Log-concave Gorenstein sequences

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Abstract

We show here that codimension three Artinian Gorenstein sequences are log-concave, and that there are codimension four Artinian Gorenstein sequences that are not log-concave. We also show the log-concavity of level sequences in codimension two.

Dedicated to the memory of friend and colleague, Jacques Emsalem.

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1 Introduction.

A codimension r Gorenstein sequence is here a Hilbert function H(A) that occurs for a codimension r graded Artinian Gorenstein (AG) algebra A over an infinite field \mathbb{F} . The codimension two Gorenstein sequences are the same as those for complete intersections $A(a, b) = \mathbb{F}[x, y]/(x^a, y^b)$ - known to F.H.S. Macaulay [Mac1, Mac2]. The codimension three Gorenstein sequences are known because of the Pfaffian structure theorem of D. Buchsbaum and D. Eisenbud [BuEi], (Lemma 1.3 below, see also [St1,Di]). The Hilbert function H(A) of a graded Artin algebra is a sequence satisfying a certain condition determined by F.H.S. Macaulay (Equation (2), Lemma 1.1 below, [Mac3], and [BrHe, §4.2]). We will call such a sequence satisfying Equation (2) a Macaulay sequence. The socle of an Artinian algebra A is $(0 : \mathfrak{m}_A)$ where \mathfrak{m}_A is its maximal ideal, and the socle degree of A is the highest degree of a socle element; when A is Artinian Gorenstein, the socle degree of A is the highest degree j for which $H(A)_j \neq 0$. We recall

Lemma 1.1. [Macaulay's theorem [Mac3]] Let $H = H(A) = (1, r, h_2, ..., h_a, ...)$ be the Hilbert function of an algebra quotient of $R = \mathbb{F}[x_1, ..., x_r]$. We may write uniquely, for $a \ge 1$, the Macaulay expansion

$$h_a = \sum_{i=1}^{a} {n_i \choose i} \text{ with } n_a > n_{a-1} > \dots > n_1 \ge 0.$$
 (1)

Then we have

$$h_{a+1} \le h_a^{(a)} := \sum_{i=1}^a \binom{n_i+1}{i+1}.$$
 (2)

For further discussion see [BrHe, Lemma 4.2.6, Theorem 4.2.10]. When there is equality in Equation (2) we term this maximum Macaulay growth from degree a to degree a + 1 [BrHe, Section 4.2].

Definition 1.2. An SI sequence $H = (1, r, ..., r, 1_j)$ of socle-degree j is a sequence satisfying both

$$h_i = h_{j-i} \text{ for } 0 \le i \le j/2;$$

(ΔH) _{is a Macaulay sequence, (3)}

where $(\Delta H)_i = h_i - h_{i-1}$ (take $h_{-1} = 0$).

We have

Lemma 1.3. [BuEi, St1] A sequence $H = (1, 3, ..., h_i, ..., 3, 1)$ is a codimension three Gorenstein sequence if and only if H is an SI sequence.

The proof follows from the D. Buchsbaum and D. Eisenbud Pfaffian structure theorem for codimension three Gorenstein algebras [BuEi]; see also [Di] and [IK, Theorem 5.25].

It is well known that in any codimension, the SI sequences are a subset of the Gorenstein sequences: when $r \ge 5$, the SI sequences are a proper subset of the Gorenstein sequences, which may be non-unimodal; when r = 4 it is open whether the SI sequences might be all the Gorenstein sequences. A result of N. Altafi shows that given a finite SI sequence H, there is always a strong Lefschetz Artinian Gorenstein algebra of Hilbert function H [Alt].

Definition 1.4. We say that a finite sequence $H = (h_0, h_1, \ldots, h_i, \ldots, h_j)$ is *log-concave in a degree* $i \in [1, j - 1]$ if

$$h_{i-1} \cdot h_{i+1} \le h_i^2. \tag{4}$$

The sequence H is *log-concave* if it is log-concave in each such degree i.

See the R. Stanley 1989 survey [St2], the F. Brenti 1994 [Bre], and many more recent articles. Log-concavity has a relation with the Hodge-Riemann property of certain algebras [Ba, H, MMS, MuNaYa].

2 Codimension three Gorenstein sequences are log-concave.

Theorem 2.1. Let A be a standard graded AG algebra of socle degree j with Hilbert function $H(A) = (h_0, h_1, \ldots, h_j)$ satisfying $h_1 = 3$. Then the sequence H(A) is log-concave.

