. $\mathcal{V} = \mathcal{V}(\phi^i(x))$. To maintain this gauge choice under local supersymmetry transformations, compensating H rotations will be needed. We will also need a vielbein e_i^A as well as gauge

connections Q_i^{IJ} and Q_i^α for the tangent-space group $SO(N) \otimes \hat{H}$. These are defined by (for a systematic and complete discussion of coset spaces, see e.g. ref. [33])

$$\mathcal{V}^{-1} \partial_i \mathcal{V} = \frac{1}{2} Q_i^{IJ} X^{IJ} + Q_i^\alpha X^\alpha + e_i^A Y^A, \quad (5.5)$$

where ∂_i is the derivative with respect to the target-space coordinate ϕ^i .

The integrability condition corresponding to (5.5) are the so-called Cartan–Maurer equations. In this case they read

$$D_{[i} e_{j]}^A = \partial_{[i} e_{j]}^A + \left(\frac{1}{4} Q_{[i}^{IJ} \Gamma_{AB}^{IJ} + Q_{[i}^\alpha h_{AB}^\alpha \right) e_{j]}^B = 0, \quad (5.6)$$

$$R_{ij}^{IJ} = -\frac{1}{2} e_i^A e_j^B \Gamma_{AB}^{IJ}, \quad (5.7)$$

$$R_{ij}^\alpha = -\frac{1}{8} e_i^A e_j^B C_{\alpha\beta} h_{AB}^\beta, \quad (5.8)$$

where R_{ij}^{IJ} was already defined in (4.3), while R_{ij}^α equals

$$R_{ij}^\alpha \equiv \partial_i Q_j^\alpha - \partial_j Q_i^\alpha + f_{\beta\gamma}^\alpha Q_i^\beta Q_j^\gamma. \quad (5.9)$$

The geometrical content of the theory is fixed once we identify e_i^A as the vielbein of the coset manifold with Q_i^{IJ} and Q_i^α the spin-connection fields. The latter take their values in the algebra of the isotropy group, which is the subgroup of $SO(d)$ that acts on the tangent space with the generators $\frac{1}{2} \Gamma^{IJ}$ and h^α defined above. According to (5.6) the space is torsion-free, so that the vielbein is covariantly constant with respect to the Christoffel connection,

$$D_i e_j^A = \partial_i e_j^A - \Gamma_{ij}^k e_k^A + \left(\frac{1}{4} Q_i^{IJ} \Gamma_{AB}^{IJ} + Q_i^\alpha h_{AB}^\alpha \right) e_j^B = 0. \quad (5.10)$$

The vielbein e_i^A is related to the target-space metric of sect. 4 by

$$g_{ij}(\phi) = e_i^A(\phi) e_j^B(\phi) \eta_{AB}, \quad (5.11)$$

where η_{AB} is a symmetric and \hat{H} -invariant tensor; in case there is more than one invariant tensor, the metric is thus no longer unique. The vielbein can also be used to convert curved into flat indices in the usual fashion; for instance, the generators of \hat{H} are related to (a subset of) the matrices h_{ij}^α used previously (see eq. (4.15)) by

$$h_{ij}^\alpha = h_{AB}^\alpha e_i^A e_j^B. \quad (5.12)$$

The curvature tensor on G/H can be computed from

$$R_{ijkl} = -e_k^A e_l^B \left(\frac{1}{4} R_{ij}^{IJ} \Gamma_{AB}^{IJ} + R_{ij}^\alpha h_{AB}^\alpha \right). \quad (5.13)$$

Using eqs. (5.7) and (5.8) one thus obtains

$$R_{ABCD} = \frac{1}{8}(\Gamma_{AB}^{IJ}\Gamma_{CD}^{IJ} + C_{\alpha\beta}h_{AB}^{\alpha}h_{CD}^{\beta}), \quad (5.14)$$

which precisely coincides with (4.19). In terms of flat indices, the curvature tensor is therefore constant; moreover, the Jacobi identity (5.4) ensures the cyclicity of the Riemann tensor and is thus equivalent to (4.22).

