AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 219–232

Locating-domination and identifying codes in trees*

Mostafa Blidia   Mustapha Chellali

Department of Mathematics
University of Blida, B.P. 270, Blida
Algeria
mblidia@hotmail.com   m_chellali@yahoo.com

Frédéric Maffray   Julien Moncel

C.N.R.S., Laboratoire G-SCOP
46 Avenue Felix Viallet, 38031 Grenoble, Cedex
France
frederic.maffray@g-scop.inpg.fr   julien.moncel@g-scop.inpg.fr

Ahmed Semri

USTHB, Laboratory LAID3, Department of Mathematics
BP 32, 16111 El Alia, Algiers
Algeria
ahmedsemri@yahoo.fr

Abstract

A set D of vertices in a graph G is (a) a dominating set if every vertex of $G \setminus D$ has a neighbor in D, (b) a locating-dominating set if for every two vertices u, v of $G \setminus D$ the sets $N(u) \cap D$ and $N(v) \cap D$ are non-empty and different, and (c) an identifying code if for every two vertices x, y of G the sets $N[x] \cap D$ and $N[y] \cap D$ are non-empty and different. The minimum cardinality of a dominating set, respectively, locating-dominating set, identifying code, is denoted by $\gamma(G)$, respectively, $\gamma_L(G)$, $M(G)$. We show that for a tree T with $n \geq 4$ vertices, ℓ leaves and s support vertices, $M(T) \geq 3(n + \ell - s + 1)/7$, and for a tree of order $n \geq 3$, $(n + \ell - s)/2 \geq \gamma_L(T) \geq (n + \ell - s + 1)/3$. Moreover we characterize the trees satisfying $\gamma_L(T) = (n + \ell - s)/2$, $M(T) = \gamma(T)$ or $\gamma_L(T) = \gamma(T)$.

* This research was supported by joint Algerian-French program CMEP 05 MDU 639.

1 Introduction

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$, and the *degree* of v is $\deg_G(v) = |N(v)|$. A set $D \subseteq V$ is a *dominating set* if every vertex of $V \setminus D$ has a neighbor in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set $D \subseteq V$ is a *locating-dominating set* if it is dominating and every two vertices x, y of $V \setminus D$ satisfy $N(x) \cap D \neq N(y) \cap D$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. Locating-domination was introduced by Slater [5, 6]. A set $D \subseteq V$ is an *identifying code* if D is dominating and every two vertices x, y of V satisfy $N[x] \cap D \neq N[y] \cap D$. The minimum cardinality of an identifying code is denoted by $M(G)$. Every graph $G = (V, E)$ admits a dominating set and a locating-dominating set, since V itself is such a set. In contrast, not every graph admits an identifying code; for if a graph G has two vertices x, y such that $N[x] = N[y]$ (*twins*), then it clearly cannot have an identifying code. On the other hand, if a graph does not have twins then V is an identifying code. Moreover, since every locating-dominating set is a dominating set, and an identifying code is a locating-dominating set, every graph G that admits an identifying code satisfies the inequality

$$\gamma(G) \leq \gamma_L(G) \leq M(G).$$

This chain inequality can be strict and the difference between any two of the numbers in the inequality can be arbitrarily large, even for trees. To see this, consider the graph T_k ($k \geq 3$) formed by a path P_k on k vertices and by k paths on nine vertices (P_9) by identifying the center vertex of each P_9 with a different vertex of the path P_k. Then it is easy to check that $M(T_k) = 5k$, $\gamma_L(G) = 4k$ and $\gamma(k) = 3k$.

A tree is a connected graph that contains no cycle. A vertex of degree one is called a *pendant vertex* or a *leaf*. Every tree with at least two vertices has a leaf. The neighbor of any leaf is called a *support vertex*, and every leaf adjacent to a support vertex x is called a *leaf of x*. Here we establish sharp bounds on $\gamma_L(T)$ and $M(T)$ for trees T, improving the bound due to Slater [5] (Theorem 2.5 below) and the bound due to Bertrand, Charon, Hudry and Lobstein [1] (Theorem 3.1 below). More precisely, we show that if T is a tree of order $n \geq 4$, with ℓ leaves and s support vertices, then $M(T) \geq 3(n + \ell - s + 1)/7$, and if T is a tree of order $n \geq 3$, then $(n + \ell - s)/2 \geq \gamma_L(T) \geq (n + \ell - s + 1)/3$. We also give a characterization of trees with $\gamma_L(T) = (n + \ell - s)/2$, $M(T) = \gamma(T)$ or $\gamma_L(T) = \gamma(T)$.

We finish this section with some terminology and notation. If v is a support vertex in a graph G and there are at least two leaves of v, then v is called a *strong* support. Let $L(G)$ and $S(G)$ respectively denote the sets of leaves and support vertices in G, and let $\ell(G) = |L(G)|$ and $s(G) = |S(G)|$. We may use ℓ and s if there is no ambiguity. Whenever a tree called T' (or T'', ...) is introduced, we let n', ℓ', s' (or n'', ℓ'', s'', ...) be its order, number of leaves, and number of support vertices respectively. A *star* is a tree with $\ell \geq n - 1$. A *double star* is a tree that contains exactly two vertices that are not leaves. A double star with respectively p and q

leaves attached to each support vertex is denoted by $S_{p,q}$. A *subdivided star* SS_q is the tree obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex.

The *corona* of a graph H is the graph obtained from H by adding a new vertex v' for each vertex $v \in V(H)$ and the edge $v'v$. Note that the corona of a tree is a tree.

The *distance* $\mathrm{dist}(x, y)$ between two vertices x, y in a graph G is the length of a shortest path from x to y. The *eccentricity* of a vertex u is the maximum of $\mathrm{dist}(u, v)$ over all v's. The *diameter* of a graph G is the maximum distance over all pairs of vertices of G.

2 Locating domination in trees

In this section we give a new upper bound and a new lower bound on $\gamma_L(T)$ when T is any tree. We start with a lemma that will be very useful.