Proof. Note that it suffices to show that Equation (4) holds for $1 \le i \le \left\lfloor \frac{j}{2} \right\rfloor$, since for $i > \left\lfloor \frac{j}{2} \right\rfloor$, we have $1 \le j - i \le \left\lfloor \frac{j}{2} \right\rfloor$ and hence Equation (4) will hold for these *i* by symmetry of the Hilbert function. For each $1 \le i \le \left\lfloor \frac{j}{2} \right\rfloor$, we have $h_i \le \binom{h_1+i-1}{i} = \binom{i+2}{i}$, and if we have equality for every *i*, then H(A) is log-concave since the binomial coefficients are log-concave. Otherwise we may choose the smallest index $u, 1 \le u \le \left\lfloor \frac{j}{2} \right\rfloor$ such that $h_u < \binom{u+2}{u}$. Then of course Equation (4) holds for $1 \le i \le u - 1$ for the preceding reason, and hence we need only check Equation (4) for $u \le i \le \left\lfloor \frac{j}{2} \right\rfloor$. Let $j' == \left\lfloor \frac{j}{2} \right\rfloor$. The following two observations are key: by Lemma 1.3

(i). H(A) is an SI sequence, and hence the first difference

$$\Delta H(A)_{\leq j'} = (1, 2, \dots, \Delta H_{j'}) \tag{5}$$

is the Hilbert function for some standard graded Artinian algebra of codimension 2, and

(ii). The Hilbert function of a standard graded Artinian algebra of codimension 2 is non-increasing after the initial degree d of its defining ideal (here $d = \min\{i \mid \Delta H(A)_i < i+1\}$) so $\Delta H(A)_{\leq j'}$ in Equation (5) satisfies

$$\Delta H(A)_{j'} = (1, 2, \dots, d, \Delta_d, \dots, \Delta_{j'}), \text{ with } d \ge \Delta_d \ge \Delta_{d+1} \ge \dots \ge \Delta_{j'}.$$
(6)

The second observation is well known (see [Mac1], [I1, Lemma 1.3])¹ and can be seen as follows: Let $B = \mathbb{F}[x, y]/I$ be any standard graded Artinian algebra in codimension two, suppose that $I_p \neq 0$ and suppose that $f_1, \ldots, f_m \in I_p$ are linearly independent forms in I of degree p. Then certainly xf_1, \ldots, xf_m are linearly independent in I of degree p + 1; also, if f_1 has maximum ypower among the set of f_i , then $yf_1, xf_1, \ldots, xf_m \subset \mathbb{F}[x, y]_{p+1}$ are linearly independent, showing that $\dim_{\mathbb{F}}(I_p) < \dim_{\mathbb{F}}(I_{p+1})$, and hence G = H(B) = $(1, 2, \ldots, g_p, \ldots)$ satisfies

$$g_p = \dim_{\mathbb{F}}(\mathbb{F}[x, y]_p) - \dim_{\mathbb{F}}(I_p) = (p + 1 - \dim_{\mathbb{F}}(I_p))$$

$$\geq p + 2 - \dim_{\mathbb{F}}(I_{p+1}) = \dim_{\mathbb{F}}(\mathbb{F}[x, y]_{p+1}) - \dim_{\mathbb{F}}(I_{p+1}) = g_{p+1}.$$

Finally, for any integer *i* satisfying $d \le i \le \left\lfloor \frac{j}{2} \right\rfloor$, we must therefore have

$$h_i^2 \ge h_i^2 - (h_i - h_{i-1})^2 = (h_i - (h_i - h_{i-1}))(h_i + (h_i - h_{i-1})) \ge (h_i - (h_i - h_{i-1}))(h_i + (h_{i+1} - h_i)) = h_{i-1}h_{i+1},$$

and hence H(A) is log-concave.

3 Codimension four Gorenstein sequences that are not log-concave.

Many codimension four Gorenstein sequences, as H = (1, 4, 10, 14, 10, 4, 1), are log-concave. We first show that there are codimension four SI sequences that are not log-concave (Proposition 3.2); then we show that there are codimension four SI sequences that are not log-concave for an arbitrarily large consecutive sequence of degrees (Proposition 3.4).

