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# LOCKING-FREE ADAPTIVE DISCONTINUOUS GALERKIN FEM FOR LINEAR ELASTICITY PROBLEMS

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Abstract. An adaptive discontinuous Galerkin finite element method for linear elasticity problems is presented. We develop an a posteriori error estimate and prove its robustness with respect to nearly incompressible materials (absence of volume locking). Furthermore, we present some numerical experiments which illustrate the performance of the scheme on adaptively refined meshes.

## 1. Introduction

In structural mechanics, many applications involve a certain parameter-dependency in their corresponding mathematical formulations. It is well known (see e.g., [2]) that the numerical approximation of such problems by low-order finite element methods may significantly degrade as one or more of the parameters approach a certain critical (problem-dependent) limit. This nonrobustness of the FEM is widely termed "locking" and appears in various forms. For example, very thin domains in shell and plate models may lead to the so-called *shear locking*. Moreover, the interaction between bending and membrane energies, arising in shell theories, can cause *membrane locking*. Finally, *volume locking* is observed in applications dealing with nearly incompressible materials, such as the elasticity problems in this paper.

In order to avoid the locking effects, several approaches have been suggested. We mention here the mixed finite element methods in [8, 10], the nonconforming methods proposed in [7, 18], and the higher-order methods in [24]. All these methods have been quite extensively studied within an a priori context.

Another way of circumventing locking is the use of discontinuous Galerkin (DG, for short) finite element methods. In the recent article [12] on linear elasticity (see also [26]), a priori results for these schemes have been presented; in particular, it has been shown that they are free of volume locking, even in the low-order case. Based on this knowledge, the aim of this paper is to establish a locking-free a posteriori error analysis of discontinuous Galerkin methods for linear elasticity problems.

At present, only a small number of results on the a posteriori error estimation of discontinuous Galerkin methods for elliptic problems has been published. We mention here the articles [4, 17], where energy norm a posteriori error estimates

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for diffusion problems have been derived, the paper [21] on $L^2$-error estimation for DGFEM, and the work in [14] about energy norm a posteriori estimation of mixed DGFEM for the time-harmonic Maxwell equations.

In this paper, we apply a new error estimation technique for DG methods, which was recently proposed in [15]. We develop a computable upper bound (up to a generic constant $C$) on the (broken) $H^1$-norm of the DG error $\|\mathbf{e}_h\|_{1,h}$ in terms of the DG approximation $\mathbf{u}_h$; more precisely, we derive a function $\Phi$ (error estimator) such that

$$\|\mathbf{e}_h\|_{1,h} \leq C\Phi(\mathbf{u}_h).$$

Our main result (Theorem 4.1) states that the constant $C$ and the term $\Phi(\mathbf{u}_h)$ are uniformly bounded with respect to the incompressibility parameter arising in linear elasticity problems. We remark that the quantity $\Phi(\mathbf{u}_h)$ may be computed element-wise, thereby allowing for a standard adaptive procedure—based on automatic mesh refinement (see [23])—that may be implemented easily. Related work on nonconforming methods can be found in [9] and the references therein.

The main idea of our analysis is to rewrite the method in a nonconsistent way using a lifting operator (see e.g., [1, 20, 22]), and to decompose the discontinuous finite element spaces as described in [13]. Then, according to this decomposition, the error of the DGFEM may be split into two parts which may be analyzed separately using an appropriate inf-sup stability, some suitable interpolation properties and a recent norm equivalence result from [13].

The outline of the paper is as follows. In Section 2, the linear elasticity problem is described. Section 3 introduces the discontinuous Galerkin FEM. In Section 4, the (locking-free) a posteriori error estimate is derived. Section 5 is dedicated to a series of numerical experiments illustrating the performance of the proposed error estimator within an automatic mesh refinement procedure; in particular, we show that our numerical results are robust with respect to volume locking. Finally, we summarize our work in Section 6.

## 2. Problem formulation

In the following section, we establish an appropriate functional setting for this paper and introduce the linear elasticity problems under consideration.

2.1. **Notation.** For a bounded Lipschitz domain $D$ in $\mathbb{R}^d$, $d \geq 1$, let $L^2(D)$ be the Lebesgue space of square integrable functions, endowed with the usual norm $\|\cdot\|_{0,D}$. The standard Sobolev spaces of functions with integer regularity exponents $s \geq 0$ are denoted by $H^s(D)$. We write $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ for the corresponding norms and semi-norms, respectively, and set $H^0(D) = L^2(D)$. The trace space of $H^1(D)$ is denoted by $H^{\frac{1}{2}}(\partial D)$ and, as usual, we define $H^1_0(D)$ as the subspace of functions in $H^1(D)$ with zero trace on $\partial D$. Furthermore, for a function space $X(D)$, let $X(D)^d$ and $X(D)^{d\times d}$ be the spaces of all vector and tensor fields whose components belong to $X(D)$, respectively. Without further specification, these spaces are equipped with the usual product norms (which, for simplicity, are denoted similarly as the norm in $X(D)$). For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, and matrices $\underline{\sigma}, \underline{\tau} \in \mathbb{R}^{d\times d}$, we use the standard notation $(\nabla \mathbf{v})_{ij} = \partial_j v_i$, $(\nabla \cdot \underline{\sigma})_i = \sum_{j=1}^d \partial_j \sigma_{ij}$, and $\underline{\sigma} : \underline{\tau} = \sum_{i,j=1}^d \sigma_{ij}\tau_{ij}$. Furthermore, let $\mathbf{v} \otimes \mathbf{w}$ be the matrix whose $ij$-th component is $v_i\, w_j$. Note that the identity $\mathbf{v} \cdot \underline{\sigma} \cdot \mathbf{w} = \sum_{i,j=1}^d v_i \sigma_{ij} w_j = \underline{\sigma} : (\mathbf{v} \otimes \mathbf{w})$ holds.

2.2. **The Lamé system.** Given a polygonal domain $\Omega$ in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ and an external force $\mathbf{f} \in L^2(\Omega)^2$, the linear elasticity problem considered in this paper is to find a vector field $\mathbf{u} = (u_1, u_2) \in H_0^1(\Omega)^2$ (displacement) such that

$$-\nabla \cdot \underline{\sigma}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma. \tag{2.2}$$

Here $\nabla\cdot$ is the divergence operator, $\underline{\sigma}(\mathbf{u}) = \{\sigma_{ij}(\mathbf{u})\}_{i,j=1}^2$ is the stress tensor for homogeneous isotropic material given by

$$\underline{\sigma}(\mathbf{u}) = 2\mu\underline{\varepsilon}(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\,\mathbb{D}_{2\times 2},$$

where $\underline{\varepsilon}(\mathbf{u}) = \{\varepsilon_{ij}(\mathbf{u})\}_{i,j=1}^2$, with

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i),$$

denotes the symmetric gradient of $\mathbf{u}$, and

$$\mathbb{D}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the $2 \times 2$ identity matrix. Moreover, $\mu, \lambda > 0$ are the so-called Lamé coefficients. Standard arguments imply the unique solvability of the system (2.1)–(2.2).

Note that from the weak formulation of (2.1)–(2.2), which is to find $\mathbf{u} \in H_0^1(\Omega)^2$ such that

$$\int_\Omega \underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\mathbf{v})\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} \qquad \forall \mathbf{v} \in H_0^1(\Omega)^2, \tag{2.3}$$

the stability estimate

$$\|\mathbf{u}\|_{1,\Omega}^2 + \lambda\|\nabla \cdot \mathbf{u}\|_{0,\Omega}^2 \le C\|\mathbf{f}\|_{0,\Omega}^2,$$

easily follows, where $C > 0$ is a constant independent of $\lambda$. This implies that, for $\lambda \to \infty$, there arises the constraint $\nabla \cdot \mathbf{u} \to 0$ which corresponds to (nearly) incompressible materials. It is well known (see [2]) that this incompressibility constraint may cause a loss of uniformity (with respect to $\lambda$) in the convergence regime of low-order (conforming) finite element methods. This does not mean that the FEM does not converge; however, it may happen that the convergence begins to take place at such high numbers of degrees of freedom that the method is not feasible in practice. This nonrobustness of the FEM with respect to incompressible materials is called *volume locking*. In [12] it was shown that this effect may be overcome by the use of discontinuous Galerkin methods (see also [26]).

