# Log-concave Gorenstein sequences 

Anthony A. Iarrobino<br>Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

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#### Abstract

We show here that codimension three Artinian Gorenstein sequences are log-concave, and that there are codimension four Artinian Gorenstein sequences that are not log-concave. We also show the log-concavity of level sequences in codimension two.


Dedicated to the memory of friend and colleague, Jacques Emsalem.

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[^0]Email address: a.iarrobino@northeastern.edu

## 1 Introduction.

A codimension $r$ Gorenstein sequence is here a Hilbert function $H(A)$ that occurs for a codimension $r$ graded Artinian Gorenstein (AG) algebra $A$ over an infinite field $\mathbb{F}$. The codimension two Gorenstein sequences are the same as those for complete intersections $A(a, b)=\mathbb{F}[x, y] /\left(x^{a}, y^{b}\right)$ - known to F.H.S. Macaulay [Mac1, Mac2]. The codimension three Gorenstein sequences are known because of the Pfaffian structure theorem of D. Buchsbaum and D. Eisenbud [BuEi], (Lemma 1.3 below, see also [St1, Di]). The Hilbert function $H(A)$ of a graded Artin algebra is a sequence satisfying a certain condition determined by F.H.S. Macaulay (Equation (2), Lemma 1.1 below, [ Mac 3 ], and $[\mathrm{BrHe}, \S 4.2]$ ). We will call such a sequence satisfying Equation (2) a Macaulay sequence. The socle of an Artinian algebra $A$ is $\left(0: \mathfrak{m}_{A}\right)$ where $\mathfrak{m}_{A}$ is its maximal ideal, and the socle degree of $A$ is the highest degree of a socle element; when $A$ is Artinian Gorenstein, the socle degree of $A$ is the highest degree $j$ for which $H(A)_{j} \neq 0$. We recall

Lemma 1.1. [Macaulay's theorem [Mac3]] Let $H=H(A)=\left(1, r, h_{2}, \ldots, h_{a}, \ldots\right)$ be the Hilbert function of an algebra quotient of $R=\mathbb{F}\left[x_{1}, \ldots, x_{r}\right]$. We may write uniquely, for $a \geq 1$, the Macaulay expansion

$$
\begin{equation*}
h_{a}=\sum_{i=1}^{a}\binom{n_{i}}{i} \text { with } n_{a}>n_{a-1}>\cdots>n_{1} \geq 0 . \tag{1}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
h_{a+1} \leq h_{a}^{(a)}:=\sum_{i=1}^{a}\binom{n_{i}+1}{i+1} . \tag{2}
\end{equation*}
$$

For further discussion see [ BrHe , Lemma 4.2.6,Theorem 4.2.10]. When there is equality in Equation (2) we term this maximum Macaulay growth from degree $a$ to degree $a+1$ [ BrHe , Section 4.2].

Definition 1.2. An SI sequence $H=\left(1, r, \ldots, r, 1_{j}\right)$ of socle-degree $j$ is a sequence satisfying both

$$
\begin{equation*}
h_{i}=h_{j-i} \text { for } 0 \leq i \leq j / 2 ; \tag{3}
\end{equation*}
$$

$(\Delta H)_{\leq j / 2}$ is a Macaulay sequence,
where $(\Delta H)_{i}=h_{i}-h_{i-1}\left(\right.$ take $\left.h_{-1}=0\right)$.
We have
Lemma 1.3. [BuEi, St1] A sequence $H=\left(1,3, \ldots, h_{i}, \ldots, 3,1\right)$ is a codimension three Gorenstein sequence if and only if $H$ is an SI sequence.

The proof follows from the D. Buchsbaum and D. Eisenbud Pfaffian structure theorem for codimension three Gorenstein algebras [BuEi]; see also [Di] and [IK, Theorem 5.25].

It is well known that in any codimension, the SI sequences are a subset of the Gorenstein sequences: when $r \geq 5$, the SI sequences are a proper subset of the Gorenstein sequences, which may be non-unimodal; when $r=4$ it is open whether the SI sequences might be all the Gorenstein sequences. A result of N . Altafi shows that given a finite SI sequence $H$, there is always a strong Lefschetz Artinian Gorenstein algebra of Hilbert function $H$ [Alt].

Definition 1.4. We say that a finite sequence $H=\left(h_{0}, h_{1}, \ldots, h_{i}, \ldots, h_{j}\right)$ is log-concave in a degree $i \in[1, j-1]$ if

$$
\begin{equation*}
h_{i-1} \cdot h_{i+1} \leq h_{i}^{2} . \tag{4}
\end{equation*}
$$

The sequence $H$ is log-concave if it is log-concave in each such degree $i$.
See the R. Stanley 1989 survey [St2], the F. Brenti 1994 [Bre], and many more recent articles. Log-concavity has a relation with the Hodge-Riemann property of certain algebras [ $\mathrm{Ba}, \mathrm{H}, \mathrm{MMS}, \mathrm{MuNaYa}$ ].

