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LOCATIONS OF MEDIANS
ON STOCHASTIC NETWORKS

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Abstract

The definition of network medians is extended to the case where travel times on network links are random variables with known discrete probability distributions. Under a particular set of assumptions, it is shown that the well-known theorems of HAKIMI and of LEVY can be extended to such stochastic networks. The concepts are further extended to the case of stochastic oriented networks. A particular set of applications as well as formulations of the problem for solution using mathematical programming techniques are also discussed briefly.

LOCATIONS OF MEDIANS ON STOCHASTIC NETWORKS

The problem of locating facilities on a network has attracted wide attention in recent years (see FRANCIS AND GOLDSTEIN [2]). In its most general form the problem can be described as follows: travel in a given area is restricted to take place solely along the links of a network; demands for services are generated at a finite number of points, designated as a subset of the node set of the network; the question of where to locate service facilities so as to serve these demands in an optimal way is then considered. The solution, naturally, depends on the optimality criterion at hand.

One type of optimality criterion is the minimization of average or total level distance (or travel time or travel cost) to and/or from a specified number, say k, of facilities. These location problems are known as the network median problems. The basic theoretical results in this area are due to HAKIMI [8]. Subsequently, GOLDMAN [7], HAKIMI and MAHESHWARI [9], LEVY [11], and WENDELL and HURTER [15] have extended and generalized HAKIMI's results.

In this paper these results will be extended to the case of networks in which the "lengths" of the network's links are random variables with known probability distributions. Median problems on such stochastic networks present several peculiarities which are not present when link lengths are deterministic. After a brief description of the notation used, these peculiarities are illustrated and discussed in a somewhat informal section intended to enhance understanding of the problem. The context within which this problem was examined is also described. A theorem and a corollary analogous to ones offered earlier by HAKIMI [8] and by LEVY [11] are then proven. We also introduce a number of definitions and extensions for the case of oriented networks (deterministic and stochastic). The paper concludes with a brief discussion of computational approaches to this type of problem.

1. "States" of the Network and Notation

The notational conventions used here are the following: We consider a network G with a node set V with n members, v_i , i = 1,2,... , n. Unless otherwise indicated G will be assumed nonoriented. Associated with each node v_i is a "demand weight" h_i. A link connecting nodes v_i and v_j is indicated by (i,j) and its length (or travel time) by t(i,j). We assume the existence of ℓ links on G. The shortest distance from a point $x \varepsilon G$ to a point $y \varepsilon G$ is denoted by d(x,y). (While t(i,j) = t(j,i), and d(x,y) = d(y,x) for nonoriented networks, the same generally is not true for oriented networks.) For convenience, we use d(i,y) for $d(v_i,y)$ and d(i,j) for $d(v_i,v_j)$. Finally, if X and Y denote two sets of points in G, then d(X,Y) denotes the minimum of the shortest distances d(x,y) where $x \varepsilon X$ and $y \varepsilon Y$, while d(z,Y) is the minimum of the shortest distances d(z,y) where z is a given point in G and $y \varepsilon Y$.

The link lengths t(i,j) are assumed to be random variables with known discrete probability distributions over a finite set of values, including possibly the value of infinity. Given this assumption, it can then be asserted that the network can have only a finite number of states. Each network state differs from the others by a change in the length of at least one link. Thus, the finest-grain sample space for the network consists of a listing of the mutually exclusive and collectively exhaustive set of all m possible network states, denoted by $G_1, G_2, \ldots, G_m$, with each network state G_r having a probability of occurence $P_r, r=1,2,\ldots,m$.

In general the number of states m will depend on the degree of statistical dependence among the random variables $t(i,j)$. In the extreme case where complete statistical independence prevails we have $m = \prod_{s=1}^{\ell} m_s$, where m_s is the number of values that travel time on link s can take. Similarly, the state probabilities, P_r, are assumed to be either known (in the case when the $t(i,j)$ are not mutually independent) or easily computable (when we have statistical independence). The implications of these assumptions on the computational characteristics of the median location problems will be discussed in a later section.

When the network is stochastic, $t_r(i,j)$ denotes the particular value that random variable $t(i,j)$ takes when the network is in state G_r. Similarly $d_r(x,y)$ denotes the shortest distance between points x and y when the network state is G_r.