## 3. DG discretization

In this section, we introduce the discontinuous Galerkin FEM for the discretization of the elasticity problem (2.1)–(2.2), and discuss its well-posedness.

3.1. **Meshes and traces.** Let $\mathcal{T}_h$ be a finite element mesh on $\Omega$ consisting of triangles $\{K\}_{K\in\mathcal{T}_h}$. For each $K \in \mathcal{T}_h$, we denote by $\mathbf{n}_K$ the unit outward normal vector on the boundary $\partial K$, and by $h_K$ the elemental diameter. We assume that the triangles in $\mathcal{T}_h$ are shape regular, i.e., there exists a constant $C > 0$ such that for all $K \in \mathcal{T}_h$, $h_K \le C\rho_K$ holds, where $\rho_K$ is the radius of the largest inscribed-circle in $K$. In addition, we suppose that the intersection between two elements of $\mathcal{T}_h$ is either empty, a common vertex or a common edge. These properties imply that the local mesh sizes are of bounded variation, that is, there is a positive constant

$C$ such that $Ch_K \leq h_{K'} \leq C^{-1}h_K$, whenever $K$ and $K'$ share a common edge. As usual, we define the mesh size by $h = \max_{K\in\mathcal{T}_h} h_K$.

An interior edge of $\mathcal{T}_h$ is the (nonempty) one-dimensional interior of $\partial K^+ \cap \partial K^-$, where $K^+$ and $K^-$ are two adjacent elements of $\mathcal{T}_h$. Similarly, a boundary edge of $\mathcal{T}_h$ is the (nonempty) one-dimensional interior of $\partial K \cap \Gamma$ which consists of entire edges of $\partial K$. By $\mathcal{E}_\mathcal{I}$ we denote the union of all interior edges. Similarly, $\mathcal{E}_\mathcal{B}$ is the union of all boundary edges. We set $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{B}$.

Next, we define the trace operators that are required for the DG method. To this end, let $K^+$ and $K^-$ be two adjacent elements of $\mathcal{T}_h$, and $\mathbf{x}$ an arbitrary point on the interior edge $e = \partial K^+ \cap \partial K^- \subset \mathcal{E}_\mathcal{I}$. Furthermore, let $q$, $\mathbf{v}$, and $\underline{\tau}$ be some scalar-, vector-, and matrix-valued functions, respectively, that are smooth inside each element $K^\pm$. By $(q^\pm, \mathbf{v}^\pm, \underline{\tau}^\pm)$, we denote the traces of $(q, \mathbf{v}, \underline{\tau})$ on $e$ taken from within the interior of $K^\pm$, respectively. Then, we introduce the following averages at $\mathbf{x} \in e$:

$$\{\!\{q\}\!\} = \frac{1}{2}(q^+ + q^-), \qquad \{\!\{\mathbf{v}\}\!\} = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \qquad \{\!\{\underline{\tau}\}\!\} = \frac{1}{2}(\underline{\tau}^+ + \underline{\tau}^-).$$

Similarly, the jumps at $\mathbf{x} \in e$ are given by

$$\begin{aligned} [\![q]\!] &= q^+\,\mathbf{n}_{K^+} + q^-\,\mathbf{n}_{K^-}, \qquad & [\![\mathbf{v}]\!] &= \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-}, \\ \underline{[\![\mathbf{v}]\!]} &= \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-}, \qquad & [\![\underline{\tau}]\!] &= \underline{\tau}^+\mathbf{n}_{K^+} + \underline{\tau}^-\mathbf{n}_{K^-}. \end{aligned}$$

On boundary edges $e \subset \mathcal{E}_\mathcal{B}$, we set $\{\!\{q\}\!\} = q$, $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$, $\{\!\{\underline{\tau}\}\!\} = \underline{\tau}$, as well as $[\![q]\!] = q\mathbf{n}$, $[\![\mathbf{v}]\!] = \mathbf{v} \cdot \mathbf{n}$, $\underline{[\![\mathbf{v}]\!]} = \mathbf{v} \otimes \mathbf{n}$, and $[\![\underline{\tau}]\!] = \underline{\tau}\mathbf{n}$. Here $\mathbf{n}$ is the unit outward normal vector to the boundary $\Gamma$.

3.2. **The discontinuous Galerkin method.** On a given mesh $\mathcal{T}_h$, we define the finite element space

$$\mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in \mathcal{P}_1(K)^2,\, K \in \mathcal{T}_h\},$$

where $\mathcal{P}_1(K) = \{v : v(x_1, x_2) = a_K x_1 + b_K x_2 + c_K,\, a_K, b_K, c_K \in \mathbb{R}\}$, $K \in \mathcal{T}_h$, denotes the space of all linear polynomials on $K$. Furthermore, we consider the following discontinuous Galerkin finite element formulation for the approximation of the linear elasticity problem (2.1)–(2.2): find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) = l_h(\mathbf{v}) \tag{3.1}$$

for all $\mathbf{v} \in \mathbf{V}_h$. The bilinear forms $a_h$ and the linear functional $l_h$ are defined by

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) = &\int_\Omega \underline{\sigma}_h(\mathbf{u}) : \underline{\varepsilon}_h(\mathbf{v})\,d\mathbf{x} - \int_\mathcal{E} \left(\{\!\{\underline{\sigma}_h(\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]} + \underline{[\![\mathbf{u}]\!]} : \{\!\{\underline{\sigma}_h(\mathbf{v})\}\!\}\right) ds \\ &+ \int_\mathcal{E} \mathtt{c}\underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\,ds + \lambda \int_\mathcal{E} \mathtt{c}[\![\mathbf{u}]\!][\![\mathbf{v}]\!]\,ds \end{aligned} \tag{3.2}$$

and

$$l_h(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,d\mathbf{x}, \tag{3.3}$$

respectively; see [12]. Here $\underline{\sigma}_h$ and $\underline{\varepsilon}_h$ are the stress tensor and symmetric gradient, respectively, taken element-wise. Furthermore, the function $\mathtt{c} \in L^\infty(\mathcal{E})$ is the so-called discontinuity stabilization function which is chosen as follows: writing $\mathtt{h} \in L^\infty(\mathcal{E})$ to denote the auxiliary mesh function defined by

$$\mathtt{h}(\mathbf{x}) = \begin{cases} \min\{h_K, h_{K'}\}, & \mathbf{x} \in e = \partial K \cap \partial K' \subset \mathcal{E}_\mathcal{I}, \\ h_K, & \mathbf{x} \in e = \partial K \cap \partial\Omega \subset \mathcal{E}_\mathcal{B}, \end{cases}$$

we set

$$\mathtt{c} = \gamma \mathtt{h}^{-1}, \tag{3.4}$$

with a parameter $\gamma > 0$ that is independent of the mesh size.

*Remark* 3.1. The DG scheme (3.1), based on the bilinear form $a_h$ in (3.2), is called the *interior penalty (IP) discontinuous Galerkin method*. It was previously introduced for the discretization of closely related diffusion problems; see, e.g., [3, 11, 25]. For those problems, there exists a remarkable number of other DG methods (see [1] and the references therein) which could also be used for the approximation of the elasticity problems in this paper.