## 2 Codimension three Gorenstein sequences are log-concave.

Theorem 2.1. Let $A$ be a standard graded $A G$ algebra of socle degree $j$ with Hilbert function $H(A)=\left(h_{0}, h_{1}, \ldots, h_{j}\right)$ satisfying $h_{1}=3$. Then the sequence $H(A)$ is log-concave.
Proof. Note that it suffices to show that Equation (4) holds for $1 \leq i \leq\left\lfloor\frac{j}{2}\right\rfloor$, since for $i>\left\lfloor\frac{j}{2}\right\rfloor$, we have $1 \leq j-i \leq\left\lfloor\frac{j}{2}\right\rfloor$ and hence Equation (4) will hold for these $i$ by symmetry of the Hilbert function. For each $1 \leq i \leq\left\lfloor\frac{j}{2}\right\rfloor$, we have $h_{i} \leq\binom{ h_{1}+i-1}{i}=\binom{i+2}{i}$, and if we have equality for every $i$, then $H(A)$ is log-concave since the binomial coefficients are log-concave. Otherwise we may choose the smallest index $u, 1 \leq u \leq\left\lfloor\frac{j}{2}\right\rfloor$ such that $h_{u}<\binom{u+2}{u}$. Then of course Equation (4) holds for $1 \leq i \leq u-1$ for the preceding reason, and hence we need only check Equation (4) for $u \leq i \leq\left\lfloor\frac{j}{2}\right\rfloor$. Let $j^{\prime}==\left\lfloor\frac{j}{2}\right\rfloor$. The following two observations are key: by Lemma 1.3
(i). $H(A)$ is an SI sequence, and hence the first difference

$$
\begin{equation*}
\Delta H(A)_{\leq j^{\prime}}=\left(1,2, \ldots, \Delta H_{j^{\prime}}\right) \tag{5}
\end{equation*}
$$

is the Hilbert function for some standard graded Artinian algebra of codimension 2, and
(ii). The Hilbert function of a standard graded Artinian algebra of codimension 2 is non-increasing after the initial degree $d$ of its defining ideal (here $\left.d=\min \left\{i \mid \Delta H(A)_{i}<i+1\right\}\right)$ so $\Delta H(A)_{\leq j^{\prime}}$ in Equation (5) satisfies

$$
\begin{equation*}
\Delta H(A)_{j^{\prime}}=\left(1,2, \ldots, d, \Delta_{d}, \ldots, \Delta_{j^{\prime}}\right), \text { with } d \geq \Delta_{d} \geq \Delta_{d+1} \geq \cdots \geq \Delta_{j^{\prime}} \tag{6}
\end{equation*}
$$

The second observation is well known (see [Mac1], [I1, Lemma 1.3]) ${ }^{1}$ and can be seen as follows: Let $B=\mathbb{F}[x, y] / I$ be any standard graded Artinian algebra in codimension two, suppose that $I_{p} \neq 0$ and suppose that $f_{1}, \ldots, f_{m} \in I_{p}$ are linearly independent forms in $I$ of degree $p$. Then certainly $x f_{1}, \ldots, x f_{m}$ are linearly independent in $I$ of degree $p+1$; also, if $f_{1}$ has maximum $y$ power among the set of $f_{i}$, then $y f_{1}, x f_{1}, \ldots, x f_{m} \subset \mathbb{F}[x, y]_{p+1}$ are linearly independent, showing that $\operatorname{dim}_{\mathbb{F}}\left(I_{p}\right)<\operatorname{dim}_{\mathbb{F}}\left(I_{p+1}\right)$, and hence $G=H(B)=$ $\left(1,2, \ldots, g_{p}, \ldots\right)$ satisfies

$$
\begin{aligned}
& g_{p}=\operatorname{dim}_{\mathbb{F}}\left(\mathbb{F}[x, y]_{p}\right)-\operatorname{dim}_{\mathbb{F}}\left(I_{p}\right)=\left(p+1-\operatorname{dim}_{\mathbb{F}}\left(I_{p}\right)\right. \\
& \geq p+2-\operatorname{dim}_{\mathbb{F}}\left(I_{p+1}\right)=\operatorname{dim}_{\mathbb{F}}\left(\mathbb{F}[x, y]_{p+1}\right)-\operatorname{dim}_{\mathbb{F}}\left(I_{p+1}\right)=g_{p+1} .
\end{aligned}
$$

Finally, for any integer $i$ satisfying $d \leq i \leq\left\lfloor\frac{j}{2}\right\rfloor$, we must therefore have

$$
\begin{aligned}
h_{i}^{2} & \geq h_{i}^{2}-\left(h_{i}-h_{i-1}\right)^{2}=\left(h_{i}-\left(h_{i}-h_{i-1}\right)\right)\left(h_{i}+\left(h_{i}-h_{i-1}\right)\right) \\
& \geq\left(h_{i}-\left(h_{i}-h_{i-1}\right)\right)\left(h_{i}+\left(h_{i+1}-h_{i}\right)\right)=h_{i-1} h_{i+1},
\end{aligned}
$$

and hence $H(A)$ is log-concave.