To illustrate the above, we now define the <u>expected</u> <u>k-medians</u>, i.e., the stochastic network equivalent of HAKIMI's absolute k-medians.

<u>Definition 1</u>:

A set of the points X_k^* in a nonoriented stochastic network G is a set of expected k-medians of G if for every $X_k \varepsilon G$, $\bar{J}(X_k^*) \leq \bar{J}(X_k)$, where

$$\bar{J}(X_k) = \sum_{r=1}^{m} P_r \sum_{i=1}^{n} h_i d_r (i, X_k). \quad (1)$$

2. Assumptions, Observations and Context

The results of Sections 3 and 4 are derived under three fundamental assumptions. These assumptions can be stated informally as follows:

Assumption 1: When there are k facilities to be located on a network, G, those nodes with nonzero demand ($h_i > 0$) belong to at most a total of k components of G, under all states of G. (See the next paragraph for further explication.)

Assumption 2: The state of the network is known at all times. Moreover, the time intervals between instantaneous changes in the state of the network are much longer than the trip times on the network.

Assumption 3 (Homogeneity Assumption): The time required to travel a fraction Θ of the link (i,j) in the network state G_r is equal to $\Theta t_r(i,j)$ for all $r = 1,2,\ldots,m$.

Assumption 1 is necessary to make the concept of minimizing expected travel distance a meaningful one. If, for an obvious example, the nonzero demand nodes for a given state G_r of the network are partitioned into more than k components of G, then at least one travel distance between a facility and a demand point will be infinite (implying an infinite expected travel distance, as well). Note, however, that the condition imposed by Assumption 1 is stronger than simply requiring that for no state of G should the set of nonzero demand nodes be partitioned into more than k components. Instead, it is required that, after all the cuts (due to link failures) for all states m of G are superimposed, the resulting number of components containing one or more nonzero demand nodes must not exceed k. To indicate how strong Assumption 1 is, we note that for the single facility location problem it becomes:

Assumption 1(a): When there is a single facility to be located on a network, G, those nodes with nonzero demand ($h_i > 0$) are always connected (or strongly connected when oriented networks are considered).

Assumption 2 was motivated by the particular set of applications with which this research was concerned. It allows the choice of the shortest path available for each "trip". Moreover, it assumes that this shortest path will not change once a trip has been initiated.

As a consequence of Assumption 2, at least two characteristics of the search for optimal facility locations are peculiar to stochastic networks and should be commented on. First, a finest-grain sample space should be used due to the fact that the "minimum" operator is a nonlinear one and thus the shortest path between any two points may vary. Thus, for instance, the substitution of expected value, $\bar{t}(i,j)$, for the actual, finest-grain values, $t(i,j)$, of the random variables representing link lengths may lead to erroneous results. Unfortunately, this substitution is often used erroneously in practice.*

An example will illustrate this point. Consider the network of Figure 1. The lengths $t(1,3)$, $t(2,3)$, $t(2,4)$ and $t(3,4)$ are deterministic with values of 5,1,5 and 4 units respectively, while $t(1,2)$ takes on the value 1 with probability 0.6 and the value 16 with probability 0.4. Thus, $\bar{t}(1,2) = 7$.

Assuming for the moment that the solution to the optimal location problem is known to be at one of the nodes of the network, substitution of

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* It should be noted, however, that in the absence of any knowledge regarding the current state of the network, the use of $\bar{t}(i,j)$ as a deterministic substitute for the random variables $t(i,j)$ may be permissible since a traveller then would choose the expected shortest path whose length is determined from $\bar{t}(i,j)$ of the links of the path.