From the previous sections we know of the existence of $N - 1$ almost-complex structures f_{ij}^P (remember that $P, Q, \dots = 2, \dots, N$). In the coset formulation they can be represented by

$$f_{ij}^P = \pm(\Gamma^P\Gamma^1)_{AB}e_i^Ae_j^B, \quad (5.15)$$

and are not $\text{SO}(N)$ covariant. On the other hand, the antisymmetric tensors f_{ij}^J , which were defined in (4.1), are $\text{SO}(N)$ covariant, and take the form

$$f_{ij}^J = -\Gamma_{AB}^{IJ}e_i^Ae_j^B. \quad (5.16)$$

The tensors f_{ij}^P are only almost-complex structures; from (5.10) and the definition (5.15), we immediately deduce that

$$D_i(\Gamma^{\hat{P}})f_{jk}^P = \pm\frac{1}{4}Q_i^{JJ}[\Gamma^{JJ}, \Gamma^P\Gamma^1]_{jk} = -Q_i^Qf_{jk}^{PQ} - Q_i^{PQ}f_{jk}^Q, \quad (5.17)$$

where we made use of the definition (4.2). Relation (5.17) is nothing but the previous formula (3.28).

In the coset formulation the fermion fields do not carry target-space indices. To appreciate this feature, let us recall the supersymmetry transformation

$$\delta\phi^i = \frac{1}{2}(\bar{\epsilon}\chi^i + \bar{\epsilon}^Pf_{Pj}^i\chi^j). \quad (5.18)$$

By making use of the supersymmetry transformation with parameter ϵ , one naturally defines fermion fields that transform as the components of a target-space vector. In the coset formulation, on the other hand, one considers $\mathcal{V}^{-1}\delta\mathcal{V}$, which takes its values in the Lie algebra of G . By a suitable (field-dependent) H-transformation, this expression can be restricted to take its values in the generators Y^A . This motivates one to introduce fermion fields χ^A that transform covariantly under H, so that the supersymmetry variation takes the form

$$\mathcal{V}^{-1}\delta\mathcal{V} = \frac{1}{2}\bar{\epsilon}^I\chi^A\Gamma_{AA}^I Y^A. \quad (5.19)$$

In a given gauge the two transformations should coincide, modulo a compensating (field-dependent) H-transformation to maintain the gauge choice.

By comparing the two supersymmetry variations we can find the relation between the fermion fields χ^i and χ^A . We first observe that the direct variation of \mathcal{V} yields

$$\mathcal{V}^{-1}\delta\mathcal{V} = \delta\phi^i\mathcal{V}^{-1}\partial_i\mathcal{V} = \delta\phi^i\left(\frac{1}{2}Q_i^{IJ}X^{IJ} + Q_i^\alpha X^\alpha + e_i^A Y^A\right). \quad (5.20)$$

Obviously the first two terms correspond to infinitesimal field-dependent H-transformations. The last term should be matched with (5.19), so that

$$\left(\bar{\epsilon}\chi^i + \bar{\epsilon}^P f_{Pj}^i \chi^j\right)e_i^A = \bar{\epsilon}^I \chi^A \Gamma_{AA}^I. \quad (5.21)$$

Making use of (5.15) this relation leads to the identifications $\epsilon^I = \pm\epsilon$ and

$$\chi^A = \pm\Gamma_{AA}^1 e_i^A \chi^i. \quad (5.22)$$

With this result the variations (5.18) and (5.19) coincide, provided one adds a compensating H-transformation to (5.19) with parameters

$$\omega^{IJ} = \delta\phi^i Q_i^{IJ}, \quad \omega^\alpha = \delta\phi^i Q_i^\alpha. \quad (5.23)$$

This compensating transformation must be included in all supersymmetry variations. To see the corresponding relation for the fermions χ^i is slightly more subtle. Using (5.22) we find

$$\begin{aligned} \delta\chi^A &= \pm\Gamma_{AA}^1 (\delta e_i^A \chi^i + e_i^A \delta\chi^i) \\ &= \pm\Gamma_{AA}^1 \delta\phi^i (\partial_j e_i^A - \Gamma_{ji}^k e_k^A) \chi^j + \frac{1}{2}\Gamma_{AA}^1 e_i^A \hat{\delta}\phi^i \epsilon^I, \end{aligned} \quad (5.24)$$

where we made use of (3.25) and (5.15). Here it is important that Γ_{ji}^k is *not* the Christoffel connection, but the modified connection defined in (3.27). Using (5.10) and (3.27) shows that the first term is equal to