The codimension four Gorenstein sequences H include the SI sequences, those satisfying $\Delta H_{\leq j/2} = (1, 3, ...)$ is the Hilbert function of a codimension

¹The article [Mac1] also determines all the Hilbert functions that do occur for local algebras of codimension two, as well as for the graded algebras.

three graded Artin algebra A = R/I (Definition 1.2). We first restrict to codimension four SI sequences satisfying

$$\Delta H_{\leq j/2} = (1, 3, \dots, r_k, b, c) \text{ where } r_k = \binom{k+2}{2} = \dim_{\mathbb{F}} R_k.$$
(7)

We denote by $S = \mathbb{F}[x, y, z, w]$, and let $s_k = \dim_{\mathbb{F}} S_k = (1+3+\cdots+r_k) = \binom{k+3}{3}$. Then the sum function of $\Delta(H)_{\leq j/2}$ above satisfies

$$H_{\leq j/2} = (1, 4, \dots, s_k, s_k + b, s_k + b + c).$$

The log-concavity condition (4) here in degree k + 1 for H is

$$s_k(s_k + b + c) < (s_k + b)^2$$
 or, equivalently, $s_k(c - b) < b^2$. (8)

Keeping b, c constant with c > b then this certainly is negated for k large enough. The next idea is to choose b suitably and let $s_k + b$ in degree k + 1 to $s_k + b + c$ in degree k + 2 have maximum Macaulay growth (see Lemma 1.1).

The dimension $\dim_{\mathbb{F}}(\mathbb{F}[x, y, z, w]_k) = s_k$. For codimension three, if $h_a < r_a$ we will denote by $\delta(h_a) = h_a^{(a)} - h_a$: here, this is just the number of terms in the Macaulay expansion of Equation (1) with $n_i = i + 1$. Then, taking $a = k + 1, h_a = s_k + b, s_{k+2} = h_a + c$ with $c = h_a + \delta(h_a)$ (maximum Macaulay growth) the log-concavity condition Equation (8) becomes $\delta \cdot s_k \leq b^2$. Thus, to violate log-concavity in degree k + 1 for a Hilbert function sequence H having as its key entries $h_k = s_k, h_{k+1} = s_k + b, h_{k+2} = s_k + b + \delta$ we need only assure

$$\delta \cdot s_k > b^2. \tag{9}$$

Remark 3.1. Recall that the Gotzmann regularity degree of the constant polynomial $\{s\}$ is itself s (see [IK, Proposition C.32]). This implies for an SI sequence H that once $(\Delta H)_i \leq s$ for an integer $i \in [s, j/2]$ then $(\Delta H)_{\leq j/2}$ is non-increasing in higher degrees than i. Also, in order for $\delta = h_a^{(a)} - h_a \geq 2$ we must have $h_a \geq 2a + 1$, with equality when $h_a = \binom{a+1}{a} + \binom{a}{a-1}$. For $\delta = h_a^{(a)} - h_a \geq 3$ we need $h_a \geq 3a$, with equality when $h_a = \binom{a+1}{a} + \binom{a}{a-1} + \binom{a-1}{a-2}$. Evidently, for $\delta \geq 4$ we need

$$b = h_a \ge \delta \cdot a - (2 + \dots + (\delta - 2))$$

= $\delta \cdot a - \delta(\delta - 3)/2.$ (10)

These inequalities for $\delta \geq 2$ will greatly affect our search for small examples of SI sequences in codimension four that are not log-concave - that satisfy Equation (9). We will denote by $H_{a\to b}$ the subsequence $(h_a, h_{a+1}, \ldots, h_b)$ of H.

Proposition 3.2 (SI sequences in 4 variables that are not log-concave). We give a series of minimal examples, depending on the choice of δ .

Case $\delta = 1$. First we consider $\delta = 1$ and take $b = \binom{s+1}{s}$, in degree s = k + 1. We need $s_k > b^2 = (s+1)^2$. Taking $\delta = 1, b = 7_6, k = 5$, so $s_5 = 56 > 7^2$ we have

 $H = (1, 4, 10, 20, 35, 56, 63, 71, 63, 56, 35, 20, 10, 4, 1_{14}), \quad n = 449,$

then $h_5 \cdot h_7 = 56 \cdot 71 = 3976 > 3969 = 63^2 = h_6^2$. Taking $\delta = 1, b = 8_7, k = 6$ so $s_6 = 84 > 8^2$ we have

 $H = (1, 4, 10, 20, 35, 56, 84, 92, 101, 92, 84, 56, 35, 20, 10, 4, 1_{16}), \quad n = 705,$

then $h_6 \cdot h_8 = 84 \cdot 101 = 8484 > 8464 = 92^2 = h_7^2$. Since $84 > 9^2$ we have a second example where $\delta = 1, b = 9, H_{8\to 12} = (84, 93, 103, 93, 84)$ of socle degree 16 and length n = 709, also not log-concave in degree 7, as $h_6 \cdot h_8 = 84 \cdot 103 = 8652 > 8649 = 93^2 = h_7^2$.