Lemma 2.1 *In any tree T of order $n \geq 3$, there is a $\gamma_L(T)$-set X with the following properties:*

- *If x is a support vertex, and ℓ_x is the number of leaves of x, then X contains x and exactly $\ell_x - 1$ leaves of x;*
- *If a-b-c-d is a path where a is a leaf, b, c have degree two, and d is not a leaf, then X contains b, d and does not contain a, c.*

Proof. Let X be a $\gamma_L(T)$-set that contains as many support vertices as possible and as few leaves as possible. Let x be any support vertex and ℓ_x be the number of leaves of x. Suppose that x is not in X. Then all leaves of x are in X, for otherwise any leaf of x that is not in X would have no neighbor in X. Let y be one leaf of x. Then $(X \setminus \{y\}) \cup \{x\}$ is a $\gamma_L(T)$-set that contains more support vertices and fewer leaves than X, a contradiction. So $x \in X$. Next, suppose that two leaves y, z of x are not in X. Then y, z have the same neighbor in X, a contradiction. So at least $\ell_x - 1$ leaves of x are in X. Now, suppose that all leaves of x are in X. Let y be one leaf of x. Since X is a $\gamma_L(T)$-set, the set $X \setminus \{y\}$ is not a locating-dominating set, so there is a vertex u such that $u \notin X$ and the only neighbor of u in X is x. Note that u is not a leaf of x. Then the set $(X \setminus \{y\}) \cup \{u\}$ is a $\gamma_L(T)$-set with as many support vertices and fewer leaves than X, a contradiction. This proves the first item. Now consider the second item. By the first item, X contains b and not a. If X contains c, then $(X \setminus \{c\}) \cup \{d\}$ is a $\gamma_L(T)$-set, with as many support vertices and leaves as X, and with the property described in the second item. If X does not contain c, then it contains d, for otherwise a, c would have the same neighbor in X; and so X satisfies the desired property. Thus the lemma holds. ■

2.1 A new upper bound on γ_L for trees

Now we give an upper bound on the locating-domination number of trees.

Theorem 2.2 *If T is a tree of order $n \geq 2$ then $\gamma_L(T) \leq (n + \ell - s)/2$.*

Proof. If T has diameter one or two, then the result is easy to check (and we omit the details). So assume that T has diameter at least three. Let T^* be the tree obtained from T by removing all its leaves. So T^* has order at least 2 and admits a unique bipartition A, B into non-empty independent sets. Let C be a set of leaves of T chosen as follows: for every strong support vertex x in T, put in C all the leaves adjacent to x except one. Then $|C| = \ell - s$. Note that every leaf of T^* is a support vertex in T and every vertex of $T^* \setminus S(T)$ has degree at least two in T^*. Now the set $C \cup S(T) \cup A$ is a locating-dominating set of T since T is a tree and so no two vertices of $B \setminus S(T)$ have two common neighbors in A. Likewise $C \cup S(T) \cup B$ is a locating-dominating set of T. Therefore $\gamma_L(T) \leq \min\{|C \cup S(T) \cup A|, |C \cup S(T) \cup B|\} = |C| + |S(T)| + \min\{|A \setminus S(T)|, |B \setminus S(T)|\} \leq \ell - s + s + (n - \ell - s)/2 = (n + \ell - s)/2$. ■

For the purpose of characterizing the trees attaining the upper bound of Theorem 2.2, we define a family $\mathcal{F}$ of trees as follows. A tree T is in $\mathcal{F}$ if it can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees such that T_1 is a path $P_3 = x$-y-t or a path P_4, $T = T_k$, and, if $k \geq 2$ and $i < k$, T_{i+1} can be obtained from T_i by one of the operations defined below, where $D(T_1) = \{x, y\}$ if $T_1 = P_3$ and $D(T_1) = S(T_1)$ if $T_1 = P_4$.

- *Operation $\mathcal{F}_1$:* Add a single vertex w attached by an edge to any support vertex in T_i. Set $D(T_{i+1}) = D(T_i) \cup \{w\}$.
- *Operation $\mathcal{F}_2$:* Add a path $P_2 = u$-v attached by edge uz to any support vertex z in T_i. Set $D(T_{i+1}) = D(T_i) \cup \{u\}$.
- *Operation $\mathcal{F}_3$:* Add a subdivided star H of order at least five, with center vertex a, attached by an edge ab' to any leaf b' of a strong support vertex in T_i. Set $D(T_{i+1}) = D(T_i) \cup S(H)$.
- *Operation $\mathcal{F}_4$:* Add a $P_3 = b$-c-d and $p \geq 0$ paths $P_2 = u_j$-v_j attached by edges df and $u_j f$ for every j to any leaf f of $D(T_i)$ of a strong support vertex in T_i. Let $D(T_{i+1}) = D(T_i) \cup \{c, u_1, \ldots, u_p\}$.
- *Operation $\mathcal{F}_5$:* Add a $P_4 = a$-b-c-d and $p \geq 0$ paths $P_2 = u_j$-v_j attached by edges dy and $u_j d$ for every j to any vertex y of T_i that is not a support vertex and satisfies $\gamma_L(T_i \setminus y) = \gamma_L(T_i)$. Let $D(T_{i+1}) = D(T_i) \cup \{b, d, u_1, \ldots, u_p\}$. (Note that this operation cannot be performed on a leaf y of strong support vertex since $\gamma_L(T_i \setminus y) < \gamma_L(T_i)$.)

Lemma 2.3 *If $T \in \mathcal{F}$, then $\gamma_L(T) = (n + \ell - s)/2$.*

Proof. Let $T \in \mathcal{F}$. From the way T is constructed, it is easy to check that $D(T)$ is a locating-dominating set of T. Now we show that $D(T)$ is a $\gamma_L(T)$-set of size $(n+\ell-s)/2$. We proceed by induction on the total number of operations $\mathcal{F}_i$ performed to construct T. Clearly the property is true for $T_1 = P_3$ or P_4. Assume that the property is true for all trees of $\mathcal{F}$ constructed with $k - 1 \geq 0$ operations and let T be a tree of $\mathcal{F}$ constructed with k operations. Thus T is obtained by performing one of the five operations on a tree T' that can be obtained by $k - 1$ operations. We examine the corresponding five cases. Let X be a $\gamma_L(T)$-set. We can assume that X satisfies the properties described in Lemma 2.1.

Case 1: The operation performed on T' to obtain T is $\mathcal{F}_1$. So T is obtained from T' by attaching a single vertex w to a support vertex q in T'. So q is a strong support in T. By Lemma 2.1, X contains q and all its leaves except one, and we may assume that $w \in X$. Then $X \setminus \{w\}$ is a locating-dominating set of T', and so $\gamma_L(T') \leq \gamma_L(T) - 1$. Also since $D(T') \cup \{w\}$ is a locating-dominating set of T, we have $\gamma_L(T) \leq \gamma_L(T') + 1$, which implies equality. Then since $n = n' + 1$, $\ell = \ell' + 1$, and $s = s'$, it follows that $D(T)$ is a $\gamma_L(T)$-set of size $\gamma_L(T) = (n + \ell - s)/2$.

