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Locating and Correcting Errors in Images

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Abstract

Most image interpolation or extrapolation algorithms assume that the locations of the unknown pixels are known. In this paper we attempt to remove this restriction. More precisely, we propose an algorithm for locating the incorrect pixels of an image, assuming only partial knowledge of its Fourier transform. Note that this is a nonlinear problem: the unknown quantities are the positions and values of the (say) n erroneous pixels. We show that the positions can be evaluated in $O(n^2)$ or even $O(n \log n)$ flops by solving a set of n linear equations and computing a FFT. The determination of n is part of the algorithm, whose stability is also briefly discussed. The values of the n incorrect pixels can then be estimated using any of the interpolation methods known.

1 Notation

The complex N-dimensional space, with the usual inner product and norm, is denoted by $\mathbb{C}^N$. The conjugate transpose is denoted by *. The Fourier matrix F is the unitary $N \times N$ matrix whose elements F_{ab} are given by

$$F_{ab} = \frac{1}{\sqrt{N}} e^{-\mathrm{j}\frac{2\pi}{N}ab},$$

where j denotes the imaginary unit. The discrete Fourier transform (DFT) of a signal or image x is denoted by $\hat{x}$.

2 Outline

Many of the known image interpolation and extrapolation methods have one common characteristic: the positions of the unknown pixels are assumed to be known. In practice, this is not always the case, a fact that renders these methods of limited practical usefulness under some circumstances.

In this paper we attempt to bridge this gap, and propose an algorithm for locating the positions of the incorrect pixels in an image, assuming only partial knowledge of its Fourier transform. Note that this is a nonlinear problem, in which the unknown quantities are the values and positions of the erroneous pixels.

We show that the positions of n incorrect pixels in a $N \times N$ image can be evaluated in $O(n^2)$ or $O(n \log n)$ time by solving a set of n equations and computing a FFT.

It is not necessary to know n beforehand: the computation of n is part of the algorithm. After locating the position of the incorrect pixels, it remains to interpolate them — any of the known interpolation methods can be in principle used for this purpose.

The stability of the algorithm, which naturally depends on the pattern of the incorrect pixels, is also briefly mentioned.

3 Related work

There is a vast literature on image interpolation, extrapolation, and related issues — they include implementation aspects, stability analysis, convergence acceleration, noniterative algorithms, the sampled analog of the problem, convergence criteria, the effect of stabilizing constraints, and much more. See [1–17], among many others.

The problem discussed in this paper is also related to error-control codes: it is known that oversampling [18] is an alternative to error-control coding. A discussion concerning the relations between these two and a few other relevant issues, in the context of signal and image reconstruction, can be found in [19].

4 Locating the incorrect pixels

This section describes a simple technique that enables the location of n incorrect pixels in any row or column of a $N \times N$ image (in the following we assume that we are dealing with a row). The constraint upon which the method depends is the vanishing of $2n$ elements of the DFT of the row. Given a unconstrained row of $N - 2n$ pixels this can readily be accomplished

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by computing its DFT, padding $2n$ zeros, and computing the inverse DFT.

The set

$$U = \{i_0, i_1, \ldots i_{n-1}\}$$

is used to describe the positions of the n incorrect pixels in a row x of a $N \times N$ image. We denote by y the observed row, and by e the error

$$e = x - y,$$

which coincides with x except for the pixels whose indexes belong to U. Thus, $e(k) = 0$ for all $k \notin U$. Typically, the cardinal of U is much less than N, that is, the density n/N of the incorrect pixels is low.

Consider the polynomial

$$P(z) = \sum_{i=0}^{n} h_i z^i,$$

defined by

$$h_n = 1, \qquad P\left(e^{-\mathrm{j}\frac{2\pi}{N} i_m}\right) = 0, \quad (0 \le m < n).$$

Let $\{j_0, j_1, j_2, \ldots\}$ be a set of distinct integers. Substituting $z = e^{-\mathrm{j}\frac{2\pi}{N} i_m}$, multiplying by $e(i_m)e^{\mathrm{j}\frac{2\pi}{N} i_m j_\ell}$ and summing over m leads to

$$\sum_{k=0}^{n} h_k \, \hat{e}(k - j_\ell) = 0,$$

where $\hat{e} = Fe$ is the DFT of e. The idea is to use these equations to determine the n unknown coefficients h_k (recall that $h_n = 1$ by definition).

When $j_\ell = -r + \ell$ ($\ell = 0, 1, \ldots, n-1$), only $2n$ samples of $\hat{e}$ need to be known, and the linear equations will be Toeplitz. If the total number of pixels N of the band-limited row of the image is even, and if $r = N/2$, the matrix will be Toeplitz and Hermitian, because, for real e,

$$\hat{e}(r + i) = \hat{e}^*(-r - i) = \hat{e}^*(N - r - i) = \hat{e}^*(r - i).$$

The linear equations can be brought to the form $Ax = b$, with

$$A = \begin{bmatrix} \hat{e}(r) & \hat{e}(r-1) & \cdots \hat{e}(r-n+1) \\ \hat{e}(r+1) & \hat{e}(r) & \cdots \hat{e}(r-n+2) \\ \vdots & \vdots & \vdots \\ \hat{e}(r+n-1) \cdots & & \cdots \hat{e}(r) \end{bmatrix} \qquad (1)$$

and

$$b = [\hat{e}(r+1) \ \hat{e}(r+2) \ \cdots \ \hat{e}(r+n)]^T,$$

$$x = [h_{n-1} \ h_{n-2} \ \cdots \ h_0]^T.$$

See [20] for details.