In general, taking $\delta = 1, k >> 5$, we may choose b_{k+1} (that is, b in degree k+1) satisfying $k+2 \leq b \leq (k/6)^{3/2}$ (asymptotically, not for small b) that will satisfy the conditions of Remark 3.1 and also satisfy $s_k > b^2$, Equation 9, so we will obtain again a non log-concave H.

Case $\delta = 2$. Now taking $\delta = 2, b = 23$ we have $2s_{10} = 2(286) = 572 > 529 = 23^2$ and 23 satisfies 23 = 2(11) + 1, the lower bound from Remark 3.1. This is the lowest pair $\delta = 2, b = 23$ with a = 11 satisfying Equation (9), and leads to an example of non log-concave H of socle degree 24, whose key entries are

$$H_{10\to 14} = (286, 309, 334, 309, 286), \tag{11}$$

satisfying $h_{10} \cdot h_{12} = 286 \cdot 334 = 95524 > 95481 = 309^2 = h_{11}^2$, so *H* of length n = 2954 that is non log-concave in degree 11.

Now taking $\delta = 2, b = 25$ we have $2s_{11} = 2(364) = 728 > 625 = 25^2 = \Delta h_{12}^2$ so we have new H of socle degree 26 whose key entries are

$$H_{11\to15} = (364, 389, 416, 389, 364)$$

satisfying $h_{11} \cdot h_{13} = 364 \cdot 416 = 151424 > 151321 = 389^2 = h_{12}^2$, so *H* of length n = 4034 that is non log-concave in degree 12. Evidently, we may replace 25 by 26 as also $728 > 26^2 = 676$. Then the key entries of *H* would be

for
$$b = 26, H_{11 \to 15} = (364, 390, 418, 390, 364).$$

This sequence of examples with $\delta = 2$ can evidently be continued, the next arises from $\delta = 2, b = 27, 2s_{12} = 2(455) = 910 > 729 = (27)^2$, so gives an SI sequence H of socle degree 28 with key entries

$$H_{12\to 16} = (455, 482, 511, 482, 455),$$

satisfying $h_{12} \cdot h_{14} = 232505 > 232324 = 482^2 = h_{13}^2$. Evidently, we may replace 27 by $b \in [27, 30]$ as $910 > 30^2$.

The general $\delta = 2$ case with fixed k and lowest b = 2k + 3 will be $s_k, k \ge 10$ satisfying $2s_k > (2k+3)^2$, leading to an SI sequence $H = (1, 4, \dots, 4, 1_{2k+4})$ of socle degree 2k + 4, with key entries

$$H_{k \to (k+4)} = (s_k, s_k + 2k + 3, s_k + 4k + 8, s_k + 2k + 3, s_k),$$

of length $n = 2\binom{k+4}{4} + 3s_k + 8k + 14$, that is not log-concave in degree k + 1.

Given k, the maximum b satisfying Equation (9) is $\sqrt{2s_k}$; for k = 25this is $b = \lfloor \sqrt{2s_{25}} \rfloor = \lfloor \sqrt{6552} \rfloor = 80$. When (k,b) is a fixed pair, satisfying $2k + 3 \le b \le \sqrt{2s_k}$ the key entries in the corresponding non log-concave H of socle degree 2k + 4 are

$$H_{k \to (k+4)} = (s_k, s_k + b, s_k + 2b + 2, s_k + b, s_k),$$

and *H* has length $2\binom{k+4}{4} + 3s_k + 4b + 2$.