## 3 Codimension four Gorenstein sequences that are not log-concave.

Many codimension four Gorenstein sequences, as $H=(1,4,10,14,10,4,1)$, are log-concave. We first show that there are codimension four SI sequences that are not log-concave (Proposition 3.2); then we show that there are codimension four SI sequences that are not log-concave for an arbitrarily large consecutive sequence of degrees (Proposition 3.4).

The codimension four Gorenstein sequences $H$ include the SI sequences, those satisfying $\Delta H_{\leq j / 2}=(1,3, \ldots)$ is the Hilbert function of a codimension

[^1]three graded Artin algebra $A=R / I$ (Definition 1.2). We first restrict to codimension four SI sequences satisfying
\[

$$
\begin{equation*}
\Delta H_{\leq j / 2}=\left(1,3, \ldots, r_{k}, b, c\right) \text { where } r_{k}=\binom{k+2}{2}=\operatorname{dim}_{\mathbb{F}} R_{k} . \tag{7}
\end{equation*}
$$

\]

We denote by $S=\mathbb{F}[x, y, z, w]$, and let $s_{k}=\operatorname{dim}_{\mathbb{F}} S_{k}=\left(1+3+\cdots+r_{k}\right)=\binom{k+3}{3}$. Then the sum function of $\Delta(H)_{\leq j / 2}$ above satisfies

$$
H_{\leq j / 2}=\left(1,4, \ldots, s_{k}, s_{k}+b, s_{k}+b+c\right) .
$$

The log-concavity condition (4) here in degree $k+1$ for $H$ is

$$
\begin{equation*}
s_{k}\left(s_{k}+b+c\right)<\left(s_{k}+b\right)^{2} \text { or, equivalently, } s_{k}(c-b)<b^{2} . \tag{8}
\end{equation*}
$$

Keeping $b, c$ constant with $c>b$ then this certainly is negated for $k$ large enough. The next idea is to choose $b$ suitably and let $s_{k}+b$ in degree $k+1$ to $s_{k}+b+c$ in degree $k+2$ have maximum Macaulay growth (see Lemma 1.1).

The dimension $\operatorname{dim}_{\mathbb{F}}\left(\mathbb{F}[x, y, z, w]_{k}\right)=s_{k}$. For codimension three, if $h_{a}<r_{a}$ we will denote by $\delta\left(h_{a}\right)=h_{a}^{(a)}-h_{a}$ : here, this is just the number of terms in the Macaulay expansion of Equation (1) with $n_{i}=i+1$. Then, taking $a=k+1, h_{a}=s_{k}+b, s_{k+2}=h_{a}+c$ with $c=h_{a}+\delta\left(h_{a}\right)$ (maximum Macaulay growth) the log-concavity condition Equation (8) becomes $\delta \cdot s_{k} \leq b^{2}$. Thus, to violate $\log$-concavity in degree $k+1$ for a Hilbert function sequence $H$ having as its key entries $h_{k}=s_{k}, h_{k+1}=s_{k}+b, h_{k+2}=s_{k}+b+\delta$ we need only assure

$$
\begin{equation*}
\delta \cdot s_{k}>b^{2} \tag{9}
\end{equation*}
$$

Remark 3.1. Recall that the Gotzmann regularity degree of the constant polynomial $\{s\}$ is itself $s$ (see [IK, Proposition C.32]). This implies for an SI sequence $H$ that once $(\Delta H)_{i} \leq s$ for an integer $i \in[s, j / 2]$ then $(\Delta H)_{\leq j / 2}$ is non-increasing in higher degrees than $i$. Also, in order for $\delta=h_{a}^{(a)}-h_{a} \geq 2$ we must have $h_{a} \geq 2 a+1$, with equality when $h_{a}=\binom{a+1}{a}+\binom{a}{a-1}$. For $\bar{\delta}=$ $h_{a}^{(a)}-h_{a} \geq 3$ we need $h_{a} \geq 3 a$, with equality when $h_{a}=\binom{a+1}{a}+\binom{a}{a-1}+\binom{a-1}{a-2}$. Evidently, for $\delta \geq 4$ we need

$$
\begin{align*}
b=h_{a} & \geq \delta \cdot a-(2+\cdots+(\delta-2)) \\
& =\delta \cdot a-\delta(\delta-3) / 2 . \tag{10}
\end{align*}
$$

These inequalities for $\delta \geq 2$ will greatly affect our search for small examples of SI sequences in codimension four that are not log-concave - that satisfy Equation (9).