In contrast to [18], the matrix is Toeplitz and more importantly, its principal submatrices do not depend on n, the number of incorrect pixels. Thus, Levinson's recursion [21] can be used to find the number of errors and simultaneously to solve the system. This costs at most $O(n^2)$ flops (but methods that cost $O(n \log n)$ also exist).

The zeros of the polynomial $P(z)$ yield the positions of the unknown pixels. The FFT provides a fast and stable way of evaluating the zeros of $P(z)$ (by construction, the zeros are multiples of $e^{-\mathrm{j}\frac{2\pi}{N}}$).

5 Applying the method

The method described in the previous section can be used to locate up to n errors in any row of the image, provided that its DFT contains $2n$ zeros. If there are more than n errors in a row the method will fail.

It is always possible to pack the image data in a vector (for example, by stacking columns) and then to apply the method to this vector (possibly by dividing it into blocks, and then dealing with each block separately). Although this is possible, it does not necessarily take advantage of the image structure and the natural band-limits that may exist.

Another possibility is to apply the method to the rows *and* to the columns of the image. In this case, the redundancy will be higher (the columns must also be band-limited, and consequently a $N \times N$ image will contain only $(N - 2n)(N - 2m)$ independent data).

A two-dimensional version of the method used in the previous section is as follows. Consider the product $P(z)Q(w)$, where $P(z)$ and $Q(w)$ are polynomials whose zeros mark the positions of the incorrect pixels, which are now described by a pair of integers (i_m, j_m). Thus,

$$P\left(e^{-\mathrm{j}\frac{2\pi}{N} i_m}\right) = Q\left(e^{-\mathrm{j}\frac{2\pi}{N} j_m}\right) = 0, \quad (0 \le m < n).$$

Using an argument similar to the one used in the one-dimensional case it is possible to show that, for any u, v belonging to a certain index set,

$$\sum_{p=0}^{n} \sum_{q=0}^{n} c_{pq} \, \hat{e}(p - u, q - v) = 0,$$

where $\hat{e}$ is the two-dimensional DFT of the error image, and the c_{pq} are the coefficients of the product $P(z)Q(w)$.

If sufficiently many samples of $\hat{e}$ are known these equations can be solved for c_{pq}, and the zeros of the polynomials found by computing a two-dimensional FFT. The zeros determine the positions of the incorrect pixels, which can then be interpolated using an appropriate algorithm.

6 Stability

The condition number of a matrix M with respect to a norm $\|\cdot\|$ is

$$\kappa(M) = \|M\| \cdot \|M^{-1}\|,$$

which, in the spectral norm, becomes

$$\kappa(M) = \frac{\lambda_{\max}(M)}{\lambda_{\min}(M)},$$

where $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ are the largest and smallest eigenvalue of M in absolute value (see [21], for example).

The eigenvalues of a Hermitian $n \times n$ matrix are real. We adopt the convention that they are labeled according to nondecreasing size:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

The stability of the equations $Ax = b$ depends on the $n \times n$ matrix A (1) whose elements are $A_{ab} = \hat{e}(r + a - b)$, with $r = N/2$. As we have seen, A is Toeplitz and Hermitian if the errors e are real.

There is one important characteristic of the problem: the matrix A can be positive definite, negative definite, or indefinite, depending on the sign of the $e(i_m)$ and the parity of i_m. It is easy to show, considering the quadratic form

$$v^* A v = \sum_{a=0}^{n-1} \sum_{b=0}^{n-1} A_{ab}\, v^*(a)\, v(b),$$

that A is positive definite if and only if $e(i_m)(-1)^{i_m} > 0$ for all $m = 0, 1, \ldots, n-1$, and negative definite if and only if $e(i_m)(-1)^{i_m} < 0$ for all $m = 0, 1, \ldots, n-1$.

Denote by W the diagonal $n \times n$ matrix whose main diagonal elements are

$$W_{kk} = \sqrt{N}\, e(i_k)(-1)^{i_k}. \qquad (2)$$

Without loss of generality, let W be nonsingular. Introduce the matrix E, also $n \times n$, and such that

$$E_{pq} = \frac{1}{\sqrt{N}} e^{-\mathrm{j}\frac{2\pi}{N} i_p q}.$$

The role played by these matrices is explained by the following observation: the matrix A is *-congruent to the diagonal matrix W, or, more precisely,

$$A = E^* W E.$$

This can be used to show that the eigenvalues of A and those of the diagonal matrix W satisfy

$$\lambda_{\min}(E^*E)\lambda_k(W) \leq \lambda_k(A) \leq \lambda_{\max}(E^*E)\lambda_k(W).$$

The extreme eigenvalues of W are of course

$$e_{\max} = \max_k |e(i_k)|, \quad e_{\min} = \min_k |e(i_k)|.$$

Then, the condition number of A, denoted by $\kappa(A)$, satisfies

$$\kappa(A) \leq \frac{e_{\max}}{e_{\min}} \kappa(E^*E).$$

The eigenvalues of EE^* (hence, E^*E) have been studied before because these matrices plays an important role in the band-limited interpolation problems in which the positions of the pixels to be interpolated are known). See, for example, [17, 22–24] and the review in [25].

The congruence between A and W reduces the study of the eigenvalues of A to those of EE^*, because W is diagonal. In one word, we succeeded in reducing the study of the stability problem to two separate, decoupled problems: the eigenvalues of EE^*, which depend upon the position of the incorrect pixels, but not on the amplitude of the errors; and those of W, that depend on the amplitude of the errors, but not on their positions.

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