$$\delta_\omega\chi^A = -\frac{1}{4}\omega^{IJ}\Gamma_{AB}^{IJ}\chi^B - \omega^\alpha h_{AB}^\alpha \chi^B, \quad (5.25)$$

where $h_{AB}^\alpha = \Gamma_{AA}^1 h_{AB}^\alpha \Gamma_{BB}^1$ by virtue of (5.1). In deriving this, we also made use of (4.2) and (5.10). The terms (5.25) are precisely cancelled by the compensating transformation (5.23). The remaining variation thus takes the form

$$\delta\chi^A = \frac{1}{2}\gamma^\mu \epsilon^I \Gamma_{AA}^I \hat{P}_\mu^A, \quad (5.26)$$

where we use the notation

$$\hat{P}_\mu^A \equiv \hat{\partial}_\mu \phi^i e_i^A. \quad (5.27)$$

Finally, by similar manipulations as described above, one may verify that

$$\begin{aligned} D_\mu(\Gamma)\chi^i &= \pm \left(D_\mu(\Gamma^{\hat{a}})e_A^i \right) \Gamma_{AA}^1 \chi^A \pm e_A^i \Gamma_{AA}^1 \partial_\mu \chi^A - e_A^i \Gamma_{AA}^P Q_j^P \partial_\mu \phi^j \chi^A \\ &= \pm e_A^i \Gamma_{AA}^1 \left(\delta_{AB} \partial_\mu + \frac{1}{4} Q_\mu^{IJ} \Gamma_{AB}^{IJ} + Q_\mu^\alpha h_{AB}^\alpha \right) \chi^B, \end{aligned} \quad (5.28)$$

where

$$Q_\mu^{IJ} = \partial_\mu \phi^i Q_i^{IJ}, \quad Q_\mu^\alpha = \partial_\mu \phi^i Q_i^\alpha. \quad (5.29)$$

The modification of the fermionic connection as given in (3.27) is thus indispensable for recasting the results in such a systematic and covariant form in the coset formulation. The reader is advised to consult ref. [12] to see that these various ingredients are indeed present for the theories constructed there.

Appendix A

In this appendix, we will establish the crucial Jacobi identity (5.4) for the exceptional groups E_8 , E_7 , E_6 and F_4 . For the convenience of the reader, we here repeat formula (5.4) for $C_{\alpha\beta} = A\delta_{\alpha\beta}$

$$\Gamma_{[AB}^{IJ} \Gamma_{CD]}^{IJ} + A h_{[AB}^\alpha h_{CD]}^\alpha = 0. \quad (A.1)$$

For $G = E_8$ and F_4 , the subgroup \hat{H} is trivial, and the second term is therefore absent. For $G = E_7$ and $G = E_6$, we have $\hat{H} = \text{Sp}(1)$ and $\hat{H} = \text{U}(1)$, respectively, so the second term in (A.1) must be taken into account; with the normalization adopted in (4.17), we find $A = 2$ for E_7 and $A = 3$ for E_6 , as stated below (5.4). To prove (A.1), we will need to know the Fierz identities for matrices acting on the d -dimensional chiral spinor representations of $\text{SO}(N)$ (there is only one multiplet, so we have $d = d_N$). Since we are dealing with a *real* representation of the Clifford algebra, the standard Fierz identities for complex Γ -matrix algebras must be modified. Fierz identities for real Clifford algebras have been derived in ref. [22]; however, these are not quite suitable for our purposes, and we will therefore present an alternative formulation. We will make use of the standard definition

$$\Gamma^{I_1 \dots I_{2k}} \equiv \Gamma^{[I_1} \dots \Gamma^{I_{2k}]}. \quad (A.2)$$