We will denote by $H_{a \rightarrow b}$ the subsequence $\left(h_{a}, h_{a+1}, \ldots, h_{b}\right)$ of $H$.
Proposition 3.2 (SI sequences in 4 variables that are not log-concave). We give a series of minimal examples, depending on the choice of $\delta$.
Case $\delta=1$. First we consider $\delta=1$ and take $b=\binom{s+1}{s}$, in degree $s=k+1$. We need $s_{k}>b^{2}=(s+1)^{2}$. Taking $\delta=1, b=7_{6}, k=5$, so $s_{5}=56>7^{2}$ we have

$$
H=\left(1,4,10,20,35,56,63,71,63,56,35,20,10,4,1_{14}\right), \quad n=449
$$

then $h_{5} \cdot h_{7}=56 \cdot 71=3976>3969=63^{2}=h_{6}^{2}$.

$$
\text { Taking } \delta=1, b=8_{7}, k=6 \text { so } s_{6}=84>8^{2} \text { we have }
$$

$$
H=\left(1,4,10,20,35,56,84,92,101,92,84,56,35,20,10,4,1_{16}\right), \quad n=705,
$$

then $h_{6} \cdot h_{8}=84 \cdot 101=8484>8464=92^{2}=h_{7}^{2}$. Since $84>9^{2}$ we have a second example where $\delta=1, b=9, H_{8 \rightarrow 12}=(84,93,103,93,84)$ of socle degree 16 and length $n=709$, also not log-concave in degree 7 , as $h_{6} \cdot h_{8}=84 \cdot 103=$ $8652>8649=93^{2}=h_{7}^{2}$.

In general, taking $\delta=1, k \gg 5$, we may choose $b_{k+1}$ (that is, $b$ in degree $k+1$ ) satisfying $k+2 \leq b \leq(k / 6)^{3 / 2}$ (asymptotically, not for small b) that will satisfy the conditions of Remark 3.1 and also satisfy $s_{k}>b^{2}$, Equation 9, so we will obtain again a non log-concave $H$.

Case $\delta=2$. Now taking $\delta=2, b=23$ we have $2 s_{10}=2(286)=572>529=$ $23^{2}$ and 23 satisfies $23=2(11)+1$, the lower bound from Remark 3.1. This is the lowest pair $\delta=2, b=23$ with $a=11$ satisfying Equation (9), and leads to an example of non log-concave $H$ of socle degree 24, whose key entries are

$$
\begin{equation*}
H_{10 \rightarrow 14}=(286,309,334,309,286), \tag{11}
\end{equation*}
$$

satisfying $h_{10} \cdot h_{12}=286 \cdot 334=95524>95481=309^{2}=h_{11}^{2}$, so $H$ of length $n=2954$ that is non log-concave in degree 11.

Now taking $\delta=2, b=25$ we have $2 s_{11}=2(364)=728>625=25^{2}=\Delta h_{12}^{2}$ so we have new $H$ of socle degree 26 whose key entries are

$$
H_{11 \rightarrow 15}=(364,389,416,389,364),
$$

satisfying $h_{11} \cdot h_{13}=364 \cdot 416=151424>151321=389^{2}=h_{12}^{2}$, so $H$ of length $n=4034$ that is non log-concave in degree 12. Evidently, we may replace 25 by 26 as also $728>26^{2}=676$. Then the key entries of $H$ would be

$$
\text { for } b=26, H_{11 \rightarrow 15}=(364,390,418,390,364) \text {. }
$$

This sequence of examples with $\delta=2$ can evidently be continued, the next arises from $\delta=2, b=27,2 s_{12}=2(455)=910>729=(27)^{2}$, so gives an $S I$ sequence $H$ of socle degree 28 with key entries

$$
H_{12 \rightarrow 16}=(455,482,511,482,455)
$$

satisfying $h_{12} \cdot h_{14}=232505>232324=482^{2}=h_{13}^{2}$. Evidently, we may replace 27 by $b \in[27,30]$ as $910>30^{2}$.

The general $\delta=2$ case with fixed $k$ and lowest $b=2 k+3$ will be $s_{k}, k \geq 10$ satisfying $2 s_{k}>(2 k+3)^{2}$, leading to an SI sequence $H=\left(1,4, \ldots, 4,1_{2 k+4}\right)$ of socle degree $2 k+4$, with key entries

$$
H_{k \rightarrow(k+4)}=\left(s_{k}, s_{k}+2 k+3, s_{k}+4 k+8, s_{k}+2 k+3, s_{k}\right)
$$

of length $n=2\binom{k+4}{4}+3 s_{k}+8 k+14$, that is not log-concave in degree $k+1$.
Given $k$, the maximum $b$ satisfying Equation (9) is $\sqrt{2 s_{k}}$; for $k=25$ this is $b=\left\lfloor\sqrt{2 s_{25}}\right\rfloor=\lfloor\sqrt{6552}\rfloor=80$. When $(k, b)$ is a fixed pair, satisfying $2 k+3 \leq b \leq \sqrt{2 s_{k}}$ the key entries in the corresponding non log-concave $H$ of socle degree $2 k+4$ are

$$
H_{k \rightarrow(k+4)}=\left(s_{k}, s_{k}+b, s_{k}+2 b+2, s_{k}+b, s_{k}\right)
$$

and $H$ has length $2\binom{k+4}{4}+3 s_{k}+4 b+2$.
Case $\delta=3$. We have $3 \cdot s_{18}=3 \cdot\binom{20}{3}=3420>3249=9(19)^{2}$, so taking $(\delta, k, b)=(3,18,3 \cdot 19)$ we generate a smallest $H$ for $\delta=3$, here of socle degree $j=40$ that is not log-concave in degree 19, that has key $H$ values