*Remark* 3.2. In the case of inhomogeneous Dirichlet boundary conditions $\mathbf{u} = \mathbf{g}$ on $\Gamma$, with a datum $\mathbf{g}$, the functional $l_h$ in (3.3) must be replaced by

$$l_h(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} - \int_{\mathcal{E}_\mathcal{B}} (\mathbf{g} \otimes \mathbf{n}) : \underline{\sigma}_h(\mathbf{v})\, ds + \int_{\mathcal{E}_\mathcal{B}} \mathtt{c}\, \mathbf{g} \cdot \mathbf{v}\, ds + \lambda \int_{\mathcal{E}_\mathcal{B}} \mathtt{c}(\mathbf{g} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})\, ds.$$

3.3. **Well-posedness.** The existence of a unique solution of the DGFEM (3.1) follows directly by the coercivity of the DG bilinear form $a_h$.

**Proposition 3.3.** *There exist constants $C, \gamma_{\min} > 0$ independent of the Lamé coefficient $\lambda$ and of the mesh size such that for all $\gamma$ in* (3.4) *with $\gamma > \gamma_{\min}$, it follows that*

$$a_h(\mathbf{u}, \mathbf{u}) \geq C \|\mathbf{u}\|_{DG}^2 \qquad \forall \mathbf{u} \in \mathbf{V}_h. \tag{3.5}$$

*Here*

$$\|\mathbf{u}\|_{DG}^2 = \|\underline{\varepsilon}_h(\mathbf{u})\|_{0,\Omega}^2 + \lambda \|\nabla_h \cdot \mathbf{u}\|_{0,\Omega}^2 + \int_\mathcal{E} \mathtt{c} |\underline{[\![\mathbf{u}]\!]}|^2\, ds + \lambda \int_\mathcal{E} \mathtt{c} |[\![\mathbf{u}]\!]|^2\, ds.$$

*Proof.* This can be proved as in [12, Proposition 6]. ☐

Henceforth, we always assume that

$$\gamma > \max(1, \gamma_{\min}), \tag{3.6}$$

where $\gamma_{\min}$ is the constant from the previous Proposition 3.3.

## 4. Locking-free a posteriori error estimation

Our main result is a locking-free a posteriori error estimate for the DGFEM (3.1) with respect to the norm

$$\|\mathbf{v}\|_{1,h}^2 = \|\nabla_h \mathbf{v}\|_{0,\Omega}^2 + \int_\mathcal{E} \mathtt{c} \underline{[\![\mathbf{v}]\!]}^2\, ds,$$

where $\nabla_h$ is the element-wise gradient on $\mathcal{T}_h$.

**Theorem 4.1.** *Let $\mathbf{u} \in H_0^1(\Omega)^2$ be the exact solution of the linear elasticity problem* (2.1)–(2.2)*, and $\mathbf{u}_h$ its DG approximation obtained by* (3.1)*. Then, there exists a constant $C > 0$ independent of the Lamé coefficient $\lambda$ (incompressibility parameter), the mesh size and of the stability parameter $\gamma$, such that the following a posteriori error bound holds true:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h}^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K^2. \tag{4.1}$$

*Here the elemental error indicators* $\eta_K$, $K \in \mathcal{T}_h$, *are given by*

$$\eta_K^2 = h_K^2 \|\mathbf{f}\|_{0,K}^2 + h_K \|[\![\underline{\varepsilon}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\partial\Omega}^2 + \gamma^2 h_K^{-1} \|\underline{[\![\mathbf{u}_h]\!]}\|_{0,\partial K}^2. \tag{4.2}$$

*The error estimator, i.e., the right-hand side of* (4.1)*, is bounded independently of* $\lambda$.

*Remark* 4.2. The above a posteriori result is a computable upper bound on the energy norm of the DG error $\mathbf{u} - \mathbf{u}_h$ (in terms of the DG approximation $\mathbf{u}_h$) which may be easily integrated in an automatic mesh refinement procedure (see the adaptive algorithm in Section 5.1). We stress the fact that this upper bound is uniformly bounded with respect to the incompressibility parameter $\lambda$. This clearly confirms the absence of volume locking in the DG method which was previously studied in the a priori analyses [12, 26].

The proof of Theorem 4.1 is worked out in Section 4.4. It follows from several auxiliary results, which are developed in the ensuing Sections 4.1–4.3. The main idea is to rewrite the method using a lifting operator (see e.g., [1, 20, 22]), and to decompose the DG spaces in an appropriate way (see [13]). This approach was recently presented in [15] for the a posteriori error estimation of mixed DG discretizations for the Stokes problem.

4.1. **Lifting operator and nonconsistent DG form.** Due to the (possibly) low regularity of the exact solution $\mathbf{u} \in H_0^1(\Omega)^2$ of the elasticity problem (2.1)–(2.2), we first reformulate the DGFEM (3.1) using a lifting operator; this results in a new (nonconsistent) formulation of the DGFEM that is well defined on $H_0^1(\Omega)^2$. To do so, we introduce the spaces

$$\mathbf{V}(h) = H_0^1(\Omega)^2 + \mathbf{V}_h$$

and

$$\underline{\Sigma}_h = \{\underline{\tau} \in L^2(\Omega)^{2\times 2} : \underline{\tau}|_K \in \mathcal{P}_1(K)^{2\times 2},\, K \in \mathcal{T}_h\}.$$

Then, the lifting operator $\underline{\mathcal{L}} : \mathbf{V}(h) \to \underline{\Sigma}_h$ is defined by

$$\int_\Omega \underline{\mathcal{L}}(\mathbf{v}) : \underline{\tau}\, d\mathbf{x} = \int_{\mathcal{E}} \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds \qquad \forall \underline{\tau} \in \underline{\Sigma}_h. \tag{4.3}$$

Note that $\underline{\mathcal{L}}$ is stable (see e.g., [1]), i.e., there exists a constant $C > 0$ independent of the mesh size such that

$$\|\underline{\mathcal{L}}(\mathbf{v})\|_{0,\Omega}^2 \le C \int_{\mathcal{E}} \mathtt{h}^{-1} |\underline{[\![\mathbf{v}]\!]}|^2\, ds \tag{4.4}$$

for any $\mathbf{v} \in \mathbf{V}(h)$.

*Remark* 4.3. For $\mathbf{v} \in H_0^1(\Omega)^2$, we have $\underline{\mathcal{L}}(\mathbf{v}) = \underline{[\![\mathbf{v}]\!]} = \underline{0}$ and $[\![\mathbf{v}]\!] = 0$ on $\mathcal{E}$.

With these definitions, we can now introduce the following (nonconsistent) DG form on $\mathbf{V}(h) \times \mathbf{V}(h)$:

$$\begin{aligned}\widetilde{a}_h(\mathbf{u},\mathbf{v}) = &\int_\Omega \underline{\sigma}_h(\mathbf{u}) : \underline{\varepsilon}_h(\mathbf{v})\, d\mathbf{x} - \int_\Omega \left(\underline{\mathcal{L}}(\mathbf{v}) : \underline{\sigma}_h(\mathbf{u}) + \underline{\mathcal{L}}(\mathbf{u}) : \underline{\sigma}_h(\mathbf{v})\right) d\mathbf{x} \\ &+ \int_{\mathcal{E}} \mathtt{c}\underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\, ds + \lambda \int_{\mathcal{E}} \mathtt{c}[\![\mathbf{u}]\!][\![\mathbf{v}]\!]\, ds.\end{aligned}$$

Note that, due to (4.3), $\widetilde{a}_h(\mathbf{u},\mathbf{v}) = a_h(\mathbf{u},\mathbf{v})$, for all $\mathbf{u},\mathbf{v} \in \mathbf{V}_h$, where $a_h$ is the bilinear form (3.2). Hence, the DGFEM (3.1) is equivalent to finding $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\widetilde{a}_h(\mathbf{u}_h,\mathbf{v}) = l_h(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{4.5}$$

Here $l_h$ is again the linear functional from (3.3).