Case 2: The operation performed on T' to obtain T is $\mathcal{F}_2$. By Lemma 2.1, X contains u, z and not v. Thus $X \setminus \{u\}$ is a locating-dominating set of T' and $\gamma_L(T') \leq \gamma_L(T) - 1$. Then since $D(T') \cup \{u\}$ is a locating-dominating set of T, we have $\gamma_L(T) = \gamma_L(T') + 1$. Since $n = n' + 2$, $\ell = \ell' + 1$, and $s = s' + 1$, it follows that $D(T)$ is a $\gamma_L(T)$-set of size $\gamma_L(T) = (n + \ell - s)/2$.

Case 3: The operation performed on T' to obtain T is $\mathcal{F}_3$. Then $n = n' + 2|S(H)| + 1$, $\ell = \ell' + |S(H)| - 1$, and $s = s' + |S(H)|$. Clearly $D(T') \cup S(H)$ is a locating-dominating set of T, and so $\gamma_L(T) \leq \gamma_L(T') + |S(H)|$. By Lemma 2.1, we have $S(H) \subset X$ and $a \notin X$. Thus $X \setminus S(H)$ is a locating-dominating set of T' and $\gamma_L(T') \leq \gamma_L(T) - |S(H)|$. It follows that $\gamma_L(T) = \gamma_L(T') + |S(H)|$, and so $D(T)$ is a $\gamma_L(T)$-set of size $\gamma_L(T) = (n + \ell - s)/2$.

Case 4: The operation performed on T' to obtain T is $\mathcal{F}_4$. Then $n = n' + 3 + 2p$, $s = s' + 1 + p$, and $\ell = \ell' + p$. Clearly $D(T') \cup \{c, u_1, \ldots, u_p\}$ is a locating-dominating set of T, and so $\gamma_L(T) \leq \gamma_L(T') + 1 + p$. Equality follows from the fact that, by Lemma 2.1, there is a $\gamma_L(T')$-set that contains f, and such a set can be extended to a locating-dominating set of T by adding $\{c, u_1, \ldots, u_p\}$. Thus $\gamma_L(T) = \gamma_L(T') + 1 + p$ and $D(T)$ is a $\gamma_L(T)$-set of size $\gamma_L(T) = (n + \ell - s)/2$.

Case 5: The operation performed on T' to obtain T is $\mathcal{F}_5$. Then $n = n' + 4 + 2p$, and either $s = s' + 1 + p$, $\ell = \ell' + 1 + p$ (if y is not a leaf) or $s = s' + p$, $\ell = \ell' + p$ (if y is a leaf in T'). Then $D(T') \cup \{b, d, u_1, \ldots, u_p\}$ is a locating-dominating set of T, and so $\gamma_L(T) \leq \gamma_L(T') + 2 + p$. On the other hand, by Lemma 2.1, there exists a $\gamma_L(T)$-set X that contains $\{b, d, u_1, \ldots, u_p\}$. If $y \in X$, then $X \cap T'$ is a locating-dominating set of T' and so $\gamma_L(T') \leq \gamma_L(T) - 2 - p$. If $y \notin X$, then $X \cap T'$ is a locating-dominating set of $T' \setminus y$ and so $\gamma_L(T') = \gamma_L(T' \setminus y) \leq \gamma_L(T) - 2 - p$. In either case we obtain $\gamma_L(T) = \gamma_L(T') + 2 + p$, and so $D(T)$ is a $\gamma_L(T)$-set of size $\gamma_L(T) = (n + \ell - s)/2$. ■

We are now ready to characterize the trees T with $\gamma_L(T) = (n + \ell - s)/2$.

Theorem 2.4 *Let T be a tree with $n \geq 2$. Then $\gamma_L(T) = (n + \ell - s)/2$ if and only if $T \in \mathcal{F}$.*

Proof. If $T \in \mathcal{F}$, then, by Lemma 2.3, $D(T)$ is a $\gamma_L(T)$-set of size $(n+\ell-s)/2$. To prove the converse, we proceed by induction on the order of T. Let T be a tree of order n with $\gamma_L(T) = (n + \ell - s)/2$ and assume that every tree T' of order $n' < n$ with $\gamma_L(T') = (n' + \ell' - s')/2$ is in $\mathcal{F}$. If T has diameter 1, then T is a P_2, and clearly $\gamma_L(T) = 1 < (n + \ell - s)/2$. If T has diameter 2, then T is a star $K_{1,t}$ $(t \geq 2)$ and $\gamma_L(T) = (n + \ell - s)/2 = \ell$. Thus T is in $\mathcal{F}$ because it is obtained from $T_1 = P_3$ by using $t - 2$ operations $\mathcal{F}_1$. If T has diameter 3, then T is a double star $S_{p,q}$ and $\gamma_L(T) = (n + \ell - s)/2 = p + q$. Thus $T \in \mathcal{F}$ since it is obtained from $T_1 = P_4$ by using zero or more operations $\mathcal{F}_1$. Now we may assume that T has diameter at least 4. Let X be a $\gamma_L(T)$-set. We may assume that X satisfies the properties described in Lemma 2.1.

First suppose that T has a strong support vertex x. By Lemma 2.1, X contains x and all leaves of x except for one leaf x'' of x. Let x' be another leaf of x and let $T' = T \setminus x'$. Since $N(x'') \cap X = \{x\}$, every non-leaf neighbor of x not contained in X must have at least another neighbor in X, and so $X \setminus \{x'\}$ is a locating-dominating set of T'. Thus $\gamma_L(T') \leq \gamma_L(T) - 1$. Equality results from the fact that every $\gamma_L(T')$-set can be extended to a locating-dominating set of T by adding x'. Then since $n = n' + 1$, $s = s'$, and $\ell = \ell' - 1$, we obtain $\gamma_L(T') = (n' + \ell' - s')/2$. By the induction hypothesis on T', we have $T' \in \mathcal{F}$. Thus T is in $\mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{F}_1$. So we may now assume that there is no strong support vertex in T.