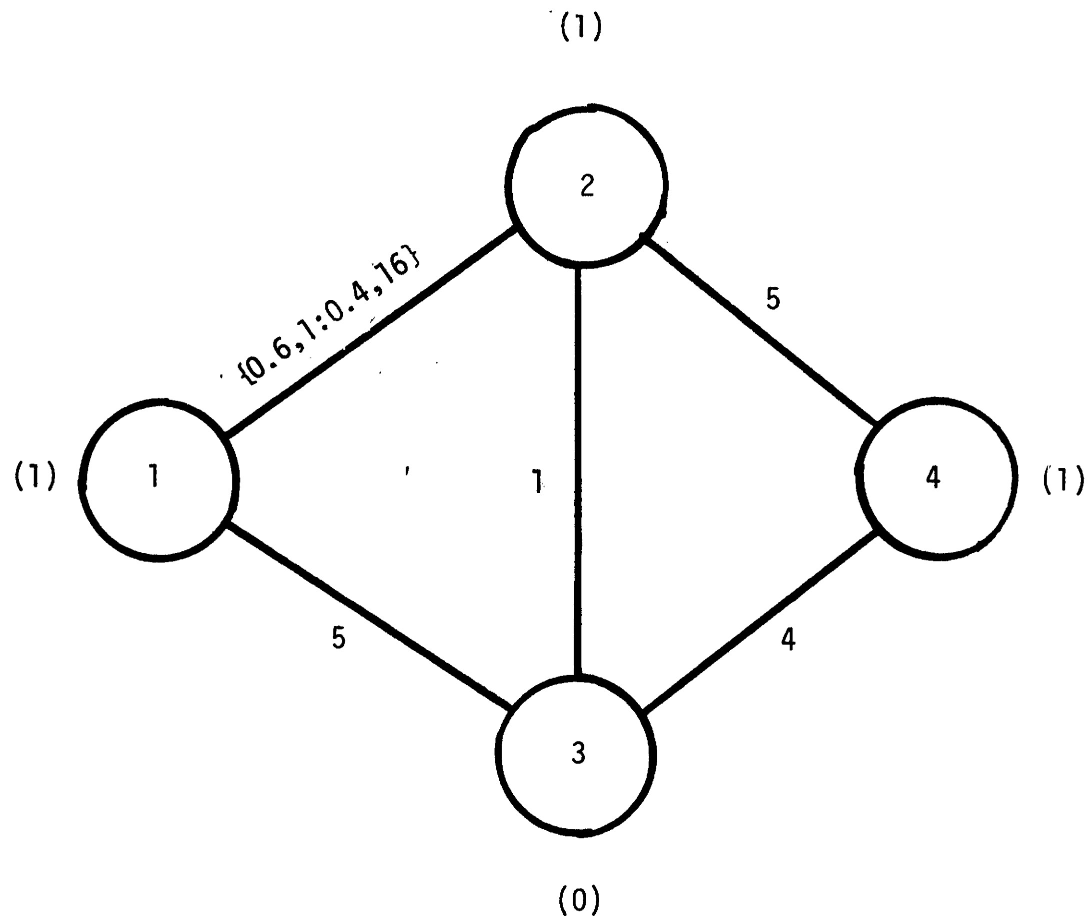

Figure 1: Example of a stochastic network. Link (1,2) has 0.6 probability of 1 unit in travel time and 0.4 probability of 16 units in travel time. The rest of the links have constant travel times. Nodes 1, 2, and 4 have demand weights of 1 unit each and node 3 of 0 units as indicated by the numbers in parentheses.

$\bar{t}(1,2)$ for the length $t(1,2)$ leads to the location of the facility at node v_3 with an expected travel distance of $3\frac{1}{3}$ units $(=\frac{1}{3}(5) + \frac{1}{3}(1) + \frac{1}{3}(4))$.

It can be seen, however, that the minimum distance from v_2 to v_1 is either 1 or 6 units (via node v_3 when $t(1,2) = 16$) with probabilities 0.6 and 0.4 respectively. The expected travel distance when the facility is located at v_2 then is equal to $2\frac{2}{3}$ units ($=0.6(\frac{1}{3} + \frac{5}{3}) + 0.4(\frac{6}{3} + \frac{5}{3})$) and v_2 is a better location than v_3. In fact, in this particular example, it turns out v_2 is the optimal location.

The second observation related to Assumption 2 is that, when there are two or more facilities on a stochastic network and assuming that any demand point will always be served by its nearest facility at the moment the demand is generated, the facility serving a particular demand point will depend on the state of the network. This point is quite obvious and it is due to the change of shortest paths with the state of the network.

Assumption 3 implies that the speed of travel on any given link is uniform - a straightforward and reasonable assumption since the network can easily be defined or constructed such that this assumption holds.