4.2. **Inf-sup stability.** In order to prove an a posteriori error estimate for the DGFEM that is robust (locking-free) with respect to nearly incompressible materials, the following stability result, which is explicit with respect to the Lamé coefficient $\lambda$, is required.

**Lemma 4.4.** *For any $\mathbf{u} \in H_0^1(\Omega)^2$, there exists a function $\mathbf{v}_\mathbf{u} \in H_0^1(\Omega)^2$ such that*

$$\widetilde{a}_h(\mathbf{u},\mathbf{v}_\mathbf{u}) = |\mathbf{u}|_{1,\Omega}^2 \tag{4.6}$$

*and*

$$|\mathbf{v}_\mathbf{u}|_{1,\Omega}^2 + \lambda^2 \|\nabla\cdot\mathbf{v}_\mathbf{u}\|_{0,\Omega}^2 \le C|\mathbf{u}|_{1,\Omega}^2, \tag{4.7}$$

*where $C > 0$ is a constant independent of $\lambda$ and of the mesh size.*

*Proof.* We define $\mathbf{v}_\mathbf{u}$ as the unique solution in $H_0^1(\Omega)^2$ of

$$\widetilde{a}_h(\mathbf{v}_\mathbf{u},\mathbf{w}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{w}\, d\mathbf{x} \qquad \forall \mathbf{w} \in H_0^1(\Omega)^2. \tag{4.8}$$

Then, taking $\mathbf{w} = \mathbf{u}$ in (4.8) and using the symmetry of the form $\widetilde{a}_h$, results in (4.6).

It remains to prove (4.7). Selecting $\mathbf{w} = \mathbf{v}_\mathbf{u}$ in (4.8) and using Remark 4.3, leads to

$$2\mu\|\underline{\varepsilon}(\mathbf{v}_\mathbf{u})\|_{0,\Omega}^2 + \lambda\|\nabla\cdot\mathbf{v}_\mathbf{u}\|_{0,\Omega}^2 = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}_\mathbf{u}\, d\mathbf{x} \le |\mathbf{u}|_{1,\Omega}|\mathbf{v}_\mathbf{u}|_{1,\Omega}.$$

Here the first Korn inequality (see [19], for example) implies that

$$\|\underline{\varepsilon}(\mathbf{v}_\mathbf{u})\|_{0,\Omega}^2 \ge C_{\mathrm{Korn}}|\mathbf{v}_\mathbf{u}|_{1,\Omega}^2,$$

for a constant $C_{\mathrm{Korn}} > 0$, and hence we have

$$2\mu C_{\mathrm{Korn}}|\mathbf{v}_\mathbf{u}|_{1,\Omega}^2 \le |\mathbf{u}|_{1,\Omega}|\mathbf{v}_\mathbf{u}|_{1,\Omega}.$$

Squaring both sides of the above inequality and dividing by $|\mathbf{v}_\mathbf{u}|_{1,\Omega}^2$ results in

$$4\mu^2 C_{\mathrm{Korn}}^2|\mathbf{v}_\mathbf{u}|_{1,\Omega}^2 \le |\mathbf{u}|_{1,\Omega}^2. \tag{4.9}$$

Furthermore, by [6, Lemma 9.2.3] (and the references therein), there exists $\phi \in H_0^1(\Omega)^2$ such that

$$\nabla\cdot\phi = \lambda\nabla\cdot\mathbf{v}_\mathbf{u}, \qquad |\phi|_{1,\Omega} \le \kappa\lambda\|\nabla\cdot\mathbf{v}_\mathbf{u}\|_{0,\Omega},$$

where $\kappa > 0$ is a constant independent of $\mathbf{v}_\mathbf{u}$, $\lambda$, and of the mesh size. Therefore, taking $\mathbf{w} = \phi$ in (4.8), leads to

$$\begin{aligned}\widetilde{a}_h(\mathbf{v}_\mathbf{u},\phi) &= \int_\Omega \nabla\mathbf{u} : \nabla\phi\, d\mathbf{x}\\ &\le |\mathbf{u}|_{1,\Omega}|\phi|_{1,\Omega}\\ &\le \kappa\lambda|\mathbf{u}|_{1,\Omega}\|\nabla\cdot\mathbf{v}_\mathbf{u}\|_{0,\Omega}\\ &\le \kappa^2|\mathbf{u}|_{1,\Omega}^2 + \frac{\lambda^2}{4}\|\nabla\cdot\mathbf{v}_\mathbf{u}\|_{0,\Omega}^2.\end{aligned} \tag{4.10}$$

Moreover, recalling again Remark 4.3, it holds that

$$\begin{aligned}\widetilde{a}_h(\mathbf{v_u},\phi) &= 2\mu\int_\Omega \underline{\varepsilon}(\mathbf{v_u}) : \underline{\varepsilon}(\phi)\,d\mathbf{x} + \lambda\int_\Omega (\nabla\cdot\mathbf{v_u})(\nabla\cdot\phi)\,d\mathbf{x}\\ &\geq -2\mu\|\underline{\varepsilon}(\mathbf{v_u})\|_{0,\Omega}\|\underline{\varepsilon}(\phi)\|_{0,\Omega} + \lambda^2\|\nabla\cdot\mathbf{v_u}\|_{0,\Omega}^2\\ &\geq -2\mu|\mathbf{v_u}|_{1,\Omega}|\phi|_{1,\Omega} + \lambda^2\|\nabla\cdot\mathbf{v_u}\|_{0,\Omega}^2\\ &\geq -2\mu\kappa\lambda|\mathbf{v_u}|_{1,\Omega}\|\nabla\cdot\mathbf{v_u}\|_{0,\Omega} + \lambda^2\|\nabla\cdot\mathbf{v_u}\|_{0,\Omega}^2\\ &\geq -4\mu^2\kappa^2|\mathbf{v_u}|_{1,\Omega}^2 + \frac{3}{4}\lambda^2\|\nabla\cdot\mathbf{v_u}\|_{0,\Omega}^2.\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), results in

$$-4\mu^2\kappa^2|\mathbf{v_u}|_{1,\Omega}^2 + \frac{\lambda^2}{2}\|\nabla\cdot\mathbf{v_u}\|_{0,\Omega}^2 \leq \kappa^2|\mathbf{u}|_{1,\Omega}^2. \tag{4.12}$$

Hence, (4.7) follows by choosing a suitable linear combination of the inequalities (4.9) and (4.12) (independently of $\lambda$). □

4.3. **The BDM interpolant.** Although the so-called Brezzi-Douglas-Marini interpolant (BDM) was originally introduced for the analysis of mixed FEM, it has been recently shown (see [12], for example) that it is also ideally suited for the analysis of DG methods for linear elasticity problems. The reason for this is that the BDM interpolant provides some essential properties for the circumvention of volume locking which are not available for conforming elements (see also [7, 26] for related considerations based on the Crouzeix-Raviart interpolant). Referring to [8], we collect some of its basic properties.