Recall the tree T^* and sets A, B, C defined in Theorem 2.2. Since there is no strong support vertex, we have $C = \emptyset$. By the proof of Theorem 2.2, we know that $S(T) \cup A$ and $S(T) \cup B$ are two locating-dominating sets of T where $\min\{|A \setminus S(T)|, |B \setminus S(T)|\} \leq (n - \ell - s)/2$; and since $\gamma_L(T) = (n + \ell - s)/2$, it must be that each of $S(T) \cup A$ and $S(T) \cup B$ is a $\gamma_L(T)$-set.

Let r, u' be two vertices of T at maximum distance (equal to the diameter of T). Root T at r, and in the rooted tree let u, v, z, y be the parents of u', u, v, z respectively. We distinguish between three cases.

Case 1: v is a support vertex. Let $T' = T \setminus \{u, u'\}$. By Lemma 2.1, X contains u, v and not u'. Then $X \setminus \{u\}$ is a locating-dominating set of T' and $\gamma_L(T') \leq \gamma_L(T) - 1$. Moreover, every $\gamma_L(T')$-set can be extended to a locating-dominating set of T by adding u. Thus $\gamma_L(T') = \gamma_L(T) - 1$. Since $n = n' + 2$, $s = s' + 1$, $\ell = \ell' + 1$, it follows that $\gamma_L(T') = (n' + \ell' - s')/2$. By the induction hypothesis, we have $T' \in \mathcal{F}$, and so $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{F}_2$.

Case 2: v is not a support vertex and has at least two children. Thus T_v is a subdivided star of order at least five. Up to symmetry we may assume that z is in A, and so v is in B (since v is not a leaf). Recall that $S(T) \cup B$ is a $\gamma_L(T)$-set. Therefore the set $Y = S(T) \cup B \setminus \{v\}$ is not a locating-dominating set. Since v itself has two neighbors in Y, it must be that z is not in Y and either (i) z has no neighbor in Y or (ii) there is a vertex $z' \notin Y$, $z' \neq z$, such that $N(z) \cap Y = N(z') \cap Y$. Since z is not in Y, it is not a support vertex, so y is not a leaf, and so $y \in B \subseteq Y$. Thus we are not in case (i), and we are in case (ii) with $N(z) \cap Y = N(z') \cap Y = \{y\}$. If z has a child $v' \neq v$, then by the same argument we have $v' \in B \subseteq Y$, so v', y are two neigbhors of z in Y, a contradiction. It follows that the only child of z is v. If z' is not a leaf in T, then either $z' \in S(T) \subseteq Y$, a contradiction, or $z' \in A \setminus S(T)$ and (as in the proof of Theorem 2.2) z' has two neighbors in B (necessarily in $B \setminus \{v\}$), again a contradiction. So z' is a leaf and consequently y is a support vertex in T. Now let $T' = T \setminus T_v$. Then z, z' are leaves of y in T', so y is a strong support vertex in T'. It can be seen that $\gamma_L(T') = \gamma_L(T) - (\deg_T(v) - 1)$, and since $n = n' + 2\deg_T(v) - 1$, $s = s' + \deg_T(v) - 1$, and $\ell = \ell' + \deg_T(v) - 2$, we obtain $\gamma_L(T') = (n' + \ell' - s')/2$. By the induction hypothesis on T', we have $T' \in \mathcal{F}$. Thus $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{F}_3$.

Case 3: v is not a support vertex and has only one child. Seeing the previous cases we may assume that every descendant of z has degree one or two. By the second item of Lemma 2.1, X contains u, z and not u', v. Let $T'' = T \setminus \{u', u, v\}$. Suppose that either z is a support vertex or there is a path z-c-b-a in T_z. In either case, by Lemma 2.1, there is a $\gamma_L(T'')$-set containing z. Such a set can be extended to a locating-dominating set of T by adding u, and so $\gamma_L(T) \leq \gamma_L(T'') + 1$. Since $n = n'' + 3$, $s = s'' + 1$ and $\ell = \ell'' + 1$, it follows that $(n + \ell - s)/2 = \gamma_L(T) \leq (n'' + \ell'' - s'')/2 + 1 = (n - 1 + \ell - s)/2$, a contradiction. Therefore every child of z (if any) other than v is a support vertex. Call Z the set of support vertices adjacent to z in T_z (if any), and put $|Z| = p$. We consider two subcases:

Subcase 3.1: y is a support vertex in T. Let $T' = (T \setminus T_z) \cup \{z\}$. Thus y is a strong support vertex in T' and $n' \geq 3$. Since $z \in X$, the set $X \setminus (\{u\} \cup Z)$ is a locating-dominating set of T' and so $\gamma_L(T') \leq \gamma_L(T) - 1 - p$. By Lemma 2.1, there exists a $\gamma_L(T')$-set that contains z. Such a set can be extended to a locating-dominating set of T by adding $Z \cup \{u\}$, and so $\gamma_L(T) \leq \gamma_L(T') + 1 + p$. Thus $\gamma_L(T') = \gamma_L(T) - 1 - p$. Clearly $n' = n - 3 - 2p$, $s' = s - 1 - p$ and $\ell' = \ell - p$, implying that $\gamma_L(T') = (n' + \ell' - s')/2$, where $X \setminus (\{u\} \cup Z)$ is a $\gamma_L(T')$-set that contains z. Now by induction on T', we have $T' \in \mathcal{F}$. Thus T is in $\mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{F}_4$.

Subcase 3.2: y is not a support vertex in T. Suppose that y has degree 2 and its parent is a support vertex, and let $T' = T \setminus T_y$. Then $n' \geq 2$. If $n' = 2$ then $\gamma_L(T) \neq (n + \ell - s)/2$ since $\ell = s = p + 2$ and n is odd, a contradiction. So $n' \geq 3$. Then every $\gamma_L(T')$-set can be extended to a locating-dominating set of T by adding $Z \cup \{u, z\}$, and so $\gamma_L(T) \leq \gamma_L(T') + 2 + p$. Since $n' = n - 5 - 2p$, $s' = s - 1 - p$ and $\ell' = \ell - 1 - p$, it follows that $(n + \ell - s)/2 = \gamma_L(T) \leq (n' + \ell' - s')/2 + 2 + p = (n - 1 + \ell - s)/2$, a contradiction. Thus either $\deg_T(y) \geq 3$ or $\deg_T(y) = 2$ and the parent of y is not a support vertex. Let $T' = T \setminus T_z$. Clearly $n' = n - 4 - 2p \geq 3$, and either $s' = s - 1 - p$ and $\ell' = \ell - 1 - p$, or $s' = s - p$ and $\ell' = \ell - p$, depending on whether y has degree at least three or has degree two, respectively. Now since every

