Loewner Theory in Several Complex Variables and Related Problems

by

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Abstract

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The first part of the thesis deals with aspects of Loewner theory in several complex variables. First we show that a Loewner chain with minimal regularity assumptions ($Df(0,\cdot)$ of local bounded variation) satisfies an associated Loewner equation. Next we give a way of renormalizing a general Loewner chain so that it corresponds to the same increasing family of domains. To do this we will prove a generalization of the converse of Carathéodory's kernel convergence theorem. Next we address the problem of finding a Loewner chain solution to a given Loewner chain equation. The main result is a complete solution in the case when the infinitesimal generator satisfies $Dh(0,t) = A$ where $\inf\{\mathrm{Re}\langle Az, z\rangle : \|z\| = 1\} > 0$. We will see that the existence of a bounded solution depends on the real resonances of A, but there always exists a polynomially bounded solution. Finally we discuss some properties of classes of biholomorphic mappings associated to A-normalized Loewner chains. In particular we give a characterization of the compactness of the class of spirallike mappings in terms of the resonance of A.

The second part of the thesis deals with the problem of finding examples of extreme points for some classes of mappings. We see that straightforward generalizations of one dimensional extreme functions give examples of extreme Carathéodory mappings and extreme starlike mappings on the polydisc, but not on the ball. We also find examples of extreme Carathéodory mappings on the ball starting from a known example of extreme Carathéodory function in higher dimensions.

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Contents

Chapter 1

Loewner Theory

1.1 Introduction

Loewner's theory of infinitesimal methods for univalent functions, as originally introduced by C. Loewner and later developed by P. P. Kufarev and C. Pommerenke, is one of the main tools in geometric function theory. For a detailed history of the evolution of Loewner's theory and its applications see [ABCDM10]. The study of Loewner's theory in several complex variables, originated by J. Pfaltzgraff [Pfa74], is naturally motivated by its success in one variable. For an account of the development of the higher dimensional theory from 1974 to 2003 see [GK03] and [GKP04].

To discuss the more recent developments let us first recall the basic objects of Loewner theory. Let B^n be the Euclidean unit ball in $\mathbb{C}^n$. If f and g are two mappings defined on B^n with values in $\mathbb{C}^n$ we say that f is subordinated to g, denoted by $f \prec g$, if $f(0) = g(0) = 0$ and $f(B^n) \subseteq g(B^n)$. We say that a mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is a *subordination chain* if $f(0,t) = 0$, $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$, whenever $0 \leq s \leq t < \infty$. A subordination chain is called a *Loewner chain* if in addition $f(\cdot, t)$ is biholomorphic on B^n for all $t \geq 0$. We will sometimes use the notation $f_t(z) := f(z,t)$. If f is a Loewner chain then the mapping defined by $v(z,s,t) = f_t^{-1}(f_s(z))$ is called the *transition mapping* associated to the Loewner chain. The main property of a Loewner chain is that it satisfies a partial differential equation, called the Loewner chain equation,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)\,h(z,t)\,,\ z \in B^n,\ a.e.\,t \in [0,\infty), \tag{1.1.1}$$

where D denotes Fréchet differentiation with respect to the complex variable. The mapping h is called the *infinitesimal generator* of the Loewner chain. The transition mapping associated to the chain satisfies an ordinary differential equation (with an initial value), called the Loewner equation,

$$\frac{\partial v}{\partial t} = -h(v,t) \quad a.e.\,t \geq s, v(z,s,s) = z,\ s \geq 0. \tag{1.1.2}$$

In order to obtain these equations and other results about Loewner chains we need to impose a regularity condition on the chain. It turns out that it is enough to impose a regularity condition on $Df(0,t)$. In analogy with the one dimensional theory, Loewner chains in higher dimensions were initially studied using the normalization $Df(0,t) = e^t I$ (or $Df(0,t) = \phi(t)\,I$ where ϕ is such that it can be reparametrized to e^t). Unlike the one variable situation, in higher dimensions it is not true that the case of non-normalized Loewner chains can be reduced, through a reparametrization of t, to the $Df(0,t) = e^t I$ case (see [DGHK10, p 413]). This justifies the study of Loewner chains with more general normalization or, even better, with no normalization at all. In particular, in [GHKK08a] Loewner chains are studied assuming that $Df(0,t) = e^{tA}$ and in [GHKK08b] Loewner chains are studied assuming that $Df(0,t) = e^{\int_0^t A(\tau)d\tau}$ (with certain restrictions on A and $A(\cdot)$). Very recently Loewner chains have also been studied on abstract complex manifolds without any normalization in [ABHK10] (see also [BCDM09]).

The purpose of this chapter is to add to the study of Loewner chains in [GHKK08b] and [GHKK08a]. We will discuss aspects that are not covered by the more general work from [ABHK10] and [BCDM09]. In Section 1.2 we collect some basic facts that will be needed throughout the first part of the chapter. In Section 1.3 we show that a Loewner chain with minimal regularity assumptions ($Df(0,\cdot)$ is of local bounded variation) satisfies an associated Loewner equation. This requires a discussion of the connection between the regularities of $Df(0,\cdot)$, f and v. In Section 1.4 we give a way of renormalizing a general Loewner chain so that it corresponds to the same increasing family of domains. To do this we will prove a generalization of the converse of Carathéodory's kernel convergence theorem. We then concentrate on the problem of finding a Loewner chain solution to a given Loewner chain equation. This also includes giving conditions that determine a solution uniquely. We start with the general case in Section 1.5, where we prove results that are very similar to those in [GHKK08b] but using less assumptions. The main result of the chapter is a complete solution, in Section 1.6, for the case when the infinitesimal generator satisfies $Dh(0,t) = A$ where $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ is such that $m(A) > 0$. Finally, in Section 1.7, we discuss some properties of classes of biholomorphic mappings associated to Loewner chains that satisfy $Df(0,t) = e^{tA}$. The last two sections of this chapter are based on the results published in [Vod11].

1.2 Preliminaries

We start with a short review of some useful facts about the nonstationary (homogeneous) abstract Cauchy problem in Banach spaces following [DK74, Chapter III]. Consider the equation

$$\frac{dx}{dt} = A(t)x \tag{1.2.1}$$

satisfied at almost all points of an interval J (finite or infinite), where x takes values in a Banach space $\mathcal{B}$ and $A(t)$ takes values in $L(\mathcal{B},\mathcal{B})$ (which is also a Banach space). We will be assuming that $A(t)$ is strongly measurable and Bochner integrable on the finite subintervals of J (in the finite dimensional case this condition on $A(t)$ amounts to its being measurable and locally integrable in the Lebesgue sense). Under these assumptions, the equation (1.2.1) has a unique locally absolutely continuous solution on J satisfying the initial condition $x(t_0) = x_0$. Furthermore we have

$$\max_{s\in[a,b]} \|x(s)\| \leq \|x(t_0)\| \, e^{\int_a^b \|A(\tau)\| d\tau}, t_0 \in [a,b] \subseteq J. \tag{1.2.2}$$

In particular this applies to the equation

$$\frac{dU}{dt} = A(t)U, \quad U(t_0) = I \tag{1.2.3}$$

where U takes values in $X := L(\mathcal{B}, \mathcal{B})$. Note that in this situation the operator $A(t)$ from equation (1.2.1) is in fact $A_l(t)$ taking values in $L(X, X)$, defined by $A_l(t)(U) = A(t)U$ for all $U \in X$. It is not hard to see that $\|A(t)\| = \|A_l(t)\|$ and that measurability and integrability are equivalent for $A(t)$ and $A_l(t)$. The same goes for $A_r(t) = UA(t)$. The estimate (1.2.2) and the above discussion yields that

$$\|U(t)\| \leq e^{\int_{t_0}^t \|A(\tau)\| d\tau}, \, t \geq t_0. \tag{1.2.4}$$

One can also consider the so called *adjoint associate equation* to (1.2.3):

$$\frac{dV}{dt} = -VA(t), \quad V(t_0) = I.$$

It is not difficult to verify that $U(t)^{-1}$ exists and $V(t) = U(t)^{-1}$. Since $\|-A(\tau)\| = \|A(\tau)\|$ we also get that

$$\|V(t)\| \leq e^{\int_{t_0}^t \|A(\tau)\| d\tau}, \, t \geq t_0. \tag{1.2.5}$$

Using the uniqueness of solutions for (1.2.1), one can see that the linear evolution family associated to $A(t)$, i.e. the family of linear operators $U(s,t)$ satisfying

$$\frac{\partial U(s,t)}{\partial t} = A(t)U(s,t) \quad a.e. \, t \geq s, U(s,s) = I \tag{1.2.6}$$

is given by $U(s,t) = U(t)U(s)^{-1}$ where U solves (1.2.3). Note that estimates (1.2.4) and (1.2.5) give us

$$\|U(s,t)\| \leq e^{\int_s^t \|A(\tau)\| d\tau} \, (s \leq t) \tag{1.2.7}$$

and

$$\|U(s,t)^{-1}\| \leq e^{\int_s^t \|A(\tau)\| d\tau} \, (s \leq t). \tag{1.2.8}$$

Next we specialize some of the above estimates to the case of $\mathbb{C}^n$ with Euclidean norm. For a given operator $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ let $m(A) := \inf\{\operatorname{Re}\langle A(z), z\rangle : \|z\| = 1\}$, $k(A) := \sup\{\operatorname{Re}\langle \mathrm{A(z), z}\rangle : \|z\| = 1\}$ and $|V(A)| := \sup\{|\langle A(z), z\rangle| : \|z\| = 1\}$. If we let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be measurable and locally integrable and U the solution to $dU/dt = A(t)U$, with $U(0) = I$ then estimates (1.5) and (1.6) from [GHKK08b] become:

Proposition 1.2.1.

$$e^{\int_0^t m(A(\tau))d\tau} \leq \|U(t)\| \leq e^{\int_0^t k(A(\tau))d\tau}, \quad t \in [0, \infty)$$

and

$$e^{-\int_0^t k(A(\tau))d\tau} \leq \|U(t)^{-1}\| \leq e^{-\int_0^t m(A(\tau))d\tau}, \quad t \in [0, \infty).$$

Proof. The first claim follows exactly as (1.5) in [GHKK08b]. For the second claim, let $V(t) = U(t)^{-1}$. From the discussion above we know that we have $dV/dt = -VA(t)$. We cannot apply the same idea as for the first claim to V (unless we assume that V commutes with $A(t)$), but if we consider the adjoint V^* of V we have that $dV^*/dt = -A^*(t)V^*$. Now the result follows from the first claim because $\|V^*\| = \|V\|$, $m(-A^*(t)) = -k(A^*(t)) = -k(A(t))$ and $k(-A^*(t)) = -m(A^*(t)) = -m(A(t))$ by properties of the adjoint. $\square$

Remark 1.2.2. It can be easily deduced that we also have the more general estimate

$$e^{\int_s^t m(A(\tau))d\tau} \leq \|U(s,t)\| \leq e^{\int_s^t k(A(\tau))d\tau}, \quad t \in [s, \infty). \tag{1.2.9}$$

Also, the same result holds if $dU/dt = UA(t)$ (rather than $A(t)U$).

We will see that general Loewner chains have infinitesimal generators whose values are in the class

$$\mathcal{N}_0 = \{h \in H(B^n) : h(0) = 0, \operatorname{Re}\langle h(z), z\rangle \geq 0, z \in B^n \backslash \{0\}\}.$$

rather than in the class

$$\mathcal{N} = \{h \in H(B^n) : h(0) = 0, \operatorname{Re}\langle h(z), z\rangle > 0, z \in B^n \backslash \{0\}\}.$$

We record some basic properties of $\mathcal{N}_0$.

Proposition 1.2.3. *(i)* $\mathcal{N}_0 = \overline{\mathcal{N}}$

(ii) If $h \in \mathcal{N}_0$ *then* $m(Dh(0)) \geq 0$ *and if* $h \in \mathcal{N}$ *then* $m(Dh(0)) > 0$.

(iii) If $h \in \mathcal{N}_0$ *and* $A = Dh(0)$ *then*

$$\operatorname{Re}\langle A(z), z\rangle \frac{1-\|z\|}{1+\|z\|} \leq \operatorname{Re}\langle h(z), z\rangle \leq \operatorname{Re}\langle A(z), z\rangle \frac{1+\|z\|}{1-\|z\|}, z \in B^n \tag{1.2.10}$$

and

$$\|h(z)\| \leq \frac{4\|z\|}{(1-\|z\|)^2} |V(A)| \leq \frac{4\|z\|}{(1-\|z\|)^2} \|A\|, \; z \in B^n. \tag{1.2.11}$$

Proof. (i) is clear because of the simple remark that given $h \in \mathcal{N}_0$ we have that $h + \epsilon I \in \mathcal{N}$ for all $\epsilon > 0$.

For (ii) suppose that $h \in \mathcal{N}_0$ and $m(Dh(0)) < 0$. So there exists z_0, $\|z_0\| = 1$ such that $\operatorname{Re}\langle A(z_0), z_0\rangle < 0$. Using Taylor series we see that near 0 we have that

$$\begin{aligned} \frac{1}{|\zeta|^2}\operatorname{Re}\langle h(\zeta z_0), \zeta z_0\rangle &= \frac{1}{|\zeta|^2}\operatorname{Re}\left\langle \zeta\left[A(z_0) + \frac{o(\|\zeta z_0\|)}{\zeta}\right], \zeta z_0\right\rangle \\ &= \operatorname{Re}\langle A(z_0), z_0\rangle + \operatorname{Re}\left\langle \frac{o(\|\zeta z_0\|)}{\zeta}, z_0\right\rangle. \end{aligned}$$

So, for small enough ζ we get that $\operatorname{Re}\langle h(\zeta z_0), \zeta z_0\rangle < 0$ contradicting $h \in \mathcal{N}_0$.

The result for $h \in \mathcal{N}$ follows immediately from the result above and [GK03, Lemma 6.1.30]

The estimates from (iii) are known for the class $\mathcal{N}$. For (1.2.10) see the proof of [Gur75, Lemma 1] and for (1.2.11) see [GHKK08a, Lemma 1.2]. The results for $\mathcal{N}_0$ follow from the results for $\mathcal{N}$ by using (i) and the continuity of the terms involving A. □

A family of holomorphic mappings that is locally uniformly bounded is also locally uniformly Lipschitz. This simple fact will be used throughout this chapter and we provide a proof bellow.

Lemma 1.2.4. *Let* f *be a holomorphic mapping on* B^n *such that* $\|f(z)\| \leq M_r$, $\|z\| \leq r$. *Then there exists a constant* C_r *(depending on* M *but not on* f*) such that*

$$\|f(z) - f(w)\| \leq C_r \|z - w\|, \; \|z\|, \|w\| \leq r.$$

Proof. If $\|z\| \le r < R < 1$ and $\|h\| \le 1$ then from Cauchy's integral formula we have

$$Df(z)\,h = \frac{1}{2\pi i}\int_{|\zeta|=R-r}\frac{f(z+\zeta h)}{\zeta}d\zeta$$

and hence

$$\|Df(z)\| \le \frac{M_R}{R-r},\ \|z\| \le r,\ 1 > R > r.$$

Using the integral formula for the remainder of the Taylor series and the above inequality we get

$$\begin{aligned} \|f(z)-f(w)\| &= \left\|\int_0^1 Df(w+t(z-w))(z-w)dt\right\| \\ &\le \frac{M_R}{R-r}\|z-w\|,\ \|z\|,\|w\| \le r,\ 1>R>r. \end{aligned}$$

The conclusion follows by letting $R = (1+r)/2$. □

We will also need some results related to positive definite matrices. For convenience we quote them from [HJ90]. M_n will denote the space of $n \times n$ complex matrices.

Proposition 1.2.5. *([HJ90] Theorem 7.2.1) A Hermitian matrix $A \in M_n$ is positive semidefinite if and only if all of its eigenvalues are nonnegative. It is positive definite if and only if all of its eigenvalues are positive.*

Let $A, B \in M_n$ be Hermitian matrices. We write $A \ge B$ ($A > B$) if the matrix $A - B$ is positive semidefinite (positive definite).

For a matrix $A \in M_n$, $\rho(A)$ denotes its spectral radius.

Theorem 1.2.6. *([HJ90] Theorem 7.7.3) Let $A, B \in M_n$ be Hermitian matrices, and suppose A is positive definite and B is positive semidefinite. Then $A \ge B$ if and only if $\rho(BA^{-1}) \le 1$, and $A > B$ if and only if $\rho(BA^{-1}) < 1$.*

Corollary 1.2.7. *([HJ90] Corollary 7.7.4) If $A, B \in M_n$ are positive definite, then:*

1. *$A \ge B$ if and only if $B^{-1} \ge A^{-1}$;*

2. *If $A \geq B$, then* $\det A \geq \det B$ *and* $\mathrm{tr} A \geq \mathrm{tr} B$*;*

3. *More generally, $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, 2, \ldots, n$ if the respective eigenvalues of A and B are arranged in the same (increasing or decreasing) order.*

Theorem 1.2.8. *([HJ90] Corollary 7.3.3) If $A \in M_n$, then it may be written in the form $A = PU$ where P is positive semidefinite and U is unitary. The matrix P is always uniquely determined as $P = (AA^*)^{1/2}$; if A is nonsingular , then U is uniquely determined as $U = P^{-1}A$. If A is real, then P and U may be taken to be real.*

1.3 Loewner chains without normalization

If $f(z,t)$ is a Loewner chain and $v(z,s,t)$ is its transition mapping then from the definitions it immediately follows that $v(\cdot, s, t)$ is a Schwarz mapping (i.e. a self-map of B^n fixing the origin) such that $Dv(0,s,t) = Df(0,t)^{-1} Df(0,s)$. Furthermore it is easy to check that the transition mapping satisfies the following semigroup property

$$v(v(z,s,t_1), t_1, t_2) = v(z,s,t_2),\ z \in B^n,\ 0 \leq s \leq t_1 \leq t_2. \tag{1.3.1}$$

First we want to prove Proposition 1.3.4 that collects some useful estimates relating a Loewner chain and its transition mapping (these are inspired by the proof of [GK03, Theorem 8.1.8]). For this we will need a consequence of the following result.

Proposition 1.3.1. *Let f and g be holomorphic mappings on B^n such that $f(0) = g(0) = 0$ and $Dg(0)$ is invertible. If $f \prec g$ then $P_f \leq P_g$, where P_f and P_g are the unique positive semidefinite matrices from the polar decomposition of f and respectively g.*

Proof. From $f \prec g$ we know that there exists a Schwarz mapping w such that $f = g \circ w$. Hence $Dg(0)^{-1}Df(0) = Dw(0)$. Let $Df(0) = P_f U_f$ and $Dg(0) = P_g U_g$ be polar decompositions. Since $Dg(0)$ is invertible, P_g is in fact positive definite. We have that $U_g^* P_g^{-1} P_f U_f = Dw(0)$ and hence $P_g^{-1} P_f = U_g Dw(0) U_f^*$. From the Carathéodory-Cartan-Kaup-Wu theorem (see [Kra01, Theorem 11.3.1]) applied to $U_g \circ w \circ U_f^*$ it follows that $\rho\left(P_g^{-1}P_f\right) \leq 1$. Now the result follows from Theorem 1.2.6 (we are using the known fact that AB and BA have the same eigenvalues and hence the same spectral radius). □

Corollary 1.3.2. *If $f(z,t)$ is a Loewner chain then $Df(0,\cdot)^{-1}$ is bounded and there exists a constant C such that*

$$\left\|Df(0,t_1)^{-1} - Df(0,t_2)^{-1}\right\| \leq C\left\|Df(0,t_1) - Df(0,t_2)\right\|.$$

Proof. Let $Df(0,t) = P_tU_t$ be the polar decomposition of $Df(0,t)$. From the previous proposition we have that $P_{t_1} \leq P_{t_2}$ whenever $t_1 \leq t_2$. In particular we have that $P_0 \leq P_t$ and hence $P_t^{-1} \leq P_0^{-1}$. This implies that $\left\|P_t^{-1}\right\| \leq \left\|P_0^{-1}\right\|$. Since $\left\|Df(0,t)^{-1}\right\| = \left\|U_t^*P_t^{-1}\right\| = \left\|P_t^{-1}\right\|$ we have proved the first claim. The second claim follows from the first one and the identity

$$Df(0,t_1)^{-1} - Df(0,t_2)^{-1} = Df(0,t_1)^{-1}\left(Df(0,t_2) - Df(0,t_1)\right)Df(0,t_2)^{-1}.$$

□

Remark 1.3.3. The proof of the above result relied on the particular choice of matrix norm (we used the unitary invariance of the norm), but clearly the results also hold for any other norm (since all norms on M_n are equivalent).

Proposition 1.3.4. *Let $f(z,t)$ be a Loewner chain and let $v(z,s,t)$ be its transition mapping. Then the following estimates hold:*

(i) For all $\|z\| \leq r$, $0 \leq s \leq t_1, t_2$ we have that

$$\|v(z,s,t_1) - v(z,s,t_2)\| \leq C_r\|Df(0,t_1) - Df(0,t_2)\|.$$

(ii) For all $\|z\| \leq r$, $0 \leq s_1 \leq s_2 \leq t$ we have that

$$\begin{aligned}\|v(z,s_1,t) - v(z,s_2,t)\| &\leq C_r\|v(z,s_1,s_1) - v(z,s_1,s_2)\| \\ &\leq C_r\|Df(0,s_1) - Df(0,s_2)\|.\end{aligned}$$

(iii) For all $\|z\| \le r$, $0 \le t_1, t_2 \le T$ *we have that*

$$\begin{aligned}\|f(z,t_1) - f(z,t_2)\| &\le C_{r,T} \|v(z,t_1,T) - v(z,t_2,T)\| \\ &\le C_{r,T} \|Df(0,t_1) - Df(0,t_2)\|.\end{aligned}$$

Proof. Suppose $0 \le t_1 \le t_2$. Since $v(\cdot, t_1, t_2)$ is a Schwarz mapping we have that

$$\operatorname{Re}\langle z - v(z,t_1,t_2), z\rangle = \|z\|^2 - \operatorname{Re}\langle v(z,t_1,t_2), z\rangle \ge \|z\|^2 - \|v(z,t_1,t_2)\| \|z\| \ge 0.$$

Hence $z - v(z,t_1,t_2)$ is a mapping in $\mathcal{N}_0$ and using the estimate (1.2.11) we get that

$$\begin{aligned}\|z - v(z,t_1,t_2)\| &\le C_r \|I - Dv(0,t_1,t_2)\| = C_r \left\|I - Df(0,t_2)^{-1}Df(0,t_1)\right\| \\ &\le C_r \left\|Df(0,t_2)^{-1}\right\| \|Df(0,t_2) - Df(0,t_1)\| \\ &\le C_r \|Df(0,t_1) - Df(0,t_2)\|, \|z\| \le r.\end{aligned}$$

For the last inequality we used Proposition 1.3.1. Since $v(\cdot, s, t_1)$ is a Schwarz mapping we can replace z by $v(z,s,t_1)$ in the above inequality and we get the estimate (i) (by using the semigroup property of the transition mapping).

Using the fact that the mappings $v(\cdot,s,t)$, $0 \le s \le t$ are Schwarz mappings, hence locally uniformly bounded, we get that (see Lemma 1.2.4)

$$\|v(z,s,t) - v(w,s,t)\| \le C_r \|z - w\|, \; \|z\|, \|w\| \le r, \; 0 \le s \le t.$$

Replace s by s_2 and w by $v(z,s_1,s_2)$ in the above inequality and use the semigroup property for the transition mapping and the estimate from (i) to get that

$$\begin{aligned}\|v(z,s_2,t) - v(z,s_1,t)\| &\le C_r \|v(z,s_1,s_1) - v(z,s_1,s_2)\| \\ &\le C_r \|Df(0,s_1) - Df(0,s_2)\|, \; \|z\| \le r.\end{aligned}$$

Using the fact that $f(\cdot, T)$ is locally Lipschitz we easily get the last estimate

$$\begin{aligned}\|f(z,t_1) - f(z,t_2)\| &= \|f(v(z,t_1,T),T) - f(v(z,t_2,T),T)\| \\ &\le C_{r,T} \|v(z,t_1,T) - v(z,t_2,T)\|, \; \|z\| \le r.\end{aligned}$$

$\square$

Now we can see how imposing regularity conditions on $Df(0,\cdot)$ yields the regularity of the entire Loewner chain.