**Lemma 4.5.** *Let $K \in \mathcal{T}_h$ and $\mathbf{v} \in H^1(K)^2$. Then, the BDM interpolant*

$$\mathbf{I}_K : H^1(K)^2 \to \mathcal{P}_1(K)^2$$

*on $K$ is uniquely defined by*

$$\int_e (\mathbf{v} - \mathbf{I}_K\mathbf{v})\cdot\mathbf{n}_e q\,ds = 0 \qquad \forall q \in \mathcal{P}_1(e),\ \forall e \in \mathcal{E}_K. \tag{4.13}$$

*Here $\mathcal{E}_K = \{e \in \mathcal{E} : e \subset \partial K\}$ is the set of all edges of $K$, $\mathbf{n}_e$ is the unit outward vector of $K$ on $e \in \mathcal{E}_K$, and $\mathcal{P}_1(e)$ is the space of all polynomials on $e \in \mathcal{E}_K$ of degree at most* 1*. The following approximation properties hold true:*

(a)
$$h_K^{-1}\|\mathbf{v} - \mathbf{I}_K\mathbf{v}\|_{0,K} + |\mathbf{v} - \mathbf{I}_K\mathbf{v}|_{1,K} \leq C|\mathbf{v}|_{1,K};$$

(b)
$$\|\mathbf{v} - \mathbf{I}_K\mathbf{v}\|_{0,\partial K} \leq C h_K^{1/2}|\mathbf{v}|_{1,K};$$

(c)
$$\|\nabla\cdot(\mathbf{v} - \mathbf{I}_K\mathbf{v})\|_{0,K} \leq C\|\nabla\cdot\mathbf{v}\|_{0,K}.$$

*Proof.* This follows directly from the results in [8, §III.3]. □

We are now ready to derive a locking-free a posteriori error estimator for the DG method (3.1).

4.4. **Proof of Theorem 4.1.** We start with the proof of the a posteriori error estimate (4.1). To do so, we proceed in several steps.

*Step 1:* We first decompose the discontinuous finite element space $\mathbf{V}_h$ into two orthogonal parts: a conforming part and a purely nonconforming part. To this end, let $\mathbf{V}_h^c = \mathbf{V}_h \cap H_0^1(\Omega)^2$. The orthogonal complement in $\mathbf{V}_h$ of $\mathbf{V}_h^c$ with respect to the norm $\|\cdot\|_{1,h}$, is denoted by $\mathbf{V}_h^\perp$; i.e.,

$$\mathbf{V}_h = \mathbf{V}_h^c \oplus \mathbf{V}_h^\perp. \tag{4.14}$$

Then, decomposing the solution $\mathbf{u}_h \in \mathbf{V}_h$ of the DGFEM (3.1) into $\mathbf{u}_h = \mathbf{u}_h^c \oplus \mathbf{u}_h^\perp$, according to (4.14), we can write the error of the DG approximation as

$$\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h = \mathbf{e}_h^c - \mathbf{u}_h^\perp, \tag{4.15}$$

where $\mathbf{e}_h^c = \mathbf{u} - \mathbf{u}_h^c \in H_0^1(\Omega)^2$, and it holds that

$$\|\mathbf{e}_h\|_{1,h}^2 \leq 2\big(\|\mathbf{e}_h^c\|_{1,h}^2 + \|\mathbf{u}_h^\perp\|_{1,h}^2\big). \tag{4.16}$$

Furthermore, refering to [13, Theorem 5.3], we have the estimate

$$\|\mathbf{u}_h^\perp\|_{1,h}^2 \leq C \int_{\mathcal{E}} \mathsf{c}|\underline{[\![\mathbf{u}_h]\!]}|^2\, ds, \tag{4.17}$$

for a constant $C > 0$ independent of the mesh size and of the stability constant $\gamma$. Thus, using (3.6) and (4.16), we obtain

$$\|\mathbf{e}_h\|_{1,h}^2 \leq C\Big(\|\mathbf{e}_h^c\|_{1,h}^2 + \int_{\mathcal{E}} \mathsf{c}|\underline{[\![\mathbf{u}_h]\!]}|^2\, ds\Big) \leq C\Big(\|\mathbf{e}_h^c\|_{1,h}^2 + \gamma^2 \sum_{K\in\mathcal{T}_h} h_K^{-1}|\underline{[\![\mathbf{u}_h]\!]}|_{0,\partial K}^2\Big), \tag{4.18}$$

and it remains to bound the term $\|\mathbf{e}_h^c\|_{1,h}$.

*Step 2:* Since $\mathbf{e}_h^c \in H_0^1(\Omega)^2$, we notice that $\|\mathbf{e}_h^c\|_{1,h} = |\mathbf{e}_h^c|_{1,\Omega}$ (cf. Remark 4.3). Furthermore, we can apply Lemma 4.4 to find $\mathbf{w} \in H_0^1(\Omega)^2$ such that

$$\widetilde{a}_h(\mathbf{e}_h^c, \mathbf{w}) = |\mathbf{e}_h^c|_{1,\Omega}^2$$

and

$$|\mathbf{w}|_{1,\Omega}^2 + \lambda^2\|\nabla\cdot\mathbf{w}\|_{0,\Omega}^2 \leq C|\mathbf{e}_h^c|_{1,\Omega}^2. \tag{4.19}$$

Hence, due to (4.15), we have

$$|\mathbf{e}_h^c|_{1,\Omega}^2 = \widetilde{a}_h(\mathbf{e}_h, \mathbf{w}) + \widetilde{a}_h(\mathbf{u}_h^\perp, \mathbf{w}).$$

Moreover, due to Remark 4.3, the exact solution $\mathbf{u} \in H_0^1(\Omega)^2$ of (2.1)–(2.2), respectively (2.3), satisfies $\widetilde{a}_h(\mathbf{u}, \mathbf{v}) = l_h(\mathbf{v})$ for all $\mathbf{v} \in H_0^1(\Omega)^2$, and therefore, it follows that

$$|\mathbf{e}_h^c|_{1,\Omega}^2 = \widetilde{a}_h(\mathbf{u}, \mathbf{w}) - \widetilde{a}_h(\mathbf{u}_h, \mathbf{w}) + \widetilde{a}_h(\mathbf{u}_h^\perp, \mathbf{w}) = l_h(\mathbf{w}) - \widetilde{a}_h(\mathbf{u}_h, \mathbf{w}) + \widetilde{a}_h(\mathbf{u}_h^\perp, \mathbf{w}).$$

Now let $\mathbf{I}_h : H_0^1(\Omega)^2 \to \mathbf{V}_h$ be the global BDM interpolant defined by $\mathbf{I}_h|_K = \mathbf{I}_K$ $\forall K \in \mathcal{T}_h$, where $\mathbf{I}_K$, $K \in \mathcal{T}_h$, is the local BDM interpolant from Lemma 4.5. Then, using (4.5), we obtain that

$$|\mathbf{e}_h^c|_{1,\Omega}^2 = l_h(\mathbf{w} - \mathbf{I}_h\mathbf{w}) - \widetilde{a}_h(\mathbf{u}_h, \mathbf{w} - \mathbf{I}_h\mathbf{w}) + \widetilde{a}_h(\mathbf{u}_h^\perp, \mathbf{w}) = E_1 - E_2 + E_3. \tag{4.20}$$

*Step 3:* We analyze the terms $E_1, E_2, E_3$ separately. For convenience, we write $\xi_h = \mathbf{w} - \mathbf{I}_h\mathbf{w}$. Using the definition of $l_h$ results in

$$E_1 = \sum_{K\in\mathcal{T}_h} \int_K \mathbf{f}\cdot\xi_h\, d\mathbf{x} \leq \sum_{K\in\mathcal{T}_h} \|\mathbf{f}\|_{0,K}\|\xi_h\|_{0,K}.$$