This work was performed as part of an extensive research project concerned with planning for urban emergency service systems. Thus, the context within which the problem was viewed is that of vehicles moving on a grid of metropolitan area or regional highways. It is possible to identify at least three factors that contribute to random variations in travel times in such networks: (i) the random fluctuations in traffic density and the attendant variations in travel speeds causing what could be termed "routine" randomness; (ii) changes in the average value of the volume of traffic during hourly, weekly, and seasonal cycles creating "hour-of-the-day" variations in travel times; and (iii) accidents, changes in weather conditions, and other events causing "nonroutine" randomness.

With decisions on the location of facilities being of the strategic type, i.e., ones concerned with making a good choice in the long-term sense, it is variations of the second kind that we are particularly concerned with here. Thus, the states of the network, G_r, in this context would reflect the travel conditions that exist for different time periods of a daily cycle and the probabilities, P_r, the relative duration of each set of conditions (or the weight assigned to it). (In a more detailed model a set of states might be used for each period of time to indicate the whole range of possibilities for that period, i.e., variations of types (i) and (iii), and the corresponding probabilities would be adjusted accordingly.) Vehicle dispatching strategies in such a system - both in the sense of what facility serves incidents at a given location and in the sense of the route travelled to reach a demand point - will depend on the state of the network. Hence the use of Assumption 2.

The need to check on whether the conditions of Assumption 1 are satisfied in each case arises, not only as a consequence of the obvious possibility that certain roads may become impassable (links fail) during different periods of the day or under particular sets of conditions, but also because of possible constraints imposed by local requirements. For instance, in connection with a facility location problem for an emergency medical service system for 35 municipalities in the Bel-O-Mar region of West Virginia, JARVIS, STEVENSON and WILLEMAIN [10] have reported that one of the requirements posed by local officials was that no vehicles should be forced to cross the Ohio River in responding to calls. In addition, some of the municipalities involved refused cooperation with facilities situated at other specific towns.

Finally, the need for extensions to the case of oriented networks is evident in this context, especially with respect to facility location in urban environments.

3. Expected Optimal k-Medians

The k-median problem arises when k facilities must be located on a nonoriented deterministic network, with the objective of minimizing the average travel time [8]. A more general situation would involve a nonlinear utility function for travel times and a stochastic network. We thus define the expected optimal k medians as follows:

Definition 2:

A set of k points X_k^* in a nonoriented stochastic network G is a set of expected optimal k-medians of G if for every $X_k \varepsilon G$

$$J_u(X_k^*) \geq J_u(X_k)$$

where

$$J_u(X_k) = \sum_{r=1}^{m} P_r \sum_{i=1}^{n} h_i \, u(d_r(i,X_k)) \tag{2}$$

and u(t) is the utility function of travel time t.

For the expected optimal k-medians to exist, either Assumption 1 or the assumption that the utility function is always finite is necessary. We can now obtain a generalization of Hakimi's main result for the criterion of maximizing the expected utility when the utility function if convex.

The assumption of convexity of the utility function, incidentally, is a reasonable one in most transportation contexts since, in general, long trips tend to have lower per-mile (real or perceived) costs. That is, each successive mile or minute of travel is no more "costly" than its predecessor.

We now state first some facts which can be easily proven from the properties of concave and convex functions.

Fact 1: A linear function is concave (and also convex).

Fact 2: Multiplying a convex (concave) function by a nonnegative constant yields a convex (concave) function.

Fact 3: If $g_i(\Theta)$, $i = 1,2,\ldots,I$, are concave functions of Θ, then $g(\Theta) = \min_i (g_i(\Theta))$ is concave in Θ. (If $g_i(\Theta)$ are convex, then $\max_i(g_i(\Theta))$ is convex in Θ.)

Fact 4: If $f(\Theta)$ is concave in Θ and $u(x)$ is convex and non-increasing then $u(f(\Theta))$ is convex in Θ. If $f(\Theta)$ is linear, $u(x)$ need not be non-increasing for convex $u(f(\Theta))$.

Fact 5: The sum of convex (concave) functions is convex (concave).

Fact 6: If $f(\Theta)$ is a convex function defined on the closed interval $0 \leq \Theta \leq 1$ then

$$\max_{0 \leq \Theta \leq 1} f(\Theta) = \max(f(0), f(1)).$$

We can now prove the following theorem:

Theorem 1: At least one set of expected optimal k-medians exists on the nodes in a nonoriented stochastic network if the utility function for travel time is convex.