Notice that we consider only matrices built out of an even number of Γ -matrices, which do not mix the d -dimensional chiral subspaces. For brevity, we will denote these matrices by $\Gamma^{(2k)}$ below, so that $\Gamma_{AB}^{(2k)} \equiv \Gamma_{AB}^{I_1 \dots I_{2k}}$. The matrices $\Gamma^{(2k)}$ are

symmetric for even k , and antisymmetric for odd k . Let us first record the important formulas

$$\mathrm{Tr}\left(\Gamma^{I_1 \dots I_{2k}} \Gamma_{J_1 \dots J_{2k}}\right) = d(-)^k (2k)! \delta_{J_1 \dots J_{2k}}^{I_1 \dots I_{2k}}, \quad (\text{A.3})$$

and

$$\Gamma^{IJ} \Gamma^{K_1 \dots K_{2p}} \Gamma^{IJ} = \left(N - (N - 4p)^2\right) \Gamma^{K_1 \dots K_{2p}}, \quad (\text{A.4})$$

which are valid for arbitrary N (traces are understood to be over the chiral subspace labeled by the indices $A, B, \dots = 1, \dots, d$). From the explicit representation of the Γ -matrices in sect. 2, it is not difficult to check that, for $N = 4 \bmod 4$, the matrix \tilde{F} in (2.5) can be taken equal to the identity matrix. Since the fermion number operator \mathbf{F} is also unity in the chiral subspace, the matrices $\Gamma^{(2k)}$ and $\Gamma^{(N-2k)}$ are related to each other by duality, hence linearly dependent; for $2k = N/2$ there are thus only $\frac{1}{2} \binom{N}{2}$ linearly independent matrices. For $N = 2 \bmod 4$, we find $\tilde{F} = e$; therefore, duality now relates $\Gamma^{(N-2k)}$ and $e\Gamma^{(2k)}$. For odd N , on the other hand, all matrices are linearly independent.

For $N = 8n$, we have $d = 2^{4n-1}$ from table 1. Elementary counting arguments show that the matrices $\mathbf{1}, \Gamma^{(2)}, \Gamma^{(4)}, \dots, \Gamma^{(4n)}$ form a complete and linearly independent set of (real) $d \times d$ matrices (for the matrices $\Gamma^{(4n)}$, one must not forget to take into account the self-duality constraint, as we just explained). The relevant Fierz identity for an antisymmetric matrix M_{AB} (which is all we need for (A.1)) therefore reads

$$M_{AB} = -\frac{1}{d} \sum_{k=1, 3, \dots, 2n-1} \frac{1}{(2k)!} \Gamma_{AB}^{(2k)} \mathrm{Tr}(M\Gamma^{(2k)}). \quad (\text{A.5})$$

Summation over the $2k$ indices I_1, \dots, I_{2k} is implied in (A.5) and similar formulas below. For $N = 16$, this sum evidently contains only two terms. Evaluating (A.5) for the matrix $M_{AB} = \Gamma_{C|A}^{IJ} \Gamma_{B|D}^{IJ}$, we obtain

$$\Gamma_{C|A}^{IJ} \Gamma_{B|D}^{IJ} = \frac{1}{128} \frac{1}{2!} \Gamma_{AB}^{(2)} (\Gamma^{IJ} \Gamma^{(2)} \Gamma^{IJ})_{CD} + \frac{1}{128} \frac{1}{6!} \Gamma_{AB}^{(6)} (\Gamma^{IJ} \Gamma^{(6)} \Gamma^{IJ})_{CD}. \quad (\text{A.6})$$

From (A.4), we get $\Gamma^{IJ} \Gamma^{(2)} \Gamma^{IJ} = -128\Gamma^{(2)}$ and $\Gamma^{IJ} \Gamma^{(6)} \Gamma^{IJ} = 0$, so (A.6) reduces to

$$\Gamma_{C|A}^{IJ} \Gamma_{B|D}^{IJ} = -\frac{1}{2} \Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ}, \quad (\text{A.7})$$

from which the desired relation (A.1) follows directly (with $A = 0$).

For $N = 4 + 8n$, we have $d = 2^{2+4n}$. In contrast to the previous case, a complete set of real $d \times d$ matrices now cannot be constructed from the Γ -matrices alone, as one can quickly verify by counting the number of such matrices. In addition,

however, there are now three complex structures represented by the antisymmetric matrices h_{AB}^α for $\alpha = 1, 2, 3$, which generate the centralizer subgroup $\text{Sp}(1)$. With the normalization (4.17), we have $(h^\alpha)^2 = -2$ (no summation over α) and

$$[h^\alpha, h^\beta] = 2\sqrt{2}\epsilon_{\alpha\beta\gamma}h^\gamma. \quad (\text{A.8})$$