$\gamma_L(T')$-set can be extended to a locating-dominating set of T by adding $Z \cup \{u, z\}$, we obtain:

$$(n+\ell-s)/2 = \gamma_L(T) \leq \gamma_L(T')+2+p \leq (n'+\ell'-s')/2+2+p = (n+\ell-s)/2. \quad (1)$$

It follows that $\gamma_L(T') = (n'+\ell'-s')/2$ and, by induction, $T' \in \mathcal{F}$. Let us prove that $\gamma_L(T' \setminus y) = \gamma_L(T')$. Since $\gamma_L(T') = (n'+\ell'-s')/2$ then, as remarked before (see Theorem 2.2), T' admits two $\gamma_L(T')$-sets, namely $Q_1 = C(T') \cup S(T') \cup A(T')$ and $Q_2 = C(T') \cup S(T') \cup B(T')$. If y is a leaf in T', then $y \notin C(T')$ since its parent is not a strong support vertex which belongs to Q_1, and so Q_1 is a locating-dominating set of $T' \setminus y$. Now if $\deg_T(y) \geq 3$, then without loss of generality $y \in B'(T')$ and so Q_1 is a locating-dominating set of $T' \setminus y$. In each case, we have $\gamma_L(T' \setminus y) \leq |Q_1| = \gamma_L(T')$. Assume now that $\gamma_L(T' \setminus y) < \gamma_L(T')$. Then every $\gamma_L(T' \setminus y)$-set can be extended to a locating-dominating set of T by adding $Z \cup \{u, z\}$, where y is locating dominated by z. So we have $\gamma_L(T) \leq \gamma_L(T' \setminus y) + 2 + p < \gamma_L(T') + 2 + p$, a contradiction to (1). Thus $\gamma_L(T' \setminus y) = \gamma_L(T')$, and $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{F}_5$. This completes the proof. ■

2.2 A new lower bound on γ_L for trees

The following lower bound on the locating domination number in trees is due to Slater [5].

Theorem 2.5 (Slater [5]) *For every tree T, $\gamma_L(T) > n/3$.*

Here we improve the lower bound of Theorem 2.5.

Theorem 2.6 *If T is a tree with order $n \geq 3$, then $\gamma_L(T) \geq (n+\ell-s+1)/3$ and this bound is sharp.*

Proof. If there is no strong support vertex in T, then $\ell = s$ and by Theorem 2.5, the result is valid. Thus assume that T contains at least one strong support vertex. Clearly the result holds for stars of order at least three. Let T be the smallest tree that does not satisfy the theorem, that is $\gamma_L(T) < (n+\ell-s+1)/3$, and let S be a $\gamma_L(T)$-set and u a strong support vertex. Let T' be the tree obtained from T by removing a leaf u' adjacent to u. Then without loss of generality, S contains u and all its leaves except one leaf $u'' \neq u'$. Then $S \setminus \{u'\}$ is a locating-dominating set of T', and so $\gamma_L(T') \leq \gamma_L(T) - 1$. Since T' has order less that T, then $\gamma_L(T') \geq (n'+\ell'-s'+1)/3$. Also since $n' = n-1$, $\ell' = \ell - 1$, and $s' = s$, we obtain:

$$\gamma_L(T) \geq \gamma_L(T') + 1 \geq (n'+\ell'-s'+1)/3 + 1 \geq (n+\ell-s+2)/3,$$

contradicting our assumption.

The fact that this bound is sharp was already proved in [5] with trees that satisfy $s = \ell$. Another example consists of the caterpillar (tree such that the removal of the leaves produces a path) with $\ell = s = k \geq 2$ and where the distance between any two consecutive support vertices is two. (The path P_5 is the smallest such tree). Then T has order $n = 2k + k - 1$, and $\gamma_L(T) = k = (n + \ell - s + 1)/3$. This completes the proof. ■

2.3 Trees T with $\gamma_L(T) = \gamma(T)$

In order to characterize the trees T that satisfy $\gamma_L(T) = \gamma(T)$, we define a family $\mathcal{T}$ of trees as follows. A tree T is in $\mathcal{T}$ if it can be obtained through a sequence T_1, T_2, …, T_k ($k \geq 1$) of trees such that T_1 is the path $P_2 = xy$, $T = T_k$, and, if $k \geq 2$, T_{i+1} can be obtained from T_i by one of the operations defined below. Let one the vertices of T_1 be considered a support, say x and the other a leaf, say y.

- *Operation* $\mathcal{T}_1$*:* Pick one support vertex z in T_i and add a path u-v attached by an edge uz.
- *Operation* $\mathcal{T}_2$*:* Pick one arbitrary vertex b of T_i and add a subdivided star $H = SS_p$, $p \geq 2$, with center vertex a attached by an edge ab.
- *Operation* $\mathcal{T}_3$*:* Pick one vertex c of T_i that belongs to a $\gamma_L(T_i)$-set and add a path u-v-w attached by an edge uc.

Lemma 2.7 *If T is a tree with $\gamma_L(T) = \gamma(T)$, then T has no strong support vertex.*

Proof. For suppose that x is a strong support, and let y, z be two leaves adjacent to x. Let X be a $\gamma_L(T)$-set. By Lemma 2.1, we may assume that X contains x and y. But then $X \setminus \{y\}$ is dominating set that is strictly smaller than X, so T has a dominating set of cardinality $\gamma(T) - 1$, a contradiction. Thus the lemma holds. ■

We now are ready to characterize the trees T with $\gamma_L(T) = \gamma(T)$.

Theorem 2.8 *Let T be a tree with $n \geq 2$. Then $\gamma_L(T) = \gamma(T)$ if and only if $T \in \mathcal{T}$.*

Proof. First suppose that T is a tree in $\mathcal{T}$. We prove that $\gamma_L(T) = \gamma(T)$ by induction on the number of operations performed to construct T. Clearly the property is true for $T_1 = P_2$. Assume that the property is true for all trees of $\mathcal{T}$ constructed with $k - 1 \geq 0$ operations and let T be a tree of $\mathcal{T}$ constructed with k operations. Thus T is obtained by performing one operation of $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ on a tree T' obtained by $k - 1$ operations. Let X be a $\gamma_L(T)$-set of T. By Lemma 2.1, we may assume that X contains all support vertices.