Case $\delta = 3$. We have $3 \cdot s_{18} = 3 \cdot {\binom{20}{3}} = 3420 > 3249 = 9(19)^2$, so taking $(\delta, k, b) = (3, 18, 3 \cdot 19)$ we generate a smallest H for $\delta = 3$, here of socle degree j = 40 that is not log-concave in degree 19, that has key H values

$$H_{18\to 22} = (1140, 1197, 1257, 1197, 1140).$$

Case $\delta \geq 4$. To satisfy Remark 3.1 and Equation (9), we must have $\delta \cdot s_k \geq b^2$, or $\delta \cdot \binom{k+3}{k} > b^2$, where also $b \geq \delta k$. Solving for b_{\max} , and approximating $\binom{k+3}{k}$ by $k^3/6$, we have $\delta k^3/6 \leq b_{\max}^2$, so if b is in the interval

$$b \in [\delta \cdot k, \, \delta^{1/2} \cdot k^{3/2} / 6^{3/2}],\tag{12}$$

then the triple (δ, k, b) will produce a codimension four SI sequence H with key entries

$$H_{k \to k+4} = (s_k, s_k + \delta \cdot b, s_k + 2\delta \cdot b, s_k + \delta \cdot b, s_k)$$

and socle degree 2k + 4 that is non log-concave in degree k + 1.

Example 3.3 (Lengthening the non log-concave examples H). The lowest $\delta = 2$ example is Equation (11) of socle degree j = 24, where $H_{10\to 14} = (286, 309, 334, 309, 286)$ and $\Delta H_{10\to 12} = (r_{10}, r_{10} + 23, r_{10} + 25) = (66, 89, 91)$. We may lengthen this by adding a further sequence after $\Delta H_{12} = 114$ that satisfies the Macaulay inequalities Equation (2) for codimension three. For example we may adjoin to ΔH the sequence $(16_{13}, 17_{14}, 8_{15}, 7_{16})$, leading to a key sequence

 $H'_{10\to22} = (286, 309, 334, 350, 367, 375, 382, 375, 367, 350, 334, 309, 286),$

of an AG Hilbert function H' of socle degree j = 32 that is still non log-concave in degree 11.

In the next Proposition, we extend the maximum growth portion of ΔH , by a distance ℓ , obtaining in some cases, especially when $s_b >> b^2$ a new AG sequence $H'(\ell)$ having a longer consecutive subset of non log-concave adjacent triples, whose length we can specify. Further, we specify $h_{k+u}h_{k+u+2} - h_{k+u+1}^2$ for each $u \in [0, \ell]$, showing that it is a sequence with linear first differences.

Proposition 3.4 (Sequences *H* having multiple non log-concave places). Assume that (δ, k, b) is a triple as in Proposition 3.2 for which $\delta \cdot s_k > b^2$, and let *H* be the related codimension four Hilbert function of socle degree j = 2k + 4, whose key part is

$$H_{k \to k+4} = (s_k, s_k + b, s_k + 2b + \delta, s_k + b, s_k)$$

and which is non log-concave in degree k + 1. Let ℓ be a positive integer. We define an extended sequence $H'(\ell)$ identical to H in degrees $i \in [0, k + 2]$, and satisfying

$$\Delta H'(\ell)_t = \Delta T_t \text{ for } t \le k+1 \text{ and for } t > k+\ell+2;$$

$$\Delta H'(\ell)_{k+1\to k+\ell+2} = (b, b+\delta, b+2\delta, \dots, b+(\ell-1)\delta, b+(\ell)\delta, b+(\ell+1)\delta).$$
(13)

Then $H'(\ell)$ has socle degree $2k + 4 + 2\ell$, and we have for each $u \in [1, \ell + 2]$ that $h'_{k+u} = s_k + bu + \delta(1 + 2 + \dots + (u - 1))$. Letting $\delta = 1$ we have for each such u

$$h'_{k+u} \cdot h'_{k+u+2} - (h'_{k+u+1})^2 = s_k - b^2 - ((b+1) + \dots + (b+u)).$$
(14)

Assume now that

$$b\ell + \frac{\ell(\ell+1)}{2} < s_k - b^2.$$
(15)

Then $H'(\ell)$ is non log-concave in the positions $k+1, k+2, \ldots, k+\ell, k+\ell+1$.

Proof. We generalize the proof of Equations (8) and (9). Beginning with $(\delta = 1, b, k)$ then using the definition of $\Delta H'$ in Equation (13) and recalling that $H_k = s_k$ we conclude that $H' = (1, h'_1, \ldots)$ satisfies

for
$$u \in [0, \ell]$$
, $h'_{k+u} = s_k + bu + \sum_{\nu=1}^{u-1} \nu$.