$$
H_{18 \rightarrow 22}=(1140,1197,1257,1197,1140)
$$

Case $\delta \geq 4$. To satisfy Remark 3.1 and Equation (9), we must have $\delta \cdot s_{k} \geq b^{2}$, or $\delta \cdot\binom{k+\overline{3}}{k}>b^{2}$, where also $b \geq \delta k$. Solving for $b_{\max }$, and approximating $\binom{k+3}{k}$ by $k^{3} / 6$, we have $\delta k^{3} / 6 \leq b_{\max }^{2}$, so if $b$ is in the interval

$$
\begin{equation*}
b \in\left[\delta \cdot k, \delta^{1 / 2} \cdot k^{3 / 2} / 6^{3 / 2}\right] \tag{12}
\end{equation*}
$$

then the triple $(\delta, k, b)$ will produce a codimension four SI sequence $H$ with key entries

$$
H_{k \rightarrow k+4}=\left(s_{k}, s_{k}+\delta \cdot b, s_{k}+2 \delta \cdot b, s_{k}+\delta \cdot b, s_{k}\right)
$$

and socle degree $2 k+4$ that is non log-concave in degree $k+1$.

Example 3.3 (Lengthening the non log-concave examples $H$ ). The lowest $\delta=2$ example is Equation (11) of socle degree $j=24$, where $H_{10 \rightarrow 14}=$ $(286,309,334,309,286)$ and $\Delta H_{10 \rightarrow 12}=\left(r_{10}, r_{10}+23, r_{10}+25\right)=(66,89,91)$. We may lengthen this by adding a further sequence after $\Delta H_{12}=114$ that satisfies the Macaulay inequalities Equation (2) for codimension three. For example we may adjoin to $\Delta H$ the sequence $\left(16_{13}, 17_{14}, 8_{15}, 7_{16}\right)$, leading to a key sequence

$$
H_{10 \rightarrow 22}^{\prime}=(286,309,334,350,367,375,382,375,367,350,334,309,286),
$$

of an AG Hilbert function $H^{\prime}$ of socle degree $j=32$ that is still non log-concave in degree 11 .

In the next Proposition, we extend the maximum growth portion of $\Delta H$, by a distance $\ell$, obtaining in some cases, especially when $s_{b} \gg b^{2}$ a new AG sequence $H^{\prime}(\ell)$ having a longer consecutive subset of non log-concave adjacent triples, whose length we can specify. Further, we specify $h_{k+u} h_{k+u+2}-h_{k+u+1}^{2}$ for each $u \in[0, \ell]$, showing that it is a sequence with linear first differences.

Proposition 3.4 (Sequences $H$ having multiple non log-concave places). Assume that $(\delta, k, b)$ is a triple as in Proposition 3.2 for which $\delta \cdot s_{k}>b^{2}$, and let $H$ be the related codimension four Hilbert function of socle degree $j=2 k+4$, whose key part is

$$
H_{k \rightarrow k+4}=\left(s_{k}, s_{k}+b, s_{k}+2 b+\delta, s_{k}+b, s_{k}\right)
$$

and which is non log-concave in degree $k+1$. Let $\ell$ be a positive integer. We define an extended sequence $H^{\prime}(\ell)$ identical to $H$ in degrees $i \in[0, k+2]$, and satisfying

$$
\begin{align*}
\Delta H^{\prime}(\ell)_{t} & =\Delta T_{t} \text { for } t \leq k+1 \text { and for } t>k+\ell+2 ; \\
\Delta H^{\prime}(\ell)_{k+1 \rightarrow k+\ell+2} & =(b, b+\delta, b+2 \delta, \ldots, b+(\ell-1) \delta, b+(\ell) \delta, b+(\ell+1) \delta) . \tag{13}
\end{align*}
$$

Then $H^{\prime}(\ell)$ has socle degree $2 k+4+2 \ell$, and we have for each $u \in[1, \ell+2]$ that $h_{k+u}^{\prime}=s_{k}+b u+\delta(1+2+\cdots+(u-1))$. Letting $\delta=1$ we have for each such $u$

$$
\begin{equation*}
h_{k+u}^{\prime} \cdot h_{k+u+2}^{\prime}-\left(h_{k+u+1}^{\prime}\right)^{2}=s_{k}-b^{2}-((b+1)+\cdots+(b+u)) . \tag{14}
\end{equation*}
$$

Assume now that

$$
\begin{equation*}
b \ell+\frac{\ell(\ell+1)}{2}<s_{k}-b^{2} . \tag{15}
\end{equation*}
$$

Then $H^{\prime}(\ell)$ is non log-concave in the positions $k+1, k+2, \ldots, k+\ell, k+\ell+1$.