Corollary 1.3.5. *Let $f(z,t)$ be a Loewner chain and $v(z,s,t)$ be its transition mapping. Then the following statements are equivalent:*

(i) *$Df(0,\cdot)$ is continuous (of local bounded variation, locally absolutely continuous, locally Lipschitz) on $[0,\infty)$.*

(ii) *$f(z,\cdot)$ is continuous (of local bounded variation, locally absolutely continuous, locally Lipschitz) on $[0,\infty)$, locally uniformly with respect to z.*

(iii) *For all $s\geq 0$, $v(z,s,\cdot)$ is continuous (of local bounded variation, locally absolutely continuous, locally Lipschitz) on $[s,\infty)$, locally uniformly with respect to z, uniformly with respect to s.*

(iv) *For all $t>0$, $v(z,\cdot,t)$ is continuous (of bounded variation, absolutely continuous, Lipschitz) on $[0,t]$, locally uniformly with respect to z.*

Proof. By the estimates from Proposition 1.3.4 it is clear that (i) implies all the other statements. Using Cauchy's formula it immediately follows that (ii) implies (i) . So it is also true that (ii) implies (iii). (iii) implies (iv) and (iv) implies (i) are immediate consequences of the estimates (ii) and respectively (iii) from Proposition 1.3.4. □

Now we are able to check that a Loewner chain with minimal regularity assumptions satisfies an associated Loewner chain equation. Let $\mathcal{H}_0$ be the class of mappings $h: B^n\times[0,\infty)\to\mathbb{C}^n$ such that $h(z,\cdot)$ is measurable on $[0,\infty)$ for all $z\in B^n$ and $h(\cdot,t)\in\mathcal{N}_0$ for all $t\in[0,\infty)$. Let $\mathcal{H}$ be the class of mappings $h\in\mathcal{H}_0$ such that $h(\cdot,t)\in\mathcal{N}$ for all $t\in[0,\infty)$.

Proposition 1.3.6. *Let $f(z,t)$ be a Loewner chain such that $Df(0,\cdot)$ is of local bounded variation. Then $\partial_t f(\cdot,t)$ exists and is holomorphic on B^n for a.e. $t\geq 0$ and*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t),\ z \in B^n,\ a.e.\, t \geq 0 \tag{1.3.2}$$

where h is a mapping in $\mathcal{H}_0$.

Proof. Let $v(z,s,t)$ be the transition mapping associated to $f(z,t)$. We have that

$$\frac{f(z,t+\epsilon)-f(z,t)}{\epsilon} = \frac{f\left(z,t+\epsilon\right)-f\left(v\left(z,t,t+\epsilon\right),t+\epsilon\right)}{\epsilon} = A(z,t,\epsilon)\left(\frac{z-v(z,t,t+\epsilon)}{\epsilon}\right)$$

where by Taylor's formula

$$A(z,t,\epsilon) = \int_0^1 Df\left(z+\tau(v\left(z,t,t+\epsilon\right)-z),t+\epsilon\right)d\tau.$$

We claim that as $\epsilon \to 0+$ we have that $A\left(z,t,\epsilon\right) \to Df\left(z,t\right)$ for all $z \in B^n$ and for $a.e.\, t \geq 0$ (in fact the exceptional set is countable). Indeed, we have that

$$\begin{aligned}\left\|A\left(z,t,\epsilon\right)-Df\left(z,t\right)\right\| \leq \int_0^1 &\left\|Df\left(z+\tau(v\left(z,t,t+\epsilon\right)-z),t+\epsilon\right)-Df\left(z,t+\epsilon\right)\right\|d\tau \\ &+\left\|Df\left(z,t+\epsilon\right)-Df\left(z,t\right)\right\|.\end{aligned}$$

Fix $T > t$. Then from Cauchy's integral formula and Proposition 1.3.4 it follows that

$$\left\|Df\left(z,t+\epsilon\right)-Df\left(z,t\right)\right\| \leq C_{r,T}\left\|Df\left(0,t+\epsilon\right)-Df\left(0,t\right)\right\|,\ \|z\| \leq r, \epsilon \leq T-t.$$

It is easy to see that the family $\left\{f\left(\cdot,t\right)\right\}_{t\in[0,T]}$ is locally uniformly bounded. Indeed, if M_r is such that $\left\|f\left(z,T\right)\right\| \leq M_r$ for $\|z\| \leq r$, then using the fact that the $v\left(\cdot,t,T\right)$ are Schwarz mappings we also have that

$$\left\|f\left(z,t\right)\right\| = \left\|f\left(v\left(z,t,T\right),T\right)\right\| \leq M_r,\ \|z\| \leq r.$$

Now Cauchy's integral formula and Lemma 1.2.4 imply that

$$\left\|Df\left(w,t\right)-Df\left(z,t\right)\right\| \leq C_{r,T}\left\|z-w\right\|,\ \|z\|,\|w\| \leq r, t \in \left[0,T\right].$$

Using Proposition 1.3.4 again we can conclude that

$$\left\|A\left(z,t,\epsilon\right)-Df\left(z,t\right)\right\| \leq C_{r,T}\left\|Df\left(0,t+\epsilon\right)-Df\left(0,t\right)\right\|,\ \|z\| \leq r, \epsilon \leq T-t.$$

This shows that $A(z,t,\epsilon) \to Df(z,t)$ for all $z \in B^n$ whenever t is a point of continuity for $Df(0,\cdot)$. This proves the claim, since it is known that functions of bounded variation are continuous except for a countable set.

From the estimates in Proposition 1.3.4 we see that the quotients $(f(z,t+\epsilon)-f(z,t))/\epsilon$ and $(z-v(z,t,t+\epsilon))/\epsilon$ are locally uniformly bounded in z in neighborhoods of points t where $Df(0,\cdot)$ is differentiable. But $Df(0,\cdot)$ is of bounded variation and so it has a derivative almost everywhere. By the same arguments as in [GK03, Theorem 8.1.9] we get that, due to the fact that $f(z,\cdot)$ and $v(z,s,\cdot)$ are of local bounded variation and hence have derivatives in t *a.e.*, the limits

$$h(z,t) := \lim_{\epsilon\to 0}\frac{z-v(z,t,t+\epsilon)}{\epsilon} \tag{1.3.3}$$

$$\frac{\partial f}{\partial t}(z,t) := \lim_{\epsilon\to 0}\frac{f(z,t+\epsilon)-f(z,t)}{\epsilon} \tag{1.3.4}$$

exist for almost all $t \in [0,\infty)$ locally uniformly in z (to apply Vitali's theorem as in [GK03, Theorem 8.1.9] we need first to restrict attention to points t where $Df(0,\cdot)$ is differentiable). Now we have that equation (1.3.2) is satisfied at all points where $A(z,t,\epsilon) \to Df(z,t)$ and where the limits (1.3.3) and (1.3.4) exist, hence it holds almost everywhere. We just need to check that $h(z,t)$ has the claimed properties. By defining $h(\cdot,t)$ to be some arbitrary mapping from $\mathcal{N}_0$ at points where the limit doesn't exist, we get that for each $z \in B$, $h(z,\cdot)$ is a measurable mapping on $[0,\infty)$ (because it is the pointwise *a.e.* limit of a sequence of measurable mappings). Since $v(\cdot,s,t)$ is a Schwarz mapping we get that $\mathrm{Re}\,\langle z-v(z,t,t+\epsilon),z\rangle \geq 0$ and hence $h(\cdot,t) \in \mathcal{N}_0$. $\square$

Let $F(z,t) = (f(z_1,t),z_2)$ where $f(z_1,t)$ is a normalized Loewner chain ($f'(0,t)=e^t$). It is easy to see that the transition mapping is defined by $V(z,s,t) = (v(z_1,s,t),z_2)$ and the infinitesimal generator is $H(z,t) = (h(z_1,t),0)$ where $v(z_1,s,t)$ and $h(z_1,t)$ are the transition mapping and respectively the infinitesimal generator for $f(z_1,t)$. This example shows that for Loewner chains in general we need to allow for the infinitesimal generators to be in $\mathcal{H}_0$ rather than in $\mathcal{H}$ (as considered in [GHKK08b]).

1.4 Loewner chains with positive definite normalization

The goal of this section is to provide an analogue in higher dimensions of the fact that in one variable we can renormalize a Loewner chain so that it satisfies $Df(0,t) = e^t$. We will be able to do so after we generalize one of the implications of the Carathéodory kernel convergence theorem. We will need the following lemma.

Lemma 1.4.1. *Let G be a domain biholomorphic to the unit ball B^n and let $w \in G$. Then there exists a unique biholomorphism $f : B^n \to G$ such that $f(0) = w$ and $Df(0) > 0$.*

Proof. Let g be a biholomorphism between B^n and G and ϕ an automorphism of B^n such that $\phi(0) = g^{-1}(w)$. If U is a unitary matrix then $f := g \circ \phi \circ U$ is a biholomorphism between B^n and G such that $f(0) = w$ and furthermore by Theorem 1.2.8 U can be chosen so that $Df(0) > 0$.

Suppose that f and g are two biholomorphisms between B^n and G such that $f(0) = g(0) = w$ and $Df(0), Dg(0) > 0$. Then $g^{-1} \circ f$ is an automorphism of B^n that fixes 0, hence $g^{-1} \circ f = U$ where U is a unitary mapping. This yields that $Df(0) = Dg(0)U$. By the uniqueness part of Theorem 1.2.8 it follows that $U = I$ and hence $f = g$. □

Let $\{G_k\}$ be a sequence of open subsets of $\mathbb{C}^n$. We say that G is the *kernel* of $\{G_k\}$ if G is the largest open set such that for any compact set $K \subset G$ there exists $k_0 = k_0(K)$ such that $K \subset G_k$ for all $k \geq k_0$. We say that $\{G_k\}$ converges in kernel to G, denoted $G_k \to G$, if every subsequence of $\{G_k\}$ has the same kernel G. It is known that if we have a sequence of biholomorphic mappings $\{f_k\}$ on B^n that converge locally uniformly to a biholomorphism f on B^n then $f_k(B^n) \to f(B^n)$ (this is the direct implication of Carathéodory's kernel convergence theorem). See [ABHK10, Theorem 3.5] for the most general version of this result. We will prove a converse of this result.

Proposition 1.4.2. *Let $\{G_k\}$ be a sequence of domains containing* 0 *and which are biholomorphic to B^n. Furthermore, assume that $G_k \to G$ where G is also biholomorphic to B^n. Let $\{f_k\}$ be the sequence of biholomorphisms $f_k : B^n \to G_k$ such that $f_k(0) = 0$ and $Df_k(0) > 0$ and let f be the biholomorphism $f : B^n \to G$ such that $f(0) = 0$ and $Df(0) > 0$. If $\{Df_k(0)^{-1} f_k\}$ is locally uniformly bounded on B^n then $f_k \to f$ locally uniformly on B^n.*

Proof. Let $0 < \lambda_{k,1} \leq \cdots \leq \lambda_{k,i} \leq \cdots \leq \lambda_{k,n}$ denote the eigenvalues of $Df_k(0)$. We claim that $\lambda_1 := \inf\{\lambda_{k,1}\} > 0$ and $\lambda_n := \sup\{\lambda_{k,n}\} < \infty$. First we show that the conclusion follows if the claim is true and then we will check the claim.

We will denote by B^n_r the ball centered at the origin and of radius r and by $\overline{B}^n_r$ its closure. From the assumption that $\left\{Df_k(0)^{-1} f_k\right\}$ is locally uniformly bounded it follows that given $r \in (0,1)$ there exists $R = R(r)$ such that $Df_k(0)^{-1} f_k(B^n_r) \subseteq B^n_R$. Then

$$f_k(B^n_r) = Df_k(0)\, Df_k(0)^{-1} f_k(B^n_r) \subseteq Df_k(0)(B^n_R) \subseteq \lambda_{k,n} B^n_R \subseteq \lambda_n B^n_R.$$

This shows that $\{f_k\}$ is locally uniformly bounded.

Let $\{f_{k_l}\}$ be a convergent subsequence. We want to show that its limit is f. Let $g = \lim_{l\to\infty} f_{k_l}$. We know that

$$\langle Df_k(0)z, z\rangle \geq \lambda_{k,1} \geq \lambda_1 > 0, \|z\| = 1.$$

Hence $\langle Dg(0)z, z\rangle \geq \lambda_1 > 0$, $\|z\| = 1$, and so $Dg(0) > 0$. In particular $Dg(0)$ is nonsingular and using Hurwitz's Theorem we get that g is a biholomorphism. By the direct implication of Carathéodory's kernel convergence theorem and the definition of kernel convergence we have that $g(B^n) = G$. Now, Lemma 1.4.1 implies that $g = f$.

In conclusion, since $\{f_k\}$ is locally uniformly bounded we get that every subsequence has a converging subsequence, and by the above argument the limit of each such subsequence is f. Hence we must have that $f_k \to f$.

Now we just have to check that $\lambda_1 > 0$ and $\lambda_n < \infty$. Suppose that $\lambda_1 = 0$. Let $r > 0$ be such that $\overline{B}^n_r \subset G$. From the definition of kernel convergence we know that there exists k_0 such that $\overline{B}_r^n \subset f_k(B^n)$ for all $k \geq k_0$. This means we can consider the restrictions $f_k^{-1}|_{B_r^n} : B_r^n \to B^n$, $k \geq k_0$. Since $f_k^{-1}(0) = 0$ we get (by Schwarz's lemma) that $f_k^{-1}(\overline{B}_{r/2}^n) \subset \overline{B}_{1/2}^n$, $k \geq k_0$. Equivalently, $\overline{B}_{r/2}^n \subset f_k(\overline{B}_{1/2}^n)$ for all $k \geq k_0$. Let z_k with $\|z_k\| = r/2$ be an eigenvector of $Df_k(0)^{-1}$ associated to the eigenvalue $1/\lambda_{k,1}$. Then

$$\frac{1}{\lambda_{k,1}} z_k = Df_k(0)^{-1}(z_k) \in Df_k(0)^{-1} f_k(\overline{B}_{\frac{1}{2}}^n), k \geq k_0.$$

But $\lambda_1 = 0$ implies that $\sup\{1/\lambda_{k,1}\} = \infty$ and so the above contradicts the uniform boundedness of $\left\{f_k(\overline{B}_{1/2}^n)\right\}$. So, we must have that $\lambda_1 > 0$.

Now suppose that $\lambda_n = \infty$. Up to a subsequence we may assume that $\lambda_{k,n} \nearrow \infty$. From the local uniform boundedness of $\{Df_k(0)^{-1} f_k\}$ we get that there exists $r > 0$ such that $B_r^n \subset Df_k(0)^{-1} f_k(B^n)$ for all k. Indeed, let $S(B^n)$ be the class of biholomorphic mappings f on B^n such that $f(0) = 0$ and $Df(0) = I$. Then it is easy to check that the covering radius functional $\mathcal{C} : S(B^n) \to \mathbb{R}$ is continuous with respect to the locally uniform convergence topology ($\mathcal{C}(f)$ is the radius of the largest ball centered at the origin contained in $f(B^n)$). It is enough to choose r to be the minimum of $\mathcal{C}$ on the closure of the set $\left\{Df_k(0)^{-1} f_k\right\}$, which is a compact subset of $S(B^n)$ (by Montel's theorem). Now we have that $Df_k(0)(B_r^n) \subset f_k(B^n)$ for all k. Since $Df_k(0)$ is Hermitian it is unitarily diagonalizable, so there exists a unitary matrix U_k such that $Df_k(0) = U_k D_k U_k^*$, where $D_k = \mathrm{diag}(\lambda_{k,1}, \ldots, \lambda_{k,n})$. Since unitary matrices form a compact subset of M_n it follows that up to a subsequence there is a unitary matrix U such that $U_k \to U$. Thinking of the sets $\mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k_0,n})(B_r^n)$ geometrically as ellipsoids and of the sets $U_k\mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k_0,n})(B_r^n)$ as rotations of said ellipsoids it is easy to see that if we fix k_0 there exists k_1 such that

$$U_k\mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k_0,n})(B_r^n) \supset \frac{1}{2} U\mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k_0,n})(B_r^n), k \geq k_1.$$

Alternatively one can use the direct implication of Carathéodory's kernel convergence theorem to reach the same conclusion. Note that for the above inclusion it was essential that $\lambda_1 > 0$. At the same time we have

$$f_k\left(B^n\right) \supset Df_k(0)(B_r^n) = U_k D_k U_k^*(B_r^n) = U_k \mathrm{diag}\left(\lambda_{k,1}, \ldots, \lambda_{k,n}\right)\left(B_r^n\right)$$
$$\supseteq U_k \mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k_0,n})(B_r^n), k \geq k_0$$

(remember that we are assuming $\lambda_{k,n}$ is an increasing sequence). We can now conclude that for sufficiently large k we have that

$$\frac{1}{2} U \mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k_0,n})(B_r^n) \subset f_k(B^n)$$

and from the definition of kernel convergence and the fact that k_0 is arbitrary, we get that

$$\frac{1}{2} U \mathrm{diag}(\lambda_1, \ldots, \lambda_1, \lambda_{k,n})(B_r^n) \subset G$$

for all k. Since $\lambda_{k,n} \nearrow \infty$ we got that G contains the complex line passing through 0 and Ue_n ($e_n = (0, \ldots, 0, 1)$) and hence G cannot be biholomorphic to B^n (or any other bounded domain). We arrived at a contradiction so it means we must have that $\lambda_n < \infty$. This concludes the proof. □

Now we are able to provide a way of renormalizing a general Loewner chain. If $Df\left(0, \cdot\right)$ is continuous it is not hard to see, using the direct implication of Carathéodory's kernel convergence theorem, that the family $\{f\left(B^n, t\right)\}$ is an increasing family of domains such that $f\left(B^n, t\right) \to f\left(B^n, t_0\right)$ whenever $t \to t_0$. The goal is to preserve the geometric picture, hence the family of domains $\{f\left(B^n, t\right)\}$. Also, geometrically it makes sense to restrict ourselves to families of domains such that each domain appears only once.

Proposition 1.4.3. *Let $\{G_t\}_{t\geq 0}$ be a family of domains containing 0 which are biholomorphic to B^n and such that $G_s \subset G_t$ for every $0 \leq s \leq t$ and $G_t \to G_{t_0}$ as $t \to t_0$ for every $t \geq t_0$. Let $f(z,t)$ be such that f_t are biholomorphisms of B^n onto G_t and further satisfying $f(0,t) = 0$ and $Df(0,t) > 0$ for all $t \geq t_0$. Then $f(z,t)$ is a Loewner chain such that $Df\left(0, \cdot\right)$ is continuous and of local bounded variation. Furthermore*

$\alpha(t) := \mathrm{tr}Df(0,t)$ is increasing and $g(z,t) = f(z, \alpha^{-1}(e^t \mathrm{tr}Df(0,0)))$ is a Loewner chain such that $\mathrm{tr}Dg(0,t) = e^t \mathrm{tr}Df(0,0)$.

Proof. The fact that $f(z,t)$ is a Loewner chain is immediate and the continuity of $Df(0,\cdot)$ follows easily from Proposition 1.4.2. We know from Proposition 1.3.1 that $Df(0,t_2) - Df(0,t_1) \geq 0$ whenever $t_2 > t_1$, hence

$$\|Df(0,t_2) - Df(0,t_1)\| = \lambda_n \leq \mathrm{tr}(Df(0,t_2) - Df(0,t_1)), t_2 > t_1$$

where λ_n is the largest eigenvalue of $Df(0,t_2) - Df(0,t_1)$ (for the inequality we are using the fact that all the eigenvalues are nonnegative). From Corollary 1.2.7 we know that $\mathrm{tr}Df(0,\cdot)$ is a nondecreasing function on $[0,\infty)$. This and the above inequality are enough to conclude that $Df(0,\cdot)$ is of local bounded variation.

Furthermore if for $t_2 \geq t_1$ we have that $\mathrm{tr}Df(0,t_2) = \mathrm{tr}Df(0,t_1)$ then $Df(0,t_1)$ and $Df(0,t_2)$ have the same eigenvalues (we are using Corollary 1.2.7) and hence $\det Df(0,t_1) = \det Df(0,t_2)$ which implies that $\det Dv(0,t_1,t_2) = 1$. From the Carathéodory-Cartan-Kaup-Wu theorem we get that $v(\cdot,t_1,t_2) \in \mathrm{Aut}(B^n)$ (in fact one can show that $v(\cdot,t_1,t_2) = id_{B^n}$) and hence $G_{t_1} = f(B^n,t_1) = f(B^n,t_2) = G_{t_2}$. This means $\mathrm{tr}Df(0,t_2) = \mathrm{tr}Df(0,t_1)$ if and only if $t_1 = t_2$. We can conclude that $\mathrm{tr}Df(0,t)$ is increasing on $[0,\infty)$. This insures that α^{-1} is well defined and the last assertion is tautological. □

The above proposition shows that if we are interested in the geometrical aspect of Loewner chains then it is enough to consider Loewner chains for which $Df(0,\cdot)$ is locally Lipschitz.

1.5 Solutions of the Loewner chain equation when $Dh(0,t) = A(t)$

In this section we start addressing the problem of finding a Loewner chain corresponding to a given infinitesimal generator $h \in \mathcal{H}_0$. Note that in order to be able to solve (1.1.1) and

(1.1.2) we need to impose some integrability condition on h. Without loss of generality we will assume from now on that $Df(0,0) = I$.

It will be convenient to have the following notations. If $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ is locally Lebesgue integrable on $[0,\infty)$ we will denote by $V(t)$ and $U(t)$ the unique locally absolutely continuous solutions to the equations

$$\frac{dV}{dt} = -A(t)V \quad a.e.\, t \geq 0, V(0) = I \tag{1.5.1}$$

and respectively

$$\frac{dU}{dt} = UA(t) \quad a.e.\, t \geq 0, U(0) = I. \tag{1.5.2}$$

As noted in the preliminaries, we have that $U(t) = V(t)^{-1}$ and $V(s,t) := V(t)V(s)^{-1} = U(t)^{-1} U(s)$ solves

$$\frac{\partial V(s,t)}{\partial t} = -A(t)V(s,t) \quad a.e.\, t \geq s, V(s,s) = I. \tag{1.5.3}$$

Solutions to the Loewner chain equation will be constructed from solutions to the Loewner equation. We will use the notation $A(t) := Dh(0,t)$. We have the following result about solutions to the Loewner equation.