Proof: Let the set of expected optimal k-medians be located at a set of points $Y_k = \{y_1, y_2, \ldots, y_k\}$. Let the point y_s, location of the s^{th} expected optimal k-median, be on (p,q) and, using Assumption 3, let Θ_s be defined by

$$\Theta_s = \frac{t_r(p,y_s)}{t_r(p,q)} , \quad 0 \leq \Theta_s \leq 1. \tag{3}$$

Then

$$d_r(i,y_s) = \min[d_r(i,p) + \Theta_s t_r(p,q),\ d_r(i,q) + (1 - \Theta_s)t_r(p,q)] \ , \quad (4)$$

and by Facts 1 and 3, $d_r(i,y_s)$, is concave in Θ_s. By Fact 4, $u(d_r(i,y_s))$ is convex in Θ_s. For $j \neq s$, $u(d_r(i,y_j))$ is constant with respect to Θ_s and hence, convex in Θ_s. Therefore, by Fact 3,

$$u(d_r(i,Y_k)) = \max_{y_j \in Y_k} u(d_r(i,y_j)) \quad (5)$$

is also convex in Θ_s.

By repeated application of Facts 2 and 5, $\bar{J}_k(Y_k)$ (defined by (3)) is convex in Θ_s and so, by Fact 6, is maximized over $0 \leq \Theta_s \leq 1$ at either $\Theta_k = 0$ or $\Theta_k = 1$. That is, y_s is replaced in Y_k by the appropriate one of v_p or v_q. Applying this argument, in succession, to each point in Y_k which is not a node we establish the existence of a set of nodes $X_k = \{x_1, x_2, \ldots, x_3\}$ such that $\bar{J}_u(X_k) \geq \bar{J}_u(Y_k)$. (End of Proof.)

Note that when m = 1 we have a deterministic network and hence the results of LEVY [11] who showed that facilities which minimize transportation costs must be located at the nodes in the network when the transportation costs are concave.

The following follows directly from Theorem 1 when a linear utility function is assumed.

<u>Corollary 1</u>: At least one set of expected k-medians exists on the nodes in a nonoriented stochastic network.

Again for the case m=1 the above corollary reduces to the absolute k-median theorem first proved by HAKIMI [8].

4. Oriented Networks and Extensions

In oriented networks we can have different values for t(i,j) and t(j,i). In fact, if link (i,j) is a one-way link then the value of travel time in the "wrong way" is infinite. Thus, when a deterministic network is oriented, then d(y,i) is not, in general , equal to d(i,y). If the travel time response of a system refers to the time of traveling from an incident to the facility (for example, a hospital) then the average travel time is minimized by locating the facility at $x^* \varepsilon G$ such that

$$J(x^*) \leq J(x)$$

where

$$J(x) = \sum_{i=1}^{n} h_i \, d(i,x). \tag{6}$$

Conversely, if one wishes to minimize average travel time from the facility (for example, a fire station) to the incidents then the facility should be located at y^* such that

$$J'(y^*) \leq J'(y)$$

where

$$J'(y) = \sum_{i=1}^{n} h_i \, d(y,i). \tag{7}$$

One often refers to "travel time response" of an emergency medical system, which is often taken to mean the travel time of the ambulance from the facility to the incident at v_i plus the time of travel from v_i back to the facility. In this case the average travel time is minimized by locating the facility at z^* such that for every $z \varepsilon G$

$$J''(z^*) \leq J''(z)$$

where

$$J''(z) = \sum_{i=1}^{n} h_i(C_1 d(z,i) + C_2 d(i,z)). \tag{8}$$

The non-negative constants C_1 and C_2 reflect the difference in travel speed when the ambulance is not and is transporting a patient respectively.