A complete and linearly independent set of antisymmetric matrices is given by $h^\alpha, \Gamma^{(2)}, h^\alpha\Gamma^{(4)}, \dots, h^\alpha\Gamma^{(4n)}, \Gamma^{(4n+2)}$, while the symmetric matrices are $\mathbf{1}, h^\alpha\Gamma^{(2)}, \Gamma^{(4)}, \dots, \Gamma^{(4n)}, h^\alpha\Gamma^{(4n+2)}$. Instead of writing down the general formula, let us immediately specialize to $N = 12$, so that $d = 64$; in this case, the relevant identities are

$$\begin{aligned} & \Gamma_{C[A}^{IJ} \Gamma_{B]D}^{IJ} \\ &= \frac{1}{64} \left\{ \frac{1}{2} h_{AB}^\alpha (\Gamma^{IJ} h^\alpha \Gamma^{IJ})_{CD} + \frac{1}{2} \Gamma_{AB}^{(2)} (\Gamma^{IJ} \Gamma^{(2)} \Gamma^{IJ})_{CD} \right. \\ & \quad \left. + \frac{1}{2 \cdot 4!} (h^\alpha \Gamma^{(4)})_{AB} (\Gamma^{IJ} h^\alpha \Gamma^{(4)} \Gamma^{IJ})_{CD} + \frac{1}{2 \cdot 6!} \Gamma_{AB}^{(6)} (\Gamma^{IJ} \Gamma^{(6)} \Gamma^{IJ})_{CD} \right\} \\ &= \frac{1}{64} \left\{ -66 h_{AB}^\alpha h_{CD}^\alpha - 26 \Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ} - \frac{1}{12} (h^\alpha \Gamma^{(4)})_{AB} (h^\alpha \Gamma^{(4)})_{CD} + \frac{1}{120} \Gamma_{AB}^{(6)} \Gamma_{CD}^{(6)} \right\}, \quad (\text{A.9}) \end{aligned}$$

and

$$\begin{aligned} & h_{C[A}^\alpha h_{B]D}^\alpha \\ &= \frac{1}{64} \left\{ \frac{1}{2} h_{AB}^\beta (h^\alpha h^\beta h^\alpha)_{CD} + \frac{1}{2} \Gamma_{AB}^{(2)} (h^\alpha \Gamma^{(2)} h^\alpha)_{CD} \right. \\ & \quad \left. + \frac{1}{2 \cdot 4!} (h^\beta \Gamma^{(4)})_{AB} (h^\alpha h^\beta \Gamma^{(4)} h^\alpha)_{CD} + \frac{1}{2 \cdot 6!} \Gamma_{AB}^{(6)} (h^\alpha \Gamma^{(6)} h^\alpha)_{CD} \right\} \\ &= \frac{1}{64} \left\{ h_{AB}^\alpha h_{CD}^\alpha - 3 \Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ} + \frac{1}{24} (h^\alpha \Gamma^{(4)})_{AB} (h^\alpha \Gamma^{(4)})_{CD} - \frac{1}{240} \Gamma_{AB}^{(6)} \Gamma_{CD}^{(6)} \right\}, \quad (\text{A.10}) \end{aligned}$$

where (A.3) was used (the extra factor of $\frac{1}{2}$ in front of the terms containing $\Gamma^{(6)}$ is due to the self-duality constraint, which was explained above). It is now straightforward to check that

$$\Gamma_{C[A}^{IJ} \Gamma_{B]D}^{IJ} + 2h_{C[A}^\alpha h_{B]D}^\alpha = -\frac{1}{2} (\Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ} + 2h_{AB}^\alpha h_{CD}^\alpha), \quad (\text{A.11})$$

so that (A.1) is satisfied with $A = 2$.

For $N = 2 + 8n$, we read off $d = 2^{1+4n}$ from table 1. There is now only one complex structure represented by the antisymmetric matrix h_{AB} , which generates the group $\text{U}(1)$ and is again normalized such that $(h)^2 = -2$. The antisymmetric