Proposition 1.5.1. *Let $h \in \mathcal{H}_0$ be such that $A(t)$ is locally Lebesgue integrable (locally Lipschitz). Then the initial value problem (1.1.2) has a unique solution $v(z,s,t)$ such that $v(\cdot,s,t)$ is a univalent Schwarz mapping, $v(z,s,\cdot)$ is locally absolutely continuous (locally Lipschitz) on $[s,\infty)$ locally uniformly with respect to $z \in B^n$ and the following relations hold:*

$$\frac{\|v(z,s,t)\|}{(1-\|v(z,s,t)\|)^2} \leq e^{-\int_s^t m(A(\tau))d\tau} \frac{\|z\|}{(1-\|z\|)^2}, z \in B^n, t \geq s \geq 0; \tag{1.5.4}$$

$$\frac{\|v(z,s,t)\|}{(1+\|v(z,s,t)\|)^2} \geq e^{-\int_s^t k(A(\tau))d\tau} \frac{\|z\|}{(1+\|z\|)^2}, z \in B^n, t \geq s \geq 0. \tag{1.5.5}$$

Proof. The first part is just a particular case of [BCDM09, Proposition 3.1]. The estimates (1.5.4) and (1.5.5) follow exactly as in [GHKK08b, Theorem 2.1] (using part (iii) of Proposition 1.2.3). $\square$

We can now find a solution to the Loewner chain equation provided that $A(t)$ satisfies a certain condition. We will say that a solution $f(z,t)$ of (1.1.2) is polynomially bounded (bounded) if $\left\{Df(0,\cdot)^{-1} f(\cdot,t)\right\}_{t\geq 0}$ is locally polynomially bounded (locally uniformly bounded), i.e. for any compact set $K \subset B^n$ there exists a constant C_K and a polynomial (constant polynomial) P such that

$$\left\|Df(0,t)^{-1} f(z,t)\right\| \leq C_K P(t), \ z \in K, \ t \in [0,\infty).$$

Proposition 1.5.2. *Let $h \in \mathcal{H}_0$ be such that $A(t)$ is locally Lebesgue integrable and let $v(z,s,t)$ be the unique locally absolutely continuous solution of the initial value problem* (1.1.2)*. Assume that*

$$\sup_{s\geq 0} \int_s^\infty \|A(t)\| \|V(s,t)^{-1}\| e^{-2\int_s^t m(A(\tau))d\tau} dt < \infty. \tag{1.5.6}$$

Then the limit

$$\lim_{t\to\infty} V(t)^{-1} v(z,s,t) = f(z,s) \tag{1.5.7}$$

exists locally uniformly in z for each $s \geq 0$. Moreover, $f(z,t)$ is a bounded Loewner chain solution of (1.1.1)*.*

Proof. The proof is just an adjustment of the proof for [GHKK08b, Theorem 2.3].

Fix $s \geq 0$ and let $u(z,s,t) = V(t)^{-1} v(z,s,t)$ for $z \in B^n$ and $t \geq s$. Let $g(z,t) = h(z,t) - A(t)(z)$ for $z \in B^n$ and $t \geq 0$.

We know that $U(t) = V(t)^{-1}$ satisfies the initial value problem (1.5.2) , so at points where both equations (1.5.2) and (1.1.2) are satisfied we have

$$\frac{\partial u}{\partial t} = \frac{dU}{dt} v + U\frac{\partial v}{\partial t} = UA(t)v + U(-h(v,t)) = -Ug(v,t). \tag{1.5.8}$$

To use the fundamental theorem of calculus on u we will prove that it is locally absolutely continuous with respect to t.Fix $s \geq 0$, $T > s$ and $t_1, t_2 \in [s,T]$. Then

$$\|u(z,s,t_1) - u(z,s,t_2)\| \leq \|U(t_1)\| \, \|v(z,s,t_1) - v(z,s,t_2)\| + \|v(z,s,t_2)\| \, \|U(t_1) - U(t_2)\| .$$

We know that $U(\cdot)$ is locally absolutely continuous and hence also locally bounded on $[0,\infty)$. Also, from Proposition 1.5.1 we know that $v(z,s,\cdot)$ is locally bounded and locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to z. All these and the above inequality show that $u(z,s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to z.

Now we can write

$$\begin{aligned} \|u(z,s,t_1)-u(z,s,t_2)\| &= \left\|\int_{t_1}^{t_2} -U(t)g(v(z,s,t),t)dt\right\| \\ &\le \int_{t_1}^{t_2} \|U(t)\|\,\|g(v(z,s,t),t)\|\,dt \qquad (1.5.9)\\ &= \int_{t_1}^{t_2} \left\|V(0,t)^{-1}\right\|\,\|g(v(z,s,t),t)\|\,dt. \end{aligned}$$

Using the integral formula for the remainder of the Taylor series, Cauchy's integral formula and the estimate (1.2.11) it is not hard to check that

$$\|g(z,t)\| \le C_r\,\|z\|^2\,\|A(t)\|\,, z\in \overline{B}_r^n, t\ge 0. \qquad (1.5.10)$$

Using estimates (1.5.9), (1.5.10) and (1.5.4) we get that

$$\begin{aligned} \|u(z,s,t_1)-u(z,s,t_2)\| &\le \int_{t_1}^{t_2} \left\|V(0,t)^{-1}\right\| C_r\,\|v(z,s,t)\|^2\,\|A(t)\|\,dt \\ &\le C_r e^{2\int_0^s m(A(\tau))d\tau} \int_{t_1}^{t_2} \|A(t)\|\left\|V(0,t)^{-1}\right\| e^{-2\int_0^t m(A(\tau))d\tau}dt. \end{aligned}$$

From the assumptions we have that $\int_0^\infty \|A\left(t\right)\| \left\|V\left(0,t\right)^{-1}\right\| e^{-2\int_0^t m(A(\tau))d\tau}dt < \infty$ and now it is easy to conclude that the limit (1.5.7) exists locally uniformly. In view of the uniqueness of solutions to the initial value problem (1.1.2) it is easy to see that v satisfies the semigroup property (1.3.1). This and (1.5.7) imply that f is a subordination chain with v as transition mapping. Since $Dv(0,s,t) = V\left(s,t\right)$ we get that $Df(0,s) = \lim_{t\to\infty} V(t)^{-1}Dv(0,s,t) = V(s)^{-1}$ and hence $f(\cdot,s)$ is biholomorphic (by Hurwitz's theorem).

Now we have that $f(z,t)$ is a Loewner chain and we just want to prove that $\{V(t)f(\cdot,t)\}$ is a locally uniformly bounded family. To this end, we consider the mapping $V(s)u(z,s,t)$

and notice that it is locally absolutely continuous in t and satisfies (see (1.5.8))

$$\frac{\partial}{\partial t}(V(s)u(z,s,t)) = -V(s)V(t)^{-1}g(v(z,s,t),t) = -V(s,t)^{-1}g(v(z,s,t),t).$$

Let $s \leq t_1 \leq t_2$ and $z \in B_r^n$, $r \in (0,1)$. Then using the estimates (1.5.10) and (1.5.4) we have that

$$\begin{aligned}
\|V(s)u(z,s,t_1) - V(s)u(z,s,t_2)\| &= \|\int_{t_1}^{t_2} -V(s,t)^{-1}g(v(z,s,t),t)dt\| \\
&\leq C_r \int_{t_1}^{t_2} \|A(t)\| \left\|V(s,t)^{-1}\right\| e^{-2\int_s^t m(A(\tau))d\tau} dt \\
&\leq C_r \sup_{s\geq 0} \int_s^{\infty} \|A(t)\| \left\|V(s,t)^{-1}\right\| e^{-2\int_s^t m(A(\tau))d\tau} dt.
\end{aligned}$$

The locally uniform boundedness of $\{V(t)f(\cdot,t)\}$ now follows by letting $t_1 = s$ and $t_2 \to \infty$ in the above inequality. $\square$

Remark. Using the estimate (1.2.9) and the condition (1.5.6) we get the more restrictive, but "nicer looking" condition

$$\sup_{s\geq 0} \int_s^{\infty} \|A(t)\| e^{\int_s^t [k(A(\tau))-2m(A(\tau))]d\tau} dt < \infty.$$

Remark. Note that in the case when $A(\cdot)$ is bounded (rather than just locally bounded), similarly to [GHKK08b], we can replace the condition (1.5.6) by

$$\sup_{s\geq 0} \int_s^{\infty} \left\|V(s,t)^{-1}\right\| e^{-2\int_s^t m(A(\tau))d\tau} dt < \infty. \tag{1.5.11}$$

Also, note that if $A(t) \to 0$ as $t \to \infty$ then the $\|A(t)\|$ factor contributes to the convergence of condition (1.5.6). Hence, even though in the case when $A(\cdot)$ is bounded the factor $\|A(t)\|$ can be dropped, in general we get in fact a better condition.

Next we see that the requirement that the solution to the Loewner chain equation is bounded determines it uniquely when A satisfies certain conditions.

Proposition 1.5.3. *Assume $A(\cdot)$ is bounded and satisfies the condition (1.5.11). Let $f : B^n \times [0,\infty) \to \mathbb{C}^n$ be such that $f(\cdot,t) \in H(B^n)$, $f(0,t) = 0$ and $f(z,\cdot)$ is locally absolutely* *continuous locally uniformly with respect to z. Let $h(z,t)$ satisfy the assumptions of Proposition 1.5.1. Assume that $f(z,t)$ is a bounded solution of (1.1.2). Then $f(z,t)$ is a Loewner chain with transition mapping $v(z,s,t)$ and*

$$f(z,s) = \lim_{t\to\infty} V(t)^{-1} v(z,s,t)$$

locally uniformly on B^n for $s \geq 0$, where $v(z,s,t)$ is as in Proposition 1.5.1.

Proof. We just follow the same steps as in the proof of [GHKK08b, Theorem 2.5].

First, one lets $f(z,s,t) := f(v(z,s,t),t)$ and proves that $f(z,s,t) = f(z,s,s)$, i.e. $f(z,t)$ is a subordination chain with transition mapping $v(z,s,t)$. This follows from the absolute continuity of f and the fact that it follows from (1.1.1) and (1.1.2) that $\partial_t f(z,s,t) = 0$ for almost every $t \geq s$. Next, one wants to show the mappings $f(\cdot,t)$ are univalent. For this it suffices to prove that there exists a sequence $\{t_k\}$, $t_k \to \infty$ such that

$$\lim_{k\to\infty} V(t_k)^{-1} v(z,s,t_k) = f(z,s)$$

locally uniformly on B^n. Indeed, since $\{V(t)f(\cdot,t)\}$ is a normal family, we have that

$$\|V(t)f(z,t)\| \leq C_r, \|z\| \leq r, t \geq 0.$$

From the integral formula for the remainder of the Taylor series and Cauchy's integral formula we get

$$\|V(t)f(z,t) - z\| \leq C_r \|z\|^2, \|z\| \leq r, t \geq 0.$$

Replacing z by $v(z,s,t)$ in the above inequality and using (1.5.4) we obtain that

$$\begin{aligned}
\|f(z,s) - V(t)^{-1}v(z,s,t)\| &= \|V(0,t)^{-1}[V(t)f(v(z,s,t),t) - v(z,s,t)]\| \\
&\leq \|V(0,t)^{-1}\| C_r \|v(z,s,t)\|^2 \\
&\leq C_r e^{2\int_0^s m(A(\tau))d\tau} \left\|V(0,t)^{-1}\right\| e^{-2\int_0^t m(A(\tau))d\tau}.
\end{aligned}$$

Since $\int_0^\infty \|V(0,t)^{-1}\| e^{-2\int_0^t m(A(\tau))d\tau} dt < \infty$, there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ and $\|V(0,t_k)^{-1}\| e^{-2\int_0^{t_k} m(A(\tau))d\tau} \to 0$ as $m \to \infty$. From the above inequality we see that

$\{t_k\}$ is the sequence we were looking for. All the other claims follow immediately using the fact that we can apply Proposition 1.5.2. $\square$

The next result shows that all the solutions of the Loewner chain equation can be obtained from one particular solution.

Proposition 1.5.4. *Let $f(z,t)$ be a Loewner chain that is locally absolutely continuous in t, locally uniformly in z and let $g(z,t)$ be a subordination chain that is locally absolutely continuous in t, locally uniformly with respect to z. Suppose that both chains satisfy the same Loewner chain equation. Then $g(z,t) = \phi(f(z,t))$ where ϕ is a holomorphic mapping on $\cup f_t(B^n)$. Furthermore, $g(z,t)$ is a Loewner chain if and only if ϕ is a biholomorphism.*

Proof. From the fact that f and g satisfy (1.1.1) it follows easily that f and g have the same transition mapping (see the first part of the proof of Proposition 1.5.3). The rest can be proved the same way as in [ABHK10, Theorem 4.9] $\square$

1.6 Solutions of the Loewner chain equation when $Dh(0,t) = A$

Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be a linear operator such that $m(A) > 0$. Let $\mathcal{H}_A$ be the class of mappings $h \in \mathcal{H}$ such that $Dh(0,t) = A$ (note that $m(A) > 0$ is necessary for $h(\cdot,t) \in \mathcal{N}$). We will study the solutions of the Loewner chain equation when the infinitesimal generator is from $\mathcal{H}_A$.

Let $k_+(A) := \max\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} = \lim_{t\to\infty} \ln \left\|e^{tA}\right\|/t$ be the upper exponential (Lyapunov) index of A ($\sigma(A)$ denotes the spectrum of A). Throughout this and the next section we let $n_0 := [k_+(A)/m(A)]$. The situation when $n_0 = 1$ has been studied in [GHKK08a]. We will deal with the case when $n_0 \geq 2$, but our approach also applies to the case $n_0 = 1$.

From Proposition 1.5.1 (see also [GHKK08a, Theorem 2.1]) we know that the Loewner equation for the transition mapping has a solution regardless of the value of n_0. Furthermore we know that

$$\frac{\|v(z,s,t)\|}{(1-\|v(z,s,t)\|)^2} \leq e^{m(A)(s-t)} \frac{\|z\|}{(1-\|z\|)^2},\ z \in B^n,\ t \geq s \geq 0. \tag{1.6.1}$$

If $f(z,t)$ is a Loewner chain satisfying (1.1.1) with $h \in \mathcal{H}_A$ then $Df(0,t) = e^{tA}$. Loewner chains with such a normalization will be called *A-normalized*. Let

$$\begin{aligned} f(z,t) &= e^{tA}\left(z + \sum_{k=2}^{\infty} F_k\left(z^k, t\right)\right) \\ h(z,t) &= Az + \sum_{k=2}^{\infty} H_k\left(z^k, t\right) \end{aligned}$$

where $F_k(\cdot,t)$ and $H_k(\cdot,t)$ are homogeneous polynomial mappings of degree k. We will denote the Banach space of homogeneous polynomial mappings of degree k from $\mathbb{C}^n$ to $\mathbb{C}^n$ by $\mathcal{P}^k(\mathbb{C}^n)$.

Equating coefficients on both sides of (1.1.1) we get

$$\frac{dF_k}{dt}(z^k,t) = B_k\left(F_k\left(z^k,t\right)\right) + N_k\left(z^k,t\right),\ \text{a.e. } t \in [0,\infty) \tag{1.6.2}$$

where B_k is a linear operator on $\mathcal{P}^k(\mathbb{C}^n)$ defined by

$$B_k\left(Q_k\left(z^k\right)\right) = kQ_k\left(Az, z^{k-1}\right) - AQ_k\left(z^k\right)$$

and $N_k(\cdot,t) \in \mathcal{P}^k(\mathbb{C}^n)$ is defined by

$$N_k\left(z^k,t\right) = H_k\left(z^k,t\right) + \sum_{j=2}^{k-1} jF_j\left(H_{k-j+1}\left(z^{k-j+1},t\right), z^{j-1}, t\right).$$

The solutions of (1.6.2) will be regarded as functions $F_k : [0,\infty) \to \mathcal{P}^k(\mathbb{C}^n)$. Consequently we will say that such F_k are polynomially bounded (bounded) if there exists a polynomial (constant polynomial) P such that $\|F_k(t)\| \leq P(t)$, $t \geq 0$.

Proposition 1.6.1 will show that an A-normalized polynomially bounded solution of (1.1.1) can be recovered from its first n_0 coefficients and the solution of (1.1.2). Conversely, Theorem 1.6.8 will show that by finding polynomially bounded solutions to the first n_0 coefficient equations (1.6.2) we can find an A-normalized solution of (1.1.1). These results generalize Poreda [Por91, Theorem 4.1 and Theorem 4.4]. Finally, after a discussion about the existence of polynomially bounded solutions to the coefficient equations we will obtain the main result, Theorem 1.6.11, that guarantees the existence of a Loewner chain solution for (1.1.1). This result has also been independently obtained through a different method in [Aro10].

Note that the solutions of (1.1.1), (1.1.2), (1.6.2) and (1.7.2) will be assumed to be locally absolutely continuous in t, locally uniformly with respect to z.

We will repeatedly use the fact that given $\epsilon > 0$ there exists a constant C_ϵ such that

$$\left\|e^{tA}\right\| \leq C_\epsilon e^{t(k_+(A)+\epsilon)},\ t \geq 0 \tag{1.6.3}$$

(this follows immediately from the definition of $k_+(A)$). In fact we can find a polynomial P_A such that

$$\left\|e^{tA}\right\| \leq P_A(t)\, e^{tk_+(A)},\ t \geq 0 \tag{1.6.4}$$

(see for example [DK74, p 61, Exercise 16]). Furthermore, if A is normal then

$$\left\|e^{tA}\right\| = e^{tk_+(A)},\ t \geq 0.$$

Indeed, if we write $A = UDU^*$ where D is a diagonal matrix and U is a unitary matrix then

$$\left\|e^{tA}\right\| = \left\|Ue^{tD}U^*\right\| = \left\|e^{tD}\right\| = e^{tk_+(D)} = e^{tk_+(A)}$$

(for non-Euclidean norms we just get $\left\|e^{tA}\right\| \leq C_A e^{tk_+(A)}$).

The following is a generalization of [Por91, Theorem 4.1].

Proposition 1.6.1. *If $f(z,t)$ is a polynomially bounded solution of* (1.1.1) *such that*

$$f(z,t) = e^{tA}\left(z + \sum_{k=2}^{\infty} F_k\left(z^k, t\right)\right)$$

then

$$f(z,s) = \lim_{t\to\infty} e^{tA}\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\left(v(z,s,t)^k, t\right)\right)$$

and the limit is locally uniform in z.

Proof. It is not hard to check that if $f(z,t)$ is a solution of (1.1.1) and v is the solution of the initial value problem (1.1.2) then $f(z,t)$ is a subordination chain with transition mapping v (see [GHKK08a, Theorem 2.6]). Hence

$$
\begin{aligned}
&f(z,s) = f(v(z,s,t),t) \\
&\qquad = e^{tA}\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\left(v(z,s,t)^k, t\right)\right) + e^{tA}R(v(z,s,t),t)
\end{aligned}
$$

where $R(z,t) = \sum_{k=n_0+1}^{\infty} F_k\left(z^k, t\right)$. From the assumption on f, the formula for the remainder of the Taylor series and Cauchy's formula, we easily get that $\{R(\cdot,t)\}_{t\geq 0}$ is locally polynomially bounded and in fact

$$\|R(z,t)\| \leq C_r P(t)\|z\|^{n_0+1},\ \|z\| \leq r.$$

From the above, (1.6.1) and (1.6.3) we get

$$
\begin{aligned}
\left\|e^{tA}R(v(z,s,t),t)\right\| &\leq C_{\epsilon,r}e^{t(k_+(A)+\epsilon)}P(t)\|v(z,s,t)\|^{n_0+1} \\
&\leq C_{\epsilon,r,s}e^{t(k_+(A)+\epsilon-(n_0+1)m(A))}P(t),\ \|z\| \leq r.
\end{aligned}
$$

Hence, taking ϵ small enough we can conclude that $e^{tA}R(v(z,s,t),t) \to 0$ locally uniformly. □

In order to prove Theorem 1.6.8 we will need the following lemmas.

Lemma 1.6.2. *If $Q_k \in \mathcal{P}^k(\mathbb{C}^n)$ then the following identities hold for $t \in \mathbb{R}$*

$$e^{tA}e^{tB_k}Q_k\left(\left(e^{-tA}z\right)^k\right) = Q_k\left(z^k\right) \tag{1.6.5}$$

$$e^{tA}Q_k\left(\left(e^{-tA}z\right)^k\right) = e^{-tB_k}Q\left(z^k\right) \tag{1.6.6}$$

$$e^{tA}e^{tB_k}Q_k\left(z^k\right) = Q_k\left(\left(e^{tA}z\right)^k\right) \tag{1.6.7}$$

.

Proof. Define A_k on $\mathcal{P}^k(\mathbb{C}^n)$ by $A_k\left(Q_k\left(z^k\right)\right) = AQ_k\left(z^k\right)$. One easily sees that $e^{tA_k}\left(Q_k\left(z^k\right)\right) = e^{tA}Q_k\left(z^k\right)$, $A_kB_k = B_kA_k$ and

$$(A_k + B_k)\left(Q_k\left(z^k\right)\right) = kQ_k\left(Az, z^{k-1}\right).$$

For (1.6.5) it is enough to check that $\phi(t) = e^{t(A_k+B_k)}Q_k\left(\left(e^{-tA}z\right)^k\right)$ satisfies $\phi'(t) = 0$. Indeed

$$\phi'(t) = e^{t(A_k+B_k)}\left[(A_k + B_k)\left(Q_k\left(\left(e^{-tA}z\right)^k\right)\right) - kQ_k\left(e^{-tA}Az, \left(e^{-tA}z\right)^{k-1}\right)\right] = 0.$$

The last two identities follow immediately from the first one. $\square$

Lemma 1.6.3. *If $F_k, G_k : [0,\infty) \to \mathcal{P}^k(\mathbb{C}^n)$, $k = 2,\ldots,m$ are solutions of (1.6.2) and the limits*

$$\begin{aligned} f(z,s) &:= \lim_{t\to\infty} e^{tA}\left(v(z,s,t) + \sum_{k=2}^{m} F_k\left(v(z,s,t)^k, t\right)\right) \\ g(z,s) &:= \lim_{t\to\infty} e^{tA}\left(v(z,s,t) + \sum_{k=2}^{m} G_k\left(v(z,s,t)^k, t\right)\right) \end{aligned}$$

exist locally uniformly in $z \in B^n$ for some $s \geq 0$, then $f(\cdot,s) = g(\cdot,s)$ if and only if $F_k = G_k$, $k = 2,\ldots,m$.

Proof. Assume that $f(\cdot,s) = g(\cdot,s)$ and that there exists $k \in \{2,\ldots,m\}$ such that $F_k \neq G_k$. Let k_0 be the minimal such k.

Equating the k_0-th coefficients of $f(\cdot,s)$ and $g(\cdot,s)$ and taking into account the minimality of k_0 we get

$$\lim_{t\to\infty} e^{tA}\left(F_{k_0}\left(\left(e^{(s-t)A}z\right)^{k_0}, t\right) - G_{k_0}\left(\left(e^{(s-t)A}z\right)^{k_0}, t\right)\right) = 0. \tag{1.6.8}$$

Using (1.6.6) and the fact that F_{k_0} is a solution of (1.6.2) we get

$$\begin{aligned} e^{tA}F_{k_0}\left(\left(e^{(s-t)A}z\right)^{k_0}, t\right) &= e^{sA}e^{(s-t)B_k}F_{k_0}\left(z^{k_0}, t\right) \\ &= e^{sA}e^{sB_k}\left(F_{k_0}\left(z^{k_0}, 0\right) + \int_0^t e^{-sB_{k_0}}N_{k_0}\left(z^{k_0}, s\right)ds\right). \end{aligned}$$

We can get an analogous identity for G_{k_0} and then (1.6.8) becomes

$$e^{sA}e^{sB_k}\left(F_{k_0}\left(z^{k_0},0\right)-G_{k_0}\left(z^{k_0},0\right)\right)=0.$$

Since F_{k_0} and G_{k_0} satisfy the same differential equation with the same initial condition we have $F_{k_0}=G_{k_0}$, thus reaching a contradiction. □

Lemma 1.6.4. *If P is a polynomial such that $P(t)\geq 0$ for $t\geq s$ then*

$$\int_s^\infty P(t)\left\|e^{(t-s)A}\right\|\frac{\|v(z,s,t)\|^{n_0+1}}{\left(1-\|v(z,s,t)\|\right)^2}dt\leq\frac{Q_{\epsilon,A,P}(s)}{\left(1-\|z\|\right)^{2\frac{k_+(A)}{m(A)}+\epsilon}},\ \epsilon>0$$

where $Q_{\epsilon,A,P}$ is a polynomial of the same degree as P.