Proof. We generalize the proof of Equations (8) and (9). Beginning with ( $\delta=1, b, k$ ) then using the definition of $\Delta H^{\prime}$ in Equation (13) and recalling that $H_{k}=s_{k}$ we conclude that $H^{\prime}=\left(1, h_{1}^{\prime}, \ldots\right)$ satisfies

$$
\text { for } u \in[0, \ell], h_{k+u}^{\prime}=s_{k}+b u+\sum_{\nu=1}^{u-1} \nu .
$$

Then, similarly to Equation (8) we have for $u \in[1, \ell+1]$ and $i=k+u$

$$
\begin{align*}
{h_{i-1}^{\prime}}^{h_{i+1}^{\prime}-h_{i}^{\prime 2}} & \left.=\left(h_{i}^{\prime}-(b+u-1)\right)\right)\left(h_{i}^{\prime}+b+u\right)-h_{i}^{\prime 2} \\
& =h_{i}^{\prime}-(b+u-1)(b+u) \\
& =s_{k}-b^{2}-b(u-1)-\sum_{\nu=1}^{u-1} \nu \\
& =s_{k}-b^{2}-b(u-1)-\frac{u^{2}-u}{2} . \tag{16}
\end{align*}
$$

The last statement of the Proposition concerning non log-concavity for $H^{\prime}(\ell)$ for $\ell$ satisfying Equation (15) follows.

Example 3.5. We take from Proposition 3.2 the first sequence $H$ where $\delta=$ $1, k=8, b=10_{9}$ with key entries $H_{8 \rightarrow 12}=(1658,175,186,175,165)$ and recall that $h_{8} h_{10}-h_{9}^{2}=30690-30625=65=s_{8}-b^{2}=165-10^{2}$. We choose $\ell$ as in Equation (14), the maximum such that for $b=10$

$$
(b+1)+(b+2)+\cdots+(b+\ell)<65, \text { from which we have } \ell=4 .
$$

We now consider $\Delta H^{\prime}(4)_{\leq j / 2}=\left(1,3,6,10,15,21,28,36,45,10_{9}, 11,12,13,14,15\right)$, so

$$
\begin{align*}
H^{\prime}(4)= & \left(1,4,10,20,35,56,84,120,165,1759,186,198,211,225,240_{14},\right. \\
& \left.225,211,198,186,175,165,120,84,56,35,20,10,4,1_{28}\right) . \tag{17}
\end{align*}
$$

of socle degree 28 , which is non log-concave in degrees $9,10,11,12,13$ with successive differences (writing $h_{i}$ for $t^{\prime}{ }_{i}$ )

$$
h_{i-1} h_{i+1}-h_{i}^{2}=\left(65_{9}, 54_{10}, 42_{11}, 29_{12}, 15_{13}\right),
$$

whose second differences are $(11,12,13,14)$, consistent with Equations (13) and (14). Note that $H^{\prime}(5)_{13,14,15}=\left(225,240_{14}, 256\right)$ whence $h_{i-1} h_{i+1}-h_{i}^{2}=0$ for $i=16$, so $H^{\prime}(5)$ begins log-concavity in degree 16 ; this is again consistent with Equation (14) and shows that the maximum length of a non log-concave sequence for $H^{\prime}$ arising from this $H$ is five.

Note, that combining with the idea of Example 3.3, we can make codimension four Gorenstein sequences $H^{\prime}$ with certain specified consecutive subsequences of degrees where $H^{\prime}$ is non log-concave, but that may have gaps between them of degrees where $H^{\prime}$ is log-concave.
Remark 3.6. According to [H, Theorem 3] these non log-concave $h$-vectors cannot be the $h$-vectors of a matroid representable in characteristic zero. Chris McDaniel poses the question of whether a related $f$-vector: $f(t)=h(1+t)$ of one of these non log-concave $h$ might be non-unimodal.

## 4 Higher codimension Gorenstein sequences, level sequences.

In codimension at least five, Gorenstein sequences need not be unimodal. For example, R. Stanley used an idealization construction [Na] or [IMM, §3.1] to show $H=(1,13,12,13,1)$ is a Gorenstein sequence for an algebra $A \times$ $\operatorname{Hom}(A, \mathbb{F})$ where $A=\mathbb{F}[x, y, z] /(x, y, z)^{4}$ of Hilbert function $(1,3,6,10)$ [St1], Analogous examples, sometimes requiring higher socle degree $j$, can be made in codimensions at least five: in codimension five the lowest socle-degree nonunimodal Gorenstein sequence known has $j=16$ and contains the subsequence $\left(h_{7}, h_{8}, h_{9}\right)=(91,90,91)[\mathrm{BeI}]$ (see also [MZ, MNZ1] for further discussion of unimodality for Gorenstein sequences).