Then, similarly to Equation (8) we have for $u \in [1, \ell + 1]$ and i = k + u

$$h'_{i-1}h'_{i+1} - h'_{i}^{2} = (h'_{i} - (b+u-1))(h'_{i} + b+u) - {h'_{i}}^{2}$$

= $h'_{i} - (b+u-1)(b+u)$
= $s_{k} - b^{2} - b(u-1) - \sum_{\nu=1}^{u-1} \nu$
= $s_{k} - b^{2} - b(u-1) - \frac{u^{2} - u}{2}.$ (16)

The last statement of the Proposition concerning non log-concavity for $H'(\ell)$ for ℓ satisfying Equation (15) follows.

Example 3.5. We take from Proposition 3.2 the first sequence H where $\delta = 1, k = 8, b = 10_9$ with key entries $H_{8\to12} = (165_8, 175, 186, 175, 165)$ and recall that $h_8h_{10} - h_9^2 = 30690 - 30625 = 65 = s_8 - b^2 = 165 - 10^2$. We choose ℓ as in Equation (14), the maximum such that for b = 10

$$(b+1) + (b+2) + \dots + (b+\ell) < 65$$
, from which we have $\ell = 4$

We now consider $\Delta H'(4)_{\leq j/2} = (1, 3, 6, 10, 15, 21, 28, 36, 45, 10_9, 11, 12, 13, 14, 15)$, so

$$H'(4) = (1, 4, 10, 20, 35, 56, 84, 120, 165, 175_9, 186, 198, 211, 225, 240_{14}, 225, 211, 198, 186, 175, 165, 120, 84, 56, 35, 20, 10, 4, 1_{28}).$$
(17)

of socle degree 28, which is non log-concave in degrees 9, 10, 11, 12, 13 with successive differences (writing h_i for t'_i)

$$h_{i-1}h_{i+1} - h_i^2 = (65_9, 54_{10}, 42_{11}, 29_{12}, 15_{13}),$$

whose second differences are (11, 12, 13, 14), consistent with Equations (13) and (14). Note that $H'(5)_{13,14,15} = (225, 240_{14}, 256)$ whence $h_{i-1}h_{i+1} - h_i^2 = 0$ for i = 16, so H'(5) begins log-concavity in degree 16; this is again consistent with Equation (14) and shows that the maximum length of a non log-concave sequence for H' arising from this H is five.

Note, that combining with the idea of Example 3.3, we can make codimension four Gorenstein sequences H' with certain specified consecutive subsequences of degrees where H' is non log-concave, but that may have gaps between them of degrees where H' is log-concave.

Remark 3.6. According to [H, Theorem 3] these non log-concave *h*-vectors cannot be the *h*-vectors of a matroid representable in characteristic zero. Chris McDaniel poses the question of whether a related *f*-vector: f(t) = h(1+t) of one of these non log-concave *h* might be non-unimodal.

4 Higher codimension Gorenstein sequences, level sequences.

In codimension at least five, Gorenstein sequences need not be unimodal. For example, R. Stanley used an idealization construction [Na] or [IMM, §3.1] to show H = (1, 13, 12, 13, 1) is a Gorenstein sequence for an algebra $A \times$ $\operatorname{Hom}(A, \mathbb{F})$ where $A = \mathbb{F}[x, y, z]/(x, y, z)^4$ of Hilbert function (1, 3, 6, 10) [St1], Analogous examples, sometimes requiring higher socle degree j, can be made in codimensions at least five: in codimension five the lowest socle-degree nonunimodal Gorenstein sequence known has j = 16 and contains the subsequence $(h_7, h_8, h_9) = (91, 90, 91)$ [BeI] (see also [MZ, MNZ1] for further discussion of unimodality for Gorenstein sequences).

4.1 Higher codimension SI sequences.

SI sequences by construction are unimodal. We note that, as in codimension four, some SI sequences in codimension five or larger are log-concave, and some are not. We illustrate both with the next example.