Proof. Let

$$\alpha=\frac{k_+(A)}{m(A)}+\frac{\epsilon}{2}.$$

We can restrict to the case when ϵ is small enough so that $\alpha<n_0+1$. Using (1.6.1) we see that

$$\frac{\|v(z,s,t)\|^{n_0+1}}{\left(1-\|v(z,s,t)\|\right)^2}\leq\frac{\|v(z,s,t)\|^{\alpha}}{\left(1-\|v(z,s,t)\|\right)^2}\leq\frac{e^{(s-t)\alpha m(A)}}{\left(1-\|z\|\right)^{2\alpha}}.$$

Let ϵ' be small enough so that

$$\left\|e^{(t-s)A}\right\|\leq C_{\epsilon'}e^{(t-s)(k_+(A)+\epsilon')}$$

and $\delta:=\alpha m(A)-k_+(A)-\epsilon'>0$. Then

$$\int_s^\infty P(t)\left\|e^{(t-s)A}\right\|\frac{\|v(z,s,t)\|^{n_0+1}}{\left(1-\|v(z,s,t)\|\right)^2}dt\leq\frac{C_{\epsilon'}}{\left(1-\|z\|\right)^{2\alpha}}\int_s^\infty P(t)\,e^{-\delta(t-s)}dt$$

and it is not hard to see that

$$Q_{\epsilon,A,P}(s):=C_{\epsilon'}\int_s^\infty P(t)\,e^{-\delta(t-s)}dt$$

satisfies our requirements. □

Remark 1.6.5. When A is normal, P is constant and $k_+(A)/m(A)>1$ we can sharpen the above bound by letting $\epsilon=0$.

Let $\beta = n_0 + 1 - \alpha$, then using (1.6.1) we get

$$\frac{\|v(z,s,t)\|^{n_0+1}}{(1-\|v(z,s,t)\|)^2} \leq \frac{e^{(s-t)\alpha m(A)}}{(1-\|z\|)^{2\alpha}} \|v(z,s,t)\|^{\beta} (1-\|v(z,s,t)\|)^{2(\alpha-1)}.$$

If A is normal we know that

$$\left\| e^{(t-s)A} \right\| = e^{(t-s)k_+(A)}$$

and hence we get

$$\int_s^{\infty} \left\| e^{(t-s)A} \right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1-\|v(z,s,t)\|)^2} dt$$
$$\leq \frac{1}{(1-\|z\|)^{2\alpha}} \int_s^{\infty} \|v(z,s,t)\|^{\beta} (1-\|v(z,s,t)\|)^{2(\alpha-1)} dt.$$

From the proof of [GHKK08a, Theorem 2.1] we know that

$$-\frac{1+\|v(z,s,t)\|}{1-\|v(z,s,t)\|} \frac{1}{\|v(z,s,t)\|} \frac{d\|v(z,s,t)\|}{dt} \geq m(A).$$

Using the above inequality it is easy to conclude that

$$\int_s^{\infty} \|v(z,s,t)\|^{\beta} (1-\|v(z,s,t)\|)^{2(\alpha-1)} dt$$
$$\leq -\frac{2}{m(A)} \int_s^{\infty} \|v(z,s,t)\|^{\beta-1} (1-\|v(z,s,t)\|)^{2\alpha-3} \frac{d\|v(z,s,t)\|}{dt} dt$$
$$= \frac{2}{m(A)} \int_0^{\|z\|} u^{\beta-1} (1-u)^{2\alpha-3} du$$
$$\leq \frac{2}{m(A)} \int_0^1 u^{\beta-1} (1-u)^{2\alpha-3} du.$$

The last integral converges because by our assumptions $\beta - 1 > -1$ and $2\alpha - 3 > -1$. This completes the proof of our claim.

Lemma 1.6.6. *If $F_k : [0,\infty) \to \mathcal{P}^k(\mathbb{C}^n)$, $k = 2,\dots,m$ are polynomially bounded and the limit*

$$f(z,s) = \lim_{t\to\infty} e^{tA} \left(v(z,s,t) + \sum_{k=2}^{m} F_k\left(v(z,s,t)^k, t \right) \right)$$

exists locally uniformly in $z \in B^n$ for some $s \geq 0$ then $f(\cdot,s)$ is univalent.

Proof. First note that if $Q \in \mathcal{P}^k(\mathbb{C}^n)$ then

$$\begin{aligned} \left\| Q\left(z^k\right) - Q\left(w^k\right) \right\| &= \left\| \sum_{j=0}^{k-1} Q\left(z-w, z^j, w^{k-1-j}\right) \right\| \\ &\leq \|Q\| \, \|z-w\| \sum_{j=0}^{k-1} \|z\|^j \, \|w\|^{k-1-j} . \end{aligned} \tag{1.6.9}$$

Using the above and (1.6.1) we see that

$$\begin{aligned} &\left\| \sum_{k=2}^{m} F_k\left(v\left(z_1,s,t\right)^k, t\right) - \sum_{k=2}^{m} F_k\left(v\left(z_2,s,t\right)^k, t\right) \right\| \\ &\qquad \leq C_r P(t)\, e^{(s-t)m(A)} \left\| v\left(z_1,s,t\right) - v\left(z_2,s,t\right) \right\|, \ \|z_1\|, \|z_2\| \leq r \end{aligned}$$

where P is a polynomial bound on F_k, $k = 2, \ldots, m$. For sufficiently large t we get

$$\begin{aligned} &\left\| \sum_{k=2}^{m} F_k\left(v\left(z_1,s,t\right)^k, t\right) - \sum_{k=2}^{m} F_k\left(v\left(z_2,s,t\right)^k, t\right) \right\| \\ &\qquad < \left\| v\left(z_1,s,t\right) - v\left(z_2,s,t\right) \right\|, \ \|z_1\|, \|z_2\| \leq r \end{aligned}$$

which implies that for sufficiently large t

$$v(z,s,t) + \sum_{k=2}^{m} F_k\left(v(z,s,t)^k, t\right)$$

is univalent on the ball $\|z\| \leq r$. Now the conclusion follows easily. □

The following consequence together with Theorem 1.6.8 generalizes [GHKK08a, Theorem 2.6].

Corollary 1.6.7. *All A-normalized polynomially bounded solutions of (1.1.1) are Loewner chains.*

Proof. This follows from Proposition 1.6.1 and Lemma 1.6.6. □

Theorem 1.6.8. *If F_k, $k = 2, \ldots, n_0$ are polynomially bounded solutions of (1.6.2) then*

$$g(z,s) := \lim_{t\to\infty} e^{tA} \left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\left(v(z,s,t)^k, t\right) \right)$$

exists locally uniformly with respect to z and is a polynomially bounded Loewner chain solution of (1.1.1)*. If $F(t)$ is a polynomial bound for F_k, $k = 2, \ldots, n_0$ then, given $\epsilon > 0$, there exists a polynomial $Q_{\epsilon,A,F}$ of the same degree as F such that*

$$\left\|e^{-tA} g(z,t)\right\| \leq \frac{Q_{\epsilon,A,F}(t)}{(1-\|z\|)^{2\frac{k_+(A)}{m(A)}+\epsilon}},\ z \in B^n, t \geq 0.$$

Furthermore, if

$$g(z,t) = e^{tA}\left(z + \sum_{k=2}^{\infty} G_k\left(z^k, t\right)\right)$$

then $G_k = F_k$, $k = 2, \ldots, n_0$.

Proof. Let

$$u(z,s,t) = e^{tA}\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\left(v(z,s,t)^k, t\right)\right).$$

We begin by showing that $\lim_{t\to\infty} u(z,s,t)$ exists locally uniformly.

It is easy to see that $u(z,s,t)$ is locally absolutely continuous in t, so

$$\|u(z,s,t_1) - u(z,s,t_2)\| = \left\|\int_{t_1}^{t_2} \frac{\partial u}{\partial t}(z,s,t)\,dt\right\| \leq \int_{t_1}^{t_2}\left\|\frac{\partial u}{\partial t}(z,s,t)\right\| dt,\ s \leq t_1 \leq t_2. \tag{1.6.10}$$

Now

$$\begin{aligned}\frac{\partial u}{\partial t}(z,s,t) = e^{tA}A\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\left(v(z,s,t)^k, t\right)\right) - e^{tA}\Bigg(h(v(z,s,t),t) + \\ \sum_{k=2}^{n_0} kF_k\left(v(z,s,t)^{k-1}, h(v(z,s,t),t), t\right) - \sum_{k=2}^{n_0}\frac{dF_k}{dt}\left(v(z,s,t)^k, t\right)\Bigg).\end{aligned}$$

Let

$$R(z,t) = h(z,t) - Az - \sum_{k=2}^{n_0} H_k\left(z^k, t\right).$$

Similarly to the proof of [Por91, Theorem 4.4], a straightforward computation using the assumption that F_k, $k = 2, \ldots, n_0$ satisfy (1.6.2), leads to

$$\begin{aligned}\frac{\partial u}{\partial t}(z,s,t) &= -e^{tA}\left(R(v(z,s,t),t) + \sum_{k=2}^{n_0} kF_k\left(v(z,s,t)^{k-1}, R(v(z,s,t),t)\right)\right) \\ &\quad -e^{tA}\left(\sum_{k=2}^{n_0}\sum_{l=n_0-k+2}^{n_0} kF_k\left(v(z,s,t)^{k-1}, H_l\left(v(z,s,t)^l, t\right), t\right)\right).\end{aligned} \tag{1.6.11}$$

Using the ideas from the proof of [GHK02, Theorem 1.2] (cf. [GHKK08a, Lemma 1.2]) we get

$$\|R(z,t)\| \le C_A \frac{\|z\|^{n_0+1}}{(1-\|z\|)^2} \tag{1.6.12}$$

From (1.6.11), (1.6.12) and (1.6.1) we get

$$\left\|\frac{\partial u}{\partial t}(z,s,t)\right\| \le P(t)\left\|e^{tA}\right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1-\|v(z,s,t)\|)^2}$$

where P is a polynomial depending only on F and A (because the bounds on H_k can be chosen to depend only on A). Substituting this estimate into (1.6.10) we get

$$\|u(z,s,t_1)-u(z,s,t_2)\| \le \left\|e^{sA}\right\| \int_{t_1}^{t_2} P(t)\left\|e^{(t-s)A}\right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1-\|v(z,s,t)\|)^2}dt.$$

Using Lemma 1.6.4 we can now conclude that $\lim_{t\to\infty} u(z,s,t)$ exists uniformly on compact subsets.

From the semigroup property for v we immediately get that

$$g(v(z,s,t),t) = g(z,s),\ 0 \le s \le t \tag{1.6.13}$$

and by Lemma 1.6.6 we can conclude that $g(z,t)$ is a Loewner chain. Differentiating (1.6.13) with respect to t and then letting $s \nearrow t$ we see that g is a solution of (1.1.1).

By the same considerations as above we get

$$\left\|e^{-sA}(u(z,s,t_1)-u(z,s,t_2))\right\| \le \int_{t_1}^{t_2} P(t)\left\|e^{(t-s)A}\right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1-\|v(z,s,t)\|)^2}dt.$$

Letting $t_1 = s$ and $t_2 \to \infty$ and using Lemma 1.6.4 we get

$$\begin{aligned}\left\|e^{-sA}g(z,s)\right\| &\le \|z\| + \sum_{k=2}^{n_0}\left\|F_k\left(z^k,s\right)\right\| + \int_s^\infty P(t)\left\|e^{(t-s)A}\right\| \frac{\|v(z,s,t)\|^{n_0+1}}{(1-\|v(z,s,t)\|)^2}dt \\ &\le \frac{Q_{\epsilon,A,F}(s)}{(1-\|z\|)^{2\frac{k_+(A)}{m(A)}+\epsilon}},\ \epsilon>0 \end{aligned} \tag{1.6.14}$$

(P depends on A and F). The above guarantees that the solution is polynomially bounded.

The last statement follows from Proposition 1.6.1 and Lemma 1.6.3. □

For the proof of Theorem 1.6.11 we will need some basic facts about ordinary differential equations. We will use [DK74] for this, but we are interested only in the finite dimensional case.

Let X be a finite dimensional Banach space and L be a bounded linear operator on X. We consider the equation

$$\frac{dx}{dt} = Lx + f(t), \text{ a.e. } t \geq 0 \tag{1.6.15}$$

where $f : [0, \infty) \to X$ is a locally Lebesgue integrable function. We know that any (locally absolutely continuous) solution of (1.6.15) is of the form

$$x(t) = e^{tL} x(0) + \int_0^t e^{(t-s)L} f(s)\, ds.$$

Note that the local Lebesgue integrability of f is needed to ensure the differentiability of the solution above, which follows from the Lebesgue differentiation theorem (see [Fol99, Theorem 3.21]).

We will use the following notations

$$\begin{aligned}
\sigma_+(L) &= \{\lambda \in \sigma(L) : \mathrm{Re}\lambda > 0\} \\
\sigma_\leq(L) &= \{\lambda \in \sigma(L) : \mathrm{Re}\lambda \leq 0\} \\
\sigma_0(L) &= \{\lambda \in \sigma(L) : \mathrm{Re}\lambda = 0\}.
\end{aligned}$$

P^+, $P^\leq$ and P^0 will denote the spectral projections corresponding to $\sigma_+(L)$, $\sigma_\leq(L)$ and $\sigma_0(L)$ respectively (see e.g. [DK74, p 19]).

We say that f is polynomially bounded if there exists a polynomial P such that $\|f(t)\| \leq P(t)$, $t \geq 0$

Following the proof of [DK74, Chapter II, Theorem 4.2] it is straightforward to check that if f is polynomially bounded then to each element $x_0^\leq \in P^\leq(X)$ there corresponds a unique polynomially bounded solution of (1.6.15) that satisfies $P^\leq x(0) = x_0^\leq$. This solution is given by the formula

$$x(t) = e^{(t-t_0)L} x_0^\leq + \int_0^\infty G_L(t-s) f(s)\, ds \tag{1.6.16}$$

where

$$G_L(t) = \begin{cases} e^{tL}P^{\leq} & , t \geq 0 \\ -e^{tL}P^{+} & , t < 0 \end{cases}. \tag{1.6.17}$$

Furthermore if f is bounded and $\sigma_0(L) = \emptyset$ then the solution (1.6.16) is bounded. Note that in order to obtain a polynomial bound on the solution one needs to use (1.6.4) rather than (1.6.3).

Remark 1.6.9. We are only interested in the case when $X = \mathcal{P}^k(\mathbb{C}^n)$, but the above considerations also apply to the case when X is not finite dimensional if we replace polynomially bounded by subexponential. We say that f is subexponential if for any $\epsilon > 0$ there exists C_ϵ such that $\|f(t)\| \leq C_\epsilon e^{\epsilon t}$, $t \geq 0$. To be able to define the spectral projections we would also need to require that $\sigma_+(L)$, $\sigma_\leq(L)$ lie in different connected components of $\sigma(L)$.

We will apply the above results to the coefficient equations (1.6.2) ($X = \mathcal{P}^k(\mathbb{C}^n)$ and we regard the coefficients as functions $F_k : [0,\infty) \to \mathcal{P}^k(\mathbb{C}^n)$), hence we need information about the spectra of the operators B_k. It is known (e.g. [Arn88, pp. 182-183]) that if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a vector whose components are the (not necessarily distinct) eigenvalues of A then the eigenvalues of B_k are

$$\{\langle m, \lambda\rangle - \lambda_s : |m| = k, \, s \in \{1, \ldots, n\}\}$$

where $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $|m| = m_1 + \ldots + m_n$. Furthermore, if A is a diagonal matrix then B_k is also diagonal and $z^m e_s$ is an eigenvector corresponding to $\langle m, \lambda\rangle - \lambda_s$ (e_i, $i = 1, \ldots, n$ denote the elements of the standard basis of $\mathbb{C}^n$).

Following the terminology from [Arn88] we will say that A is nonresonant if $0 \notin \sigma(B_k)$ for all k (i.e. if the eigenvalues of A are nonresonant; see [Arn88, p 180]). Otherwise we say that A is resonant.

Remark 1.6.10. $0 \notin \sigma(B_k)$ for all $k > n_0$ (i.e. there are no resonances of order greater than n_0). Indeed, if $m \in \mathbb{N}^n$, $|m| = k$ and λ is as above then

$$\operatorname{Re}\left(\langle m, \lambda\rangle - \lambda_s\right) \geq (n_0 + 1)\, k_-(A) - k_+(A) > 0$$

where $k_-(A) = \min\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$. For the last inequality we used the fact that $k_-(A) \geq m(A)$ and the definition of n_0 (which implies that $k_+(A) < (n_0+1)\, m(A)$). In particular note that if $n_0 = 1$ then A is nonresonant.

We will use P_k^+, $P_k^{\leq}$, P_k^0 to denote the projections associated with B_k. For $Q \in \mathcal{P}^k(\mathbb{C}^n)$ we will let $Q^+ := P_k^+ Q$, $Q^{\leq} := P_k^{\leq} Q$ and $Q^0 := P_k^0 Q$.

Theorem 1.6.11. *The equation (1.1.1) always has an A-normalized polynomially bounded Loewner chain solution that is uniquely determined by the values of $F_k^{\leq}\left(z^k, 0\right)$, $k = 2, \ldots, n_0$, which can be prescribed arbitrarily. Furthermore, if $A + \bar{A}$ is nonresonant then the solution can be chosen to be bounded.*

Proof. This is an immediate consequence of Theorem 1.6.8 and of the considerations on solutions of ordinary differential equations from above. Note that $A + \bar{A}$ is nonresonant if and only if $\sigma_0(B_k) = \emptyset$, $k \geq 2$. □

Remark 1.6.12. It is not hard to see that if $A + \bar{A}$ is resonant, in general, one cannot find a bounded solution (though it will be possible to do this for particular choices of the infinitesimal generator h). For example, one can choose A such that $\sigma_0(B_2) = \{0\}$ and 0 is a simple eigenvalue for B_2. In this case we would have $e^{B_2}|_{P_2^0(\mathcal{P}^2(\mathbb{C}^n))} = I_{P_2^0(\mathcal{P}^2(\mathbb{C}^n))}$ and so

$$F_2^0\left(z^2, t\right) = F_2^0\left(z^2, 0\right) + \int_0^t H_2^0\left(z^2, s\right) ds.$$

In order to get a solution that is not bounded it is enough to choose h such that $\phi(t) := \int_0^t H_2^0(z^2, s)\, ds$ is not bounded on $[0, \infty)$.

Definition 1.6.13. Let $\mathcal{F} \subset \prod_{k=2}^{n_0} P_k^{\leq}\left(\mathcal{P}^k(\mathbb{C}^n)\right)$. We define $S_A^{\mathcal{F}}(B^n)$ to be the family of mappings $f(z) = z + \sum_{k=2}^{\infty} F_k\left(z^k\right) \in S(B^n)$ that can be embedded as the first element of a polynomially bounded Loewner chain and such that $\left(F_k^{\leq}\right)_{k=2,\ldots,n_0} \in \mathcal{F}$.

We want to study the compactness of the class $S_A^{\mathcal{F}}(B^n)$. For this we need the following lemma that can be proved using similar arguments to those in the proof of [GKK03, Lemma 2.8] (cf. [GHKK08a, Lemma 2.14])

Lemma 1.6.14. *Every sequence of Loewner chains* $\{f_k(z,t)\}$ *such that* $Df_k(0,t) = e^{tA}$ *and*

$$\left\| e^{-tA} f_k(z,t) \right\| \leq C_r P(t), \; \|z\| \leq r < 1, \; t \geq 0,$$

where $P(t)$ *is a polynomial, has a subsequence that converges locally uniformly on* B^n *to a polynomially bounded Loewner chain* $f(z,t)$ *for* $t \geq 0$.

Theorem 1.6.15. *If* $\mathcal{F} \subset \prod_{k=2}^{n_0} P_k^{\leq}\left(\mathcal{P}^k(\mathbb{C}^n)\right)$ *is bounded (compact) then* $S_A^{\mathcal{F}}(B^n)$ *is normal (compact). Furthermore, given* $\epsilon > 0$ *there exists a constant* $C_{\epsilon,A,\mathcal{F}}$ *such that*

$$\|f(z)\| \leq \frac{C_{\epsilon,A,\mathcal{F}}}{(1-\|z\|)^{2\frac{k_+(A)}{m(A)}+\epsilon}}, \; f \in S_A^{\mathcal{F}}(B^n).$$

Proof. Let $f \in S_A^{\mathcal{F}}(B^n)$ and $f(z,t)$ be a polynomially bounded Loewner chain such that $f(z,0) = f(z)$. Suppose that

$$f(z,t) = e^{tA}\left(z + \sum_{k=2}^{\infty} F_k\left(z^k, t\right)\right).$$

We know that (see (1.6.2) and (1.6.16))

$$F_k\left(z^k,t\right) = e^{tB_k} F_k^{\leq}\left(z^k,0\right) + \int_0^{\infty} G_{B_k}(t-s)\, N_k\left(z^k,s\right) ds.$$

Now it is straightforward to check that if $\mathcal{F}$ is bounded then F_k, $k = 2,\ldots,n_0$ can be bounded by a polynomial F that doesn't depend on f (it depends only on $\mathcal{F}$ and A). By Theorem 1.6.8 we have

$$\left\| e^{-tA} f(z,t) \right\| \leq \frac{Q_{\epsilon,A,F}(t)}{(1-\|z\|)^{2\frac{k_+(A)}{m(A)}+\epsilon}}.$$

When $t = 0$ the above inequality proves the fact that $S_A^{\mathcal{F}}(B^n)$ is normal. Furthermore, if $\mathcal{F}$ is also closed we can now argue by contradiction using the previous Lemma to see that $S_A^{\mathcal{F}}(B^n)$ is also closed. $\square$

Remark 1.6.16. It is not hard to see that the results of this section (except for 1.6.5) remain true for any norm on $\mathbb{C}^n$. Furthermore, with appropriate modifications (see Remark 1.6.9 and [HK04]) the results can be extended to reflexive complex Banach spaces.

1.7 Spirallikeness, parametric representation, asymptotical spirallikeness

We now consider what happens to the various classes of univalent mappings that are related to Loewner chains. For convenience we recall the definitions of the classes that we are considering, as given in [GHKK08a], where the case $n_0 = 1$ is treated.

Let $\Omega \subset \mathbb{C}^n$ be a domain containing the origin.

Definition 1.7.1. We say that Ω is spirallike with respect to A if $e^{-tA}w \in \Omega$ for any $w \in \Omega$ and $t \geq 0$.

Definition 1.7.2. We say that Ω is A-asymptotically spirallike if there exists a mapping $Q = Q(z,t) : \Omega \times [0,\infty) \to \mathbb{C}^n$ that satisfies the following conditions:

(i) $Q(\cdot,t)$ is a holomorphic mapping on Ω, $Q(0,t) = 0$, $DQ(0,t) = A$, $t \geq 0$, and the family $\{Q(\cdot,t)\}_{t\geq 0}$ is locally uniformly bounded on Ω;

(ii) $Q(z,\cdot)$ is measurable on $[0,\infty)$ for all $z \in \Omega$;

(iii) the initial value problem

$$\frac{\partial w}{\partial t} = -Q(w,t) \text{ a.e. } t \geq s,\ w(z,s,s) = z \tag{1.7.1}$$

has a unique solution $w = w(z,s,t)$ for each $z \in \Omega$ and $s \geq 0$, such that $w(\cdot,s,t)$ is a holomorphic mapping of Ω into Ω for $t \geq s$, $w(z,s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to $z \in \Omega$ for $s \geq 0$, and $\lim_{t\to\infty} e^{tA}w(z,0,t) = z$ locally uniformly on Ω.