For the estimation of $E_2$, we integrate by parts. Notice that $\nabla_h \cdot \underline{\sigma}_h(\mathbf{u}_h) \equiv 0$ (since $\mathbf{u}_h$ is element-wise linear) on $\Omega$ and recall the definition of the lifting operator $\underline{\mathcal{L}}$. This leads to

$$
\begin{aligned}
-E_2 = &- \sum_{K\in\mathcal{T}_h} \int_{\partial K} \underline{\sigma}_h(\mathbf{u}_h) : (\xi_h \otimes \mathbf{n}_K)\, ds \\
&+ \int_\Omega \left(\underline{\mathcal{L}}(\xi_h) : \underline{\sigma}_h(\mathbf{u}_h) + \underline{\mathcal{L}}(\mathbf{u}_h) : \underline{\sigma}_h(\xi_h)\right) d\mathbf{x} \\
&- \int_{\mathcal{E}} \mathsf{c}\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\xi_h]\!]}\, ds - \lambda \int_{\mathcal{E}} \mathsf{c}[\![\mathbf{u}_h]\!][\![\xi_h]\!]\, ds \\
= &- \int_{\mathcal{E}_{\mathcal{I}}} \{\!\{\xi_h\}\!\} \cdot [\![\underline{\sigma}_h(\mathbf{u}_h)]\!]\, ds + \int_\Omega \underline{\mathcal{L}}(\mathbf{u}_h) : \underline{\sigma}_h(\xi_h)\, d\mathbf{x} \\
&- \int_{\mathcal{E}} \mathsf{c}\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\xi_h]\!]}\, ds - \lambda \int_{\mathcal{E}} \mathsf{c}[\![\mathbf{u}_h]\!][\![\xi_h]\!]\, ds.
\end{aligned}
\tag{4.21}
$$

Since $\nabla_h \cdot \mathbf{u}_h$ is element-wise constant on $\mathcal{E}$, it holds, with the definition of $\mathbf{I}_h$, that

$$
\int_{\mathcal{E}_{\mathcal{I}}} \{\!\{\xi_h\}\!\} \cdot [\![(\nabla_h \cdot \mathbf{u}_h)\mathbb{D}_{2\times 2}]\!]\, ds = \sum_{e\in\mathcal{E}_{\mathcal{I}}} \int_e \{\!\{\xi_h\}\!\} \cdot [\![\nabla_h \cdot \mathbf{u}_h]\!]\, ds = 0,
$$

and thus,

$$
\int_{\mathcal{E}_{\mathcal{I}}} \{\!\{\xi_h\}\!\} \cdot [\![\underline{\sigma}_h(\mathbf{u}_h)]\!]\, ds = 2\mu \int_{\mathcal{E}_{\mathcal{I}}} \{\!\{\xi_h\}\!\} \cdot [\![\underline{\varepsilon}_h(\mathbf{u}_h)]\!]\, ds.
$$

Moreover, observing that, for $e \in \mathcal{E}$, $[\![\mathbf{u}_h]\!]|_e \in \mathcal{P}_1(e)$, and applying again the definition of $\mathbf{I}_h$, yields

$$
\int_{\mathcal{E}} \mathsf{c}[\![\mathbf{u}_h]\!][\![\xi_h]\!]\, ds = \sum_{e\in\mathcal{E}} \int_e \mathsf{c}[\![\mathbf{u}_h]\!][\![\xi_h]\!]\, ds = 0.
$$

Substituting these identities into (4.21) results in

$$
\begin{aligned}
-E_2 &= -2\mu \int_{\mathcal{E}_{\mathcal{I}}} \{\!\{\xi_h\}\!\} \cdot [\![\underline{\varepsilon}_h(\mathbf{u}_h)]\!]\, ds + \int_\Omega \underline{\mathcal{L}}(\mathbf{u}_h) : \underline{\sigma}_h(\xi_h)\, d\mathbf{x} - \int_{\mathcal{E}} \mathsf{c}\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\xi_h]\!]}\, ds \\
&\le C \sum_{K\in\mathcal{T}_h} \big(\|\xi_h\|_{0,\partial K} \|[\![\underline{\varepsilon}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma} \\
&\qquad + \|\underline{\mathcal{L}}(\mathbf{u}_h)\|_{0,K} \|\underline{\sigma}_h(\xi_h)\|_{0,K} + \gamma h_K^{-1} \|\underline{[\![\mathbf{u}_h]\!]}\|_{0,\partial K} \|\xi_h\|_{0,\partial K}\big).
\end{aligned}
$$

Finally, in order to bound $E_3$, we use the fact that $\mathbf{w} \in H_0^1(\Omega)^2$ and Remark 4.3:

$$
\begin{aligned}
E_3 &= \int_\Omega \underline{\sigma}_h(\mathbf{u}_h^\perp) : \underline{\varepsilon}(\mathbf{w})\, d\mathbf{x} - \int_\Omega \underline{\mathcal{L}}(\mathbf{u}_h^\perp) : \underline{\sigma}(\mathbf{w})\, d\mathbf{x} \\
&= \int_\Omega \underline{\varepsilon}_h(\mathbf{u}_h^\perp) : \underline{\sigma}(\mathbf{w})\, d\mathbf{x} - \int_\Omega \underline{\mathcal{L}}(\mathbf{u}_h^\perp) : \underline{\sigma}(\mathbf{w})\, d\mathbf{x} \\
&\le \sum_{K\in\mathcal{T}_h} \left(\|\underline{\varepsilon}_h(\mathbf{u}_h^\perp)\|_{0,K} + \|\underline{\mathcal{L}}(\mathbf{u}_h^\perp)\|_{0,K}\right) \|\underline{\sigma}(\mathbf{w})\|_{0,K}.
\end{aligned}
$$

*Step 4:* Inserting the above estimates for $E_1, E_2, E_3$ into (4.20), applying Cauchy-Schwarz inequalities, recalling (3.6) and the stability (4.4) of the lifting operator $\underline{\mathcal{L}}$, and using the fact that $\underline{[\![\mathbf{u}_h^\perp]\!]} = \underline{[\![\mathbf{u}_h]\!]}$ on $\mathcal{E}$, we conclude that

$$|\mathbf{e}_h^c|_{1,\Omega}^2 \le C\Big( \sum_{K\in\mathcal{T}_h} h_K^2\|\mathbf{f}\|_{0,K}^2 + h_K\|[\![\underline{\varepsilon}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma}^2 + \gamma^2 h_K^{-1}\|\underline{[\![\mathbf{u}_h]\!]}\|_{0,\partial K}^2 + \|\underline{\varepsilon}(\mathbf{u}_h^\perp)\|_{0,K}^2\Big)^{1/2} \times \Big( \sum_{K\in\mathcal{T}_h} h_K^{-2}\|\xi_h\|_{0,K}^2 + \|\underline{\sigma}_h(\xi_h)\|_{0,K}^2 + h_K^{-1}\|\xi_h\|_{0,\partial K}^2 + \|\underline{\sigma}(\mathbf{w})\|_{0,K}^2\Big)^{1/2}.$$

The term $\|\underline{\varepsilon}(\mathbf{u}_h^\perp)\|_{0,K}^2$ is estimated using (4.17) as follows:

$$\sum_{K\in\mathcal{T}_h} \|\underline{\varepsilon}(\mathbf{u}_h^\perp)\|_{0,K}^2 \le C\|\mathbf{u}_h^\perp\|_{1,h}^2 \le C\int_{\mathcal{E}} \mathtt{c}|\underline{[\![\mathbf{u}_h]\!]}|^2\, ds.$$

By the definition of $\underline{\sigma}_h$, this leads to

$$|\mathbf{e}_h^c|_{1,\Omega}^2 \le C\Big( \sum_{K\in\mathcal{T}_h} \eta_K^2\Big)^{1/2}\Big( \sum_{K\in\mathcal{T}_h} h_K^{-2}\|\xi_h\|_{0,K}^2 + h_K^{-1}\|\xi_h\|_{0,\partial K}^2 + |\xi_h|_{1,K}^2 + \lambda^2\|\nabla\cdot\xi_h\|_{0,K}^2 + |\mathbf{w}|_{1,K}^2 + \lambda^2\|\nabla\cdot\mathbf{w}\|_{0,K}^2\Big)^{1/2}.$$