In view of the above, we define x* as the inward absolute median, y* as the outward absolute median, and z* as the absolute median of the given oriented network. We have similar definitions for inward expected median, outward expected median, and expected median for stochastic oriented networks. For multiple facility locations on stochastic oriented networks, where a node is served by its closest facility, we have the following general definition:

Definition 3: A set of k points Z_k^* in an oriented stochastic network G is a set of expected optimal k-medians of G if for every $Z_k \varepsilon G$

$$\bar{J}_u''(Z_k^*) \geq \bar{J}_u''(Z_k)$$

where

$$\bar{J}_u''(Z_k) = \sum_{r=1}^{m} P_r \sum_{i=1}^{n} h_i \, u_i \, (d_r(Z_k, i), \, d_r(i, Z_k)).$$

If the utility functions $u_i(.,.)$ are convex in their arguments , we can then show:

Theorem 2:

At least one set of expected optimal k-medians exist on the nodes in an oriented stochastic network if the utility function for travel time is convex.

The proof of this theorem is similar to that of Theorem 1. However, because the utility functions are now associated with two arguments (travel time to and from the facilities) we need to use Fact 4(a) instead of Fact 4.

Fact 4(a):

If $u(x_1,x_2)$ is convex and non-increasing in each argument and if x_1 and x_2 are concave functions of Θ, then $u(x_1,x_2)$ is convex in Θ .

(In connection with median problems on oriented networks, it should also be noted that in order to go from a node v_i to a point x on an one-way link (p,q) one has to pass through node v_p. Thus, there is no more a choice between v_p and v_q. This introduces a slight modification in the proof of Theorem 2.)

As special cases of Theorem 2, we can show that inward and outward expected optimal k-medians also exist at the nodes in an oriented stochastic network. Note that when m=1 we have a deterministic oriented network and results similar to the nonoriented deterministic case are also obtained.

Table I presents a classification of all the median problems for which the "locate-facilities-at-nodes" result can be proven. For a detailed discussion and the proofs see MIRCHANDANI [12]. This classification is along three lines: (i) according to the number of facilities required, i.e., single vs. multiple facilities; (ii) according to the type of network involved, i.e., oriented vs. nonoriented networks; and (iii) according to the objective function at hand, i.e., minimization of average travel time vs. maximization of the expectation of a convex utility function for travel time (when we add on the word "optimal").

There are two other types of median problems which are worth mentioning. The "additional median problem" is the problem of locating new independent facilities on a network, when there are already several existing facilities (perhaps not optimally located). The "supporting median problem" is concerned with locating supporting facilities (such as midtown air terminals where

	NONORIENTED NETWORKS		ORIENTED NETWORKS	
Single Facility	Deterministic	Stochastic	Deterministic	Stochastic
Demand - Facility	Absolute Median Optimal Median	Expected Median Expected Optimal Med.	Inward Absolute Inward Optimal	Inward Expected Inward Expected Opt.
Facility - Demand	same as above	same as above	Outward Absolute Outward Optimal	Outward Expected Outward Expected Opt.
Facility - Demand - Facility	same as above	same as above	Absolute Median Optimal Median	Expected Median Expected Opt. Med.
Multiple Facilities				
Demand - k Facilities	Absolute k-Median	Expected k-Median	Inward Abs. k-Median	Inward Expected k-Med
	Optimal k-Median	Expected Opt. k-Med.	Inward Opt. k-Median	Inward Expected Opt. k-Median
k Facilities - Demand	same as above	same as above	Outward Abs. k-Med.	Outward Expected k-Median
			Outward Opt. k-Med.	Outward Expected Opt. k-Median
k Facilities - Demand - k Facilities	same as above	same as above	Absolute k-Median Optimal k-Median	Expected k-Median Expected Opt. k-Median

TABLE I: Classification of Medians on Nonoriented and Oriented (Deterministic and Stochastic) Networks (Med.=Median, Opt.=Optimal, Abs.=Absolute).

passengers may be preprocessed and then transferred to an airport via a rapid transportation link) when there already exist one or more major facilities (e.g., the airport). When the objective is to maximize the expected utility of travel time and the utility function is convex, the "locate-facilities-at-nodes" result holds for both these problems. The analogous results for deterministic networks have been derived by GOLDMAN [7], and HAKIMI and MAHESHWARI [9].

It is also worth noting that FRANK [3,4] has examined another type of stochastic network, one for which the node weights, h_i are random variables. Under a different criterion of optimality (involving maximization of the probability that the average travel distance is below a specified value) he has shown that the "locate-facilities-at-nodes" result does not hold in this case.