Let $f : B^n \to \mathbb{C}^n$ be a normalized univalent mapping, i.e. such that $f(0) = 0$ and $Df(0) = I$. $S(B^n)$ will denote the class of all such mappings.

Definition 1.7.3. We say that f is spirallike with respect to A if $f(B^n)$ is spirallike. We will use $\hat{S}_A(B^n)$ to denote the class of mappings that are spirallike with respect to A.

Definition 1.7.4. We say that f is A-asymptotically spirallike if $f(B^n)$ is A-asymptotically spirallike. $S_A^a(B^n)$ will denote the class of A-asymptotically spirallike mappings.

Definition 1.7.5. We say that f has A-parametric representation if there exists a mapping $h \in \mathcal{H}_A(B^n)$ such that $f(z) = \lim_{t\to\infty} e^{tA} v(z,t)$ locally uniformly on B^n, where v is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \ a.e.\, t \geq 0,\ v(z,0) = z,\ z \in B^n. \tag{1.7.2}$$

$S_A^0(B^n)$ will denote the class of mappings with A-parametric representation.

We start by answering [GK03, Open Problem 6.4.13].

Theorem 1.7.6. *$\hat{S}_A(B^n)$ is compact if and only if A is nonresonant.*

Proof. If $f \in \hat{S}_A(B^n)$ we know that

$$f(z,t) := e^{tA} f(z) = e^{tA}\left(z + \sum_{k=2}^{\infty} F_k\left(z^k\right)\right) \tag{1.7.3}$$

is a Loewner chain (this follows easily from the definitions). It is clear that $\hat{S}_A(B^n) \subset S_A^{\mathcal{F}}(B^n)$, where

$$\mathcal{F} := \left\{\left(F_k^{\leq}\right)_{k=2,\dots,n_0} : f(z) = z + \sum_{k=2}^{\infty} F_k\left(z^k\right) \in \hat{S}_A(B^n)\right\}.$$

It is easy to see that $\hat{S}_A(B^n)$ is closed by using the analytic characterization (1.7.4) and the fact that $\mathcal{N}_A$ is compact. Now, by Theorem 1.6.15, if the coefficients F_k, $k = 2, \dots, n_0$ can be bounded independently of f then $\hat{S}_A(B^n)$ is compact. For our particular Loewner chain (1.7.3) the coefficient equations (1.6.2) take the simple form $0 = B_k F_k + N_k$.

If A is nonresonant then the operators B_k are invertible and hence $F_k = -B_k^{-1} N_k$. Now it is straightforward to see that we can choose bounds for F_k, $k = 2, \ldots, n_0$ that don't depend on f, thus yielding compactness of $\hat{S}_A(B^n)$.

If A is resonant then let $k_0 \leq n_0$ be the largest k such that B_k is singular (by 1.6.10 B_k is not singular for $k > n_0$). Let $h(z) = Az + H_{k_0}\left(z^{k_0}\right) \in \mathcal{N}_A$, where H_{k_0} is chosen such that $B_{k_0} F_{k_0} + H_{k_0} = 0$ has a solution. Note that for our particular h we have $N_k = 0$, $k = 2, \ldots, k_0 - 1$ and $N_{k_0} = H_{k_0}$. Since B_k, $k > k_0$ are nonsingular there is no problem in solving for F_k, $k > k_0$ and then, using Theorem 1.6.8, we get that

$$f(z,s) = \lim_{t\to\infty} e^{tA}\left(v(z,s,t) + \sum_{k=2}^{n_0} F_k\left(v(z,s,t)^k\right)\right)$$

is a Loewner chain solution of (1.1.1) with $h(z,t) = h(z)$. Since h doesn't depend on t we have $v(z,s,t) = v(z,0,t-s)$ and this yields that $f(z,s) = e^{sA} f(z,0)$. Hence $f(\cdot,0) \in \hat{S}_A(B^n)$ and by Theorem 1.6.8 its k_0-th coefficient is F_{k_0}.

This construction works with any F_{k_0} that is a solution of $B_{k_0} F_{k_0} + H_{k_0} = 0$. Since B_{k_0} is singular, the solutions of the equation form a non-trivial affine subspace of $\mathcal{P}^{k_0}(\mathbb{C}^n)$, so in particular there exist solutions of arbitrarily large norm. Now we can conclude that there exist spirallike mappings with arbitrarily large k_0-th coefficient. This proves that $\hat{S}_A(B^n)$ is not compact when A is resonant. □

Remark 1.7.7. Let $h \in \mathcal{N}_A$. By the same ideas as in the proof of the previous theorem we can conclude that if A is nonresonant then the equation

$$Df(z)\,h(z) = Af(z) \tag{1.7.4}$$

has a unique holomorphic solution, which is in fact biholomorphic (because of Corollary 1.6.7). By 1.6.10 this generalizes [DGHK10, Corollary 4.8]. On the other hand, if A is resonant, there either is no holomorphic solution (for example if $H_2 \notin B_2(\mathcal{P}^2(\mathbb{C}^n))$) or the holomorphic solutions (in fact, biholomorphic) are not unique.

Remark 1.7.8. As a consequence of the proof of Theorem 1.7.6 and of Theorem 1.6.15 we have the following bound for mappings in $\hat{S}_A(B^n)$:

$$\|f(z)\| \leq \frac{C_{\epsilon,A}}{(1-\|z\|)^{2\frac{k_+(A)}{m(A)}+\epsilon}},\ z \in B^n,\ \epsilon > 0,\ f \in \hat{S}_A(B^n)$$

(cf. [HK01, Theorem 3.1] and [CKK10, Theorem 12]). Furthermore, if A is normal the above estimate holds with $\epsilon = 0$ (the case $k_+(A)/m(A) > 1$ follows using 1.6.5, while the case $k_+(A)/m(A) = 1$ is covered by [HK01, Corollary 3.1])

Remark 1.7.9. Let $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ ($\operatorname{Re}\lambda_i > 0$) and $m \in \mathbb{N}^n$ with $m_i = 0$, $i = 1, \ldots, s$, where $1 \leq s < n$. Then it is easy to compute that for $f(z) = z + az^m e_s$ we have

$$h(z) = [Df(z)]^{-1} Af(z) = Az + a(\lambda_s - \langle m, \lambda\rangle) z^m e_s.$$

If $\lambda_s - \langle m, \lambda\rangle = 0$ we get that $f \in \hat{S}_A(B^n)$ for any $a \in \mathbb{C}^n$ generalizing an example from [HK01, p 57]. If $\lambda_s - \langle m, \lambda\rangle \neq 0$ then $f \in \hat{S}_A(B^n)$ for any a such that

$$|a| \leq \frac{m(A)}{|\lambda_s - \langle m, \lambda\rangle|}.$$

This example suggests that in the case when A is nonresonant a sharp upper growth bound on $\hat{S}_A(B^n)$ would have to depend on the entire spectrum of A.

Next we extend [GHKS02, Corollary 2.2]. For simplicity we only treat the 2-dimensional case. $S^*(B^n) = \hat{S}_I(B^n)$ denotes the class of normalized starlike mappings. For a more general result obtained by a different approach see [Eli10, Theorem 5.1].

Proposition 1.7.10. *Let* $A = \operatorname{diag}(1, \lambda)$, $\operatorname{Re}\lambda \geq 1$. *Define* $\Phi_{\alpha,\beta} : S(B^1) \to S(B^2)$ *by*

$$\Phi_{\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} z_2\right).$$

If $\alpha \in [0, \operatorname{Re}\lambda]$ *and* $\beta \in [0, 1/2]$ *such that* $\alpha + \beta \leq \operatorname{Re}\lambda$ *then* $\Phi_{\alpha,\beta}(S^*(B^1)) \subset \hat{S}_A(B^2)$.

Proof. We follow the proof of [GHKS02, Theorem 2.1]. Let $f \in S^*(B^1)$ and define

$$F(z,t) = e^{tA}\Phi_{\alpha,\beta}(f)(z) = \left(e^t f(z_1), e^{\lambda t}\left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} z_2\right).$$

It is sufficient to check that $F(z,t)$ is a Loewner chain. Because of the particular form of F and by Corollary 1.6.7 it is enough to check that F satisfies a Loewner chain equation, i.e. that

$$h(\cdot,t) := [DF(\cdot,t)]^{-1}\frac{\partial F}{\partial t}(\cdot,t) \in \mathcal{H}_A\left(B^2\right), \text{ a.e. } t \geq 0.$$

Let $p(z_1) = f(z_1)/(z_1 f'(z_1))$. Straightforward computations yield that

$$h(z,t) = (z_1 p(z_1), z_2(\lambda - \alpha - \beta + (\alpha+\beta)p(z_1) + \beta z_1 p'(z_1))).$$

The same arguments as in the proof of [GHKS02, Theorem 2.1] (we are using the fact that $f \in S^*(B^1)$ implies that $\mathrm{Re}p > 0$) show that it is sufficient to check that

$$q(x) = (\mathrm{Re}\lambda - \alpha - \beta)x^2 - 2\beta x + \alpha + \beta$$

is non-negative on $[0,1]$. This follows by elementary analysis. □

Remark 1.7.11. Let A be as in the above proposition. For $\alpha = \mathrm{Re}\lambda - 1/2$, $\beta = 1/2$ and $f(z) = z/(1-z)^2$ we can see that $\Phi_{\alpha,\beta}(f) \in \hat{S}_A(B^2)$ attains the asymptotic growth bound from 1.7.8.

Next we consider the class of mappings with A-parametric representation. Unlike the class of spirallike mappings, the class $S_A^0(B^n)$ is not compact when $n_0 > 1$, as we can see from the following example.

Example 1.7.12. Let $A = \mathrm{diag}(\lambda, 1)$, $\mathrm{Re}\lambda \geq 2$ and define

$$h(z,t) = \left(\lambda z_1 + a(t) z_2^2, z_2\right), \; z = (z_1, z_2) \in B^2.$$

If for example $|a(t)| \leq 1$, $t \geq 0$ it is easy to check that $h(\cdot,t) \in \mathcal{N}_A$, $t \geq 0$. Then

$$v(z,t) = \left(e^{-\lambda t}\left(z_1 - \left(\int_0^t a(s) e^{(\lambda-2)s} ds\right) z_2^2\right), e^{-t} z_2\right)$$

is the solution of (1.7.2). When $\lim_{t\to\infty} e^{tA} v(z,t)$ exists locally uniformly on B^n we get that $f(z) = \left(z_1 - \left(\int_0^\infty a(s) e^{(\lambda-2)s} ds\right) z_2^2, z_2\right) \in S_A^0(B^2)$. Since the second coefficient of the Taylor series expansion can be made arbitrarily large by an appropriate choice of $a(\cdot)$ we conclude that $S_A^0(B^2)$ is not compact.

This example can be generalized for any A by considering $h(z,t) = Az + a(t) H_2(z^2) \in \mathcal{H}_A(B^n)$ such that $H_2^{\leq} \neq 0$.

Next we consider the class $S_A^a(B^n)$. The following characterization of A-asymptotically spirallike mappings is derived from the proofs of [GHKK08a, Theorem 3.1 and Theorem 3.5].

Proposition 1.7.13. *Let $f : B^n \to \mathbb{C}^n$ be a holomorphic mapping and*

$$f(z) = z + \sum_{k=2}^{\infty} F_k\left(z^k\right).$$

Then f is A-asymptotically spirallike if and only if there exists $h \in \mathcal{H}_A(B^n)$ such that

$$f(z) = \lim_{t\to\infty} e^{tA}\left(v(z,t) + \sum_{k=2}^{n_0} F_k\left(v(z,t)^k\right)\right) \tag{1.7.5}$$

locally uniformly on B^n, where v is the solution of (1.7.2).

Proof. First assume that f is A-asymptotically spirallike. Hence there exists a mapping $Q : f(B^n) \times [0,\infty) \to \mathbb{C}^n$ satisfying the assumptions from Definition 1.7.2. Let ν be the solution of the initial value problem (1.7.1). By definition it will satisfy

$$\lim_{t\to\infty} e^{tA}\nu(f(z),0,t) = f(z)$$

locally uniformly on B^n.

Let v be defined by $v(z,s,t) = f^{-1}(\nu(f(z),s,t))$, $z \in B^n$, $t \geq s$. Also, let $h(z,t) = [Df(z)]^{-1} Q(f(z),t)$, $z \in B^n$, $t \geq 0$. With the same proof as in [GHKK08a, Theorem 3.5] one sees that $h \in \mathcal{H}_A(B^n)$ and that v is the solution of (1.1.2).

We have

$$f(z) = \lim_{t\to\infty} e^{tA}\nu(f(z),0,t) = \lim_{t\to\infty} e^{tA} f(v(z,0,t))$$

locally uniformly on B^n. Like in the proof of Proposition 1.6.1 we also see that

$$\lim_{t\to\infty} e^{tA} f(v(z,0,t)) = \lim_{t\to\infty} e^{tA}\left(v(z,0,t) + \sum_{k=2}^{n_0} F_k\left(v(z,0,t)^k\right)\right)$$

yielding the desired conclusion (the fact that f is univalent follows from Lemma 1.6.6).

Now assume that (1.7.5) holds. The conclusion follows exactly as in the proof of [GHKK08a, Theorem 3.1]. □

Remark 1.7.14. From the above characterization of $S_A^a(B^n)$ it is easy to see that $S_A^a(B^n) \neq S_A^0(B^n)$ when $n_0 > 1$.

In Proposition 1.7.17 we obtain a partial result about the normality of the class $S_A^a(B^n)$, but first we need the following lemmas.

Lemma 1.7.15. *Let $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $h \in \mathcal{H}_A(B^n)$. If $v = (v_1, \ldots, v_n)$ is the solution of (1.7.2) then*

$$\|v_i(z,t)\| \leq C \begin{cases} e^{-\mathrm{Re}\lambda_i t} & , \mathrm{Re}\lambda_i < 2m(A) \\ (1+t)e^{-\mathrm{Re}\lambda_i t} & , \mathrm{Re}\lambda_i = 2m(A) \\ e^{-2m(A)t} & , \mathrm{Re}\lambda_i > 2m(A) \end{cases}$$

where C is a constant that depends on A, λ_i and $\|z\|$.

Proof. Writing $h = (h_1, \ldots, h_n)$ and $\tilde{h}_i = h_i - \lambda_i z_i$, (1.7.2) yields

$$\frac{dv_i}{dt} = -\lambda_i v_i - \tilde{h}_i(v,t).$$

Integrating we get

$$e^{t\lambda_i} v_i = z_i - \int_0^t e^{s\lambda_i} \tilde{h}_i(v,s)\, ds.$$

Hence

$$\begin{aligned}\left\|e^{t\lambda_i}v_i\left(z,t\right)\right\| &\leq |z_i| + \int_0^t e^{s\mathrm{Re}\lambda_i}\left\|h\left(v\left(z,s\right),s\right) - Av\left(z,s\right)\right\| ds \\ &\leq |z_i| + C_{A,\|z\|}\int_0^t e^{s\mathrm{Re}\lambda_i}\left\|v\left(z,s\right)\right\|^2 ds \\ &\leq |z_i| + C_{A,\|z\|}\int_0^t e^{s\left(\mathrm{Re}\lambda_i - 2m(A)\right)} ds \\ &\leq \begin{cases} C_{A,\|z\|,\lambda_i} & ,\ \mathrm{Re}\lambda_i < 2m\left(A\right) \\ C_{A,\|z\|}(1+t) & ,\ \mathrm{Re}\lambda_i = 2m\left(A\right) \\ C_{A,\|z\|,\lambda_i}e^{\left(\mathrm{Re}\lambda_i - 2m(A)\right)t} & ,\ \mathrm{Re}\lambda_i > 2m\left(A\right)\end{cases}\end{aligned}$$

(the second inequality follows from (1.6.12) and the third estimate follows from (1.6.1)).

$\square$

Lemma 1.7.16. *Let* $\lambda \in \mathbb{C}$ *such that* $\mathrm{Re}\lambda \geq 0$*,* $a \in \mathbb{C}$ *and* $h : [0,\infty) \to \mathbb{C}$ *such that* $|h\left(t\right)| \leq C$*,* $t \geq 0$*. If*

$$\lim_{t\to\infty}\int_0^t e^{s\lambda}\left(h\left(s\right)+a\right) ds = 0 \tag{1.7.6}$$

then $|a| \leq C$*.*

Proof. We argue by contradiction. Assume that $|a| > C$. Then

$$\mathrm{Re}\left(\left(h\left(s\right)+a\right)\bar{a}\right) \geq |a|^2 - |a|\left|h\left(s\right)\right| \geq |a|\left(|a| - C\right) =: \delta > 0.$$

If $\mathrm{Im}\lambda = 0$ we get

$$\mathrm{Re}\left(\left(\int_0^t e^{s\lambda}\left(h\left(s\right)+a\right) ds\right)\bar{a}\right) \geq t\delta$$

contradicting (1.7.6).

If $\mathrm{Im}\lambda \neq 0$ we can find $\tau > 0$ such that

$$\mathrm{Re}\left(e^{s\lambda}\left(h\left(s\right)+a\right)\bar{a}\right) \geq \frac{\delta}{2},\ s \in \left[\frac{2k\pi}{\mathrm{Im}\lambda}, \frac{2k\pi}{\mathrm{Im}\lambda} + \tau\right],\ k \geq 0,\ k \in \mathbb{Z}.$$

From (1.7.6) we get that

$$\lim_{t\to\infty}\int_t^{t+\tau} e^{s\lambda}\left(h\left(s\right)+a\right) ds = 0.$$

This is contradicted by

$$\mathrm{Re}\left(\left(\int_{t_k}^{t_k+\tau} e^{s\lambda}\left(h\left(s\right)+a\right)ds\right)\bar{a}\right)\geq\frac{\delta\tau}{2},$$

where $t_k = 2k\pi/\mathrm{Im}\lambda$. Thus we must have that $|a| \leq C$. □

Proposition 1.7.17. *Suppose that A is normal, nonresonant and $n_0 = 2$. Then $S_A^a\left(B^n\right)$ is a normal family. Furthermore, if $f \in S_A^a\left(B^n\right)$ has $h \in \mathcal{H}_A\left(B^n\right)$ as an infinitesimal generator (see Proposition 1.7.13) then f can be embedded as the first element of a bounded Loewner chain with infinitesimal generator h.*

Proof. If U is a unitary matrix, $f \in S_A^a\left(B^n\right)$ and $h \in \mathcal{H}_A\left(B^n\right)$ is an infinitesimal generator for f then it is straightforward to check that $U^*fU \in S_{U^*AU}^a\left(B^n\right)$ and that $U^*hU \in \mathcal{H}_{U^*AU}\left(B^n\right)$ is an infinitesimal generator for U^*fU. This allows us to assume without loss of generality that $A = \mathrm{diag}\left(\lambda_1,\ldots,\lambda_n\right)$ and $\mathrm{Re}\lambda_1 \geq \ldots \geq \mathrm{Re}\lambda_n > 0$ (note that $m\left(A\right) = \mathrm{Re}\lambda_n$).

Let f be an A-asymptotically spirallike mapping and $h \in \mathcal{H}_A\left(B^n\right)$ be an infinitesimal generator for f. Let v be the solution of (1.7.2). Also, assume that f, $h\left(\cdot,t\right)$ and $v\left(\cdot,t\right)$ have the following Taylor series expansions:

$$\begin{aligned} f\left(z\right) &= z+F_2\left(z^2\right)+\ldots \\ h\left(z,t\right) &= Az+H_2\left(z^2,t\right)+\ldots \\ v\left(z,t\right) &= e^{-tA}z+V_2\left(z^2,t\right)+\ldots. \end{aligned}$$

From (1.7.2) and then (1.6.6) one easily gets that

$$\begin{aligned} e^{tA}V_2\left(z^2,t\right) &= -\int_0^t e^{sA}H_2\left(\left(e^{-sA}z\right)^2,s\right)ds \\ &= -\int_0^t e^{-sB_2}H_2\left(z^2,s\right)ds. \end{aligned}$$

As a consequence of Proposition 1.7.13 , the above equality and (1.6.6) we have

$$\begin{aligned} F_2\left(z^2\right) &= \lim_{t\to\infty}\left(e^{tA}V_2\left(z^2,t\right)+e^{tA}F_2\left(\left(e^{-tA}z\right)^2\right)\right) \\ &= \lim_{t\to\infty}\left(-\int_0^t e^{-sB_2}H_2\left(z^2,s\right)ds+e^{-tB_2}F_2\left(z^2\right)\right). \quad (1.7.7) \end{aligned}$$

We want to show that $F_2^{\leq}$ can be bounded independently of $f \in S_A^a(B^n)$. For this we will show that each of the coefficients f_{ij}^k of the monomials $z_i z_j e_k$ from $F_2^{\leq}$ can be bounded independently of $f \in S_A^a(B^n)$.

We know that

$$B_2(z_i z_j e_k) = (\lambda_i + \lambda_j - \lambda_k) z_i z_j e_k$$

and so

$$e^{tB_2}(z_i z_j e_k) = e^{t(\lambda_i+\lambda_j-\lambda_k)} z_i z_j e_k.$$

Projecting (1.7.7) on the subspace generated by $z_i z_j e_k$ we get

$$\begin{aligned} f_{ij}^k &= \lim_{t\to\infty}\left(-\int_0^t e^{-s(\lambda_i+\lambda_j-\lambda_k)} h_{ij}^k(s)\,ds + e^{-t(\lambda_i+\lambda_j-\lambda_k)} f_{ij}^k\right) \\ &= \lim_{t\to\infty}\left(-\int_0^t e^{-s(\lambda_i+\lambda_j-\lambda_k)}\left(h_{ij}^k(s) + (\lambda_i+\lambda_j-\lambda_k) f_{ij}^k\right) ds + f_{ij}^k\right) \end{aligned}$$

($h_{ij}^k(s)$ are the coefficients of the monomials $z_i z_j e_k$ from $H_2(z^2, s)$). Hence

$$\lim_{t\to\infty}\int_0^t e^{-s(\lambda_i+\lambda_j-\lambda_k)}\left(h_{ij}^k(s) + (\lambda_i+\lambda_j-\lambda_k) f_{ij}^k\right) ds = 0. \tag{1.7.8}$$

For the coefficients of the monomials of $F_2^{\leq}$ we have that $\operatorname{Re}(\lambda_i+\lambda_j-\lambda_k) \leq 0$, hence we can use Lemma 1.7.16 and the fact that $\lambda_i+\lambda_j-\lambda_k \neq 0$ (since A is nonresonant) to conclude that the coefficients of $F_2^{\leq}$ are bounded independently of f (h_{ij}^k are bounded because $\mathcal{N}_A$ is compact).

Let $f(z,t)$ denote the polynomially bounded Loewner chain with infinitesimal generator h and such that $F_2^{\leq}(z^2, 0) = F_2^{\leq}(z^2)$ (see Theorem 1.6.11). We will see that $f = f(\cdot, 0)$. This will show that $S_A^a(B^n) \subset S_A^{\mathcal{F}}(B^n)$ where

$$\mathcal{F} = \left\{F_2^{\leq} : f(z) = z + F_2\left(z^2\right) + \ldots \in S_A^a(B^n)\right\}.$$

By Theorem 1.6.15 this yields the normality of $S_A^a(B^n)$.