*Step 5:* Recalling that $\xi_h|_K = \mathbf{w} - \mathbf{I}_K\mathbf{w}$, $K\in\mathcal{T}_h$, where $\mathbf{I}_K$ is the BDM interpolant (4.13), and applying the approximation properties (a)–(c) from Lemma 4.5, leads to

$$|\mathbf{e}_h^c|_{1,\Omega}^2 \le C\Big( \sum_{K\in\mathcal{T}_h} \eta_K^2\Big)^{1/2}\Big(|\mathbf{w}|_{1,\Omega}^2 + \lambda^2\|\nabla\cdot\mathbf{w}\|_{0,\Omega}^2\Big)^{1/2}.$$

Estimating the second factor on the right-hand side of the above bound using (4.19), and dividing both sides of the resulting inequality by $|\mathbf{e}_h^c|_{1,\Omega}$, completes, together with (4.18), the proof of (4.1).

Finally, we need to show that the a posteriori error estimator $\sum_{K\in\mathcal{T}_h}\eta_K^2$ is bounded independently of $\lambda$. To do so, we note that it holds the inverse estimate

$$\sum_{K\in\mathcal{T}_h} h_K\|[\![\underline{\varepsilon}_h(\mathbf{u}_h)]\!]\|_{0,\partial K}^2 \le C\|\underline{\varepsilon}_h(\mathbf{u}_h)\|_{0,\Omega}^2,$$

and hence, we have

$$\sum_{K\in\mathcal{T}_h} \eta_K^2 \le C(\|\mathbf{u}_h\|_{DG}^2 + \|\mathbf{f}\|_{0,\Omega}^2),$$

where $\|\cdot\|_{DG}$ is the norm from Proposition 3.3. Therefore, it is sufficient to show that

$$\|\mathbf{u}_h\|_{DG} \le C\|\mathbf{f}\|_{0,\Omega}, \tag{4.22}$$

for a constant $C>0$ independent of $\lambda$.

Indeed, by the coercivity (3.5) of $a_h$, the definition (3.1) of $\mathbf{u}_h$, and the Cauchy-Schwarz inequality, we have

$$\|\mathbf{u}_h\|_{DG}^2 \le Ca_h(\mathbf{u}_h,\mathbf{u}_h) \le C\|\mathbf{f}\|_{0,\Omega}\|\mathbf{u}_h\|_{0,\Omega}.$$

Moreover, applying a broken Korn inequality (see [5]), implies

$$\|\mathbf{u}_h\|_{0,\Omega} \le C\Big(\|\underline{\varepsilon}_h(\mathbf{u}_h)\|_{0,\Omega}^2 + \int_{\mathcal{E}} \mathtt{h}^{-1}|\underline{[\![\mathbf{u}_h]\!]}|^2\Big)^{1/2},$$

and thus, we conclude

$$\|\mathbf{u}_h\|_{DG}^2 \leq C\|\mathbf{f}\|_{0,\Omega}\Big(\|\underline{\varepsilon}_h(\mathbf{u}_h)\|_{0,\Omega}^2 + \int_{\mathcal{E}} \mathtt{h}^{-1}|\underline{[\![\mathbf{u}_h]\!]}|^2\Big)^{1/2} \leq C\|\mathbf{f}\|_{0,\Omega}\|\mathbf{u}_h\|_{DG}.$$

Hence, dividing both sides by $\|\mathbf{u}_h\|_{DG}$, implies (4.22).

This completes the proof of Theorem 4.1. □

## 5. Numerical experiments

The goal of this section is to illustrate the practical performance of the a posteriori error estimator from Theorem 4.1 within an automatic mesh refinement procedure.

5.1. **Adaptive algorithm.** Starting from an initial (coarse) grid $\mathcal{T}_h^{(0)}$, our adaptive mesh refinement process operates in the following (widely used) way (see [23]):

(1) `Set` $k=0$.
(2) `Compute the DG solution` $\mathbf{u}_h^{(k)}$ `from` (3.1) `on` $\mathcal{T}_h^{(k)}$.
(3) `Compute the local error indicators` $\eta_K$, `according to` (4.2).
(4) `If` $\sum_{K\in\mathcal{T}_h}\eta_K^2$ `is sufficiently small then stop. Otherwise, find` $\eta_{\max}=\max_{K\in\mathcal{T}_h^{(k)}}\eta_K$, `and refine those elements` $K\in\mathcal{T}_h$ `with` $\eta_K>\theta\eta_{\max}$. `Set` $k=k+1$ `and go to (2).`

Here $0<\theta<1$ is a fixed threshold. In our numerical experiments, we always set $\theta=0.5$.

5.2. **Model problem.** Let $\Omega$ be the polygonal domain in Figure 1 with a reentrant corner at the origin $\mathcal{O}=(0,0)$. We set $\mu=1$ and $\mathbf{f}=\mathbf{0}$. Then, using polar coordinates $(r,\phi)$, we impose an appropriate boundary condition for $\mathbf{u}$ on $\partial\Omega$ such that

$$\mathbf{u}(r,\theta)=\frac{1}{2\mu}r^\alpha\begin{pmatrix}-(\alpha+1)\cos((\alpha+1)\theta)+(C_2-(\alpha+1))C_1\cos((\alpha-1)\theta)\\ (\alpha+1)\sin((\alpha+1)\theta)+(C_2+\alpha-1)C_1\sin((\alpha-1)\theta)\end{pmatrix}.$$

Here $\alpha=0.544484...$ is the solution of the equation

$$\alpha\sin(2\omega)+\sin(2\omega\alpha)=0,$$

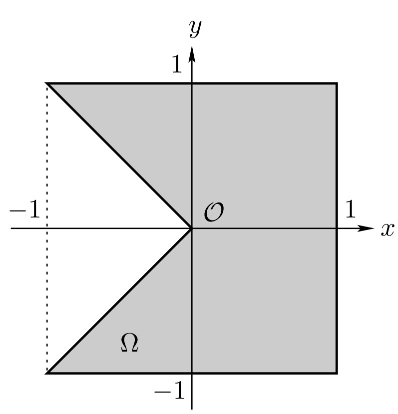


Figure 1. Polygonal domain $\Omega$

with $\omega = {}^{3\pi}/_{4}$, and $C_1, C_2$ are constants given by

$$C_1 = -\frac{\cos((\alpha+1)\omega)}{\cos((\alpha-1)\omega)}, \qquad C_2 = \frac{2(\lambda+2\mu)}{\lambda+\mu}.$$

Note that $\mathbf{u}$ is analytic inside the domain $\Omega$. However, $\nabla\mathbf{u}$ is singular at the origin; especially, $\mathbf{u} \notin H^2(\Omega)^2$. This solution exhibits the typical (singular) behavior of solutions of linear elasticity problems near reentrant corners.

5.3. **Discussion of the numerical results.** In all our computations, we set the stability constant $\gamma$ from (3.4) equal to 10 which, for the example under consideration, is sufficiently large to ensure the stability of the DG method (cf. Proposition 3.3). The adaptively refined meshes are constructed using the red-blue-green refinement strategy (see [23]). In order to incorporate the inhomogeneous boundary conditions on $\partial\Omega$, the jump terms $h_K^{-1}\|\underline{[\![\mathbf{u}_h]\!]}\|^2_{0,\partial K}$ in the local error indicators $\eta_K$ (for all $K \in \mathcal{T}_h$ with $\partial K \cap \partial\Omega \neq \emptyset$) have been modified appropriately.