Suppose that T is obtained from T' by operation $\mathcal{T}_1$. Clearly $\gamma(T) = \gamma(T') + 1$. Every $\gamma_L(T')$-set of T' can be extended to a locating-dominating set of T by adding u, and so $\gamma_L(T) \leq \gamma_L(T') + 1$. By Lemma 2.1, we have $u, z \in X$, and every neighbor of z besides its leaf is either in X or adjacent to another vertex of X. Thus $X \setminus \{u\}$

is a locating-dominating set of T' and so $\gamma_L(T') \leq \gamma_L(T)-1$. It follows that $\gamma_L(T) = \gamma_L(T')+1$ and, by induction on T', $\gamma_L(T') = \gamma(T')$, hence $\gamma_L(T) = \gamma(T)$.

Suppose that T is obtained from T' by operation $\mathcal{T}_2$. Then it can be seen easily that $\gamma(T) = \gamma(T') + p$ and $\gamma_L(T) = \gamma_L(T') + p$. By induction on T' we obtain $\gamma_L(T) = \gamma(T)$.

Suppose that T is obtained from T' by operation $\mathcal{T}_3$. Then $\gamma(T) = \gamma(T')+1$ and $\gamma_L(T) \leq \gamma_L(T')+1$ because every $\gamma_L(T')$-set that contains c can be extended to a locating-dominating set of T by adding v. By Lemma 2.1, X contains v and not u. Thus $\gamma_L(T') \leq \gamma_L(T) - 1$ and the equality is obtained. By induction on T', we have $\gamma_L(T') = \gamma(T')$, hence $\gamma_L(T) = \gamma(T)$.

To prove the converse, we proceed by induction on the order of T. Let T be a tree of order n with $\gamma_L(T) = \gamma(T)$, let D be a $\gamma_L(T)$-set, and assume that every tree T' of order $n' < n$ with $\gamma_L(T') = \gamma(T')$ is in $\mathcal{T}$. By Lemma 2.7, T has no strong support vertex. Thus, by Lemma 2.1, we may assume that

$$D \text{ contains every support vertex and no leaf.} \tag{2}$$

If T has diameter 1, then $T = P_2$ and $T \in \mathcal{T}$. By Lemma 2.7, T cannot have diameter 2. If T has diameter 3, then, by Lemma 2.7, T is a P_4, which is a member of $\mathcal{T}$ (a P_4 can be obtained from T_1 by Operation $\mathcal{T}_1$). Now we may assume that T has diameter at least 4. Let r, u' be two vertices of T at maximum distance (thus equal to the diameter of T). Root T at r and in the rooted tree let u, v, z be the parents of u', u, v respectively. We distinguish between three cases.

Case 1: v is a support vertex. Let $T' = T \setminus \{u, u'\}$. Then $\gamma(T) = \gamma(T') + 1$ and $\gamma_L(T) \leq \gamma_L(T') + 1$. We have $u, v \in D$, and so $D \setminus \{u\}$ is a locating-dominating set of T' because every vertex adjacent to v besides its leaf is either in D or adjacent to at least two vertices of D. Thus $\gamma_L(T') \leq \gamma_L(T) - 1$, which implies that $\gamma_L(T) = \gamma_L(T') + 1$. It follows that $\gamma_L(T') = \gamma(T')$ and by induction on T', we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ since it is obtained from T' by using Operation $\mathcal{T}_1$.

Case 2: v is not a support vertex and has at least two children. Let T_v be the subtree of T rooted at v. Note that every child of v is a support vertex, and so T_v is a subdivided star. Let $T' = T \setminus T_v$. Then T' is nontrivial since T has diameter at least 4. Moreover, it is easy to see that $\gamma(T) = \gamma(T') + \deg_T(v) - 1$ and $\gamma_L(T) = \gamma_L(T') + \deg_T(v) - 1$. It follows that $\gamma_L(T') = \gamma(T')$ and, by the induction hypothesis on T', we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ since it is obtained from T' by using Operation $\mathcal{T}_2$.

Case 3: v is not a support vertex and has only one child. So $\deg_T(v) = 2$. Let $T' = T \setminus \{u', u, v\}$. Then $\gamma(T) = \gamma(T') + 1$ and $\gamma_L(T') \leq \gamma_L(T) - 1$ since $u \in D$ and $v \notin D$ (else replace v by z) and $z \in D$. Suppose that $\gamma_L(T') < \gamma_L(T) - 1$, then $\gamma_L(T') < (\gamma(T')+1) - 1$, a contradiction. Thus $\gamma_L(T') = \gamma_L(T) - 1$, and $D \setminus \{u\}$ is a $\gamma_L(T')$-set containing z. By induction on T', we have $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$ because it is obtained from T' by using Operation $\mathcal{T}_3$. ∎

3 Identifying codes in trees

3.1 A new lower bound on M for trees

In [1], Bertrand, Charon, Hudry and Lobstein established the following lower bound on the minimum cardinality of an identifying code in trees.

Theorem 3.1 (Bertrand, Charon, Hudry, Lobstein [1]) *If T is a tree of order $n \geq 3$, then $M(T) \geq 3(n+1)/7$, and this bound is sharp for infinitely many values of n.*

Our next result improves the lower bound of Theorem 3.1.

Theorem 3.2 *If T is a tree of order $n \geq 4$, then $M(T) \geq 3(n+\ell-s+1)/7$, and this bound is sharp for infinitely many values of n.*

Proof. We prove the theorem by induction on n. The result holds clearly when $n = 4$. Now let T be a tree with $n \geq 5$ vertices. If $\ell = 2$, then T is a path P_n with $n \geq 4$ and $s = 2$, and by Theorem 3.1 the result is valid. Thus we may assume that $\ell \geq 3$. Choose an $M(T)$-set D that contains as few leaves of T as possible.

For each support vertex x, let L_x be the set that consists of x and all its leaves. We observe that at most one vertex of L_x is not in D; for otherwise, either x and some leaf y of x are not in D, and then $N[y] \cap D = \emptyset$, or two leaves y, z of x are not in D, and then $N[y] \cap D = N[z] \cap D$, in either case a contradiction to the fact that D is an identifying code. Moreover, the sets L_x are disjoint for different x's in $S(T)$. Thus D contains at least ℓ vertices of $L(T) \cup S(T)$. Consider the set $W = V \setminus (D \cup L(T) \cup S(T))$. If $W = \emptyset$, then $V \setminus (S(T) \cup L(T)) \subset D$, and so, by the preceding observation, we have $|D| \geq n-(\ell+s)+\ell = n-s$. We obtain $|D| \geq n-s \geq 3(n+\ell-s+1)/7$ as desired (where the last inequality follows from $\ell \geq 3$). Now let $W \neq \emptyset$, and pick any $w \in W$.