It is enough to check that

$$0 = f(z) - f(z,0) = \lim_{t\to\infty} e^{tA}\left(F_2\left(v(z,t)^2\right) - F_2\left(v(z,t)^2, t\right)\right). \tag{1.7.9}$$

We know that (see (1.6.2), (1.6.16), (1.6.17))

$$
\begin{aligned}
F_2\left(z^2,t\right) &= e^{tB_2}F_2^{\leq}\left(z^2\right)+\int_0^{\infty}G_{B_2}\left(t-s\right)N_2\left(z^2,s\right)ds\\
&= e^{tB_2}F_2^{\leq}\left(z^2\right)+\int_0^{t}e^{(t-s)B_2}H_2^{\leq}\left(z^2,s\right)ds-\int_t^{\infty}e^{(t-s)B_2}H_2^{+}\left(z^2,s\right)ds\\
&= F_2^{\leq}\left(z^2\right)+e^{tB_2}\int_0^{t}e^{-sB_2}\left(H_2^{\leq}\left(z^2,s\right)+B_2F_2^{\leq}\left(z^2\right)\right)ds\\
&\qquad -\int_0^{\infty}e^{-sB_2}H_2^{+}\left(z^2,s+t\right)ds.
\end{aligned}
$$

Substituting the above in (1.7.9) we need to verify that

$$
\lim_{t\to\infty}e^{tA}e^{tB_2}\int_0^{t}e^{-sB_2}\left(H_2^{\leq}\left(v\left(z,t\right)^2,s\right)+B_2F_2^{\leq}\left(v\left(z,t\right)^2\right)\right)ds=0 \tag{1.7.10}
$$

and

$$
\lim_{t\to\infty}e^{tA}\int_0^{\infty}e^{-sB_2}\left(H_2^{+}\left(v\left(z,t\right)^2,s\right)-H_2^{+}\left(v\left(z,t\right)^2,s+t\right)\right)ds=0. \tag{1.7.11}
$$

Using (1.6.7), (1.7.10) becomes

$$
\lim_{t\to\infty}\int_0^{t}e^{-sB_2}\left(H_2^{\leq}\left(\left(e^{tA}v\left(z,t\right)\right)^2,s\right)+B_2F_2^{\leq}\left(\left(e^{tA}v\left(z,t\right)\right)^2\right)\right)ds=0. \tag{1.7.12}
$$

Let $v=(v_1,\dots,v_n)$. Separating the monomials in (1.7.12) it is enough to prove that

$$
\lim_{t\to\infty}e^{t\lambda_i}v_i\left(z,t\right)e^{t\lambda_j}v_j\left(z,t\right)\int_0^{t}e^{-s(\lambda_i+\lambda_j-\lambda_k)}c_{ij}^{k}\left(s\right)ds=0
$$

($c_{ij}^{k}\left(s\right)$ are the coefficients of the monomials in the polynomial $H_2^{\leq}\left(\cdot,s\right)+B_2F_2^{\leq}$) provided that $\mathrm{Re}\left(\lambda_i+\lambda_j-\lambda_k\right)\leq 0$ and (because of (1.7.8))

$$
\lim_{t\to\infty}\int_0^{t}e^{-s(\lambda_i+\lambda_j-\lambda_k)}c_{ij}^{k}\left(s\right)ds=0.
$$

It is now enough to check that $e^{t\lambda_i}v_i\left(z,t\right)$ and $e^{t\lambda_j}v_j\left(z,t\right)$ are bounded on $\{z\}\times[0,\infty)$, assuming that $\mathrm{Re}\left(\lambda_i+\lambda_j-\lambda_k\right)\leq 0$. This follows from Lemma 1.7.15 provided that $\mathrm{Re}\lambda_i,\mathrm{Re}\lambda_j<2\mathrm{Re}\lambda_n$. Assume that this is not the case, so for example $\mathrm{Re}\lambda_i\geq 2\mathrm{Re}\lambda_n$. This implies that

$$
\mathrm{Re}\left(\lambda_i+\lambda_j-\lambda_k\right)\geq\mathrm{Re}\left(3\lambda_n-\lambda_1\right)>0
$$

which contradicts $\operatorname{Re}(\lambda_i + \lambda_j - \lambda_k) \leq 0$. For the last inequality we used the hypothesis $n_0 = 2$ which implies that $2 \leq \operatorname{Re}\lambda_1/\operatorname{Re}\lambda_n < 3$.

Separating the monomials in (1.7.11) it is enough to prove that

$$\lim_{t\to\infty} e^{t\lambda_k} v_i(z,t)\, v_j(z,t) \int_0^\infty e^{-s(\lambda_i+\lambda_j-\lambda_k)} d_{ij}^k(s,t)\, ds = 0$$

provided that $\operatorname{Re}(\lambda_i + \lambda_j - \lambda_k) > 0$ ($d_{ij}^k(s,t)$ are the coefficients of the monomials in the polynomial $H_2^+(\cdot,s) - H_2^+(\cdot,s+t)$). Since $H_2^+(\cdot,s) - H_2^+(\cdot,s+t)$ can be bounded independently of s and t we see that $\int_0^\infty e^{-s(\lambda_i+\lambda_j-\lambda_k)} d_{ij}^k(s,t)\, ds$ can be bounded independently of t. Hence it is enough to check that

$$\lim_{t\to\infty} e^{t\lambda_k} v_i(z,t)\, v_j(z,t) = 0.$$

If $\operatorname{Re}\lambda_i, \operatorname{Re}\lambda_j \leq 2\operatorname{Re}\lambda_n$ then using Lemma 1.7.15 we have

$$\left\| e^{t\lambda_k} v_i(z,t)\, v_j(z,t) \right\| \leq C(1+t)^2 e^{-t\operatorname{Re}(\lambda_i+\lambda_j-\lambda_k)}.$$

If $\operatorname{Re}\lambda_i > 2\operatorname{Re}\lambda_n$ or $\operatorname{Re}\lambda_j > 2\operatorname{Re}\lambda_n$ then using Lemma 1.7.15 again we get

$$\left\| e^{t\lambda_k} v_i(z,t)\, v_j(z,t) \right\| \leq C e^{-t\operatorname{Re}(3\lambda_n-\lambda_k)}.$$

Since $\operatorname{Re}(\lambda_i + \lambda_j - \lambda_k) > 0$ and $\operatorname{Re}(3\lambda_n - \lambda_k) \geq \operatorname{Re}(3\lambda_n - \lambda_1) > 0$ the above inequalities prove the desired limit. This completes the proof. $\square$

Remark 1.7.18. It is not clear whether $S_A^a(B^n)$ is closed under the assumptions of the above theorem. Suppose that $\{f_k\}$ is a sequence in $S_A^a(B^n)$ converging to some $f \in S(B^n)$. Let $\{f_k(z,t)\}$ be polynomially bounded Loewner chains such that $f_k(z,0) = f_k(z)$. Then by Lemma 1.6.14 we have that up to a subsequence $\{f_k(z,t)\}$ converges to a polynomially bounded Loewner chain $f(z,t)$ such that $f(z,0) = f(z)$. In order to conclude that $f \in S_A^a(B^n)$ it would be natural to have that f satisfies (1.7.5) with v satisfying $f(v(z,t),t) = f(z)$. Unfortunately one can find examples when this doesn't happen.

Remark 1.7.19. (1.7.8) gives a necessary condition for a mapping $h \in \mathcal{H}_A$ to be the infinitesimal generator associated to some $f \in S_A^a(B^n)$. It is possible to choose h such that (1.7.8) is not satisfied for any $f \in S_A^a(B^n)$. This means that unlike the $n_0 = 1$ case, there exist polynomially bounded Loewner chains for which the first element is not from $S_A^a(B^n)$. We will give an example of such h in the case when $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $\mathrm{Re}\lambda_1 \geq \ldots \geq \mathrm{Re}\lambda_n > 0$, $\mathrm{Re}(2\lambda_n - \lambda_1) \leq 0$. Let

$$h(z,t) = Az + (ae^{t(2\lambda_n - \lambda_1)})z_n^2 e_1,\ t \geq 0$$

where $a \in \mathbb{C} \setminus \{0\}$ is sufficiently small so that $h \in \mathcal{H}_A(B^n)$ (it is easy to see that such a's exist since we are assuming that $\mathrm{Re}(2\lambda_n - \lambda_1) \leq 0$). Using the notation of the above proposition it is straightforward to check that for our choice of h,

$$\lim_{t\to\infty} \int_0^t e^{-s(\lambda_n + \lambda_n - \lambda_1)} \left(h_{nn}^1(s) + (\lambda_n + \lambda_n - \lambda_1) f_{nn}^1\right) ds \neq 0$$

for any $f_{nn}^1 \in \mathbb{C}$.

Chapter 2

Extreme Points

2.1 Introduction

Solving extremal problems for compact classes of biholomorphic functions has been one of the main goals of geometric function theory in one variable. Starting in the 1970s the application of linear methods to study the extreme points and support points of the various classes began playing an important role. See [MW02] for a detailed account and references.

In higher dimensions very little is known about the extreme points and support points of the various classes of mappings. More specifically, in [MS06] examples are given of extreme points for the class of convex mappings on B^n and in [GKP07] a relation between extreme/support points and Loewner chains is studied. The goal of this chapter is to provide further examples of extreme points.

Consider the class $\mathcal{P} = \{f \in H(B^n, \mathbb{C}) : \mathrm{Re} f(z) > 0,\ f(0) = 1\}$ of Carathéodory functions. In one variable this class plays a pivotal role due to its ubiquity in the analytic characterizations of various subclasses and due to its relation to Loewner chains. The main result on Carathéodory functions is the integral representation formula due to Herglotz. This formula is very important for solving various extremal problems for the class

$\mathcal{P}$, which in turn contributes to solving extremal problems for classes of biholomorphic functions. In particular, this integral representation easily yields that the extreme points of the class $\mathcal{P}$ are the functions of the form $\left(1+e^{i\theta}\zeta\right)/\left(1-e^{i\theta}\zeta\right)$.

The class $\mathcal{P}$ has also been studied in higher dimensions, but with very limited success. Of importance to us is the work of F. Forelli [For75, For79], who found examples of extreme points for the class $\mathcal{P}$. Based on this we will be able to give a class of examples of extreme points for the class $\mathcal{M}$ on B^n.

For geometric function theory in higher dimensions the role of the class $\mathcal{P}$ is played by the class $\mathcal{M}$. Thus, it is natural to study extremal problems and extreme points for this class. It is known that in one dimension functions h from the class $\mathcal{M}$ can be written as $h(\zeta)=\zeta p(\zeta)$ where p is a function in the class $\mathcal{P}$. This justifies asking whether mappings of the form $(z_1p_1(z_1),\ldots,z_np_n(z_n))$, where p_i are extreme points for the class $\mathcal{P}$, are extreme points for the class $\mathcal{M}$. We will see that this is indeed the case on the polydisc but not on B^n.

2.2 Extreme points on P^n

We will be using the fact that for a convex set (such as the Carathéodory class) in order to prove that a point h is extreme it is enough to show that there is no nonzero point g such that $h\pm g$ is in the set. Indeed, suppose that there exist $\alpha\in(0,1)$ and $f_1,\ f_2$ points in the set such that $h=\alpha f_1+(1-\alpha)f_2$. Suppose that $\alpha\geq 1/2$. Then it is easy to see that we can replace f_2 by another point on the segment between f_1 and f_2 (we are using the convexity of the set), which we also denote by f_2, such that $h=(f_1+f_2)/2$. Choosing $g=(f_1-f_2)/2$ we have that $h+g=f_1$ and $h-g=f_2$ are points in the set. By the assumption it follows that $g=0$, which implies that $f_1=f_2$, thus proving that h is an extreme point.

On the polydisc $P^n=\{z\in\mathbb{C}^n:\ \|z\|_\infty<1,\ i=1,\ldots,n\}$ the class of Carathéodory mappings is defined by

$$\mathcal{M}(P^n) = \left\{ h \in H(P^n) : \operatorname{Re}\left[\frac{h_j(z)}{z_j}\right] > 0,\ \|z\|_\infty = |z_j| > 0,\ j = 1, \ldots, n;\ h(0) = 0,\ Dh(0) = I \right\}.$$

The distinguished boundary of P^n is $\partial P^n = \{z \in \mathbb{C}^n : \|z\|_\infty = 1,\ i = 1, \ldots, n\}$.

Proposition 2.2.1. *Let $h(z) = (z_1 p_1(z_{i_1}), \ldots, z_n p_n(z_{i_n}))$ where $i_1, \ldots, i_n \in \{1, \ldots, n\}$ and p_i, $i = 1, \ldots, n$ are extreme points for the class $\mathcal{P}$. Then h is an extreme point for the class $\mathcal{M}(P^n)$.*

Proof. Suppose that $g \in H(P^n)$ is such that $h \pm g \in \mathcal{M}(P^n)$. We just need to show that $g = 0$. Note that we necessarily have $g(0) = 0$ and $Dg(0) = 0$.

Let $z \in \partial P^n$ and $j \in \{1, \ldots, n\}$. From the definition of the class $\mathcal{M}(P^n)$ we have that

$$\operatorname{Re}\left[\frac{h_j(\zeta z) \pm g_j(\zeta z)}{\zeta z_j}\right] = \operatorname{Re}\left[p_j\left(\zeta z_{i_j}\right) \pm \frac{g_j(\zeta z)}{\zeta z_j}\right] > 0,\ \zeta \in B^1 \setminus \{0\}.$$

Since $p_j\left(\cdot z_{i_j}\right)$ is an extreme point of the class $\mathcal{P}$ and $g_j(\cdot z)/(\cdot z_j)$ is analytic on B^1 (because $g(0) = 0$ and $Dg(0) = 0$) we can conclude that $g_j(\cdot z)/(\cdot z_j) = 0$. Since z and j were arbitrarily chosen we have in fact that $g = 0$. Note that we used the fact that an analytic function which is zero on the distinguished boundary of the polydisc is identically zero on the polydisc. This concludes the proof □

Let X be either P^n or B^n. We will denote by $S^*(X)$ the class of normalized starlike mappings on X, which can be defined as mappings which are spirallike with respect to I. It is known that any starlike mapping $f \in S^*(X)$ satisfies $Df(z)h(z) = f(z)$ for some $h \in \mathcal{M}(X)$ called the infinitesimal generator. Conversely, given $h \in \mathcal{M}(X)$ there is a unique $f \in S^*(X)$ satisfying $Df(z)h(z) = f(z)$. Note that the class of Carathéodory mappings on B^n is defined by

$$\mathcal{M}(B^n) = \{h \in H(B^n) : \operatorname{Re}\langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\};\ h(0) = 0,\ Dh(0) = I\}.$$

We will use the notation $\mathcal{M} := \mathcal{M}(B^n)$.

Next we show that the starlike mappings that have as infinitesimal generators the mappings from the previous proposition are also extreme points.

Proposition 2.2.2. *Let $f \in S^*(P^n)$ such that its infinitesimal generator is one of the extreme points from Proposition 2.2.1. Then f is an extreme point for the class $S^*(P^n)$.*

Proof. Suppose that there exist $\alpha \in (0,1)$ and $f^1, f^2 \in S^*(P^n)$ such that $f = \alpha f^1 + (1-\alpha) f^2$. We need to show that $f^1 = f^2$. Let $h^i \in \mathcal{M}(P^n)$, $i = 1,2$ be such that $Df^i(z) h^i(z) = f^i(z)$. Also let $h \in \mathcal{M}(P^n)$ be such that $Df(z) h(z) = f(z)$. Equating the degree two coefficients on both sides of the analytical characterizations for f, f^1 and f^2 yields $H_2 = -F_2$, $H_2^1 = -F_2^1$ and $H_2^2 = -F_2^2$. Hence

$$H_2 = \alpha H_2^1 + (1-\alpha) H_2^2. \tag{2.2.1}$$

Let $z \in \partial P^n$ and $j \in \{1, \ldots, n\}$. From the definition of the class $\mathcal{M}(P^n)$ it follows that $p^i(\zeta) = h_j^i(\zeta z)/\zeta z_j$, $i = 1,2$ are functions in the class $\mathcal{P}$. If the j-th component of H_2^i is denoted by $H_{2,j}^i$ then $p^i(\zeta) = 1 + \left(H_{2,j}^i(z)/z_j\right)\zeta + \ldots$. By the coefficient bounds for the class $\mathcal{P}$ we get that

$$\left|H_{2,j}^i(z)/z_j\right| \leq 2. \tag{2.2.2}$$

By the assumptions we have that $h_j(\zeta z)/\zeta z_j = \left(1 + z_{i_j} e^{i\theta_j}\zeta\right)/\left(1 - z_{i_j} e^{i\theta_j}\zeta\right)$ for some $\theta_j \in [0, 2\pi]$. Then the j-th component of H_2, denoted $H_{2,j}$, is of the form $2 z_j z_{i_j} e^{i\theta_j}$. On the other hand, from (2.2.1), we have that $H_{2,j} = \alpha H_{2,j}^1 + (1-\alpha) H_{2,j}^2$. Since $|H_{2,j}(z)| = 2$ we can now conclude (based on (2.2.2)) that $H_{2,j}(z) = H_{2,j}^1(z) = H_{2,j}^2(z)$. Since z and j are arbitrary we get $H_2 = H_2^1 = H_2^2$. Now we have that $p^i(\zeta) = 1 + 2 z_{i_j} e^{i\theta_j}\zeta + \ldots$ which implies (see [Pom75, Corollary 2.3]) that $p^i(\zeta) = \left(1 + z_{i_j} e^{i\theta_j}\zeta\right)/\left(1 - z_{i_j} e^{i\theta_j}\zeta\right) = h_j(\zeta z)/\zeta z_j$. Since z and j are arbitrary we can deduce that $h^i = h$, $i = 1,2$ and hence $f^1 = f^2 = f$. This concludes the proof. □

In general it is not clear what the expression of the extreme starlike mappings from the above proposition is. Nonetheless, in the particular case when the infinitesimal generator is $h(z) = (z_1 p_1(z_1), \ldots, z_n p_n(z_n))$ we have that the associated starlike mapping is $f(z) = (k_1(z_1), \ldots, k_n(z_n))$, where k_i are the Koebe maps associated to p_i.

The extreme points we obtained in Proposition 2.2.1 are not all the possible extreme points. If this were the case then Choquet's theorem (see [Lax02, Theorem 13.4.5]) would imply that all the mappings in $\mathcal{M}(P^n)$ are of the form $h(z) = (z_1 p_1(z), \ldots, z_n p_n(z))$ where p_i are some holomorphic functions. But this is clearly not the case. For example $(z_1 + a z_2^2, z_2)$ is not of this form but for small enough a it belongs to $\mathcal{M}(P^n)$.

2.3 Extreme points on B^n

We start by showing that the mappings from the previous section are not extreme on B^n. For simplicity we will only consider some particular mappings on $\mathbb{C}^2$.

Proposition 2.3.1. *Let $h(z) = (z_1(1+z_1)/(1-z_1), z_2(1+z_2)/(1-z_2))$ and $g(z) = (a z_1 z_2^2, 0)$ where $a \in \mathbb{C}$. If a is small enough then $h \pm g \in \mathcal{M}$.*

Proof. Straightforward computations yield

$$\mathrm{Re}\langle h(z) \pm g(z), z\rangle = |z_1|^2 \left(\mathrm{Re}\frac{1+z_1}{1-z_1} \pm a\mathrm{Re} z_2^2\right) + |z_2|^2 \,\mathrm{Re}\frac{1+z_2}{1-z_2}.$$

To have that $h \pm g \in \mathcal{M}$ it is sufficient to have that

$$\mathrm{Re}\frac{1+z_1}{1-z_1} = \frac{1-|z_1|^2}{|1-z_1|^2} > |a|\,|z_2|^2\,,\ z \in B^2.$$

This is equivalent to

$$|z_1|^2 + |a|\,|1-z_1|^2\,|z_2|^2 < 1,\ z \in B^2.$$

The above inequality clearly holds if $|a| \le 1/4$. This concludes the proof. □

Proposition 2.3.2. *Let $f(z) = \left(z_1/(1-z_1)^2, z_2/(1-z_2)^2\right)$ and $g(z) = \left(a z_1 z_2^2 (1+z_2)^2, 0\right)$ where $a \in \mathbb{C}$. If a is small enough then $f \pm g \in S^*$.*

Proof. Let $h(z) = [D(f+g)(z)]^{-1}(f+g)(z)$. Straightforward computations yield

$$[D(f+g)(z)]^{-1} = \frac{1}{\left(\frac{1+z_1}{(1-z_1)^3} + az_2^2(1+z_2)^2\right)\frac{1+z_2}{(1-z_2)^3}} \begin{bmatrix} \frac{1+z_2}{(1-z_2)^3} & -2az_1z_2(1+z_2)(1+2z_2) \\ 0 & \frac{1+z_1}{(1-z_1)^3} + az_2^2(1+z_2)^2 \end{bmatrix}$$

$$h(z) = \left(\frac{\frac{z_1}{(1-z_1)^2} + az_1z_2^2(1+z_2)^2 - 2az_1z_2^2(1-z_2)(1+2z_2)}{\frac{1+z_1}{(1-z_1)^3} + az_2^2(1+z_2)^2}, z_2\frac{1-z_2}{1+z_2} \right).$$

For h to be in $\mathcal{M}$ it is sufficient that $\operatorname{Re}[h_1(z)\overline{z_1}] \geq 0$ on B^2. This is equivalent to showing that

$$\operatorname{Re}\left[\left(\frac{1}{(1-z_1)^2} + az_2^2(1+z_2)^2 - 2az_2^2(1-z_2)(1+2z_2)\right) \overline{\left(\frac{1+z_1}{(1-z_1)^3} + az_2^2(1+z_2)^2\right)}\right] \geq 0. \quad (2.3.1)$$

Note that

$$\operatorname{Re}\left[\frac{1}{(1-z_1)^2}\overline{\left(\frac{1+z_1}{(1-z_1)^3}\right)}\right] = \frac{|1+z_1|^2}{|1-z_1|^6}\operatorname{Re}\frac{\frac{1}{(1-z_1)^2}}{\frac{1+z_1}{(1-z_1)^3}} = \frac{|1+z_1|^2}{|1-z_1|^6}\operatorname{Re}\frac{1-z_1}{1+z_1} = \frac{1-|z_1|^2}{|1-z_1|^6}.$$

Now it can be seen (by separating the above term in (2.3.1)) that to have (2.3.1) it suffices to have

$$\frac{1-|z_1|^2}{|1-z_1|^6} \geq \Bigg(\left|az_2^2(1+z_2)^2 - 2az_2^2(1-z_2)(1+2z_2)\right|\left|\frac{1+z_1}{(1-z_1)^3}\right| + \left|\frac{1}{(1-z_1)^2} + az_2^2(1+z_2)^2 - 2az_2^2(1-z_2)(1+2z_2)\right|\left|az_2^2(1+z_2)^2\right|\Bigg).$$

The left hand side can be bounded below by $|z_2|^2/|1-z_1|^6$, whereas the right hand side can be bounded above by $Ca|z_2|^2/|1-z_1|^3$. Hence we can choose small enough a for (2.3.1) to hold. Thus we proved that for small enough a we have that $f+g \in S^*$. This also holds if we replace a by $-a$ and so we also have that $f-g \in S^*$, thus concluding the proof. □

Next we will find examples of extreme points for the class $\mathcal{M}$ on B^n. Let $f(z) = z_1^2 + z_2^2$. F. Forelli [For75] showed that $\phi = (1+f)/(1-f)$ is an extreme point for

$\mathcal{P}$. One of the key ingredients in the proof is the use of the particular form of the slice functions $\phi(\zeta z)$ for certain $z \in S^2$. It is natural to consider the mapping $h(z) = \phi(z)z$ as a possible extreme point for $\mathcal{M}$. This is justified by the fact that the slice functions associated to h are related to the slices of ϕ by

$$\frac{\langle h(\zeta z), z \rangle}{\zeta} = \phi(\zeta z), \; z \in S^2,$$

which allows us to use some of Forelli's work.