Example 4.1 (Codimension five). (i). Consider the codimension five SI sequence $H(j) = (1, 5, t_2, ...)$ having socle degree j with

$$\Delta H(j) = (1, 4, 10, 14, 20, 27, 35, 44, \ldots)$$

where $\Delta H(j)_k = \binom{k+2}{k} + \binom{k}{k-1} + \binom{k-2}{k-2}$ for $3 \le k \le j/2$ and $j \ge 6$. Then $H(j)_{\le j/2} = (1, 5, 15, 29, 49, 76, 111, 155, \dots,)$ where

$$H(j)_k = -1 + \binom{k+3}{k} + \binom{k+1}{k-1} + \binom{k-1}{k-2} \text{ for } 3 \le k \le j/2,$$

and $H(j)_k = H(j)_{j-k}$ for $k \ge j/2$. It is easy to verify that for $j \ge 6$ each such H(j) is log-concave. The reason is that the sequence $H(j)_k$ for $k \le j/2$ is dominated by its fast-growing term $\binom{k+3}{k}$ which is cubic in k.

(ii). This suggests that the way to create non log-concave examples of SI sequences is to imitate the construction in codimension four, that is, set $\Delta H' = (1, 4, r_2, \ldots, r_{u-1}, \delta_u, \ldots)$ where the non-zero terms of the Macaulay expansion of δ_u have the form $\binom{w+1}{w}$ - that is, we assume that after a certain degree u the first difference $\Delta H'$ has maximal Macaulay growth, and is linear in k. Consider then $\Delta H' = (1, 4, 10, 20, 5, 6, 7, \ldots)$ so that $H' = (1, 5, 15, 35, 40, 46, 53, 61, 70, \ldots$ Here $35 \cdot 46 = 1610 > 40^2$, so taking j = 10 and $H' = (1, 5, 15, 35, 40, 46, 63, 53, 51, 55, 1_{10})$ gives a non-log-concave sequence. Likewise, $40 \cdot 53 > 46^2$, so taking j = 12 the SI sequence $H' = (1, 5, 15, 35, 40, 46, 53, 46, 40, 35, 15, 5, 1_{12})$ is non-logconcave in two adjacent places. But it stops there, as $46 \cdot 61 = 2806 < 2809 = 53^2$.

To get a longer non-log-concave consecutive subsequence, it suffices to begin the minimal linear growth later. For example, taking $\Delta H' = (1, 4, 10, 20, 35, 6, 7, 8, ...)$ so

$$H'_{(18)$$

we have $h'_{k-1} \cdot h'_{k+1} > {h'_k}^2$ for k = 4, 5, 6, 7, 8 with equality for k = 9, so taking

 $H' = (1, 5, 15, 35, 70, 76, 83, 91, 100, 110, 100, 91, 83, 76, 70, 35, 15, 5, 1_{18})$

gives 5 adjacent non-log-concave entries centered in degrees 4 to 8. Taking j = 20 will give equality in the log-concavity equation in degree 9.

4.2 Log-concavity of level sequences.

Recall that the *socle* of an Artinian algebra A is $(0 : \mathfrak{m}_A)$ where \mathfrak{m}_A is its maximal ideal. A *level sequence* is a Hilbert function possible for an Artinian algebra $A = R/I, R = \mathbb{F}[x_1, \ldots, x_r]$ having socle all in the same degree, that is, $H(A) = (1, r, \ldots, t_j, 0)$ and the socle of $A = (0 : \mathfrak{m}_A) = A_j$: we say Ahas *type* t_j . The Hilbert functions of level algebras are well known in codimension 2, essentially due to F.H.S. Macaulay [Mac1] but they are not known for type t in higher codimension except for the Gorenstein codimension three case (r, t) = (3, 1) (for some partial results see [ChoI, Za1]). Given a Hilbert function sequence $H = (\ldots, h_i, \ldots)$, we denote by $e_i = h_{i-1} - h_i = -\Delta(H)_i$.

We have [I3, Proposition 2.6, Lemma 2.15]

Lemma 4.2. The Hilbert function H = H(A) of a level quotient A = R/I of $R = \mathbb{F}[x, y]$ satisfies

$$H = (1, 2, \dots, d, t_d, t_{d+1}, \dots, t_j, 0) \text{ where } t_j = e_{j+1} \ge e_j \ge e_{j-1} \ge \dots d - t_d.$$
(19)

A *compressed* level algebra of codimension r, socle degree j and type t is one having Hilbert function

$$H(A)_i = \min\{r_i, tr_{j-i}\},$$
 (20)

where $r_i = \dim_{\mathbb{F}} R_i$ and $R = \mathbb{F}[x_1, \ldots, x_r]$ (see [I2] and [Bo, Definition 1.3]). For example, with (r, t, j) = (3, 2, 5) the compressed level Hilbert function H = (1, 3, 6, 10, 6, 2).