For every neighbor z of w, let T_z be the component of $T \setminus zw$ (the graph obtained from T by removing the edge zw) that contains z. Let n_z, ℓ_z, s_z denote respectively the order, number of leaves and number of support vertices in the tree T_z. If $n_z = 1$ then w is a support vertex in T, a contradiction. If $n_z = 2$, then, since $w \notin D$, the two vertices z, z' satisfy $N[z] \cap D = N[z'] \cap D$, a contradiction. So $n_z \geq 3$. Suppose that $n_z = 3$. Then T_z is a path with vertex set $\{z, z', z''\}$. If $\{z, z', z''\} \subset D$, then one of z' or z'' is a leaf, say z', but then $\{w\} \cup D \setminus \{z'\}$ is an $M(T)$-set that contains fewer leaves than D, a contradiction. Thus D contains at most two vertices of $\{z, z', z''\}$. Actually D must contain exactly two non-adjacent vertices of $\{z, z', z''\}$, for otherwise either some vertex x of T_z satisfies $N[x] \cap D = \emptyset$ or some two vertices x, y of T_z satisfy $N[x] \cap D = N[y] \cap D$. Thus, if $n_z = 3$, then either T is a path z-z'-z'' with $z, z'' \in D$ and $z' \notin D$, or T is a path z'-z-z'' with $z', z'' \in D$ and $z \notin D$.

Now suppose that w has a neighbor v in $V \setminus D$. Let T_w be the component of $T \setminus vw$ that contains w, and let n_w, ℓ_w, s_w denote respectively the order, number of leaves and number of support vertices in T_w. If $n_w \le 3$, then either w is a leaf or a support vertex in T, a contradiction, or $n_w = 3$ and T_w is a path w''-w'-w, and then $N[w''] \cap D = N[w'] \cap D$, again a contradiction. So $n_w \ge 4$. Since $v \notin D$, the set $D_w = D \cap T_w$ is an identifying code of T_w, so, by the induction hypothesis, we have $|D_w| \ge 3(n_w + \ell_w - s_w + 1)/7$. Recall that $n_v \ge 3$. Also the set $D_v = D \cap T_v$ is an identifying code of T_v because $w \notin D$. If $n_v = 3$, then, by the preceding paragraph, T_v is a path v'-v-v'' with $v', v'' \in D$, so $|D_v| = 2$. Moreover, we have $n_w = n - 3$, $\ell_w \ge \ell - 2$ and $s_w = s - 1$. So $|D| = |D_w| + 2 \ge 3(n_w + \ell_w - s_w + 1)/7 + 2 \ge 3(n-3+\ell-2-s+1+1)/7+2 = 3(n+\ell-s+1)/7-12/7+2 \ge 3(n+\ell-s+1)/7$ as desired. If $n_v \ge 4$, then, by the induction hypothesis, we have $|D_v| \ge 3(n_v+\ell_v-s_v+1)/7$. So $|D| = |D_w|+|D_v| \ge 3(n_w+\ell_w-s_w+1)/7+3(n_v+\ell_v-s_v+1)/7$. Let $q = s_w+s_v-s$. Then we have $0 \le q \le 2$ and $\ell_w+\ell_v \ge \ell+q$. It follows that $|D| \ge 3(n+\ell-s+2)/7$.

Hence we may assume that all neighbors of w are in D. If $n_z = 3$ for every neighbor z of w, then, by the paragraph before the preceding one, T is a tree obtained from a star $K_{1,k}$ with center w by subdividing each edge twice. Then $n = 3k+1$, $\ell = s = k$ (so $k \ge 3$), and $|D| = 2k > 3(n+\ell-s+1)/7$ as desired. Therefore we may assume that $n_y \ge 4$ for some neighbor y of w. Suppose that $\deg_T(w) \ge 3$. Since w is not a support vertex, the component T_1 of $T \setminus yw$ that contains w has $n_1 \ge 5$ vertices. Since $\deg_T(w) \ge 3$, the set $D \cap T_1$ is an identifying code of T_1, and since $w \notin D$, the set $D \cap T_y$ is an identifying code of T_y. Then the rest of the proof is similar to the preceding paragraph. Finally we may assume that $\deg_T(w) = 2$ and call x, y the two neighbors of w, where $n_y \ge 4$. If $n_x = 3$, then T_x is a path x-x'-x'' with $x, x'' \in D$ and $x' \notin D$. Let $T' = T \setminus \{x', x''\}$. Then $D \cap T'$ is an identifying code of T and so, by the induction hypothesis, we have $|D| \ge |D \cap T'| + 1 \ge 3(n'+\ell'-s'+1)/7+1$. Since $n' = n-2$, $\ell' = \ell$ and $s' = s$, the result follows. Now assume $n_x \ge 4$. Then $D_x = D \cap T_x$ and $D_y = D \cap T_y$ are identifying codes for T_x and T_y respectively. Let $q = s_x + s_y - s$. Then $0 \le q \le 2$, $n_x + n_y = n - 1$ and $\ell_x + \ell_y \ge \ell + q$. Thus, by the induction hypothesis, we have $|D| = |D_x|+|D_y| \ge 3(n_x+\ell_x-s_x+1)/7+3(n_y+\ell_y-s_y+1)/7 \ge 3(n+\ell-s+1)/7$.

The fact that this bound is sharp was already proved in [1] with trees that satisfy $s = \ell$. Another example consists of the tree T_k ($k \ge 2$) obtained from k disjoint coronas of P_3 with centers c_i by adding $k-1$ new vertices x_i and the edges $x_i c_i$ and $x_i c_{i+1}$, $i = 1, \ldots, k-1$. Then in T_k we have $n = 7k-1$, $\ell = s = 3k$ and $M(T_k) = 3k = 3(n+\ell-s+1)/7$. (For instance, the set of support vertices is a minimum identifying codes of T_k). ■

3.2 Trees T with $M(T) = \gamma(T)$

Let us first recall that a set $A \subseteq V(G)$ is a *packing set* of G if $N[x] \cap N[y] = \emptyset$ holds for any two distinct vertices $x, y \in A$. It is well known (see [3]) that every graph G satisfies $|A| \le \gamma(G)$, for every packing set A of G.