Let $g \in H(B^n, \mathbb{C}^n)$ be such that $h \pm g \in \mathcal{M}$. We will show that such g needs to be of a certain restricted form, though not necessarily identically zero (Proposition 2.3.3). Thus, we will see that h is not an extreme point for $\mathcal{M}$ (Corollary 2.3.4), but we will be able to take advantage of the restricted form of g to produce a family of extreme points of $\mathcal{M}$ (Proposition 2.3.5).

Let $c = (c_1, c_2)$ be a point in $\mathbb{C}^2$ and let

$$g_c(z) := \frac{1}{1 - f(z)} \left(2(\mathrm{Re}c_1)z_1^2 + \overline{c_2}z_1z_2 + c_1z_2^2, c_2z_1^2 + \overline{c_1}z_1z_2 + 2(\mathrm{Re}c_2)z_2^2\right).$$

Note that the dependence on c is real linear. Let

$$\Omega = \left\{c \in \mathbb{C}^2 : \max_{\theta \in [0, \pi/2]} |\sin\theta \mathrm{Re}c_1 + \cos\theta \mathrm{Re}c_2| \leq 1/2\right\}.$$

Note that the maximum from the definition of Ω can be explicitly computed (the answer depends on the sign of $\mathrm{Re}c_1\mathrm{Re}c_2$). The mappings g_c and the set Ω will appear naturally in the proof of the following proposition.

Proposition 2.3.3. *If a holomorphic mapping g satisfies $h \pm g \in \mathcal{M}$ then $g = g_c$ for some $c \in \overline{B}^2 \cap \Omega$. Furthermore $h + g_c$ is a mapping in $\mathcal{M}$ if and only if $c \in \overline{B}^2 \cap \Omega$.*

In particular this proposition shows that h is not an extreme point of $\mathcal{M}$. In fact now we can completely characterize the extremality in $\mathcal{M}$ of mappings of the form $h + g_c$.

Corollary 2.3.4. *If $c \in B^2 \cap \Omega$ then $h + g_c$ is not an extreme point of $\mathcal{M}$.*

Proof. Since $c \in B^2$ we can choose $a \neq 0$ such that $\|c \pm a\| < 1$. Furthermore we can choose such a that also satisfies $\mathrm{Re}a_1 = \mathrm{Re}a_2 = 0$. This guarantees that $c \pm a \in B^2 \cap \Omega$. By the second part of Proposition 2.3.3 we have that $h + g_{c\pm a} = (h + g_c) \pm g_a \in \mathcal{M}$. Hence $h + g_c$ is not an extreme point. $\square$

Proposition 2.3.5. *If $c \in S^2 \cap \Omega$ then $h + g_c$ is an extreme point for $\mathcal{M}$.*

Proof. Assume there exists a holomorphic mapping g such that $h + g_c \pm g \in \mathcal{M}$. Since $-c \in S^2 \cap \Omega$ we know from the second part of Proposition 2.3.3 that $h - g_c \in \mathcal{M}$. From this we can easily deduce that

$$\mathrm{Re}\,\langle 2h\,(z) \pm g\,(z)\,, z\rangle > \mathrm{Re}\,\langle h\,(z) + g_c\,(z) \pm g\,(z)\,, z\rangle > 0,\ z \in B^2 \setminus \{0\}\,.$$

Hence $h \pm g/2 \in \mathcal{M}$. Using Proposition 2.3.3 we get that there exists $a \in \overline{B}^2 \cap \Omega$ such that $g = g_{2a}$. Now we have that $h + g_{c\pm 2a} \in \mathcal{M}$. The second part of Proposition 2.3.3 implies in particular that we must have $\|c \pm 2a\| \leq 1$ which can only happen if $a = 0$ (since $\|c\| = 1$). This concludes the proof. $\square$

Now we just have to prove Proposition 2.3.3. We start by discussing the basic tools we need for the proof. The main idea is to notice restrictions on slice functions and use them to get restrictions on the related mapping.

The slice functions that we are considering will be analytic with positive real part. Suppose that $\phi\,(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$ is an analytic function on B^1 that has positive real part. Then, necessarily, $\mathrm{Re}c_0 > 0$ and as a consequence of the coefficient inequalities for the class $\mathcal{P}$ we have that $|c_k| \leq 2\mathrm{Re}c_0$ for $k \geq 1$. We will also need the fact that if $c_0 = 1$ and $c_2 = 2$ then $c_{2k} = 2$ and $c_{2k-1} = \mathrm{const}$ for $k \geq 1$. Indeed, we know that there exists a positive Radon measure σ on the unit circle such that

$$\phi\,(\zeta) = \int_{|\xi|=1} \frac{\xi + \zeta}{\xi - \zeta} d\sigma\,(\xi)\,.$$

Then having $c_0 = 1$ and $c_2 = 2$ is equivalent to $\hat{\sigma}\,(0) = \hat{\sigma}\,(2) = 1$, where $\hat{\sigma}\,(n) = \int_{|\xi|=1} \xi^n d\sigma\,(\xi)$. Now it is known that the measure σ is supported on the points -1

and 1 (see [For79, Proposition 2.2]) hence $c_{2k} = \sigma(\{1\}) + \sigma(\{-1\}) = 1$ and $c_{2k-1} = \sigma(\{1\}) - \sigma(\{-1\})$ for $k \geq 1$.

The following two results will be used to get restrictions on the mapping based on the restrictions obtained for the slice functions. Let $S^n_+ = \{z \in S^n : z_i \geq 0,\, i = 1, \ldots, n\}$. Let $\mathcal{P}^k(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of homogeneous polynomials of degree k on $\mathbb{C}^n$ with values in $\mathbb{C}^m$. Then from [For75, Corollary 3.2] we get the following proposition.

Proposition 2.3.6. *If $P \in \cup_{k=1}^{\infty} \mathcal{P}^k(\mathbb{C}^n, \mathbb{C})$ and if $P = 0$ on S^n_+ then $P \equiv 0$.*

We will need the following consequence of the above result.

Corollary 2.3.7. *If $P \in \mathcal{P}^k(\mathbb{C}^2, \mathbb{C}^2)$ is such that $\langle P(z), z\rangle = 0$ for $z \in S^2_+$ then there exists $Q \in \mathcal{P}^{k-1}(\mathbb{C}^2, \mathbb{C})$ such that $\langle P(z), z\rangle = \operatorname{Im}(\overline{z_1} z_2)\, Q(z)$ for all z.*

Proof. Let $P = (P_1, P_2)$ and $P_i(z) = \sum_{|\alpha|=k} c^i_\alpha z^\alpha$.

Since $z = \overline{z}$ on S^2_+ we have that $\langle P(z), \overline{z}\rangle = 0$ on S^2_+. Applying Proposition 2.3.6 to $\langle P(z), \overline{z}\rangle \in \mathcal{P}^k(\mathbb{C}^2, \mathbb{C})$ yields $\langle P(z), \overline{z}\rangle \equiv 0$, hence $c^1_{(2,0)} = 0$, $c^2_{(0,2)} = 0$ and if $\alpha + e_1 = \beta + e_2$ then $c^1_\alpha + c^2_\beta = 0$. Now it is easy to check that $\langle P(z), z\rangle = (\overline{z_1} z_2 - z_1 \overline{z_2}) \sum c^1_{\alpha+e_2} z^\alpha$ and the conclusion follows immediately. ∎

One important feature of f with respect to the above results is that $f = 1$ on S^2_+. For example, suppose that $P_1 \in \mathcal{P}^k(\mathbb{C}^2, \mathbb{C}^2)$ and $P_2 \in \mathcal{P}^{k+2}(\mathbb{C}^2, \mathbb{C}^2)$ are such that $P_1 = P_2$ on S^+_2. We cannot apply Proposition 2.3.6 directly because $P_1 - P_2$ is not a homogeneous polynomial. We need to first notice that we also have that $P_1 f = P_2$ on S^2_+ and then we can apply Proposition 2.3.6 to get that $P_1 f \equiv P_2$.

We now begin the proof of Proposition 2.3.3 by showing that we must have

$$\langle g(z), z\rangle = \frac{\langle G(z), z\rangle + \operatorname{Im}(\overline{z_1} z_2)\, \psi(z)}{1 - f(z)},\ z \in B^2,$$

where $G \in \mathcal{P}^2(\mathbb{C}^2, \mathbb{C}^2)$ is the degree two term in the Taylor series expansion of g and $\psi \in H(B^2, \mathbb{C})$ is such that $\psi(0) = 0$ and $D\psi(0) = 0$.

The proof is a straightforward adjustment of the ideas from [For75, Proposition 3.12] and [For79, Proposition 2.3] based on Corollary 2.3.7.

Let $p_z^{\pm}(\zeta) = \langle h(\zeta z) \pm g(\zeta z), z\rangle / \zeta$ for all $\zeta \in B^1$ and all $z \in S^2$. If $g(z) = \sum_{k=2}^{\infty} G_k(z)$ is the Taylor series expansion of g, then

$$p_z^{\pm}(\zeta) = \zeta + \sum_{k=1}^{\infty} \left(\pm \langle G_{2k}(z), z\rangle \zeta^{2k-1} + \left(2f(z)^k \pm \langle G_{2k+1}(z), z\rangle\right) \zeta^{2k}\right).$$

Since $p_z^{\pm} \in \mathcal{P}$ we have that $|2f(z) \pm \langle G_{2k+1}(z), z\rangle| \leq 2$ for all $k \geq 1$. But $f(z) = 1$ on S_+^2 so we can conclude that $\langle G_{2k+1}(z), z\rangle = 0$ on S_2^+ for all $k \geq 1$. Corollary 2.3.7 implies that

$$\langle G_{2k+1}(z), z\rangle = \operatorname{Im}(\overline{z_1} z_2)\, Q_{2k}(z), \ k \geq 1.$$

If we let $p_z^+(\zeta) = \sum_{k=0}^{\infty} c_{z,k}\zeta^k$ then we have that $c_{z,0} = 1$ and $c_{z,2} = 2$ for $z \in S_+^2$. Hence, for $z \in S_+^2$ and $k \geq 1$ it follows that $c_{z,2k+1} = c_{z,1}$, that is $\langle G_{2k+2}(z), z\rangle = \langle G_2(z), z\rangle$. This implies that for $z \in S_+^2$ we have

$$\left\langle G_{2k+2}(z) - f(z)^k G_2(z), z \right\rangle = 0.$$

Now we can apply Corollary 2.3.7 to get

$$\langle G_{2k+2}(z), z\rangle = \langle G_2(z), z\rangle f(z)^k + \operatorname{Im}(\overline{z_1} z_2)\, Q_{2k+1}(z), \ k \geq 1.$$

The conclusion follows immediately by setting $\psi = (1-f)\sum_{k=2}^{\infty} Q_k$.

Next we will obtain some further restrictions using the same idea as [For79, Proposition 2.4]. We have that

$$\begin{aligned}
\operatorname{Re}\langle h(z) \pm g(z), z\rangle &= \operatorname{Re}\frac{(1+f(z))\|z\|^2 \pm (\langle G(z), z\rangle + \operatorname{Im}(\overline{z_1}z_2)\psi(z))}{1-f(z)} \\
&= \frac{\left(1-|f(z)|^2\right)\|z\|^2 \pm \operatorname{Re}\left[\left(1-\overline{f}(z)\right)(\langle G(z), z\rangle + \operatorname{Im}(\overline{z_1}z_2)\psi(z))\right]}{|1-f(z)|^2}.
\end{aligned}$$

Let $q^{\pm}(z) = \left(1-|f(z)|^2\right)\|z\|^2 \pm \operatorname{Re}\left[\left(1-\overline{f}(z)\right)(\langle G(z), z\rangle + \operatorname{Im}(\overline{z_1}z_2)\psi(z))\right]$. Then for $z \in B^2 \setminus \{0\}$ we have that $q^{\pm}(z) > 0$. The slice functions $q^{\pm}(\zeta z)$ are not harmonic (we want to use properties of positive harmonic functions). We want to find harmonic functions $r_z^{\pm}$ such that $r_z^{\pm}(\zeta) = q^{\pm}(\zeta z)$ when $|\zeta| = 1$. If $|\zeta| = 1$ then

$$
\begin{aligned}
q^{\pm}(\zeta z) &= \left(1 - |f(z)|^2\right) \|z\|^2 \pm \mathrm{Re}\left[\left(1 - \overline{f}(z)\,\overline{\zeta}^2\right)\left(\langle G(z), z\rangle \zeta + \mathrm{Im}\left(\overline{z_1} z_2\right) \sum_{k=2}^{\infty} \Psi_k(z)\,\zeta^k\right)\right] \\
&= \mp \mathrm{Re}\left[\overline{f}(z)\,\langle G(z), z\rangle\,\overline{\zeta}\right] + \left(1 - |f(z)|^2\right)\|z\|^2 \mp \mathrm{Re}\left[\overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_2(z)\right] \\
&\quad \pm \mathrm{Re}\left[\left(\langle G(z), z\rangle - \overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_3(z)\right)\zeta\right] \\
&\quad \pm \mathrm{Re}\left[\sum_{k=2}^{\infty} \mathrm{Im}\left(\overline{z_1} z_2\right)\left(\Psi_k(z) - \overline{f}(z)\,\Psi_{k+2}(z)\right)\zeta^k\right].
\end{aligned}
$$

So

$$
\begin{aligned}
r_z^{\pm}(\zeta) &= \left(1 - |f(z)|^2\right)\|z\|^2 \mp \mathrm{Re}\left[\overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_2(z)\right] \\
&\quad \pm \mathrm{Re}\left[\left(\langle G(z), z\rangle - f(z)\,\overline{\langle G(z), z\rangle} - \overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_3(z)\right)\zeta\right] \\
&\quad \pm \mathrm{Re}\left[\sum_{k=2}^{\infty} \mathrm{Im}\left(\overline{z_1} z_2\right)\left(\Psi_k(z) - \overline{f}(z)\,\Psi_{k+2}(z)\right)\zeta^k\right]
\end{aligned}
$$

are the functions we are looking for. Since $r_z^{\pm}(\zeta) > 0$ when $|\zeta| = 1$ it follows that $r_z^{\pm}(\zeta) > 0$ when $|\zeta| \leq 1$. For this it is necessary that

$$
\left(1 - |f(z)|^2\right)\|z\|^2 \mp \mathrm{Re}\left[\overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_2(z)\right] > 0
$$

and therefore we have

$$
\left(1 - |f(z)|^2\right)\|z\|^2 \mp \mathrm{Re}\left[\overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_2(z)\right] < 2\left(1 - |f(z)|^2\right)\|z\|^2. \tag{2.3.2}
$$

Using the above and coefficient inequalities we get that for $z \in B^2 \setminus \{0\}$ and $k \geq 2$ we have

$$
\begin{aligned}
\left|\langle G(z), z\rangle - f(z)\,\overline{\langle G(z), z\rangle} - \overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_3(z)\right| &< 4\left(1 - |f(z)|^2\right)\|z\|^2 \\
\left|\mathrm{Im}\left(\overline{z_1} z_2\right)\left(\Psi_k(z) - \overline{f}(z)\,\Psi_{k+2}(z)\right)\right| &< 4\left(1 - |f(z)|^2\right)\|z\|^2.
\end{aligned}
$$

Letting z converge to points from S^2 in the above inequalities yields

$$
\left|\langle G(z), z\rangle - f(z)\,\overline{\langle G(z), z\rangle} - \overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_3(z)\right| \leq 4\left(1 - |f(z)|^2\right),\ z \in S^2 \tag{2.3.3}
$$

$$
\left|\mathrm{Im}\left(\overline{z_1} z_2\right)\left(\Psi_k(z) - \overline{f}(z)\,\Psi_{k+2}(z)\right)\right| \leq 4\left(1 - |f(z)|^2\right),\ z \in S^2, k \geq 2. \tag{2.3.4}
$$

Let $z = \left(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}\right)$ be a point from S^2. Then a straightforward computation using the fact that $r_1^4 + r_2^4 + 2r_1^2 r_2^2 = 1$ yields $1 - |f(z)|^2 = 4r_1^2 r_2^2 \sin^2(\theta_1 - \theta_2)$. Indeed

$$\begin{aligned} 1 - |f(z)|^2 &= 1 - \left(r_1^4 + r_2^4 + 2r_1^2 r_2^2 \cos 2(\theta_1 - \theta_2)\right) \\ &= 1 - \left(r_1^4 + r_2^4 + 2r_1^2 r_2^2 + 2r_1^2 r_2^2 (\cos 2(\theta_1 - \theta_2) - 1)\right) \\ &= 4r_1^2 r_2^2 \sin^2(\theta_1 - \theta_2). \end{aligned}$$

Also note that $\operatorname{Im}(\overline{z_1} z_2) = r_1 r_2 \sin(\theta_2 - \theta_1)$. After we simplify by $|\sin(\theta_1 - \theta_2)|$ the inequality (2.3.4) becomes

$$\left|\Psi_k(z) - \overline{f(z)}\Psi_{k+2}(z)\right| \leq 16 r_1 r_2 |\sin(\theta_1 - \theta_2)|, \; k \geq 2.$$

Since $f(z) = 1$ on S_+^2 the above inequality implies that $\Psi_k f = \Psi_{k+2}$ on S_+^2 and hence

$$\Psi_k f \equiv \Psi_{k+2}, \; k \geq 2 \tag{2.3.5}$$

The next step is to show that the even coefficients of ψ are zero. Note that

$$q^{\pm}(z) + q^{\pm}(-z) = 2\left(1 - |f(z)|^2\right)\|z\|^2 \pm 2\operatorname{Re}\left[\left(1 - \overline{f}(z)\right)\operatorname{Im}(\overline{z_1} z_2)\sum_{k=1}^{\infty}\Psi_{2k}(z)\right] > 0, \; z \in B^2 \backslash \{0\}.$$

Let $S_f = \{z \in S^2 : \; f(z) = 1\}$. Using (2.3.5) and letting z converge to points from $S^2 \backslash S_f$ yields

$$\left|\operatorname{Re}\left[\operatorname{Im}(\overline{z_1} z_2)\Psi_2(z)\frac{1 - \overline{f}(z)}{1 - f(z)}\right]\right| \leq 1 - |f(z)|^2, \; z \in S^2 \setminus S_f. \tag{2.3.6}$$

Let $\Psi_2(z) = az_1^2 + bz_1 z_2 + cz_2^2$. Also let $U(z) := (z_1, -z_2)$. Note that $f \circ U = f$. Replacing z by $U(z)$ in (2.3.6) gives

$$\left|\operatorname{Re}\left[-\operatorname{Im}(\overline{z_1} z_2)\left(az_1^2 - bz_1 z_2 + cz_2^2\right)\frac{1 - \overline{f}(z)}{1 - f(z)}\right]\right| \leq 1 - |f(z)|^2, \; z \in S^2 \setminus S_f. \tag{2.3.7}$$

Using (2.3.6), (2.3.7) and the triangle inequality we get

$$\left|\operatorname{Re}\left[\operatorname{Im}(\overline{z_1} z_2)\, b z_1 z_2 \frac{1 - \overline{f}(z)}{1 - f(z)}\right]\right| \leq 1 - |f(z)|^2, \; z \in S^2 \setminus S_f.$$

If $z = \left(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}\right)$ then the above becomes

$$\left|\operatorname{Re}\left[b e^{i(\theta_1 + \theta_2)}\frac{1 - \overline{f}(z)}{1 - f(z)}\right]\right| \leq 4|\sin(\theta_1 - \theta_2)|. \tag{2.3.8}$$

If we choose $\theta_1 = \theta_2 = \theta$ then (2.3.8) yields

$$0 = \mathrm{Re}\left[b e^{i2\theta} \frac{1 - e^{-i2\theta}}{1 - e^{i2\theta}} \right] = -\mathrm{Re} b.$$

Hence (2.3.8) becomes

$$|b| \left| \mathrm{Im} \left[e^{i(\theta_1+\theta_2)} \frac{1 - \overline{f}(z)}{1 - f(z)} \right] \right| \leq 4 \left| \sin(\theta_1 - \theta_2) \right|.$$

The above inequality and the following lemma imply that $b = 0$.

Lemma 2.3.8. *Let* $z = \left(r_1 e^{i\theta_1}, r_2 e^{i\theta_2} \right)$ *and* $r_1^2 + r_2^2 = 1$. *If* $\theta_1, \theta_2 \to 0$ *such that* $\theta_2/\theta_1 \to l$ *and* $l \neq -r_1^2/r_2^2$ *then*

$$\frac{1}{\sin(\theta_1 - \theta_2)} \mathrm{Im} \left(e^{i(\theta_1+\theta_2)} \frac{1 - \overline{f}(z)}{1 - f(z)} \right) \to \frac{r_1^2 - l r_2^2}{r_1^2 + l r_2^2}.$$

Proof. First note that

$$\mathrm{Im} \left(e^{i(\theta_1+\theta_2)} \frac{1 - \overline{f}(z)}{1 - f(z)} \right) = \frac{\mathrm{Im} \left(e^{i(\theta_1+\theta_2)} \left(1 - \overline{f}(z) \right)^2 \right)}{\left| 1 - f(z) \right|^2}.$$

We want to factor $\sin(\theta_1 - \theta_2)$ from the top of the right hand side. For this we expand and explicitly compute the top and then eliminate r_1 (using $r_1^2 = 1 - r_2^2$). After this the top becomes

$$\begin{aligned} \mathrm{Im} \left(e^{i(\theta_1+\theta_2)} \left(1 - \overline{f}(z) \right)^2 \right) &= \sin(\theta_1 + \theta_2) + \sin(\theta_2 - 3\theta_1) + 2\sin(\theta_1 - \theta_2) \\ &\quad - 2r_2^2 \left(\sin(\theta_2 - 3\theta_1) + 2\sin(\theta_1 - \theta_2) + \sin(\theta_1 + \theta_2) \right) \\ &\quad + r_2^4 \left(\sin(\theta_2 - 3\theta_1) + \sin(\theta_1 - 3\theta_2) + 2\sin(\theta_1 + \theta_2) \right). \end{aligned}$$

Applying the fact that

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \tag{2.3.9}$$

repeatedly, yields

$$\begin{aligned} &\mathrm{Im} \left(e^{i(\theta_1+\theta_2)} \left(1 - \overline{f}(z) \right)^2 \right) \\ &\quad = 2\sin(\theta_1 - \theta_2) \left(1 - \cos 2\theta_1 - 2r_2^2 (1 - \cos 2\theta_1) + r_2^4 (\cos 2\theta_2 - \cos 2\theta_1) \right) \\ &\quad = 4\sin(\theta_1 - \theta_2) \sin^2\theta_1 \left(1 - 2r_2^2 + r_2^4 \left(1 - \frac{\sin^2\theta_2}{\sin^2\theta_1} \right) \right). \end{aligned}$$

Next we explicitly compute the bottom and eliminate r_1. This yields

$$\begin{aligned}|1-f(z)|^2 = 2\Big(&1-\cos 2\theta_1 - r_2^2\left(1-\cos 2\theta_1+\cos 2\theta_2-\cos 2\left(\theta_1-\theta_2\right)\right)\\ &+r_2^4\left(1-\cos 2\left(\theta_1-\theta_2\right)\right)\Big)\\ &= 4\sin^2\theta_1\left(1-r_2^2\left(1+\frac{\sin^2\left(\theta_1-\theta_2\right)}{\sin^2\theta_1}-\frac{\sin^2\theta_2}{\sin^2\theta_1}\right)+r_2^4\frac{\sin^2\left(\theta_1-\theta_2\right)}{\sin^2\theta_1}\right).\end{aligned}$$

Now it is straightforward to get that

$$\begin{aligned}\frac{1}{\sin\left(\theta_1-\theta_2\right)}\mathrm{Im}\left(e^{i(\theta_1+\theta_2)}\frac{1-\overline{f}(z)}{1-f(z)}\right) &\to \frac{1-2r_2^2+r_2^4\left(1-l^2\right)}{1-r_2^2\left(1+(1-l)^2-l^2\right)+r_2^4(1-l)^2}\\ &= \frac{\left(1-(1+l)\,r_2^2\right)\left(1-(1-l)\,r_2^2\right)}{\left(1-(1-l)\,r_2^2\right)^2}=\frac{r_1^2-lr_2^2}{r_1^2+lr_2^2}.\end{aligned}$$

□

It will be convenient to use the following notation

$$E(z)=e^{i(\theta_1+\theta_2)}\frac{1-\overline{f}(z)}{1-f(z)}.$$

Note that $|E(z)|\le 1$ and $E(z)=-1$ when $\theta_1=\theta_2$.