### 4.1 Higher codimension SI sequences.

SI sequences by construction are unimodal. We note that, as in codimension four, some SI sequences in codimension five or larger are log-concave, and some are not. We illustrate both with the next example.
Example 4.1 (Codimenson five). (i). Consider the codimension five SI sequence $H(j)=\left(1,5, t_{2}, \ldots\right)$ having socle degree $j$ with

$$
\Delta H(j)=(1,4,10,14,20,27,35,44, \ldots)
$$

where $\Delta H(j)_{k}=\binom{k+2}{k}+\binom{k}{k-1}+\binom{k-2}{k-2}$ for $3 \leq k \leq j / 2$ and $j \geq 6$. Then $H(j)_{\leq j / 2}=(1,5,15,29,49,76,111,155, \ldots$,$) where$

$$
H(j)_{k}=-1+\binom{k+3}{k}+\binom{k+1}{k-1}+\binom{k-1}{k-2} \text { for } 3 \leq k \leq j / 2
$$

and $H(j)_{k}=H(j)_{j-k}$ for $k \geq j / 2$. It is easy to verify that for $j \geq 6$ each such $H(j)$ is log-concave. The reason is that the sequence $H(j)_{k}$ for $k \leq j / 2$ is dominated by its fast-growing term $\binom{k+3}{k}$ which is cubic in $k$.
(ii). This suggests that the way to create non log-concave examples of SI sequences is to imitate the construction in codimension four, that is, set $\Delta H^{\prime}=\left(1,4, r_{2}, \ldots, r_{u-1}, \delta_{u}, \ldots\right)$ where the non-zero terms of the Macaulay expansion of $\delta_{u}$ have the form $\binom{w+1}{w}$ - that is, we assume that after a certain degree $u$ the first difference $\Delta H^{\prime}$ has maximal Macaulay growth, and is linear in $k$. Consider then $\Delta H^{\prime}=(1,4,10,20,5,6,7, \ldots)$ so that $H^{\prime}=\left(1,5,15,35,40,46,53,61,70, \ldots\right.$. Here $35 \cdot 46=1610>40^{2}$, so taking $j=10$ and $H^{\prime}=\left(1,5,15,35,40,46,40,35,15,5,1_{10}\right)$ gives a non-log-concave sequence. Likewise, $40 \cdot 53>46^{2}$, so taking $j=12$ the SI sequence $H^{\prime}=\left(1,5,15,35,40,46,53,46,40,35,15,5,1_{12}\right)$ is non-logconcave in two adjacent places. But it stops there, as $46 \cdot 61=2806<$ $2809=53^{2}$.

To get a longer non-log-concave consecutive subsequence, it suffices to begin the minimal linear growth later. For example, taking $\Delta H^{\prime}=$ $(1,4,10,20,35,6,7,8, \ldots)$ so

$$
\begin{equation*}
H_{\leq j / 2}^{\prime}=(1,5,15,35,70,76,83,91,100,110,121 \ldots), \tag{18}
\end{equation*}
$$

we have $h_{k-1}^{\prime} \cdot h_{k+1}^{\prime}>{h_{k}^{\prime}}^{2}$ for $k=4,5,6,7,8$ with equality for $k=9$, so taking

$$
H^{\prime}=\left(1,5,15,35,70,76,83,91,100,110,100,91,83,76,70,35,15,5,1_{18}\right)
$$

gives 5 adjacent non-log-concave entries centered in degrees 4 to 8. Taking $j=20$ will give equality in the log-concavity equation in degree 9 .

### 4.2 Log-concavity of level sequences.

Recall that the socle of an Artinian algebra $A$ is $\left(0: \mathfrak{m}_{A}\right)$ where $\mathfrak{m}_{A}$ is its maximal ideal. A level sequence is a Hilbert function possible for an Artinian algebra $A=R / I, R=\mathbb{F}\left[x_{1}, \ldots, x_{r}\right]$ having socle all in the same degree, that is, $H(A)=\left(1, r, \ldots, t_{j}, 0\right)$ and the socle of $A=\left(0: \mathfrak{m}_{A}\right)=A_{j}$ : we say $A$ has type $t_{j}$. The Hilbert functions of level algebras are well known in codimension 2, essentially due to F.H.S. Macaulay [Mac1] but they are not known for type $t$ in higher codimension except for the Gorenstein codimension three case $(r, t)=(3,1)$ (for some partial results see [ChoI, Za1]). Given a Hilbert function sequence $H=\left(\ldots, h_{i}, \ldots\right)$, we denote by $e_{i}=h_{i-1}-h_{i}=-\Delta(H)_{i}$.

We have [I3, Proposition 2.6, Lemma 2.15]

Lemma 4.2. The Hilbert function $H=H(A)$ of a level quotient $A=R / I$ of $R=\mathbb{F}[x, y]$ satisfies

$$
\begin{equation*}
H=\left(1,2, \ldots, d, t_{d}, t_{d+1}, \ldots, t_{j}, 0\right) \text { where } t_{j}=e_{j+1} \geq e_{j} \geq e_{j-1} \geq \cdots d-t_{d} \tag{19}
\end{equation*}
$$

A compressed level algebra of codimension $r$, socle degree $j$ and type $t$ is one having Hilbert function

$$
\begin{equation*}
H(A)_{i}=\min \left\{r_{i}, t r_{j-i}\right\}, \tag{20}
\end{equation*}
$$

where $r_{i}=\operatorname{dim}_{\mathbb{F}} R_{i}$ and $R=\mathbb{F}\left[x_{1}, \ldots, x_{r}\right]$ (see [I2] and [Bo, Definition 1.3]). For example, with $(r, t, j)=(3,2,5)$ the compressed level Hilbert function $H=(1,3,6,10,6,2)$.