Theorem 3.3 *Let G be a forest. Then $M(G) = \gamma(G)$ if and only if $V(G)$ can be partitioned into two sets C, F such that: every component of C is a corona of size* *at least 6, there is a set $I \subseteq F$ such that the closed neighbourhoods of the vertices of I are pairwise disjoint, there is no edge between I and C, every vertex of $N(I)$ has a neighbour in $S(C)$ and no neighbour among the leaves of C, every vertex of $F \setminus (I \cup N(I))$ has at least two neighbours in $S(C)$, and at most one neighbour among the leaves of C. Moreover, if G has this structure, then $I \cup S(C)$ is an identifying code of T of minimum size.*

Proof. Let $G = (V, E)$ be a forest such that $M(G) = \gamma(G)$, and let D be an identifying code of size $M(G)$. So D is also a minimum dominating set. For each vertex $v \in V$, put $D(v) = N[v] \cap D$. The fact that D is an identifying code means that there are no two vertices u, v such that $D(u) = D(v)$. Let I be the set of isolated vertices in the subgraph induced by D. Note that there is no component of size 2 in the subgraph induced by D, for otherwise the two vertices u, v of such a component would satisfy $D(u) = D(v) = \{u, v\}$. So every component of $D \setminus I$ has size at least three. Let X be any component of size at least 3 of D, and let x be any vertex of X. Then x has a private neighbour in $V \setminus D$ (i.e., a vertex of $N(x) \setminus N(D \setminus x)$), for otherwise, since x has a neighbour in X, the set $D \setminus x$ would be a dominating set smaller than D. Moreover x has exactly one private neighbour, for if it had two private neighbours x', x'', then we would have $D(x') = D(x'') = \{x\}$. So let us call x' the unique private neighbour of x in $V \setminus D$, for every vertex x of every component of size at least 3 of D. Let $C_1, \ldots, C_k$ be the components of size at least 3 of D, and let $C'_1, \ldots, C'_k$ be the corresponding sets of private neighbours. So, for each $i = 1, \ldots, k$, we have $|C'_i| = |C_i|$, and the set C'_i is a stable set, for otherwise $C_i \cup C'_i$ would contain a cycle. Also $C'_1 \cup \ldots \cup C'_k$ is a stable set for otherwise, up to symmetry, there would be vertices $x \in C_1$, $y \in C_2$ such that $x'y' \in E$, and then $(D \setminus \{x, y\}) \cup \{x'\}$ would be a dominating set smaller than D. So $C_1 \cup C'_1, \ldots, C_k \cup C'_k$ induce pairwise non-adjacent coronas or size at least 6. Put $R = (V \setminus D) \setminus (C'_1 \cup \cdots \cup C'_k)$, and consider any vertex r of R. Then r has at least two neighbours in D, for otherwise, if v is its unique neighbour in D, then, if $v \in D \setminus I$ we have $D(r) = D(v')$, and if $v \in I$ we have $D(r) = D(v)$, in either case a contradiction. It follows that no vertex of I has a private neighbour, and so every vertex of I has at least two neighbours in R. Then any $r \in R$ cannot have two neighbours $x, y \in I$, for otherwise $(D \setminus \{x, y\}) \cup \{r\}$ would be a dominating set smaller than D. Therefore the sets $N[x]$ ($x \in I$) induce pairwise disjoint stars of size at least 3, and for any $x \in I$, every neighbour of x has a neighbour in $C_1 \cup \cdots \cup C_k$, If some vertex u of R has at least two neighbours x', y' in C', then $(D \setminus \{x, y\}) \cup \{u\}$ is a dominating set smaller than D, a contradiction. If some vertex u of $N(I)$ has a neighbour x' in C', with z being the neighbour of u in I, then $(D \setminus \{x, z\}) \cup \{u\}$ is a dominating set smaller than D. So the sets $C, F (F = V \setminus C), I$ have the property described in the theorem.

Conversely, suppose that G has the structure described in the theorem. Then it is easy to see that $I \cup S(C)$ is an identifying code, so $\gamma(G) \leq M(G) \leq |I \cup S(C)|$. On the other hand, the set $(C \setminus S(C)) \cup I$ is a packing set of G, and so $\gamma(G) \geq |(C \setminus S(C)) \cup I|$. Since $|C \setminus S(C)| = |S(C)|$, we have equality throughout. ■

Comments

During the submission of this manuscript, the article [4] by Haynes, Henning and Howard appeared. These authors introduce *locating-total dominating sets* and *differentiating-total dominating sets*, which are defined respectively as locating dominating and identifying sets S with the additional property that the subgraph induced by the vertices of S does not contain isolated vertices. Since locating-total dominating sets are locating sets and differentiating-total dominating sets are identifying sets, we have $\gamma_t^L(G) \geq \gamma_L(G)$ and $\gamma_t^D(G) \geq M(G)$, where $\gamma_t^L(G)$ and $\gamma_t^D(G)$ denote the minimum cardinality of a locating-total dominating set and a differentiating-total dominating set of G respectively. Haynes, Henning and Howard prove that every non-trivial tree T satisfies $\gamma_t^L(T) \geq 2(n+1)/5$ and every tree T of order $n \geq 3$ satisfies $\gamma_t^D(T) \geq 3(n+1)/7$. Clearly, since $\gamma_t^D(G) \geq M(G)$, our Theorem 3.2 improves the lower bound on $\gamma_t^D(T)$ for trees T of order $n \geq 4$.

Finally, we note that the lower bound on $\gamma_t^L(T)$ has been improved recently by the second author in [2].

References

[1] N. Bertrand, I. Charon, O. Hudry and A. Lobstein, 1-identifying codes on trees, *Australas. J. Combinatorics* 31 (2005), 21–35.

[2] M. Chellali, On locating and differentiating-total domination in trees, (submitted).

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] T.W. Haynes, M.A. Henning and J. Howard, Locating and total dominating sets in trees, *Discrete Appl. Math.* 154 (2006), 1293–1300.

[5] P.J. Slater, Domination and location in acyclic graphs, *Networks* 17 (1987), 55–64.

[6] P.J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.* 22 (1988), 445–455.

(Received 10 Oct 2006)