If $z=\left(r_1e^{i\theta_1},r_2e^{i\theta_2}\right)$ then (2.3.6) becomes

$$\left|\mathrm{Re}\left[\left(ar_1^2e^{i2\theta_1}+cr_2^2e^{i2\theta_2}\right)e^{-i(\theta_1+\theta_2)}E(z)\right]\right|\le 4r_1r_2\left|\sin\left(\theta_1-\theta_2\right)\right|. \tag{2.3.10}$$

If we choose $\theta_1=\theta_2$ in the above inequality we get that $\mathrm{Re}\left(ar_1^2+cr_2^2\right)=0$ whenever $r_1^2+r_2^2=1$ and hence $\mathrm{Re}a=\mathrm{Re}c=0$. Now we can write

$$\begin{aligned}&\mathrm{Re}\left[\left(ar_1^2e^{i2\theta_1}+cr_2^2e^{i2\theta_2}\right)e^{-i(\theta_1+\theta_2)}E(z)\right]\\ &\qquad=-r_1^2\mathrm{Im}a\mathrm{Im}\left[e^{i(\theta_1-\theta_2)}E(z)\right]-r_2^2\mathrm{Im}c\mathrm{Im}\left[e^{i(\theta_2-\theta_1)}E(z)\right]\\ &=-\left(r_1^2\mathrm{Im}a+r_2^2\mathrm{Im}c\right)\cos\left(\theta_1-\theta_2\right)\mathrm{Im}E(z)-\left(r_1^2\mathrm{Im}a-r_2^2\mathrm{Im}c\right)\sin\left(\theta_1-\theta_2\right)\mathrm{Re}E(z).\end{aligned}$$

From the above equality, (2.3.10) and Lemma 2.3.8 we get that $r_1^2\mathrm{Im}a+r_2^2\mathrm{Im}c=0$ for any r_1 and r_2 such that $r_1^2+r_2^2=1$. Hence $\mathrm{Im}a=\mathrm{Im}c=0$. Since we already have that

Rea = Re$c = 0$ we can conclude that $a = c = 0$. This ends the proof of the fact that the even coefficients of ψ are zero.

Next we will show that the odd coefficients of ψ are zero and obtain the restrictions on the coefficients of G. Let

$$\langle G(z), z\rangle = \left(a_1 z_1^2 + b_1 z_1 z_2 + c_1 z_2^2\right)\overline{z_1} + \left(a_2 z_1^2 + b_2 z_1 z_2 + c_2 z_2^2\right)\overline{z_2}. \tag{2.3.11}$$

Replacing z by $U(z) = (z_1, -z_2)$ in (2.3.3) yields

$$\left|\langle G(U(z)), U(z)\rangle - f(z)\overline{\langle G(U(z)), U(z)\rangle} + \overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\Psi_3(U(z))\right| \\ \leq 4\left(1 - |f(z)|^2\right),\ z \in S^2.$$

Using the triangle inequality, the above inequality and (2.3.3) we get

$$\left|G_U(z) - f(z)\overline{G_U(z)} - \overline{f}(z)\,\mathrm{Im}\left(\overline{z_1} z_2\right)\tilde{\Psi}_3(z)\right| \leq 4\left(1 - |f(z)|^2\right),\ z \in S^2, \tag{2.3.12}$$

where

$$G_U(z) = \frac{1}{2}\left(\langle G(z), z\rangle + \langle G(U(z)), U(z)\rangle\right) = a_1 z_1^2\overline{z_1} + c_1 z_2^2\overline{z_1} + b_2 z_1 z_2\overline{z_2}$$

and

$$\tilde{\Psi}_3(z) = \frac{1}{2}\left(\Psi_3(z) - \Psi_3(U(z))\right) = j z_1^2 z_2 + k z_2^3.$$

We will show that $\tilde{\Psi}_3 \equiv 0$, which shows that two of Ψ_3's coefficients are zero. The same reasoning, but using $V(z) = (-z_1, z_2)$ instead of U will show that the other two coefficients are also zero. From (2.3.5) we have that $\psi = \Psi_3/(1-f)$. So, by showing that $\tilde{\Psi}_3 \equiv 0$ we can conclude that $\psi \equiv 0$.

First note that (2.3.12) implies that for $z \in S_+^2$ we have $G_U(z) = \overline{G_U(z)}$. Hence

$$\left(a_1 - \overline{a_1}\right) z_1^3 + \left(c_1 + b_2 - \overline{c_1} - \overline{b_2}\right) z_1 z_2^2 = 0,\ z \in S_+^2$$

and by Proposition 2.3.6 we can conclude that a_1 and $c_1 + b_2$ are real. To simplify notation, let $a := a_1$, $c := 2ic_1$ and $d := c_1 + b_2$. With this notation we have

$$\begin{aligned} G_U(z) &= a_1 z_1^2\overline{z_1} + (b_2 + c_1) z_1 z_2\overline{z_2} + c_1\left(z_2^2\overline{z_1} - z_1 z_2\overline{z_2}\right). \\ &= a z_1^2\overline{z_1} + d z_1 z_2\overline{z_2} + c z_2\,\mathrm{Im}\left(\overline{z_1} z_2\right). \end{aligned} \tag{2.3.13}$$

Let $\tilde{G_U}(z) = az_1^2\overline{z_1} + dz_1z_2\overline{z_2}$. With this notation (2.3.12) becomes

$$\left|\tilde{G_U}(z) - f(z)\overline{\tilde{G_U}(z)} + \mathrm{Im}\left(\overline{z_1}z_2\right)\left(cz_2 - f(z)\overline{cz_2} - \overline{f}(z)\tilde{\Psi}_3(z)\right)\right| \leq 4\left(1 - |f(z)|^2\right),\ z \in S^2. \tag{2.3.14}$$

If $z \in S^2$ is such that $z = \left(r_1, r_2e^{i\theta}\right)$ then

$$\begin{aligned}\tilde{G_U}(z) - f(z)\overline{\tilde{G_U}(z)} &= \left(ar_1^3 + dr_1r_2^2\right) - \left(r_1^2 + r_2^2e^{i2\theta}\right)\left(ar_1^3 + dr_1r_2^2\right)\\ &= r_2^2\left(1 - e^{i2\theta}\right)\left(ar_1^3 + dr_1r_2^2\right)\\ &= 2r_2^2\sin\theta\left(\sin\theta - i\cos\theta\right)\left(ar_1^3 + dr_1r_2^2\right).\end{aligned}$$

Let $z = \left(r_1, r_2e^{i\theta}\right)$ in (2.3.14), simplify $\sin\theta$ on both sides, set $\theta = 0$ and use the fact that $1 = r_1^2 + r_2^2$ to get

$$-2i\left(ar_1^3 + dr_1r_2^2\right)r_2^2 + r_1r_2\left(\left(c - \overline{c}\right)r_2\left(r_1^2 + r_2^2\right) - jr_1^2r_2 - kr_2^3\right) = 0.$$

Using Proposition 2.3.6 we can conclude that

$$2ia + j = 2id + k = c - \overline{c}. \tag{2.3.15}$$

In particular we see that j and k are imaginary numbers.

Next note that

$$\begin{aligned}&q(z) + q(U(z))\\ &= 2\left(1 - |f(z)|^2\right)\|z\|^2 \pm 2\mathrm{Re}\left[\left(1 - \overline{f}(z)\right)\left(G_U(z) + \mathrm{Im}\left(\overline{z_1}z_2\right)\frac{\tilde{\Psi}_3(z)}{1 - f(z)}\right)\right] > 0,\ z \in B^2\backslash\{0\}\end{aligned}$$

(we used (2.3.5)). Letting z converge to points in $S^2 \setminus S_f$ yields

$$\left|\mathrm{Re}\left[\left(1 - \overline{f}(z)\right)\left(G_U(z) + \mathrm{Im}\left(\overline{z_1}z_2\right)\frac{\tilde{\Psi}_3(z)}{1 - f(z)}\right)\right]\right| \leq \left(1 - |f(z)|^2\right),\ z \in S^2 \setminus S_f. \tag{2.3.16}$$

Let $z \in S^2$ be such that $z = \left(r_1e^{i\theta_1}, r_2e^{i\theta_2}\right)$. Note that

$$\begin{aligned}\mathrm{Re}\left[\left(1 - \overline{f}(z)\right)\mathrm{Im}\left(\overline{z_1}z_2\right)\frac{\tilde{\Psi}_3(z)}{1 - f(z)}\right] &= r_1r_2\sin\left(\theta_2 - \theta_1\right)\mathrm{Re}\left[\tilde{\Psi}_3(z)e^{-i(\theta_1+\theta_2)}E(z)\right]\\ &= r_1r_2\sin\left(\theta_2 - \theta_1\right)\left(\mathrm{Re}\left[\tilde{\Psi}_3(z)e^{-i(\theta_1+\theta_2)}\right]\mathrm{Re}E(z) - \mathrm{Im}\left[\tilde{\Psi}_3(z)e^{-i(\theta_1+\theta_2)}\right]\mathrm{Im}E(z)\right).\end{aligned}$$

Now we can deduce from (2.3.16) that

$$\left|\frac{\mathrm{Re}\left[G_U(z)\left(1-\overline{f}(z)\right)\right]}{r_1r_2\sin^2(\theta_1-\theta_2)}+\frac{\mathrm{Re}\left[\tilde{\Psi}_3(z)\,e^{-i(\theta_1+\theta_2)}\right]\mathrm{Re}E(z)}{\sin(\theta_2-\theta_1)}-\frac{\mathrm{Im}\left[\tilde{\Psi}_3(z)\,e^{-i(\theta_1+\theta_2)}\right]\mathrm{Im}E(z)}{\sin(\theta_2-\theta_1)}\right|\leq 4r_1r_2.$$

Let $l\leq 0$ be a real number. We claim that if θ_1, $\theta_2\to 0$ so that $\theta_2/\theta_1\to l$ then the first two terms on the left-hand side of the previous inequality can be bounded independently of l. Once we prove the claim we can use Lemma 2.3.8 for l close enough to $-r_1^2/r_2^2$ ($r_2\neq 0$) to conclude that we must have

$$0=\mathrm{Im}\tilde{\Psi}_3(z)=r_2\left(r_1^2\mathrm{Im}j+r_2^2\mathrm{Im}k\right)$$

(we used the fact that j and k are imaginary numbers). Since the above must hold for any r_1 and r_2 such that $r_1^2+r_2^2=1$ we can conclude that $j=k=0$ (we already saw that j and k are imaginary numbers) and hence $\tilde{\Psi}_3\equiv 0$.

Now we just have to check the claim. First note that from (2.3.13) we get

$$\begin{aligned}\mathrm{Re}\left[G_U(z)\left(1-\overline{f}(z)\right)\right]&=\left(ar_1^2+dr_2^2\right)\mathrm{Re}\left[z_1\left(1-\overline{f}(z)\right)\right]\\&+r_1r_2\sin(\theta_2-\theta_1)\left(\mathrm{Re}c\,\mathrm{Re}\left[z_2\left(1-\overline{f}(z)\right)\right]-\mathrm{Im}c\,\mathrm{Im}\left[z_2\left(1-\overline{f}(z)\right)\right]\right).\end{aligned}\tag{2.3.17}$$

Elementary computations using (2.3.9), $r_1^2+r_2^2=1$ and

$$\cos x-\cos y=2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$$

yield

$$\mathrm{Re}\left[z_1\left(1-\overline{f}(z)\right)\right]=2r_1r_2^2\sin(\theta_2-\theta_1)\sin\theta_2\tag{2.3.18}$$

$$\mathrm{Re}\left[z_2\left(1-\overline{f}(z)\right)\right]=2r_2r_1^2\sin(\theta_1-\theta_2)\sin\theta_1\tag{2.3.19}$$

$$\mathrm{Im}\left[z_2\left(1-\overline{f}(z)\right)\right]=2r_2\left(\sin\theta_2-r_1^2\sin(\theta_2-\theta_1)\cos\theta_1\right)\tag{2.3.20}$$

$$\mathrm{Re}\left[\tilde{\Psi}_3(z)\,e^{-i(\theta_1+\theta_2)}\right]=-r_2\left(r_1^2\sin\theta_1\mathrm{Im}j+r_2^2\sin(2\theta_2-\theta_1)\,\mathrm{Im}k\right).$$

Note that when $\theta_1,\theta_2\to 0$ so that $\theta_2/\theta_1\to l$ we have

$$\frac{\sin(s\theta_1+t\theta_2)}{\sin(\theta_1-\theta_2)}\to\frac{s+tl}{1-l}.$$

Now the conclusion of the claim follows easily.

We proved that $j = k = 0$ and by (2.3.15) this implies that $2ia = 2id = c - \overline{c} = 2i\mathrm{Im}c$. Remembering that $a = a_1$, $c = 2ic_1$, $d = c_1 + b_2$ we can deduce that $a_1 = 2\mathrm{Re}c_1$ and $b_2 = \overline{c_1}$. At this point we have that $\psi \equiv 0$ and using the notation of (2.3.11) we also have that

$$G(z) = \left(2(\mathrm{Re}c_1) z_1^2 + b_1 z_1 z_2 + c_1 z_2^2, a_2 z_1^2 + \overline{c_1} z_1 z_2 + c_2 z_2^2\right).$$

Repeating the reasoning that lead to this form of G , but using $V(z) = (-z_1, z_2)$ instead of U, gives us that

$$G(z) = \left(2(\mathrm{Re}c_1) z_1^2 + \overline{a_2} z_1 z_2 + c_1 z_2^2, a_2 z_1^2 + \overline{c_1} z_1 z_2 + 2(\mathrm{Re}a_2) z_2^2\right).$$

So we proved that $g = g_{(c_1,a_2)}$.

To complete the proof we just have to prove that given $c = (c_1, c_2)$ we have that $h + g_c \in \mathcal{M}$ if and only if $c \in \overline{B}^2 \cap \Omega$. Let

$$\phi_z(\zeta) := \langle (h + g_c)(\zeta z), z\rangle / \zeta,\ \zeta \in B^1, \|z\| = 1.$$

We have that $h+g_c \in \mathcal{M}$ if and only $\phi_z \in \mathcal{P}$ for all $\|z\| = 1$. Let $Z = \{z \in S^2 : |f(z)| = 1\} = \left\{z \in S^2 : z = \left(r_1 e^{i\theta}, r_2 e^{i\theta}\right), \theta \in [0, 2\pi)\right\}$. We will show that $\phi_z \in \mathcal{P}$ for all $z \in S^2 \setminus Z$ if and only if $c \in \overline{B}^2$ and that $\phi_z \in \mathcal{P}$ for all $z \in Z$ if and only if $c \in \Omega$. This yields the desired conclusion.

Let

$$G(z) = \left(2(\mathrm{Re}c_1) z_1^2 + \overline{c_2} z_1 z_2 + c_1 z_2^2, c_2 z_1^2 + \overline{c_1} z_1 z_2 + 2(\mathrm{Re}c_2) z_2^2\right).$$

Then a short computation yields (provided that $\|z\| = 1$)

$$\mathrm{Re}\phi_z(\zeta) = \frac{1 - |f(\zeta z)|^2 + \mathrm{Re}\left[\frac{1}{\zeta}\langle G(\zeta z), z\rangle \left(1 - \overline{f}(\zeta z)\right)\right]}{|1 - f(\zeta z)|^2}. \tag{2.3.21}$$

For $z \in S^2 \setminus Z$ we have that ϕ_z extends continuously to the boundary and hence it is enough to have $\mathrm{Re}\phi_z(\zeta) \geq 0$ for $|\zeta| = 1$. By (2.3.21) it is enough to have

$$1 - |f(\zeta z)|^2 + \mathrm{Re}\left[\langle G(\zeta z), \zeta z\rangle \left(1 - \overline{f}(\zeta z)\right)\right] \geq 0,\ |\zeta| = 1.$$

From the definition of Z we can conclude that $\phi_z \in \mathcal{P}$ for all $z \in S^2 \setminus Z$ if and only if

$$1 - |f(z)|^2 + \operatorname{Re}\left[\langle G(z), z\rangle \left(1 - \overline{f}(z)\right)\right] \geq 0,\ z \in S^2 \setminus Z. \tag{2.3.22}$$

Let

$$\begin{aligned} G_U(z) &= \frac{1}{2}\left[\langle G(z), z\rangle + \langle G(U(z)), U(z)\rangle\right] \\ G_V(z) &= \frac{1}{2}\left[\langle G(z), z\rangle + \langle G(V(z)), V(z)\rangle\right] \end{aligned}$$

where U and V are as previously defined. It is easy to see that $\langle G(z), z\rangle = G_U(z) + G_V(z)$. Let $z = \left(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}\right) \in S^2$. Then by using (2.3.17) (with $a = d = \operatorname{Im} c$ and $c = 2ic_1$), (2.3.18), (2.3.19) and (2.3.20) a straightforward computation yields

$$\operatorname{Re}\left[G_U\left(1 - \overline{f}(z)\right)\right] = 4r_1^2 r_2^2 \sin^2(\theta_2 - \theta_1) \operatorname{Re}\left(c_1 r_1 e^{-i\theta_1}\right).$$

Analogously one gets

$$\operatorname{Re}\left[G_V\left(1 - \overline{f}(z)\right)\right] = 4r_1^2 r_2^2 \sin^2(\theta_1 - \theta_2) \operatorname{Re}\left(c_2 r_2 e^{-i\theta_2}\right).$$

Now (2.3.22) becomes

$$4r_1^2 r_2^2 \sin^2(\theta_2 - \theta_1)\left(1 + \operatorname{Re}\langle c, z\rangle\right) \geq 0,\ z \in S^2 \setminus Z.$$

We can conclude that for (2.3.22) to hold it is necessary and sufficient that $1 + \operatorname{Re}\langle c, z\rangle \geq 0$ for all $z \in S^2$ (we are using continuity to get that this holds on S^2 rather than a dense subset). But this is easily seen to be equivalent to $\|c\| \leq 1$. Thus we proved that $\phi_z \in \mathcal{P}$ for all $z \in S^2 \setminus Z$ if and only if $\|c\| \leq 1$.

When $z \in Z$, ϕ_z doesn't extend continuously to all boundary points and we cannot use the same approach. Instead we will take advantage of the special form of points in Z. To have that $\phi_z \in \mathcal{P}$ for all $z \in Z$ it is necessary and sufficient that

$$1 - |\zeta|^2 + \operatorname{Re}\left[\zeta \langle G(z), z\rangle \left(1 - \overline{\zeta}^2 \overline{f}(z)\right)\right] > 0,\ \zeta \in B^1, z \in Z \tag{2.3.23}$$

(we used (2.3.21), the fact that f and G are homogeneous polynomials of degree two and the definition of Z). Let $z = \left(r_1 e^{i\theta}, r_2 e^{i\theta}\right) \in Z$ and $\zeta = re^{i\psi} \in B^1$. A straightforward computation (using $r_1^2 + r_2^2 = 1$) yields

$$\langle G(z,z)\rangle = 2\left(r_1 \mathrm{Re} c_1 + r_2 \mathrm{Re} c_2\right) e^{i\theta}.$$

Hence

$$\begin{aligned}\mathrm{Re}\left[\zeta \langle G(z), z\rangle \left(1 - \overline{\zeta}^2 \overline{f}(z)\right)\right] &= 2r\left(r_1 \mathrm{Re} c_1 + r_2 \mathrm{Re} c_2\right) \mathrm{Re}\left[e^{i(\psi+\theta)}\left(1 - r^2 e^{-i2(\psi+\theta)}\right)\right] \\ &= 2r\left(r_1 \mathrm{Re} c_1 + r_2 \mathrm{Re} c_2\right)\left(1 - r^2\right)\cos(\psi+\theta).\end{aligned}$$

Using the above equality (2.3.23) becomes

$$\left(1 - r^2\right)\left(1 + 2r\left(r_1 \mathrm{Re} c_1 + r_2 \mathrm{Re} c_2\right)\cos(\psi+\theta)\right) > 0.$$

Now we can easily argue that (2.3.23) holds if and only if $2\left|r_1 \mathrm{Re} c_1 + r_2 \mathrm{Re} c_2\right| \leq 1$ for all $r_1, r_2 \geq 0$ such that $r_1^2 + r_2^2 = 1$. The desired conclusion follows immediately. This finishes the proof of Proposition 2.3.3.

We get further examples of extreme points by noticing that if $h \in \mathcal{M}$ is an extreme point and U is a unitary operator then $U^* \circ h \circ U \in \mathcal{M}$ is also an extreme point. Note that for these extreme points we can choose a constant C so that $\|h(z)\| \leq C/(1 - \|z\|)$. However there are mappings in $\mathcal{M}$ that have larger growth near the boundary of B^n. One such example is

$$H(z) = \left(z_1 \frac{1 - z_1}{1 + z_1}, z_2 \frac{1 - z_1 - z_1^2}{(1 + z_1)^2}\right)$$

as it can be easily seen that as $z \to -1$, $\|H(z)\| = \Theta\left((1 - \|z\|)^{-3/2}\right)$. This shows that there are also other types of extreme points and that, unlike the one dimensional case, not all extreme points optimize growth.

The mapping H appears in connection with a starlike mapping F which is interesting in its own right because it doesn't satisfy the expected distortion bound for starlike mappings. The mapping F is defined by

$$F(z) = \left(k(z_1), \left(\frac{k(z_1)}{z_1}\right)^{\frac{1}{2}} \left(k'(z_1)\right)^{\frac{1}{2}} z_2\right), \; z \in B^2$$

where $k(z_1) = z_1/(1-z_1)^2$. Note that by properties of extension operators F is indeed starlike and since H satisfies $DF(z)H(z) = F(z)$ it automatically follows that H is a Carathéodory mapping. We will check that F doesn't satisfy the upper distortion bound $\|DF(z)\| \leq (1+\|z\|)/(1-\|z\|)^3$. Let $e = (1,0)$ and $z_1 = x \in [0,1)$. Then

$$\|DF(z)e\|^2 = \frac{(1+x)^2}{(1-x)^6} + \frac{(3+2x)^2}{(1-x)^6(1-x^2)}|z_2|^2.$$

Let $y = \|z\|$. Hence $|z_2|^2 = y^2 - x^2$. Also let $\phi(x) = (1+x)^2/(1-x)^6$. If the upper distortion bound would hold for F, we must have that

$$\phi(x) + \frac{(3+2x)^2}{(1-x)^6(1-x^2)}\left(y^2 - x^2\right) \leq \phi(y).$$

Hence

$$\frac{(3+2x)^2}{(1-x)^6(1-x^2)}(y+x) \leq \frac{\phi(y)-\phi(x)}{y-x}$$

for all x, y such that $0 \leq x < y < 1$. Letting $y \to x$ we get

$$\frac{(3+2x)^2}{(1-x)^6(1-x^2)}2x \leq \phi'(x) = 4\frac{(1+x)(2+x)}{(1-x)^7},\ x \in [0,1).$$

After simplifications we have

$$\frac{(3+2x)^2 2x}{1+x} \leq 4(1+x)(2+x),\ x \in [0,1).$$

Letting $x \to 1$ we get that $25 \leq 24$. This is absurd, so F cannot satisfy the distortion upper bound for all $z \in B^n$.

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