Proposition 4.3. (i). Every codimension two level sequence is log-concave. (ii). Every compressed level Hilbert function in any codimension is log-concave.

Proof. The proof of (i) is essentially a translation of the proof of Theorem 2.1, with Equation (19) replacing Equation (6), there. The proof of (ii) follows from the $\log$-concavity of the sequence $\left(1, r_{1}, r_{2}, \ldots, r_{i} \ldots\right)$ where $r_{i}=\operatorname{dim}_{\mathrm{k}} R_{i}=$ $\binom{r+i-1}{r}$ - as, letting $e=e(r, t, j)$ be the highest degree such that $H(A)_{i}=r_{i}$ in Equation (20), for $i<e$ we have $h_{i}^{2}=r_{i}^{2} \geq r_{i-1} \cdot r_{i+1}=h_{i-1} \cdot h_{i+1}$; and $h_{e}^{2}=r_{e}^{2} \geq r_{e-1} r_{e+1} \geq h_{e-1} h_{e+1}$; and for $i>e+1$ we have $h_{i}=t r_{j-i}$, so likewise $h_{i}^{2} \geq h_{i-1} h_{i}$; for $i=e+1$ we have, since $t r_{j-e} \geq h_{e}$,

$$
h_{i}^{2}=t^{2} r_{j-(e+1)}^{2} \geq t r_{j-e} \cdot t r_{j-(e+2)} \geq h_{e} \cdot h_{i+1}
$$

However, there are codimension three level sequences that are not unimodal - see [We, AhS], the former giving a non-unimodal example with $r=3, t=5$.

### 4.2.1 Problems

A pure O-sequence is a Hilbert function $H=H(A)$ that can occur for a monomial level algebra $A=R / I$ of socle degree $j$ : that is, $I_{j}$ has a basis of monomials. In codimension three, a type two monomial Artinian algebra is known to be weak Lefschetz (there is a linear form $\ell$ such that $m_{\ell}: A_{i} \rightarrow A_{i+1}$ has maximum rank for each $i$ ), provided $\mathbb{F}$ has characteristic zero [BMMNZ, Theorem 6.2]. This implies that $H(A)$ is differentiable (the first difference is an O-sequence) until its maximum value, which may be repeated, then $H(A)$ is decreasing [MNZ2, Theorem 12]). Although their Hilbert functions
(codimension three, type two) are characterized in [BMMNZ, Proposition 6.1], it remains to see if these sequences are log-concave.

There is a subtle connection of pure O-sequences with algebras associated to matroids (see [CM, C-T, H, MuNaYa]).

Recall that a sequence $H=\left(1,2, \ldots, d-1, d, h_{d}, h_{d+1}, \cdots, h_{j}\right)$ is admissible of decreasing type if $\left.d \geq h_{d} \geq \cdots\right)$ and there is $b \geq d$ such that

$$
\begin{equation*}
d=h_{d-1}=\cdots=h_{b-1}>h_{b}>h_{b+1}>\cdots>h_{j} . \tag{21}
\end{equation*}
$$

Recall that an h-vector of a Gorenstein domain $D$ of dimension $s$ is the Hilbert function $H(A), A=D /\left(\ell_{1}, \ldots, \ell_{s}\right)$ where $\ell_{1}, \ldots, \ell_{s}$ is a regular sequence of linear forms. The h -vectors of Gorenstein domains are a proper subclass of Artinian Gorenstein sequences. When $H=\left(1,2, \ldots, d, h_{d}, h_{d+1}, \ldots, h_{j}\right)$ the condition for $H$ to be an $h$-vector is for $H$ to be admissible of decreasing type [GP]. The AG sequence $H=(1,3, \ldots)$ as in Lemma 1.3 is an h -vector of a Gorenstein domain if and only if the difference sequence of Equation (6) is admissible of decreasing type [DV], [Va, Theorem 2.18]. A characterization of the Hilbert function of Gorenstein domains in higher codimension $h_{1} \geq 4$ is not known. Also, it is apparently still open whether such h -vectors $\left(1, h_{1}, \ldots\right)$ of domains with $h_{1} \geq 4$ would be log-concave [Bre, Conjecture 5.2].

While the deformation properties of Artinian algebras have been widely studied (see, for example, [EmI, Di, BoI, CJN, AEI]) it appears open to check whether there are special properties, pertaining, say to smoothability - deformations to a smooth scheme - depending on the log-concavity of the Hilbert functions $H$. What can we say about the irreducible components of the variety $\mathrm{G}_{T}$ parametrizing graded Artinian algebras of Hilbert function $H$ when $r \geq 4$ and $H$ is log-concave?

Subsequent to our work, and partly inspired by it, F. Zanello has further studied which level sequences and pure O-sequences are log-concave, resolving the problem in some cases and proposing additional problems $[\mathrm{Za} 2]$.

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[^1]:    ${ }^{1}$ The article [Mac1] also determines all the Hilbert functions that do occur for local algebras of codimension two, as well as for the graded algebras.