5. Computational Considerations

The results reported in Sections 3 and 4 greatly simplify the solution of the various median problems on stochastic networks by restricting the search to the set of nodes. However, the amount of computational effort is significantly affected by the number of network states, m.

The solution procedure consists of two stages. First, shortest paths must be determined between all pairs of nodes for all the states of the network, and, second, the combinatorial problem of selecting the required number of locations among the available candidates must be solved.

With respect to the first of the two stages, FRANK [5] has clearly demonstrated the great difficulties involved in determining exact probability distributions for shortest path lengths on stochastic networks when link lengths are described by continuous probability density functions. By restricting attention to problems where link lengths can be approximated through use of discrete probability distributions with a finite set of values and by assuming knowledge of the state of the network at all times, these difficulties have been bypassed. There is now an m-fold increase, over the case of the deterministic network, in the number of shortest paths that must be computed. That is, for each of the m states of the network, all shortest paths between all pairs of nonzero demand nodes (i.e., a total of at most $m\ n^2/2$ paths for nonoriented networks) must be determined.

Turning now to the solution of the combinatorial problem, for the moment leaving aside the questions of size, median location problems on stochastic networks can be formulated as mixed integer (linear) programming problems [12] in a fashion similar to the deterministic case. Such formulations for the deterministic case have appeared in references [1], [6], [13],[14]. For the stochastic case, however, the problem is expanded m-fold due to the existence of the m states.

We have no computational experience as of now with the application of these techniques to stochastic networks. However, from the results reported in the literature to date on deterministic networks, we are led to believe that reasonable solution times may be achieved for moderate size networks ($n \leq 50$) as long as the number of network states is small ($m \leq 10$). We feel that this is adequate for handling, in an approximate fashion, many of the regional or urban service system problems which were described earlier.

References

1. CORNUEJOLS, G., M.L. FISHER, and G.L. NEMHAUSER, "Location of Bank Accounts to Optimize Float: An Analytic Study of Exact and Approximate Algorithms", Management Science, 23, 1977, 789-810.

2. FRANCIS, R.L. and J.M. GOLDSTEIN, "Location Theory: A Selective Bibliography", Operations Research, 22, 1974, 400-409.

3. FRANK,H., "Optimum Locations on a Graph with Probabilistic Demands", Operations Research, 14, 1966, 409-421.

4. FRANK,H., "Optimum Locations on a Graph with Correlated Normal Demands", Operations Research, 15, 1967, 552-557.

5. FRANK,H., "Shortest Paths in Probabilistic Graphs", Operations Research, 17, 1969, 583-599.

6. GARFINKEL, R.S., A.W. NEEBE, and M.R. RAO, "An Algorithm for the M-Median Plant Location Problem", Transportation Science, 8, No.3, August 1974, 217-236.

7. GOLDMAN, A.J., "Optimum Locations of Centers in a Network", Transportation Science, 3, No. 4, 1969, 352-360.

8. HAKIMI, S.L., "Optimum Locations of Switching Centers and the Absolute Centers and Medians of a Graph", Operations Research, 12, 1964, 450-459.

9. HAKIMI, S.L. and S.N. MAHESHWARI, "Optimum Locations of Centers in Networks", Operations Research, 20, No.5, 1972, 967-973.

10. JARVIS, J.P., K.A. STEVENSON and T.R. WILLEMAIN, "A Simple Procedure for the Allocation of Ambulances in Semi-Rural Areas", Technical Report TR-13-75, M.I.T. Operations Research Center, Cambridge, Mass. (March 1975).

11. LEVY,J., "An Extended Theorem for Location on a Network", Operations Research Quarterly, 18, No. 4, 1967, 433-442.

12. MIRCHANDANI, P.B., "Analysis of Stochastic Networks in Emergency Service Systems", Technical Report TR-15-75, M.I.T. Operations Research Center, Cambridge, Mass. (March 1975).

13. REVELLE, C. and R.W. SWAIN, "Central Facilities Location", Geographical Analysis, 2, No.1, 1970, 30-42.

14. SWAIN,R.W.,"A Decomposition Algorithm for a Class of Facility Location Problems", Ph.D Dissertation, Cornell University, Ithaca, N.Y.,June 1971.

15. WENDELL, R.E. and A.P. HURTER, "Optimal Locations on a Network", Transportation Science, 7, No.1, February 1973, 18-33.