matrices are $h, \Gamma^{(2)}, h\Gamma^{(4)}, \dots, \Gamma^{(4n-2)}, h\Gamma^{(4n)}$, while the symmetric ones are $1, h\Gamma^{(2)}, \dots, h\Gamma^{(4n-2)}, \Gamma^{(4n)}$. One checks that altogether there are $\frac{1}{4}d^2$ antisymmetric and $\frac{1}{4}d^2$ symmetric matrices, so it would seem that we cannot generate a complete set of matrices in this way. However, we now recall that the representations are *complex* for these values of N (see the discussion in sect. 2), which means that, instead of getting d^2 real matrices, we should end up with $(d/2)^2$ complex (i.e. $(d/2)^2$ hermitean and $(d/2)^2$ antihermitean) matrices; this is precisely the number of matrices just obtained. Specializing to $N = 10$ with $d = 32$, the relevant identities read

$$\begin{aligned} & \Gamma_{C[A}^I \Gamma_{B]D}^J \\ &= \frac{1}{32} \left\{ \frac{1}{2} h_{AB} (\Gamma^{IJ} h \Gamma^{IJ})_{CD} + \frac{1}{2} \Gamma_{AB}^{(2)} (\Gamma^{IJ} \Gamma^{(2)} \Gamma^{IJ})_{CD} \right. \\ & \quad \left. + \frac{1}{2 \cdot 4!} (h \Gamma^{(4)})_{AB} (\Gamma^{IJ} h \Gamma^{(4)} \Gamma^{IJ})_{CD} \right\} \\ &= \frac{1}{32} \left\{ -45 h_{AB} h_{CD} - 13 \Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ} + \frac{1}{8} (h \Gamma^{(4)})_{AB} (h \Gamma^{(4)})_{CD} \right\}, \quad (\text{A.12}) \end{aligned}$$

and

$$\begin{aligned} h_{C[A} h_{B]D} &= \frac{1}{32} \left\{ \frac{1}{2} h_{AB} (h)_{CD}^3 + \frac{1}{2} \Gamma_{AB}^{(2)} (h \Gamma^{(2)} h)_{CD} + \frac{1}{2 \cdot 4!} (h \Gamma^{(4)})_{AB} (h h \Gamma^{(4)} h)_{CD} \right\} \\ &= \frac{1}{32} \left\{ -h_{AB} h_{CD} - \Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ} - \frac{1}{24} (h \Gamma^{(4)})_{AB} (h \Gamma^{(4)})_{CD} \right\}. \quad (\text{A.13}) \end{aligned}$$

Again, it is easy to check that

$$\Gamma_{C[A}^I \Gamma_{B]D}^J + 3 h_{C[A} h_{B]D} = -\frac{1}{2} (\Gamma_{AB}^{IJ} \Gamma_{CD}^{IJ} + 3 h_{AB} h_{CD}), \quad (\text{A.14})$$

so the identity (A.1) now holds with $A = 3$.

Finally, for $N = 9$, we have $d = 16$. As for $N = 16$, there are no complex structures; a complete and linearly independent set of real antisymmetric 16×16 matrices is given by the $\binom{9}{2}$ matrices $\Gamma^{(2)}$ and the $\binom{9}{6}$ matrices $\Gamma^{(6)}$. The relevant Fierz identity now reads

$$\Gamma_{C[A}^I \Gamma_{B]D}^J = \frac{1}{16} \frac{1}{2!} \Gamma_{AB}^{(2)} (\Gamma^{IJ} \Gamma^{(2)} \Gamma^{IJ})_{CD} + \frac{1}{16} \frac{1}{6!} \Gamma_{AB}^{(6)} (\Gamma^{IJ} \Gamma^{(6)} \Gamma^{IJ})_{CD}. \quad (\text{A.15})$$

From (A.4), we now get $\Gamma^{IJ} \Gamma^{(2)} \Gamma^{IJ} = -16 \Gamma^{(2)}$ and, by another fortunate numerical coincidence, $\Gamma^{IJ} \Gamma^{(6)} \Gamma^{IJ} = 0$. Except for the different range of indices, the resulting identity is the same as (A.7), so (A.1) is again obeyed with $A = 0$.

There is no need at this point to discuss other values of N , since we know from the classification of Lie algebras that, apart from G_2 , there are no other exceptional Lie algebras besides the ones considered above. We have given a pedestrian and rather explicit construction of these algebras, not least because, except for E_8 , the relevant Fierz identities do not seem to have been discussed anywhere in the literature. From the present point of view, there exist no exceptional Lie algebras beyond E_8 because the number of terms that must cancel after the Fierz rearrangements becomes too large, so that (A.1) can no longer be satisfied.

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