Figure 2 shows the adaptively refined meshes for $\lambda = 1$ and $\lambda = 5000$ after 14 refinement steps. Both meshes have been strongly refined near the reentrant corner $\mathcal{O}$, thereby accounting for the singularity of the exact solution $\mathbf{u}$ at this point. We see that the refinement of the meshes is (nearly) circular around the origin and symmetric with respect to the $x$-axis.

The performance of the DGFEM with respect to the (global) $L^2$-norm $\|\cdot\|_{0,\Omega}$ and the DG energy norm $\|\cdot\|_{1,h}$ for $\lambda \in \{1, 10, 100, 1000, 5000\}$ is presented in Figure 3. We see that the convergence is optimal already for a reasonably low number of degrees of freedom. Moreover, we observe that the convergence regime of the DGFEM is (asymptotically) robust with respect to the parameter $\lambda$ as $\lambda \to \infty$.

In Figure 4 we compare the energy error of the DGFEM on adaptively and uniformly refined meshes. We see that, although the DGFEM still converges robustly with respect to $\lambda$ on uniformly refined meshes, the above-mentioned optimal convergence rate on adaptively refined meshes cannot be obtained due to the presence of the corner singularity; see also [26].

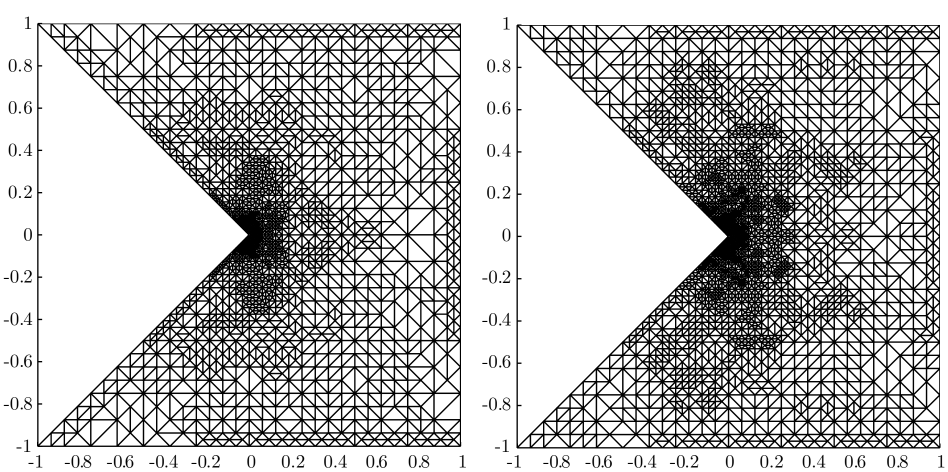


FIGURE 2. Meshes after 14 adaptive refinement steps for $\lambda = 1$ (left) and $\lambda = 5000$ (right)

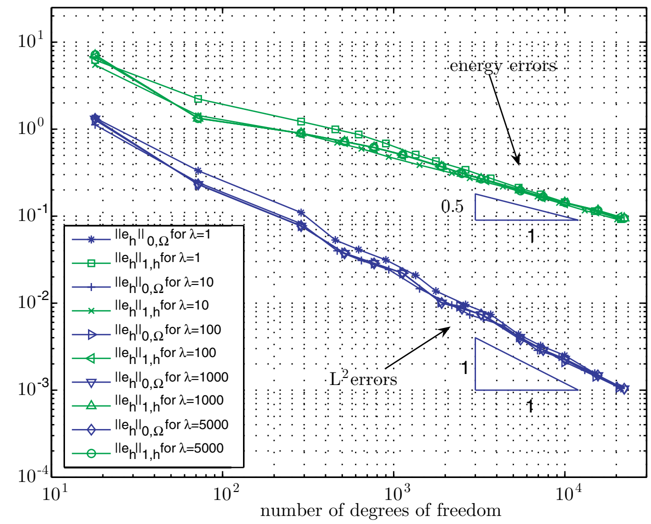


FIGURE 3. $L^2$ and DG energy norm of the error

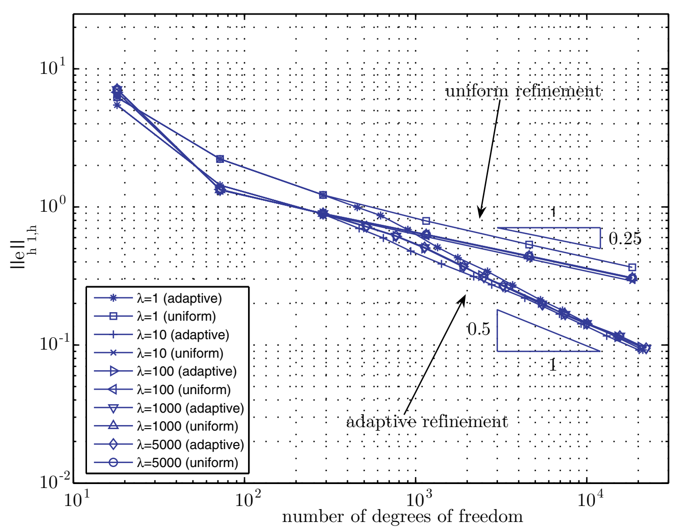


FIGURE 4. DG Energy norm of the error for uniform and adaptive mesh refinement

Finally, in Figure 5, we show a comparison of the actual and estimated DG energy norm of the error. We see that the error estimator over estimates the true error by an approximately consistent factor; indeed, we observe that the efficiency indices $i_{\text{eff}}$, defined by the ratio of the error estimator and the true error, lie in the range between 3 and 6. In particular, we see that, although the efficiency indices grow very slightly (a similar phenomenon has been observed in [15]), this range is (asymptotically) independent of the choice of $\lambda$, which again indicates the DGFEM's robustness with respect to volume locking.

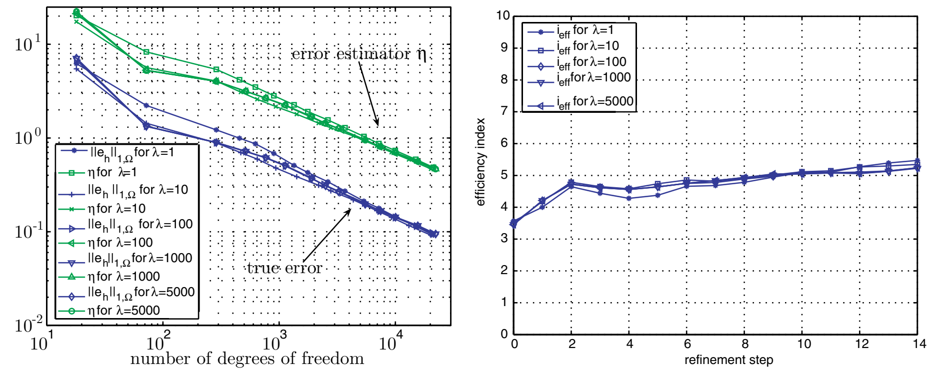


FIGURE 5. Left: Actual and estimated DG energy norm of the error. Right: Efficiency indices.

## 6. Conclusions

In this paper, we have presented an a posteriori error analysis of a discontinuous Galerkin FEM for linear elasticity problems. The main result is an a posteriori error estimate that is robust with respect to incompressible materials. Our numerical experiments show that the proposed error estimator converges to zero at the same asymptotic rate as the DG energy norm of the actual error on a sequence of (nonuniform) adaptively refined meshes. Moreover, the results confirm the absence of volume locking in the scheme.

Future work will include an extension of the low-order results in this paper to an *hp*-context; see also [16], where an *hp*-a posteriori error analysis for a mixed DG method for linear elasticity problems has been established.

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