Proposition 4.3. (i). Every codimension two level sequence is log-concave. (ii). Every compressed level Hilbert function in any codimension is log-concave.

Proof. The proof of (i) is essentially a translation of the proof of Theorem 2.1, with Equation (19) replacing Equation (6), there. The proof of (ii) follows from the log-concavity of the sequence $(1, r_1, r_2, \ldots, r_i \ldots)$ where $r_i = \dim_k R_i = \binom{r+i-1}{r}$ - as, letting e = e(r, t, j) be the highest degree such that $H(A)_i = r_i$ in Equation (20), for i < e we have $h_i^2 = r_i^2 \ge r_{i-1} \cdot r_{i+1} = h_{i-1} \cdot h_{i+1}$; and $h_e^2 = r_e^2 \ge r_{e-1}r_{e+1} \ge h_{e-1}h_{e+1}$; and for i > e+1 we have $h_i = tr_{j-i}$, so likewise $h_i^2 \ge h_{i-1}h_i$; for i = e+1 we have, since $tr_{j-e} \ge h_e$,

$$h_i^2 = t^2 r_{j-(e+1)}^2 \ge t r_{j-e} \cdot t r_{j-(e+2)} \ge h_e \cdot h_{i+1}.$$

However, there are codimension three level sequences that are not unimodal - see [We, AhS], the former giving a non-unimodal example with r = 3, t = 5.

4.2.1 Problems

A pure O-sequence is a Hilbert function H = H(A) that can occur for a monomial level algebra A = R/I of socle degree j: that is, I_j has a basis of monomials. In codimension three, a type two monomial Artinian algebra is known to be weak Lefschetz (there is a linear form ℓ such that $m_{\ell} : A_i \to A_{i+1}$ has maximum rank for each i), provided \mathbb{F} has characteristic zero [BMMNZ, Theorem 6.2]. This implies that H(A) is differentiable (the first difference is an O-sequence) until its maximum value, which may be repeated, then H(A) is decreasing [MNZ2, Theorem 12]). Although their Hilbert functions (codimension three, type two) are characterized in [BMMNZ, Proposition 6.1], it remains to see if these sequences are log-concave.

There is a subtle connection of pure O-sequences with algebras associated to matroids (see [CM, C-T, H, MuNaYa]).

Recall that a sequence $H = (1, 2, ..., d-1, d, h_d, h_{d+1}, ..., h_j)$ is admissible of decreasing type if $d \ge h_d \ge ...$ and there is $b \ge d$ such that

$$d = h_{d-1} = \dots = h_{b-1} > h_b > h_{b+1} > \dots > h_j.$$
(21)

Recall that an h-vector of a Gorenstein domain D of dimension s is the Hilbert function $H(A), A = D/(\ell_1, \ldots, \ell_s)$ where ℓ_1, \ldots, ℓ_s is a regular sequence of linear forms. The h-vectors of Gorenstein domains are a proper subclass of Artinian Gorenstein sequences. When $H = (1, 2, \ldots, d, h_d, h_{d+1}, \ldots, h_j)$ the condition for H to be an h-vector is for H to be admissible of decreasing type [GP]. The AG sequence $H = (1, 3, \ldots)$ as in Lemma 1.3 is an h-vector of a Gorenstein domain if and only if the difference sequence of Equation (6) is admissible of decreasing type [DV], [Va, Theorem 2.18]. A characterization of the Hilbert function of Gorenstein domains in higher codimension $h_1 \ge 4$ is not known. Also, it is apparently still open whether such h-vectors $(1, h_1, \ldots)$ of domains with $h_1 \ge 4$ would be log-concave [Bre, Conjecture 5.2].

While the deformation properties of Artinian algebras have been widely studied (see, for example, [EmI, Di, BoI, CJN, AEI]) it appears open to check whether there are special properties, pertaining, say to smoothability - deformations to a smooth scheme - depending on the log-concavity of the Hilbert functions H. What can we say about the irreducible components of the variety G_T parametrizing graded Artinian algebras of Hilbert function H when $r \geq 4$ and H is log-concave?

Subsequent to our work, and partly inspired by it, F. Zanello has further studied which level sequences and pure O-sequences are log-concave, resolving the problem in some cases and proposing additional problems [Za2].

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